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On Prime Z-graded Lie algebras of growth one

Consuelo Martínez*

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Abstract. We will give the structure of Z-graded prime nondegenerate algebras $L = \sum_{i \in Z} L_i$ containing the Virasoro algebra and having the dimensions of the homogeneous components, $\dim L_i$, uniformely bounded.
Mathematical Subject Index: Primary 17B60, secondary 17B70, 17C50. *Key Words and Phrases:* Z-graded Lie algebra, strongly PI, prime, nondegenerate, Virasoro algebra, loop algebra, growth, Jordan pair.

1. Introduction

Throughout the paper we consider algebras over an algebraically closed field F of zero characteristic.

By a Z-graded algebra we mean an algebra $L = \sum_{i \in Z} L_i$, $L_i L_j \subseteq L_{i+j}$, having all homogeneous components L_i finite dimensional. In [Ma1], [Ma2] (see also the earlier work [K1]) O. Mathieu classified all graded simple Lie algebras with polynomial growth of dimensions $\dim L_i$. He proved that every such algebra is a (twisted) loop algebra or an algebra of Cartan type or the Virasoro algebra Vir.

The problem of classification of Z-graded Lie superalgebras with all $\dim L_i$ uniformly bounded is still open. Of particular interest is the case when the even part of L contains Vir, that is, when L is a superconformal algebra (see [KvL]). In this paper we modify O. Mathieu's result [Ma1] to make it applicable to the study of the even part of a superconformal algebra (see [MZ1], [KMZ]).

Recall that an algebra L is called prime if for any two nonzero ideals $(0) \neq I, J \lhd L$ we have $IJ \neq (0)$. A Lie algebra L is nondegenerate if $a \in L$, $[[L,a],a] = (0)$ implies $a = 0$. Following [Z2] we say that L is a Lie algebra with finite grading if $L = \sum_{i \in Z} L_{(i)}$, $[L_{(i)}, L_{(j)}] \subseteq L_{(i+j)}$, the subspaces $L_{(i)}$ can be infinite dimensional, but $\{i | L_{(i)} \neq (0)\}$ is finite. The grading is not trivial if $\sum_{i \neq 0} L_{(i)} \neq (0)$. All Jordan algebras and their generalizations can be interpreted as Lie algebras with finite gradings (see [Z2]).

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Let $L=\sum_{i\in Z}L_i$ be a graded Lie algebra, all dimensions $\dim L_i$ are uniformly bounded and L_0 is not solvable. Then L_0 contains a copy of $sl_2(F)=Fe+Fh+Ff$, $[e,f]=h$, $[h,e]=2e$, $[h,f]=-2f$. The adjoint operator $ad(h):L\to L$ has only finitely many eigenvalues and the decomposition of L into a direct sum of eigenspaces is a finite grading on L, which is compatible with the initial Z-grading.

For a finite dimensional simple algebra $\mathcal{G}$ let $\mathcal{L}(\mathcal{G})=\mathcal{G}\otimes F[t^{-1},t]$ be its loop algebra. Every finite grading on $\mathcal{G}$ extends to a finite grading on $\mathcal{L}(\mathcal{G})$ which is compatible with the Z-grading. If $\mathcal{G}$ is graded by a finite cyclic group Z/lZ, $\mathcal{G}=\mathcal{G}_0+\cdots+\mathcal{G}_{l-1}$, then we will refer to $\sum_{i=j\,mod\,l}\mathcal{G}_i\otimes t^j$ as a twisted loop algebra.

The Virasoro algebra naturally acts on $\mathcal{L}(\mathcal{G})$ and the semidirect sum $L=\mathcal{L}(\mathcal{G})\rtimes_V ir$ is a prime nondegenerate Z-graded algebra.

Theorem 1. *Let $L=\sum_{i\in Z}L_i$ be a Z-graded prime nondegenerate algebra containing the Virasoro algebra, the dimensions* $\dim L_i$ *are uniformely bounded. Suppose that L has a nontrivial finite grading which is compatible with the Z-grading above. Then $L\simeq\mathcal{L}(\mathcal{G})\rtimes Vir$ for some finite dimensional simple Lie algebra $\mathcal{G}$.*

We prove also the following theorem on Jordan pairs (see [L]) which generalizes [MZ1] and determines the structure of Z-graded prime nondegenerated Jordan pairs having the dimensions of the homogeneous components uniformly bounded.

Theorem 2. *Let $V=(V^-,V^+)=\sum_{i\in Z}V_i$ be a prime nondegenerate Z-graded Jordan pair having all $dimV_i$ uniformly bounded. Then either V is isomorphic to a (twisted) loop pair $\mathcal{L}(W)$, where W is a finite dimensional simple Jordan pair or V is embeddable in $\mathcal{L}(W)$ and $\sum_{i\geq k}\mathcal{L}(W)_i\subseteq V\subseteq\mathcal{L}(W)$ or $\sum_{i\geq k}\mathcal{L}(W)_{-i}\subseteq V\subseteq\mathcal{L}(W)$.*

2. The strongly PI case

Let $f(x_1,\ldots,x_n)$ be a nonzero element of the free associative algebra. We say that an associative algebra A satisfies the polynomial identity $f(x_1,\ldots,x_n)=0$ if $f(a_1,\ldots,a_n)=0$ for arbitrary elements $a_1,\ldots,a_n\in A$. An algebra satisfying some polynomial identity is said to be a PI-algebra.

For an arbitrary algebra A the multiplication algebra $M(A)$ of A is the subalgebra of $\mathrm{End}_F(A)$ generated by all right and left multiplications $R(a):x\to xa$, $L(a):x\to ax$, $a\in A$.

An algebra A is *strongly PI* if its multiplications algebra $M(A)$ is PI.

An element a in a Lie algebra L over a field F is said to have rank 1 if $[[L,a],a]\subseteq Fa$.

Lemma 2.1. *([Z1]) There exists a function $R(n)$ such that an arbitrary Lie algebra generated by n-elements of rank 1 has dimension $\leq R(n)$.*

An ideal of the free Lie (resp. associative) algebra is said to be a T-ideal if it is invariant under all substitutions. For an arbitrary algebra L the ideal of all identities satisfied by L is a T-ideal.

Lemma 2.2. *Let L be a Lie algebra over a field F, $chF = 0$ and $a \in L$ an element of rank 1. Let's consider s elements $a_i = aad(x_{i1}) \cdots ad(x_{ir_i})$, $1 \le i \le s$, $x_{ij} \in L$. Let $m = 2^{r_1} + 2^{r_2} + \cdots + 2^{r_s}$ and let T be the T-ideal of all identities that are satisfied by all Lie algebras of dimension $\le R(m)$. Then the subalgebra $< a_1, \ldots, a_s >$ satisfies all identities of T*

Proof. Let's consider the Lie algebra $\tilde{L} = L((t^{-1}, t))$ of Laurent series over L. Clearly, $\tilde{L}$ is an algebra over the field of Laurent series $F((t^{-1}, t))$. The element a is an element of rank 1 in $\tilde{L}$, $[[\tilde{L}, a], a] \subseteq F((t^{-1}, t))a$.

For a series $b = \sum_i b_i t^i$, $b_i \in L$, let's denote $\min(b) = b_k$ if $b_k \neq 0$ and $b_i = 0$ for every $i < k$.

For arbitrary elements x_{ij}, $1 \le i \le s$, $1 \le j \le r_i$, we have $e^{2ad(x_{ij}t)} - e^{ad(x_{ij}t)} = ad(x_{ij})t + (\cdots)t^2$.

Therefore,
$aad(x_{i1}) \cdots ad(x_{ir_i}) = \min(a(e^{2ad(x_{i1}t)} - e^{ad(x_{i1}t)})) \cdots (e^{2ad(x_{ir_i}t)} - e^{ad(x_{ir_i}t)})$. (*)

Since $e^{ad(x_{ij}t)}$, $e^{2ad(x_{ij}t)}$ are automorphisms of $\tilde{L}$ it follows that the elements $ae^{k_1 ad(x_{i1}t)} \cdots e^{k_{r_i} ad(x_{ir_i}t)}$, $1 \le k_1, \ldots \le k_{r_i} \le 2$, are elements of rank 1 in $\tilde{L}$.

Let's denote as B the subalgebra of $\tilde{L}$ generated by m elements: $ae^{k_1 ad(x_{i1}t)} \cdots e^{k_{r_i} ad(x_{ir_i}t)}$, where $k_1, \ldots k_{r_i} \in \{1, 2\}$, $1 \le i \le s$. We have $\dim_{F((t^{-1},t))} B \le R(m)$.

Taking (*) into account, an arbitrary commutator σ in $a_1, \ldots, a_s$ is either 0 or $\min(b)$ where $b \in B$.

Let $f(x_1, \ldots, x_k) \in T$. Without loss of generality we will assume that f is multilineal. Let us consider k arbitrary commutators $\sigma_1, \ldots, \sigma_k$ in $a_1, \ldots, a_s$. If $\sigma_i = 0$ for some i, then $f(\sigma_1, \ldots, \sigma_k) = 0$. In the other case, there exist elements $b_1, \ldots, b_k \in B$ such that $\sigma_i = \min(b_i)$, $1 \le i \le s$. Hence, $f(\sigma_1, \ldots, \sigma_k) = 0$ or $f(\sigma_1, \ldots, \sigma_k) = \min f(b_1, \ldots, b_k)$. But $f(b_1, \ldots, b_k) = 0$ and so Lemma is proved.

Recall that a centroid of an algebra A is the centralizer of the multiplication algebra $M(A)$ in $\mathrm{End}_F(A)$

Lemma 2.3. *Let $A = \sum_{i \in Z} A_i$ be a graded algebra whose centroid $\Gamma = \sum_{i \in Z} \Gamma_i$ contains a homogeneous invertible element $\gamma \in \Gamma_i$ of degree $i \neq 0$. Then $A \simeq \mathcal{L}(\mathcal{G})$ is a (twisted) loop algebra.*

Proof. Let $\gamma_i \in \Gamma_i$ with $\gamma_i^{-1} = \gamma_{-i} \in \Gamma_{-i}$ and let $a_j^1, \ldots, a_j^d \in A_j$ be linearly independent elements. Then

$$\gamma_i a_j^1, \ldots, \gamma_i a_j^d \in A_{i+j}$$

are also linearly independent. Hence $\dim A_j = \dim A_{i+j} = \dim A_{-i+j}$, for arbitrary $j \in Z$.

Taking i the smallest index such that there exists an invertible γ_i, we can define a finite dimensional algebra structure in $\mathcal{G} = A_0 + A_1 + \cdots + A_{i-1}$ by the new law:

$$a_l \star b_h = \begin{cases} a_l b_h & \text{if } l+h<i \\ \gamma_i^{-1}(a_l b_h) & \text{if } l+h \geq i \end{cases}$$

It is clear that A is isomorphic to $\sum_{i=j \bmod l} \mathcal{G}_i \otimes t^j$. Lemma is proved

Lemma 2.4. *Let Λ be a subset of Z closed under addition and let $m = gcd(\Lambda)$. Then either $\Lambda = mZ$ or $m\{i \in Z, i \geq k\} \subseteq \Lambda \subseteq mZ_{\geq 0}$ or $-m\{i \in Z, i \geq k\} \subseteq \Lambda \subseteq mZ_{\leq 0}$ for some $k \geq 1$.*

Proof. Suppose at first that Λ contains both a positive element $i \geq 1$ and a negative element $-j$, $j \geq 1$. Then Λ contains the additive subgroup ijZ.

The quotient $\Lambda/ijZ \subseteq Z/ijZ$ is a sub-semigroup of a finite group, hence Λ/ijZ is a group. Hence Λ is a subgroup of Z and therefore $\Lambda = mZ$.

Now suppose that $\Lambda \subseteq Z_{\geq 0}$. Then, clearly $\Lambda \subseteq mZ_{\geq 0}$. Choose $k \geq 1$ such that $km \in \Lambda$. There exist elements $\lambda_1, \ldots, \lambda_r \in \Lambda$ and integers $k_1, \ldots, k_r$ in Z such that $k_1\lambda_1 + \cdots + k_r\lambda_r = m$.

Choose a sufficiently large integer q such that $q + ik_j \geq 0$ for all $j = 1, \ldots, r$ and for all i, $0 \leq i \leq k-1$. The element $\lambda = q(\sum_{i=1}^r \lambda_i)$ is in Λ. We claim that $\lambda + mZ_{\geq 0} \subseteq \Lambda$.

Indeed, for $0 \leq i \leq k-1$ we have $\lambda + mi \in \sum_{i=1}^r Z_{\geq 0}\lambda_i \subseteq \Lambda$.

Now it is easy to see that for an arbitrary element $\lambda' \in \Lambda$, if $\lambda', \lambda' + m, \ldots, \lambda' + (k-1)m \in \Lambda$ then $\lambda' + km \in \Lambda$ as well and therefore the element $\lambda'' = \lambda + m$ has the same property as λ'. Hence $\lambda' + mZ_{\geq 0} \subseteq \Lambda$. Lemma is proved.

Lemma 2.5. *Let $\Gamma = \sum \Gamma_i$ be a Z-graded (commutative and associative) domain over an algebraically closed field F such that the dimensions $\dim_F \Gamma_i$ are uniformly bounded. Then, either $\Gamma \simeq F[t^{-m}, t^m]$ or $\sum_{i \geq k} Ft^{mi} \subseteq \Gamma \subseteq F[t^m]$ or $\sum_{i \geq k} Ft^{-mi} \subseteq \Gamma \subseteq F[t^{-m}]$, where $m \geq 1$, $k \geq 1$.*

Proof. Let us prove first that $\dim_F \Gamma_i \leq 1$ for every i. Let $d = \max\{\dim \Gamma_i | i \in Z\}$. Choose two arbitrary nonzero elements, $a_i, b_i \in \Gamma_i$.

Since $\dim_F \Gamma_{id} \leq d$, there exists a nontrivial linear dependence relation

$$\gamma_d a_i^d + \gamma_{d-1} a_i^{d-1} b_i + \cdots + \gamma_0 b_i^d = 0.$$

The polynomial $f(x) = \gamma_d x^d + \gamma_{d-1} x^{d-1} + \cdots + \gamma_0$ can be decomposed as $f(x) = \gamma_d (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_d)$, with $\gamma_d \neq 0$, $\alpha_1, \alpha_2, \ldots, \alpha_d \in F$.

We have $0 = f(\frac{a_i}{b_i}) = \gamma_d(\frac{a_i}{b_i} - \alpha_1)(\frac{a_i}{b_i} - \alpha_2) \cdots$

Hence $a_i = \alpha_k b_i$ for some k. Now $\Lambda = \{i \in Z \,|\, \Gamma_i \neq (0)\}$ is a subsemigroup of Z and the result is a consequence of Lemma 2.4.

Let $L = \sum_{i \in Z} L_i$ be a *strongly PI* Z-graded prime nondegenerate Lie algebra. Let $d = \max_{i \in Z} \dim L_i$. Let Γ denote the centroid of L, Γ_h is the set of homogeneous elements from Γ.

Lemma 2.6. (1) $\Gamma \neq (0)$ *is an integral domain and the ring of fractions* $(\Gamma \backslash \{0\})^{-1}L$ *is a simple finite dimensional Lie algebra over the field* $K = (\Gamma \backslash \{0\})\Gamma$.

(2) *The algebra* $\tilde{L} = (\Gamma_h \setminus \{0\})^{-1}L$ *is a graded simple algebra and* $dim_F \tilde{L}_i \leq d$, *for an arbitrary* $i \in Z$.

(3) *Either* L *is isomorphic to a (twisted) loop algebra or there is a graded embedding* $\varphi : \Gamma \to F[t^{-m}, t^m]$ *such that*

$$\sum_{i \geq k} Ft^{im} \subseteq \varphi(\Gamma) \subseteq F[t^m] \text{ or } \sum_{i \geq k} Ft^{-im} \subseteq \varphi(\Gamma) \subseteq F[t^{-m}].$$

Proof. For the assertion (1) cf. see [Ro].

(2) We only need to check that $\tilde{L}$ is graded simple. Let I be a non-zero graded ideal of L. By (1), $(\Gamma \setminus \{0\})^{-1}I = (\Gamma \setminus \{0\})^{-1}L$.

Let $\dim_K (\Gamma \setminus \{0\})^{-1}L = r$ and $f_r(x_1, \ldots, x_q)$ is a multilinear central polynomial that corresponds to $r \times r$ matrices. Then $(\Gamma \setminus \{0\})^{-1}L$ is a faithful irreducible module over the multiplication algebra $M < (\Gamma \setminus \{0\})^{-1}L >$. Hence, $M < (\Gamma \setminus \{0\})^{-1}L > \simeq \mathcal{M}_r(K)$. Consequently, there exist operators $\omega_i = ad(a_{i1}) \cdots ad(a_{iq_i})$, $1 \leq i \leq q$, a_{ij} homogeneous elements of I such that $f_r(\omega_1, \ldots, \omega_q) \neq 0$. Clearly, $f_r(\omega_1, \ldots, \omega_q) \in \Gamma_h$. Now,

$$L = (Lf_r(\omega_1, \ldots, \omega_q))f_r(\omega_1, \ldots, \omega_q)^{-1} \subseteq If_r(\omega_1, \ldots, \omega_q)^{-1} \subseteq (\Gamma_h \setminus \{0\})^{-1}I.$$

This proves $(\Gamma_h \setminus \{0\})^{-1}I = (\Gamma_h \setminus \{0\})^{-1}L$ and so $\tilde{L}$ is graded simple.

In order to prove (3) we will show that $\dim \Gamma_k \leq d$ for an arbitrary k. Let's take $d+1$ arbitrary elements $\gamma_1, \ldots, \gamma_{d+1} \in \Gamma_k$ and a non zero homogeneous element $a_i \in L_i$. Since $a_i\gamma_1, a_i\gamma_2, \ldots, a_i\gamma_{d+1} \in L_{i+k}$, there exists a non trivial linear dependence relation $\sum_{j=1}^{d+1} \xi_j a_i \gamma_j = 0$, $\xi_j \in F$. Since non zero elements in Γ have zero nuclei and $a_i \in Ker \sum_{j=1}^{d+1} \xi_j \gamma_j$, it follows that $\sum_{j=1}^{d+1} \xi_j \gamma_j = 0$.

We have proved that $\dim_F \Gamma_k \leq d$ and so the assertion (3) follows from Lemmas 2.3 and 2.5.

Indeed, by Lemma 2.5, either $\Gamma \simeq F[t^{-m}, t^m]$ or there exists the wanted embedding. If $\Gamma \simeq F[t^{-m}, t^m]$, then L is a loop algebra by Lemma 2.3.

Lemma 2.7. *Let* $L = \sum_{i \in Z} L_i$ *be a prime, nondegenerate, strongly PI Lie algebra,* $dim L_i \leq d$, *as in the previous lemma. Let's assume that* $Vir = \sum_{i \in Z} Vir_i$ *can be embedded into* $Der(L)$ *as a graded algebra. Then* L *is isomorphic to a (nontwisted) loop algebra.*

Proof. If L is not isomorphic to a (twisted) loop algebra, then by Lemma 2.6 there exists a graded embedding $\varphi : \Gamma \to F[t^{-m}, t^m]$, $m \geq 1$, such that either $\sum_{i \geq k} Ft^{im} \subseteq \varphi(\Gamma) \subseteq F[t^m]$ or $\sum_{i \geq k} Ft^{-im} \subseteq \varphi(\Gamma) \subseteq F[t^{-m}]$ for some $k \geq 1$.

Let us assume that $\sum_{i \geq k} Ft^{im} \subseteq \varphi(\Gamma) \subseteq F[t^m]$. This implies that Γ is generated by a finite set of elements $\gamma_i \in \Gamma_{s_i}$, $i = 1, 2, \ldots, r$.

Let $s = \max_{1 \leq i \leq r} s_i$. The Virasoro algebra acts on Γ. For each generator γ_i the subspace $\gamma_i Vir_{-(s+1)} = (0)$, since it is contained in Γ and has negative degree.

So $Vir_{-(s+1)}$ is contained in the kernel of the action of the Virasoro algebra on the derivations of Γ. By the simplicity of the Virasoro algebra, we have that $\Gamma Vir = (0)$.

Now the Virasoro algebra acts on a finite dimensional Lie algebra $\tilde{L}_K = (\Gamma \setminus \{0\})^{-1}L$ and the action is not trivial since $Vir \subseteq Der(L)$. This leads to a contradiction, since the Virasoro algebra is not strongly PI.

We showed that L is isomorphic to a loop algebra. Let us show that this loop algebra is not twisted. Indeed, let $\Gamma \simeq F[t^{-m}, t^m]$, $m \geq 2$. Then $\Gamma Vir_1 = \Gamma Vir_{-1} = (0)$. Since $Vir_1 \neq (0)$ and the algebra Vir is simple it follows that $\Gamma Vir = (0)$. Now we can argue as above.

Lemma 2.8. *Let L be a prime nondegenerate Lie algebra and let I be a nonzero ideal of L. Then I is a prime nondegenerate algebra.*

Proof. We will prove first that I is nondegenerate. Indeed, let $0 \neq a \in I$ and $[[I, a], a] = (0)$. Since L is nondegenerate, there exists an element $x \in L$ such that $[[x, a], a] \neq 0$. Now, $Lad([[x, a], a])^2 = Lad(a)^2 ad(x)^2 ad(a)^2 \subseteq Iad(a)^2 = (0)$, (cf. [Ko]), a contradiction.

Now we will prove that I is prime. Let I', I'' be non-zero ideals of I, with $[I', I''] = (0)$. Let $id_L(I'')$ the ideal of L generated by I''. If $[id_L(I''), I'] = (0)$, then the nonzero ideal of L, $id_L(I'')$, has a non zero centralizer, which contradicts primeness of L. Hence, $J = [I', id_L(I'')]$ is a non zero ideal of I. We have

$$ad(L)ad(I')^2 \subseteq ad(I')ad(L)ad(I') + ad(I)ad(I') \subseteq ad(I')M < L > .$$

Let's choose an arbitrary nonzero element $a \in J$, $a = \sum_i a_i ad(x_{i1}) \cdots ad(x_{ir_i})$ with $a_i \in I''$, $x_{ij} \in L$, $r_i \geq 0$. So, for $r = \max_i r_i$ we have

$$aad(I')^{2r} \subseteq \sum a_i ad(I')M < L >= (0).$$

Hence, $aad(J)^{2r} = (0)$.

This proves that J has a nontrivial center, what contradicts the nondegeneracy of I and proves the lemma.

Lemma 2.9. *Let $L = \sum_{i \in Z}^{n} L_i$ be a Z-graded prime nondegenerate Lie algebra containing the Virasoro algebra and having all the dimensions $dim L_i$ uniformly bounded. Suppose that L contains a nonzero graded ideal I which is strongly PI. Then L is isomorphic to the semidirect sum of a loop algebra $\mathcal{L}(\mathcal{G})$ (for some finite dimensional simple Lie algebra $\mathcal{G}$) and the Virasoro algebra*

Proof. By Lemma 2.8 I is a prime nondegenerate algebra. Moreover, since L is prime, the action of Vir on I is faithful. Hence by Lemma 2.7 $I \simeq \mathcal{L}(\mathcal{G})$, with $\dim \mathcal{G} < \infty$. Again, since I is prime and nondegenerate it follows that the algebra $\mathcal{G}$ is simple. For an arbitrary element $a \in L$ let $ad_I(a)$ denote the linear operator $ad_I(a) : I \to I$, $x \to [x, a]$. The mapping $a \to ad_I(a)$ is an embedding of L into the Lie algebra

$$Der(\mathcal{L}(\mathcal{G})) = \mathcal{L}(\mathcal{G}) \rtimes Vir.$$

Since the Virasoro algebra is simple and not strongly PI, it follows that $Vir \cap I = (0)$. Now comparing the dimensions of the homogeneous components we conclude that the embedding $L \to Der(\mathcal{L}(\mathcal{G}))$, $a \to ad_I(a)$ is an isomorphism. The Lemma is proved

3. Lie-Jordan Connections

In this section we will study connections between Lie algebras and Jordan systems.

A Jordan pair $P = (P^-, P^+)$ is a pair of vector spaces with a pair of trilinear operations

$$\{\ ,\ ,\ \} : P^- \times P^+ \times P^- \to P^-, \qquad \{\ ,\ ,\ \} : P^+ \times P^- \times P^+ \to P^+$$

that satisfies the following identities:

(P.1) $\{x^\sigma, y^{-\sigma}, \{x^\sigma, z^{-\sigma}, x^\sigma\}\} = \{x^\sigma, \{y^{-\sigma}, x^\sigma, z^{-\sigma}\}, x^\sigma\}$,

(P.2) $\{\{x^\sigma, y^{-\sigma}, x^\sigma\}, y^{-\sigma}, u^\sigma\} = \{x^\sigma, \{y^{-\sigma}, x^\sigma, y^{-\sigma}\}, u^\sigma\}$,

(P.3) $\{\{x^\sigma, y^{-\sigma}, x^\sigma\}, z^{-\sigma}, \{x^\sigma, y^{-\sigma}, x^\sigma\}\} =$
$\{x^\sigma, \{y^{-\sigma}, \{x^\sigma, z^{-\sigma}, x^\sigma\}, y^{-\sigma}\}, x^\sigma\}$,

for every $x^\sigma, u^\sigma \in P^\sigma$, $y^{-\sigma}, z^{-\sigma} \in P^{-\sigma}$, $\sigma = \pm$ (see [L]).

If $L = \sum_{i=-n}^{n} L_{(i)}$ is a finite grading, then the pair $(L_{(-n)}, L_{(n)})$ with the operations $\{x^\sigma, y^{-\sigma}, z^\sigma\} = [[x^\sigma, y^{-\sigma}], z^\sigma]$, $\sigma = \pm$ is a Jordan pair

An element $a \in P^\sigma$ is called an *absolute zero divisor* of the pair P if $\{a, P^{-\sigma}, a\} = (0)$. A Jordan pair is said to be *nondegenerate* if it does not contain nonzero absolute zero divisors

A Jordan pair is said to be *prime* if the product of any two nonzero ideals is not zero, where an ideal of P is a pair of subspaces $I = (I^-, I^+)$ that satisfies the obvious condition.

The smallest ideal $M(P)$ of the pair P whose quotient is nondegenerate is called the McCrimmon radical of P.

An element a of a Lie algebra is a *sandwich* if $[[L, a], a] = 0$. The Kostrikin radical of a Lie algebra L is the smallest ideal $K(L)$ whose quotient is nondegenerate.

The central point in this connection is given by the following two lemmas, that reduce our original problem in Lie algebras to a Jordan pairs problem.

Lemma 3.1. *Let L be a Lie algebra with a finite grading $L = \sum_{k=-n}^{n} L_{(k)}$, $L_{(0)} = \sum_{k=1}^{n} [L_{(-k)}, L_{(k)}]$ and $L_{(n)} \neq (0)$. If L is prime and nondegenerate, then:*

(1) Every nonzero ideal of L has a nonzero intersection with $L_{(n)}$,

(2) The Jordan pair $V = (L_{(-n)}, L_{(n)})$ is prime and nondegenerate.

Proof. (1) Let $(0) \neq I \trianglelefteq L$ and suppose that $I \cap L_{(n)} = (0)$. Then, $[[I, L_{(n)}], L_{(n)}] \subseteq I \cap L_{(n)} = (0)$. Consider the subalgebra $L' = I + L_{(n)}$.

Clearly, $[[L', L_{(n)}], L_{(n)}] = (0)$. Hence, $L_{(n)}$ is in the Kostrikin radical of L' and using Lemma 2.8 and Proposition 2 of [Z1] we conclude that $[I, L_{(n)}] \subseteq K(L') \cap I = K(I) = (0)$. This contradicts primeness of L.

(2) The non-degeneracy of V follows from the fact that every absolute zero divisor of V is a sandwich of L.

Now, let us assume that I and J are nonzero ideals of V and that $I \cap J = (0)$. Let $\tilde{I}$ and $\tilde{J}$ be the ideals of L generated by I and J respectively. By (1), the nonzero ideal $\tilde{I} \cap \tilde{J}$ has nonzero intersection with V. Let $P = (\tilde{I} \cap L_{(-n)} \cap \tilde{J}, \tilde{I} \cap \tilde{J} \cap L_{(n)}) \trianglelefteq V$.

Zelmanov proved in [Z1] that the quotient pairs $\tilde{I} \cap V/I$ and $\tilde{J} \cap V/J$ coincide with their McCrimmon radicals. We will prove that this implies that $P \subseteq \mathcal{M}(V)$.

Let's recall that a sequence of elements in a Jordan pair $x_1, x_2, \ldots \in V^\sigma$, $\sigma = \pm$, is called an m-sequence if $x_{i+1} \in \{x_i, V^{-\sigma}, x_i\}$. In [Z3] it was proved that the McCrimmon radical consists of those elements x such that every m-sequence starting by x finishes in zero.

Let $x \in P^\sigma$ and let $x = x_1, x_2, \ldots$ be an m-sequence. Since $x \in \tilde{I} \cap V^\sigma$, it follows that there exists $s_1 \geq 1$ such that $x_i \in I$ for all $i \geq s_1$.

Similarly, there exists $s_2 \geq 1$ s.t. $x_j \in J$ for all $j \geq s_2$. Hence, for every $k \geq \max(s_1, s_2)$ we have that $x_k \in I \cap J = (0)$. Now, $(0) \neq P \subseteq \mathcal{M}(V)$ contradicts the nondegeneracy of V, what proves the lemma.

Lemma 3.2. *Let $L = \sum_{k=-n}^{n} L_{(k)}$ be a Lie algebra with a finite grading. Let us assume that the Jordan pair $V = (L_{(-n)}, L_{(n)})$ is prime and nondegenerate and that an arbitrary nonzero ideal of L has nonzero intersection with V. Then L is prime and nondegenerate.*

Proof. Clearly, the algebra L is prime, because if I, J are non zero ideals of L with $[I, J] = (0)$, then $I' = I \cap V$, $J' = J \cap V$ are nonzero ideals of V and $\{I'^\sigma, J'^{-\sigma}, V^\sigma\} = \{J'^{-\sigma}, I'^\sigma, V^{-\sigma}\} \subseteq I \cap J = (0)$, $\sigma = \pm$, what contradicts primeness of V.

In [Z2] it was proved that $K(L) \cap L_{(\pm n)}$ is contained in the McCrimmon radical of the pair V, hence $K(L) \cap L_{(\pm n)} = (0)$, what implies, under our assumptions, that $K(L) = (0)$ and so L is nondegenerate.

4. The Jordan Case

The last two lemmas have reduced our original problem to a problem concerning Jordan pairs. So, our aim now will be to prove Theorem 2.

We will need the following lemma

Lemma 4.1. *Let* $\mathcal{G}$ *be a simple finite dimensional Lie algebra with a* Z/lZ*-grading,* $\mathcal{G} = \sum_{i \in Z/lZ} \mathcal{G}_i$.

If dim $\mathcal{G}_0 \le d$*, then* $dim_F \mathcal{G} \le N(d) = \max(d(2d+1), 248)$.

Proof. The mapping $d : \mathcal{G} \to \mathcal{G}$, $a_i \to i a_i$ is a derivation. Since every derivation is inner, there exists an element $h \in \mathcal{G}$ such that $d = ad(h)$. So h is semisimple and is contained in some Cartan subalgebra H. Since H is abelian, the elements of H commute with h and given that $[a_i, h] = d(a_i) = i a_i$, necessarily $H \subseteq \mathcal{G}_0$. But $\dim \mathcal{G}_0 \le d$, which implies $\dim H \le d$.

Now the bound follows from the classification of simple finite dimensional Lie algebras.

Proof of Theorem 2

We will divide the proof of the theorem in three cases

Case 1. We will assume first that $\mathcal{K}(V)$ is *strongly PI* (where $\mathcal{K}(V)$ denotes the Lie algebra associated to V via the Tits-Kantor-Koecher construction).

Recall that the Tits-Kantor-Koecher Lie algebra $\mathcal{K}(V)$ can be characterized in the following way: $\mathcal{K}(V) = \mathcal{K}(V)_{-1} + \mathcal{K}(V)_0 + \mathcal{K}(V)_1$ is a Z-graded Lie algebra, $\mathcal{K}(V)_0 = [\mathcal{K}(V)_{-1}, \mathcal{K}(V)_1]$, $(\mathcal{K}(V)_{-1}, \mathcal{K}(V)_1) = V$ and $\mathcal{K}(V)_0$ does not contain nonzero ideals of $\mathcal{K}(V)$.

We will see that under our assumption, the algebra $\mathcal{K}(V)$ is prime. Let us show that every nonzero ideal of $\mathcal{K}(V)$ has non zero intersection with V^+. Since the Jordan pair V is prime, there are no elements $0 \ne x^- \in V^-$ with $[x^-, V^+, V^+] = (0)$. Similarly, there are no elements $0 \ne x^+ \in V^+$ with $[x^+, V^-, V^-] = (0)$.

If $I \cap V^+ \ne (0)$, then $(0) \ne [I \cap V^+, V^-, V^-] \subseteq I \cap V^-$. That is, for an arbitrary ideal I of V, $I \cap V^+ \ne (0)$ if and only if $I \cap V^- \ne (0)$.

Let $x = x_- + x_0 + x_+ \in I$. Let us assume that $x_- \ne 0$. Then $[x, V^+, V^+] = [x_-, V^+, V^+] \ne 0$ and $[x, V^+, V^+] \subseteq I$. So $[x, V^+, V^+] \subseteq I \cap V^+$ and $I \cap V^+ \ne (0)$.

Similarly, if $x_+ \ne 0$, then $I \cap V^- \ne (0)$.

Hence $I \subseteq [V^-, V^+]$, which implies $I = (0)$.

Now we can prove that $\mathcal{K}(V)$ is prime. Indeed, let's consider I_1, I_2 two non zero ideals of $\mathcal{K}(V)$. Then $I_1 \cap V \ne (0)$, $I_2 \cap V \ne (0)$. Since V is prime, $I_1 \cap I_2 \cap V \ne (0)$ and, in particular, $I_1 \cap I_2 \ne (0)$.

Since $L = \mathcal{K}(V)$, is a prime and strongly PI Lie algebra it follows that the centroid Γ of L is nonzero and the algebra $(\Gamma \setminus \{0\})^{-1} L$ is finite dimensional over $(\Gamma \setminus \{0\})^{-1} \Gamma$.

Let us see that Γ can be identified with the centroid of V, that is, $V^+ \Gamma \subseteq V^+$ and $V^- \Gamma \subseteq V^-$. Indeed, let's consider the derivation $d : L \to L$, $d(a_i) = i a_i$, that multiplies $V^\pm$ by ± 1 and annihilates $[V^-, V^+]$. The centroid Γ decomposes into eigenspaces with respect to the action of d : $\Gamma = \Gamma_{-2} + \Gamma_{-1} + \Gamma_0 + \Gamma_1 + \Gamma_2$. Since every element of $\cup_{i \ne 0} \Gamma_i$ is nilpotent and L is prime, we have that $\Gamma = \Gamma_0$, that is, Γ maps V^+ to V^+ and V^- to V^-.

The centroid Γ is a graded commutative domain, $\Gamma = \sum_{i\in Z}\Gamma_i$ with $\dim\Gamma_i \le 1$. If $\Gamma = \Gamma_0$, then $\Gamma = F$ and $\dim_F V < \infty$.

If there exist $i, j \ge 1$ with $\Gamma_i \ne (0) \ne \Gamma_{-j}$, then V is a (twisted) loop Jordan pair.

Let's consider finally the case when every negative component of Γ is zero (the case with all positive components of Γ equal to zero is similar).

Let γ_l be a homogeneous element of the centroid with degree l, $\gamma_l : V \to V$. Then $\mathrm{Ker}\gamma_l \trianglelefteq V$, $\mathrm{Im}\ \gamma_l \trianglelefteq V$ and they annihilate each other. Since V is prime, it follows that γ_l is injective.

From $\gamma_l(V_i) \subseteq V_{i+l}$, it follows that $\dim V_i = \dim V_i\gamma_l \le \dim V_{i+l}$. For every i, $0 \le i \le l-1$, the ascending sequence: $\cdots \dim V_i \le \dim V_{i+l} \le \dim V_{i+2l} \le \cdots$ stabilizes in some k_i, that is, $\dim V_{i+k_il} = \dim V_{i+(k_i+1)l}$.

Let $k(\gamma_l) = \max\{k_i | 0 \le i \le l-1\}$. For every $h \ge k(\gamma)$ the linear mapping $\gamma_l : V_h \to V_{h+l}$ is bijective.

Let Γ_h be the set of homogeneous elements in Γ (so $(\Gamma_h \setminus \{0\})^{-1}V$ is a graded Jordan pair over $(\Gamma_h \setminus \{0\})^{-1}\Gamma$ and an arbitrary nonzero homogeneous element of $\Gamma_h^{-1}\Gamma$ is invertible).

Let $n = \min\{l > 0 \,|\, C_l = (\Gamma_h^{-1}\Gamma)_l \ne 0\}$. If $0 \ne c_n \in C_n$, then there exist i, j, $i > j$, and $0 \ne \gamma_i \in \Gamma_i$, $0 \ne \gamma_j \in \Gamma_j$ with $c_n = \gamma_j^{-1}\gamma_i$. Let k be a multiple of n such that $k \ge \max(k(\gamma_i), k(\gamma_j))$ (let's notice that we can write $V_{h+j}\gamma_j^{-1} \subseteq V_h \subseteq V$ if $h \ge k$, even if there is no γ_j^{-1} in Γ). Hence, $V_{h+n} = V_{h+n+j}\gamma_j^{-1} = V_{h+n+j-i}\gamma_i\gamma_j^{-1} = V_h c_n$.

Let's consider the finite-dimensional vector space $\mathcal{V} = \mathcal{V}_0 + \mathcal{V}_1 + \cdots \mathcal{V}_{n-1}$ with $\mathcal{V}_h = V_{h+k}$ for $0 \le h \le n-1$.

If $0 \le r, s \le n-1$, $b^{\sigma}_{k+r} \in V^{\sigma}_{k+r}$, $b^{-\sigma}_{k+s} \in V^{-\sigma}_{k+s}$, $\sigma = \pm 1$, then

$$\{b^{\sigma}_{k+r}, b^{-\sigma}_{k+s}, b^{\sigma}_{k+r}\} \in V^{\sigma}_{3k+2r+s}.$$

Let $2k + 2r + s = ln + t$, $l \ge 0$, $0 \le t \le n-1$. Then $V_{3k+2r+s} = V_{k+ln+t} = V_{k+t}c_n^l$.

Define

$$\{b^{\sigma}_{k+r}, b^{-\sigma}_{k+s}, b^{\sigma}_{k+r}\}^{\star} = \{b^{\sigma}_{k+r}, b^{-\sigma}_{k+s}, b^{\sigma}_{k+r}\}c_n^{-l} \in V_{k+t} = \mathcal{V}_t$$

Then $\mathcal{V}$ becomes a finite-dimensional Z/nZ-graded Jordan pair with this new product and we get the wanted result.

Case 2. We will assume now that V is *finitely generated*

According to the classification of prime non-degenerated Jordan pairs by E. Zelmanov, we know that a finitely generated prime Jordan pair V is either special or strongly PI. Since the strongly PI case is already known, we only need to consider the special case.

In order to prove Theorem 2 in this case, we need to know the relation between the Gelfand Kirillov dimension of a special Jordan pair and the Gelfand Kirillov dimension of its associative enveloping algebra. We will use a result similar to the one used by Skosirskii ([SK1]) for algebras.

Lemma 4.2. *Let (P^-, P^+) be a special Jordan pair finitely generated by $a_1, a_2, \ldots, a_n$. Then every word in the associative enveloping pair can be expressed as a linear combination of elements of the form $\omega'\omega\omega''$, where ω is a Jordan word and the lengths of ω' and ω'' are not greater than 2n.*

Proof. There exists an associative algebra A (that can be assumed finitely generated by $a_1, \ldots, a_n$) such that $(P^-, P^+) \subseteq (A^-, A^+)$ and $A = A^- + (A^-A^+ + A^+A^-) + A^+$.

Let $\omega = v_1^\sigma v_2^{-\sigma} v_3^\sigma \cdots$ be a product of Jordan words v_i and the total degree of ω in $a_1, \ldots, a_n$ is N.

We will use an inverse induction on the length of v_σ, maximal among the lengths of elements v_i^σ. If the length is N, then $v = v^\sigma$. Let us assume that some $v_i^{-\sigma}$ placed to the right (similarly to the left) of the element v^σ has length ≥ 3. Using that $v_k^- v_j^+ v_i^- = \{v_k, v_j, v_i\}^- - v_i^- v_j^+ v_k^-$, we can assume, without loss of generality, that this element and v^σ are adjacent.

But

$$v^\sigma a^{-\sigma} b^\sigma a^{-\sigma} = (v^\sigma a^{-\sigma} b^\sigma + b^\sigma a^{-\sigma} v^\sigma) a^{-\sigma} - b^\sigma (a^{-\sigma} v^\sigma a^{-\sigma})$$

where elements in brackets are Jordan words of length strictly greater than the length of v^σ.

Rewrite every Jordan word v_i^σ except v^σ as an expression in the generators $a_j^\pm$, $\sigma = \sum \cdots v^\sigma a_{j1}^{-\sigma} a_{j2}^\sigma a_{j3}^{-\sigma} \cdots$.

A double occurrence of a generator $a_j^{-\sigma}$ to the right of v^σ gives rise to $a_j^{-\sigma} a_k^\sigma a_j^{-\sigma}$, the case which has been considered above.

Finally, we get that ω is of the form:

$$\omega = (\cdots) v^\sigma a_{i1}^{-\sigma} a_{i2}^\sigma a_{i3}^{-\sigma} \cdots$$

where all the generators $a_{i1}^{-\sigma}$, $a_{i3}^{-\sigma}$, ... are distinct.

Hence the length to the right of v^σ (and similarly to the left) is $\leq 2n$, where n is the number of generators.

Lemma 4.3. *If P is a finitely generated special Jordan pair and A is an associative algebra as in Lemma 4.2 with $(P^-, P^+) \subseteq (A^-, A^+)$, then $GK - \dim(P) = GK - \dim(A)$.*

Proof. Let U be a finite dimensional vector space that generates P and A.

Then

$$GK - \dim(A) = \limsup_{n\to\infty} \frac{\ln \dim U^n}{\ln n}$$

But $U^n \subseteq U'W^mU''$, where U' and U'' are subspaces of bounded dimensions (not more than C) and W^m is spanned by Jordan words in elements of U of length $\geq m = n - 4r\}$ where r is the dimension of the vector space U. So $\dim U^n \leq C^2 \dim W^m$.

Hence,

$$GK - \dim(A) = \limsup_{n\to\infty} \frac{\ln \dim U^n}{\ln n} \leq \limsup_{n\to\infty} \frac{\ln(C^2. \dim W^m)}{\ln n} =$$

$$\limsup_{m\to\infty} \frac{\ln C^2 + \ln(\dim W^m)}{\ln(m+4r)} = \limsup_{m\to\infty} \frac{\ln \dim W^m}{\ln m} = GK - \dim P$$

Now we can conclude the proof of Theorem 2 in the finitely generated case.

If the considered Jordan pair P is finitely generated and special, its associative enveloping algebra A is finitely generated and $GK - dim(A) = 1$. By the result by Small, Stafford and Warfield Jr. [SSW] we know that A is PI. Hence P is strongly PI and the result follows from Case 1.

Case 3. The General Case

Lemma 4.4. *Let $V = \sum_{i\in Z} V_i$ be a Z-graded Jordan pair having all dimensions $dim V_i$ uniformly bounded. Then the locally nilpotent radical $Loc(V)$ is equal to the McCrimmon radical $M(V)$.*

Proof. It is known that $M(V) \subseteq Loc(V)$ (see [Z4]).

Choose an arbitrary homogeneous element $v_k^\sigma \in V_k^\sigma$ and consider the homotope Jordan algebra $J = V^{-\sigma}$, $x \star y = \{x, v_k^\sigma, y\}$. Assign a new degree to homogeneous elements of J, $\deg(V_i^{-\sigma}) = i + k$. With this degree J becomes a graded Jordan algebra having all dimensions $\dim J_i$ uniformly bounded. In [MZ1] it was proved that $\mathrm{Loc}(J) = M(J)$. Since $Loc(V)^{-\sigma} \subseteq Loc(J)$ and $\{v_k^\sigma, M(J), v_k^\sigma\} \subseteq M(V)$ (see [Z4]), we conclude that $\{v_k^\sigma, Loc(V), v_k^\sigma\} \subseteq M(V)$.

In particular, an arbitrary homogeneous element of $\mathrm{Loc}(V)$ lies in $M(Loc(V)) \subseteq M(V)$. This implies that $Loc(V) \subseteq M(V)$. The Lemma is proved.

Let V be a Jordan pair satisfying the assumptions of Theorem 2 and let $\tilde{V}$ be a finitely generated graded subpair of V. The nondegenerate pair $\tilde{V}/M(\tilde{V}))$ can be approximated by finitely generated prime nondegenerate Jordan pairs. By the Case 2 each of these pairs is either $\mathcal{L}(U)$ or can be embedded into a loop pair $\mathcal{L}(U)$, where U is a simple finite dimensional pair. By Lemma 4.1, $\dim U \leq N(d)$, where $d = \max \dim V_i$.

Let T be the ideal of the free Jordan pair consisting of those elements which are identically zero in all Jordan pairs of dimension $\leq N(d)$.

We proved that for an arbitrary finitely generated subpair $\tilde{V}$ of V, the set of values $T(\tilde{V})$ lies in the locally nilpotent radical $\mathrm{Loc}(\tilde{V})$. This implies that $T(V) \subseteq Loc(V)$. By Lemma 4.4 $\mathrm{Loc}(V) = M(V) = (0)$, which implies $T(V) = (0)$. Hence the pair V is strongly PI, which is the Case 1. Theorem 2 is proved.

In the next section we will need the following lemma about loop Jordan pairs.

Let W be a simple finite dimensional Jordan pair graded by Z/lZ, $W = \sum_{i=0}^{l-1} W_i$, and let $\mathcal{L}(W) = \sum_{i=q\, mod\, l} W_i \otimes t^q$ be a (twisted) loop pair.

Lemma 4.5. *For any $k \geq 1$ we have*

1) The subpair $\sum_{i\geq k} \mathcal{L}(W)_i$ is finitely generated,

2) Every subpair $P \subseteq \mathcal{L}(W)$ containing $\sum_{i\geq k} \mathcal{L}(W)_i$ is prime and nondegenerate.

Proof. 1) We will prove that $\sum_{i\geq k} \mathcal{L}(W)_i$ is generated by $\sum_{i=k}^{3k+2l} \mathcal{L}(W)_i$.

Let $q > 3k+2l$, $a \in W_j^\sigma$, $0 \leq j \leq l-1$, $j \equiv q$ mod l and $a \otimes t^q \in \mathcal{L}(W)_q$.

We have that $W^\sigma = \{W^\sigma, W^{-\sigma}, W^\sigma\}$ (by simplicity of W), so $a = \sum_i \{a_i^{'\sigma}, b_i^{-\sigma}, a_i^{''\sigma}\}$, with $a_i^{'\sigma} \in W_{\pi(i)}^\sigma$, $b_i^{-\sigma} \in W_{\mu(i)}^{-\sigma}$, and $a_i^{''\sigma} \in W_{\rho(i)}^\sigma$, $0 \leq \pi(i)$, $\mu(i), \rho(i) \leq l-1$.

Choose integers $k \leq q_1(i), q_2(i) \leq k+l-1$ such that $q_1(i) \equiv \pi(i)$ mod l, $q_2(i) = \rho(i)$ mod l and $q_3(i) = q - q_1(i) - q_2(i)$.

From $q > 3q + 2l$, it follows that $q_3(i) > k$. Now,

$$a \otimes t^q = \sum_i \{a_i^{'\sigma} \otimes t^{q_1(i)}, b_i^{-\sigma} \otimes t^{q_3(i)}, a_i^{''\sigma} \otimes t^{q_2(i)}\},$$

that is,

$$\mathcal{L}(W)_q \subseteq \sum \{\mathcal{L}(W)_{q_1}, \mathcal{L}(W)_{q_3}, \mathcal{L}(W)_{q_2}\},$$

where $k \leq q_1, q_2, q_3 \leq q$.

2) Note that if Ω is a homogeneous operator in the multiplication algebra of $\mathcal{L}(W)$ and $(\sum_{i=k}^{k+l-1} \mathcal{L}(W)_i)\Omega = (0)$, then $\Omega = 0$

Let P be a subpair of $\mathcal{L}(W)$ with $P \supseteq \sum_{i=k}^{\infty} \mathcal{L}(W)_i$. If $a^\sigma \in P^\sigma$ is an absolute zero divisor of the pair P, then $(\sum_{i=k}^{k+l-1} \mathcal{L}(W)_i)U(a) = (0)$. This implies that $\mathcal{L}(W)U(a) = (0)$. Since $\mathcal{L}(W)$ is nondegenerate, it follows that $a = 0$. We have proved that P is nondegenerate.

Let I, J be non zero graded ideals of P with $I \cap J = (0)$.

Take $0 \neq a^\sigma \otimes t^p \in I$, $0 \neq b^\sigma \otimes t^q \in J$ and $c(x_1, \ldots, x_n, \ldots)$ an arbitrary multilineal expression in the free Jordan pair. Then

$$c(a^\sigma \otimes t^p, b^\sigma \otimes t^q, \sum_{i\geq k} \mathcal{L}(W)_i, \sum_{i\geq k} \mathcal{L}(W)_i, \ldots) = (0).$$

This implies that $c(a^\sigma, b^\sigma, W, W, \ldots) = (0)$, what contradicts primeness of W. This proves the lemma.

5. The Lie Case

Lemma 5.1. *Let A be a simple Z/lZ-graded finite dimensional algebra and let a be a homogeneous element of degree $d(a)$. Consider the loop algebra $\sum_{i=j \bmod l} A_i \otimes t^j$ and its subalgebra $\sum_{j\geq m} A_i \otimes t^j$. Choose an integer $n \geq m$ such that $n = d(a)$ mod l and let I be the ideal generated by $a \otimes t^n$ in $\sum_{j\geq m} A_i \otimes t^j$. Then $I \supseteq \sum_{j\geq p} A_i \otimes t^j$ for some $p \geq m$.*

Proof.

Let $a_1, \ldots, a_s$ be homogeneous elements of A and $b = aP(a_1)\cdots P(a_s)$, where $P = R$ or L. We choose integers $j_1, \ldots j_s \geq m$ such that $j_k = d(a_k)$ mod l, $k = 1, \ldots s$. Then $(a \otimes t^n)P(a_1 \otimes t^{j_1})\cdots P(a_s \otimes t^{j_s}) = b \otimes t^q \in I$ and for an arbitrary $k \in Z_{\geq 0}$ we have that

$$b \otimes t^{q+kl} = (a \otimes t^n)P(a_1 \otimes t^{j_1+kl})\cdots P(a_s \otimes t^{j_s}) \in I.$$

Let's take a basis $e_1, \ldots, e_r$ of A that consists of elements of the type $e_i = aR(a_{i_1})\cdots R(a_{ir_i})$, where the elements a_{ij} are homogeneous. According to what we have mentioned above, there exist integers $q_1, \ldots, q_r \geq m$ such that $e_i \otimes t^{q_i + lZ_{\geq 0}} \in I$. It suffices to take $p = \max_{1\leq i \leq r} q_i$.

Remark. The assertion of the Lemma 5.1 is true also for Z/lZ-graded simple finite dimensional Jordan pairs.

We can already prove the main result giving the structure of prime Z-graded Lie algebras.

Proof of Theorem 1

Let $L = \sum_{i\in Z} L_i = \sum_{k=-n}^{n} L_{(k)}$ be a Lie algebra that satisfies the assumptions of Theorem 1. By Lemma 3.1 and Theorem 2, we know that $V = (L_{(-n)}, L_{(n)})$ can be embedded into a loop pair $\mathcal{L}(W)$, $V \hookrightarrow \mathcal{L}(W)$, where W is a simple finite-dimensional Jordan pair and either $\sum_{i\geq k}\mathcal{L}(W)_i \subseteq V$ or $\sum_{i\geq k}\mathcal{L}(W)_{-i} \subseteq V$, for some $k \geq 1$. Let's assume that $\sum_{i\geq k}\mathcal{L}(W)_i \subseteq V$.

For an arbitrary scalar $\alpha \in F$ we define a homomorphism

$$\varphi_\alpha : W \otimes_F F[t^{-1}, t] \longrightarrow W$$

via $t \to \alpha$. Since $\varphi_\alpha(\sum_{i\geq k}\mathcal{L}(W)_i) = \varphi_\alpha(\sum_{i\geq k}\mathcal{L}(W)_{-i}) = W$, it follows that $\varphi_\alpha(V) = W$.

Let's denote $I_\alpha = Ker\varphi_\alpha \cap V$ and $\tilde{I}_\alpha$ the ideal in the Lie algebra generated by I_α. Using Lemma 14 in [Z1] we have that $\tilde{I}_\alpha \cap V = I_\alpha$.

Let $\mathcal{G}$ be the Tits-Kantor-Koecher construction associated to the Jordan pair W. A Z/lZ-graduation of W induces a Z/lZ-graduation of $\mathcal{G}$ and so $\mathcal{G}$ is $Z \times Z/lZ$-graded. The 0 component of this $Z \times Z/lZ$-graduation contains a Cartan subalgebra H.

Every $Z \times Z/lZ$-homogeneous component of $\mathcal{G}$ decomposes as a sum of eigenspaces with respect to the action of H. All the eigenspaces have dimension 1 and there exists a nonzero eigenvector x such that $[[\mathcal{G}, x], x] = Fx$. Hence, every homogeneous component $W_p^\sigma \neq (0)$, with $\sigma = \pm$, contains a non zero element a' such that $\{a', W^{-\sigma}, a'\} = Fa'$.

Choose an integer $q \geq k$, $q = p$ mod l and let $a' \otimes t^q = a \in \sum_{i\geq k}\mathcal{L}(W)_i \subseteq V$.

By Lemma 5.1 the ideal $id_V(a)$ of the Jordan pair (generated by the element a) contains a $\sum_{i\geq m}\mathcal{L}(W)_i$ for some $m \geq k$.

By Lemma 4.4(1), the subpair $\sum_{i\geq m}\mathcal{L}(W)_i$ is finitely generated. Choose, inside of the ideal $id_L(a)$ generated by a in the algebra L, a finite set of elements $a_i = aad(x_{i_1})\cdots ad(x_{i_{r(i)}})$, $1\leq i\leq s$, $x_{ij}\in L$ that are $0Z\times 0Z/lZ$-homogeneous and include generators of $\sum_{i\geq m}\mathcal{L}(W)_i$.

Consider $L' =< a_1,\ldots,a_s >$ the subalgebra generated by the elements $a_1,\ldots,a_s$, $m = 2^{r_1}+\cdots+2^{r_s}$ (as in Lemma 2.1) and T the T-ideal generated by all identities satisfied by all Lie algebras of dimension $\leq R(m)$.

For an arbitrary scalar, $0\neq\alpha\in F$, we have $\varphi_\alpha(a) = \alpha^q a'$

Hence $[[\varphi_\alpha(L),\varphi_\alpha(a)],\varphi_\alpha(a)]\subseteq\{a', W^{-\sigma}, a'\} = Fa' = F\varphi_\alpha(a)$.

By Lemma 2.1, the Lie algebra $\varphi_\alpha(L')$ satisfies all the identities of T. Since $\cap_{0\neq\alpha\in F}\tilde{I}_\alpha = (0)$ (notice that $(\cap_{0\neq\alpha\in F}\tilde{I}_\alpha)\cap V = \cap_{0\neq\alpha\in F}I_\alpha = (0)$), it follows that $T(L') = (0)$

Let $J(L')$ a $Z\times Z/lZ$-graded maximal ideal of L' such that $J(L')\cap L'_{(n)} = J(L')\cap L'_{(-n)} = (0)$ (it exists by Zorn Lemma). The Jordan pair $(L'_{(-n)}, L'_{(n)})$ is prime and nondegenerate by Lemma 4.4(1).

An arbitrary non-zero graded ideal of $L'/J(L')$ has nonzero intersection with the pair $(L'_{(-n)}, L'_{(n)})$. By Lemma 3.2, the algebra $L'/J(L')$ is prime and nondegenerate. Furthermore, $T(L'/J(L')) = (0)$, so $L'/J(L')$ is strongly PI. Using Lemma 2.6(2) and Mathieu's theorem (see [Ma2]), $(\Gamma_h(L'/J(L'))\backslash\{0\})^{-1}(L'/J(L'))$ is isomorphic to a loop algebra $\mathcal{L}(\mathcal{G})$. By Lemma 4.1, $\dim_F(\mathcal{G})\leq m = \max(d(2d+1), 248)$. Let T_m be the ideal of the free Lie that consists of all the identities that are satisfied identically in all Lie algebras of dimension $\leq m$. Then $T_m(L')\subseteq J(L')$ and so $T_m(L')\cap L_{(n)} = (0)$.

Since L' is an arbitrary finitely generated subalgebra of $id_L(a)$ containing a given (finite) subset and such subalgebras cover the ideal $id_L(a)$, we conclude that $T_m(id_L(a))\cap L_{(n)} = (0)$.

But the ideal $T_m(id_L(a))$ of $id_L(a)$ is invariant with respect to all the derivations of $id_L(a)$. Hence $T_m(id_L(a))$ is an ideal of L. By Lemma 3.1(1), $T_m(id_L(a))\cap L_{(n)} = (0)$ implies $T_m(id_L(a)) = (0)$. So the algebra $id_L(a)$ is strongly PI. Finally it suffices to apply Lemma 2.9 to finish the proof of Theorem 1.

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Consuelo Martínez
Departamento de Matemáticas
Universidad de Oviedo
C/ Calvo Sotelo, s/n
33007 Oviedo SPAIN
chelo@pinon.ccu.uniovi.es

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