Journal of Lie Theory
Volume **15** (2005) 393–414

Homomorphisms between Lie JC*-Algebras and Cauchy–Rassias Stability of Lie JC*-Algebra Derivations

Chun-Gil Park[*]

Communicated by A. Valette

Abstract. It is shown that every almost linear mapping $h: A \to B$ of a unital Lie JC*-algebra A to a unital Lie JC*-algebra B is a Lie JC*-algebra homomorphism when $h(2^n u \circ y) = h(2^n u) \circ h(y)$, $h(3^n u \circ y) = h(3^n u) \circ h(y)$ or $h(q^n u \circ y) = h(q^n u) \circ h(y)$ for all $y \in A$, all unitary elements $u \in A$ and $n = 0, 1, 2, \cdots$, and that every almost linear almost multiplicative mapping $h: A \to B$ is a Lie JC*-algebra homomorphism when $h(2x) = 2h(x)$, $h(3x) = 3h(x)$ or $h(qx)qh(x)$ for all $x \in A$. Here the numbers $2, 3, q$ depend on the functional equations given in the almost linear mappings or in the almost linear almost multiplicative mappings. Moreover, we prove the Cauchy–Rassias stability of Lie JC*-algebra homomorphisms in Lie JC*-algebras, and of Lie JC*-algebra derivations in Lie JC*-algebras.
Mathematics Subject Classification: 17B40, 39B52, 46L05, 17A36.
Keywords and Phrases: Lie JC*-algebra homomorphism. Lie JC*-algebra derivation, stability, linear functional equation.

1. Introduction

The original motivation to introduce the class of nonassociative algebras known as Jordan algebras came from quantum mechanics (see [18]). Let $L(H)$ be the real vector space of all bounded self-adjoint linear operators on H, interpreted as the (bounded) *observables* of the system. In 1932, Jordan observed that $L(H)$ is a (nonassociative) algebra via the *anticommutator product* $x \circ y := \frac{xy+yx}{2}$. A commutative algebra X with product $x \circ y$ is called a *Jordan algebra.* A unital Jordan C^*-subalgebra of a C^*-algebra, endowed with the anticommutator product, is called a *JC^*-algebra.*

A unital C^*-algebra C, endowed with the Lie product $[x, y] = \frac{xy-yx}{2}$ on C, is called a *Lie C^*-algebra.* A unital C^*-algebra C, endowed with the Lie product $[\cdot, \cdot]$ and the anticommutator product $\circ$, is called a *Lie JC^*-algebra* if $(C, \circ)$ is a JC^*-algebra and $(C, [\cdot, \cdot])$ is a Lie C^*-algebra (see [5], [6]).

[*] Supported by grant No. R05-2003-000-10006-0 from the Basic Research Program of the Korea Science & Engineering Foundation.

ISSN 0949–5932 / \$2.50

Let X and Y be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\theta \geq 0$ and $p \in [0,1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(||x||^p + ||y||^p)$$

for all $x, y \in X$. Rassias [11] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p}||x||^p$$

for all $x \in X$. Găvruta [1] generalized the Rassias' result: Let G be an abelian group and Y a Banach space. Denote by $\varphi : G \times G \to [0,\infty)$ a function such that

$$\widetilde{\varphi}(x,y) = \sum_{j=0}^{\infty} 2^{-j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in G$. Suppose that $f : G \to Y$ is a mapping satisfying

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x,y)$$

for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{2}\widetilde{\varphi}(x,x)$$

for all $x \in G$. C. Park [7] applied the Găvruta's result to linear functional equations in Banach modules over a C^*-algebra.

Jun and Lee [2] proved the following: Denote by $\varphi : X \setminus \{0\} \times X \setminus \{0\} \to [0,\infty)$ a function such that

$$\widetilde{\varphi}(x,y) = \sum_{j=0}^{\infty} 3^{-j}\varphi(3^j x, 3^j y) < \infty$$

for all $x, y \in X \setminus \{0\}$. Suppose that $f : X \to Y$ is a mapping satisfying

$$\|2f(\frac{x+y}{2}) - f(x) - f(y)\| \leq \varphi(x,y)$$

for all $x, y \in X \setminus \{0\}$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - f(0) - T(x)| \leq \frac{1}{3}(\widetilde{\varphi}(x,-x) + \widetilde{\varphi}(-x,3x))$$

for all $x \in X \setminus \{0\}$. C. Park and W. Park [9] applied the Jun and Lee's result to the Jensen's equation in Banach modules over a C^*-algebra.

Recently, Trif [17] proved the following: Let $q := \frac{l(d-1)}{d-l}$ and $r := -\frac{l}{d-l}$. Denote by $\varphi : X^d \to [0,\infty)$ a function such that

$$\widetilde{\varphi}(x_1, \cdots, x_d) = \sum_{j=0}^{\infty} q^{-j}\varphi(q^j x_1, \cdots, q^j x_d) < \infty$$

for all $x_1, \cdots, x_d \in X$. Suppose that $f : X \to Y$ is a mapping satisfying

$$\begin{aligned}\|d\ {}_{d-2}C_{l-2} f(\frac{x_1 + \cdots + x_d}{d}) &+ {}_{d-2}C_{l-1} \sum_{j=1}^{d} f(x_j) \\ &- l \sum_{1 \le j_1 < \cdots < j_l \le d} f(\frac{x_{j_1} + \cdots + x_{j_l}}{l})\| \le \varphi(x_1, \cdots, x_d)\end{aligned}$$

for all $x_1, \cdots, x_d \in X$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - f(0) - T(x)\| \le \frac{1}{l \cdot {}_{d-1}C_{l-1}} \widetilde{\varphi}(qx, \underbrace{rx, \cdots, rx}_{d-1 \text{ times}})$$

for all $x \in X$. And C. Park [8] applied the Trif's result to the Trif functional equation in Banach modules over a C^*-algebra. Several authors have investigated functional equations (see [10]–[16]).

Throughout this paper, let $q = \frac{l(d-1)}{d-l}$ and $r = -\frac{l}{d-l}$ for positive integers l, d with $2 \le l \le d-1$. Let A be a unital Lie JC^*-algebra with norm $||\cdot||$, unit e and unitary group $U(A) = \{u \in A \mid uu^* = u^*u = e\}$, and B a unital Lie JC^*-algebra with norm $\|\cdot\|$ and unit e'.

Using the stability methods of linear functional equations, we prove that every almost linear mapping $h : A \to B$ is a Lie JC^*-algebra homomorphism when $h(2^n u \circ y) = h(2^n u) \circ h(y)$, $h(3^n u \circ y) = h(3^n u) \circ h(y)$ or $h(q^n u \circ y) = h(q^n u) \circ h(y)$ for all $y \in A$, all $u \in U(A)$ and $n = 0, 1, 2, \cdots$, and that every almost linear almost multiplicative mapping $h : A \to B$ is a Lie JC^*-algebra homomorphism when $h(2x) = 2h(x)$, $h(3x) = 3h(x)$ or $h(qx) = qh(x)$ for all $x \in A$. We moreover prove the Cauchy–Rassias stability of Lie JC^*-algebra homomorphisms in unital Lie JC^*-algebras, and of Lie JC^*-algebra derivations in unital Lie JC^*-algebras.

2. Homomorphisms between Lie JC^*-algebras

Definition 2.1. A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *Lie JC^*-algebra homomorphism* if $H : A \to B$ satisfies

$$\begin{aligned} H(x \circ y) &= H(x) \circ H(y), \\ H([x, y]) &= [H(x), H(y)], \\ H(x^*) &= H(x)^* \end{aligned}$$

for all $x, y \in A$.

Remark 2.1. A $\mathbb{C}$-linear mapping $H : A \to B$ is a C^*-algebra homomorphism if and only if the mapping $H : A \to B$ is a Lie JC^*-algebra homomorphism.

Assume that H is a Lie JC^*-algebra homomorphism. Then

$$\begin{aligned} H(xy) &= H([x, y] + x \circ y) = H([x, y]) + H(x \circ y) \\ &= [H(x), H(y)] + H(x) \circ H(y) = H(x)H(y) \end{aligned}$$

for all $x, y \in A$. So H is a C^*-algebra homomorphism.

Assume that H is a C^*-algebra homomorphism. Then

$$H([x,y] = H(\frac{xy-yx}{2}) = \frac{H(x)H(y)-H(y)H(x)}{2} = [H(x), H(y)],$$
$$H(x \circ y) = H(\frac{xy+yx}{2}) = \frac{H(x)H(y)+H(y)H(x)}{2} = H(x) \circ H(y)$$

for all $x, y \in A$. So H is a Lie JC^*-algebra homomorphism.

We are going to investigate Lie JC^*-algebra homomorphisms between Lie JC^*-algebras associated with the Cauchy functional equation.

Theorem 2.1. *Let $h : A \to B$ be a mapping satisfying $h(0) = 0$ and $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $y \in A$, all $u \in U(A)$ and $n = 0, 1, 2, \cdots$, for which there exists a function $\varphi : A^4 \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y, z, w) := \sum_{j=0}^{\infty} 2^{-j} \varphi(2^j x, 2^j y, 2^j z, 2^j w) < \infty, \tag{2.i}$$
$$\|h(\mu x + \mu y + [z, w]) - \mu h(x) - \mu h(y) - [h(z), h(w)]\| \leq \varphi(x, y, z, w), \tag{2.ii}$$
$$\|h(2^n u^*) - h(2^n u)^*\| \leq \varphi(2^n u, 2^n u, 0, 0) \tag{2.iii}$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$, all $u \in U(A)$, all $x, y, z, w \in A$ and $n = 0, 1, 2, \cdots$. Assume that (2.iv) $\lim_{n\to\infty} \frac{h(2^n e)}{2^n} = e'$. *Then the mapping $h : A \to B$ is a Lie JC^*-algebra homomorphism.*

Proof. Put $z = w = 0$ and $\mu = 1 \in \mathbb{T}^1$ in (2.ii). It follows from Găvruta's Theorem [1] that there exists a unique additive mapping $H : A \to B$ such that

$$\|h(x) - H(x)\| \leq \frac{1}{2}\widetilde{\varphi}(x, x, 0, 0) \tag{2.0}$$

for all $x \in A$. The additive mapping $H : A \to B$ is given by

$$H(x) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n x) \tag{2.1}$$

for all $x \in A$.

By the assumption, for each $\mu \in \mathbb{T}^1$,

$$\|h(2^n \mu x) - 2\mu h(2^{n-1} x)\| \leq \varphi(2^{n-1} x, 2^{n-1} x, 0, 0)$$

for all $x \in A$. And one can show that

$$\|\mu h(2^n x) - 2\mu h(2^{n-1} x)\| \leq |\mu| \cdot \|h(2^n x) - 2h(2^{n-1} x)\| \leq \varphi(2^{n-1} x, 2^{n-1} x, 0, 0)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$. So

$$\begin{aligned}\|h(2^n \mu x) - \mu h(2^n x)\| \leq& \|h(2^n \mu x) - 2\mu h(2^{n-1} x)\| + \|2\mu h(2^{n-1} x) - \mu h(2^n x)\| \\ \leq& \varphi(2^{n-1} x, 2^{n-1} x, 0, 0) + \varphi(2^{n-1} x, 2^{n-1} x, 0, 0)\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$. Thus $2^{-n}\|h(2^n \mu x) - \mu h(2^n x)\| \to 0$ as $n \to \infty$ for all $\mu \in \mathbb{T}^1$ and all $x \in A$. Hence

$$H(\mu x) = \lim_{n\to\infty} \frac{h(2^n \mu x)}{2^n} = \lim_{n\to\infty} \frac{\mu h(2^n x)}{2^n} = \mu H(x) \tag{2.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$.

Now let $\lambda \in \mathbb{C}$ ($\lambda \neq 0$) and M an integer greater than $4|\lambda|$. Then $|\frac{\lambda}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By [3, Theorem 1], there exist three elements $\mu_1, \mu_2, \mu_3 \in \mathbb{T}^1$ such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$. And $H(x) = H(3 \cdot \frac{1}{3}x) = 3H(\frac{1}{3}x)$ for all $x \in A$. So $H(\frac{1}{3}x) = \frac{1}{3}H(x)$ for all $x \in A$. Thus by (2.2)

$$\begin{aligned}
H(\lambda x) &= H(\frac{M}{3} \cdot 3\frac{\lambda}{M}x) = M \cdot H(\frac{1}{3} \cdot 3\frac{\lambda}{M}x) = \frac{M}{3}H(3\frac{\lambda}{M}x) \\
&= \frac{M}{3}H(\mu_1 x + \mu_2 x + \mu_3 x) = \frac{M}{3}(H(\mu_1 x) + H(\mu_2 x) + H(\mu_3 x)) \\
&= \frac{M}{3}(\mu_1 + \mu_2 + \mu_3)H(x) = \frac{M}{3} \cdot 3\frac{\lambda}{M}H(x) \\
&= \lambda H(x)
\end{aligned}$$

for all $x \in A$. Hence

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in \mathbb{C}(\zeta, \eta \neq 0)$ and all $x, y \in A$. And $H(0x) = 0 = 0H(x)$ for all $x \in A$. So the unique additive mapping $H : A \to B$ is a $\mathbb{C}$-linear mapping.

Since $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $y \in A$, all $u \in U(A)$ and $n = 0, 1, 2, \cdots$,

$$H(u \circ y) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n u \circ y) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n u) \circ h(y) = H(u) \circ h(y) \tag{2.3}$$

for all $y \in A$ and all $u \in U(A)$. By the additivity of H and (2.3),

$$2^n H(u \circ y) = H(2^n u \circ y) = H(u \circ (2^n y)) = H(u) \circ h(2^n y)$$

for all $y \in A$ and all $u \in U(A)$. Hence

$$H(u \circ y) = \frac{1}{2^n} H(u) \circ h(2^n y) = H(u) \circ \frac{1}{2^n} h(2^n y) \tag{2.4}$$

for all $y \in A$ and all $u \in U(A)$. Taking the limit in (2.4) as $n \to \infty$, we obtain

$$H(u \circ y) = H(u) \circ H(y) \tag{2.5}$$

for all $y \in A$ and all $u \in U(A)$. Since H is $\mathbb{C}$-linear and each $x \in A$ is a finite linear combination of unitary elements (see [4, Theorem 4.1.7]), i.e., $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in U(A)$),

$$\begin{aligned}
H(x \circ y) &= H(\sum_{j=1}^m \lambda_j u_j \circ y) = \sum_{j=1}^m \lambda_j H(u_j \circ y) = \sum_{j=1}^m \lambda_j H(u_j) \circ H(y) \\
&= H(\sum_{j=1}^m \lambda_j u_j) \circ H(y) = H(x) \circ H(y)
\end{aligned}$$

for all $x, y \in A$.

By (2.iv), (2.3) and (2.5),

$$H(y) = H(e \circ y) = H(e) \circ h(y) = e' \circ h(y) = h(y)$$

for all $y \in A$. So

$$H(y) = h(y)$$

for all $y \in A$.

It follows from (2.1) that

$$H(x) = \lim_{n\to\infty} \frac{h(2^{2n}x)}{2^{2n}} \tag{2.6}$$

for all $x \in A$. Let $x = y = 0$ in (2.ii). Then we get

$$\|h([z,w]) - [h(z), h(w)]\| \le \varphi(0,0,z,w)$$

for all $z, w \in A$. So

$$\begin{aligned} \frac{1}{2^{2n}}\|h([2^n z, 2^n w]) - [h(2^n z), h(2^n w)]\| &\le \frac{1}{2^{2n}}\varphi(0,0,2^n z, 2^n w) \\ &\le \frac{1}{2^n}\varphi(0,0,2^n z, 2^n w) \end{aligned} \tag{2.7}$$

for all $z, w \in A$. By (2.i), (2.6), and (2.7),

$$\begin{aligned} H([z,w]) &= \lim_{n\to\infty} \frac{h(2^{2n}[z,w])}{2^{2n}} = \lim_{n\to\infty} \frac{h([2^n z, 2^n w])}{2^{2n}} \\ &= \lim_{n\to\infty} \frac{1}{2^{2n}}[h(2^n z), h(2^n w)] = \lim_{n\to\infty} [\frac{h(2^n z)}{2^n}, \frac{h(2^n w)}{2^n}] \\ &= [H(z), H(w)] \end{aligned}$$

for all $z, w \in A$.

By (2.i) and (2.iii), we get

$$\begin{aligned} H(u^*) &= \lim_{n\to\infty} \frac{h(2^n u^*)}{2^n} = \lim_{n\to\infty} \frac{h(2^n u)^*}{2^n} = (\lim_{n\to\infty} \frac{h(2^n u)}{2^n})^* \\ &= H(u)^* \end{aligned}$$

for all $u \in U(A)$. Since $H : A \to B$ is $\mathbb{C}$-linear and each $x \in A$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^m \lambda_j u_j$ $(\lambda_j \in \mathbb{C}, u_j \in U(A))$,

$$\begin{aligned} H(x^*) &= H(\sum_{j=1}^m \overline{\lambda_j} u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j)^* \\ &= (\sum_{j=1}^m \lambda_j H(u_j))^* = H(\sum_{j=1}^m \lambda_j u_j)^* = H(x)^* \end{aligned}$$

for all $x \in A$.

Therefore, the mapping $h : A \to B$ is a Lie JC^*-algebra homomorphism, as desired. ■

Corollary 2.2. *Let $h : A \to B$ be a mapping satisfying $h(0) = 0$ and $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $y \in A$, all $u \in U(A)$ and $n = 0, 1, 2, \cdots$, for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned} \|h(\mu x + \mu y + [z, w]) - &\mu h(x) - \mu h(y) - [h(z), h(w)]\| \\ &\leq \theta(||x||^p + ||y||^p + ||z||^p + ||w||^p), \\ \|h(2^n u^*) - h(2^n u)^*\| &\leq 2 \cdot 2^{np} \theta \end{aligned}$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(A)$, all $x, y, z, w \in A$ and $n = 0, 1, 2, \cdots$. Assume that $\lim_{n\to\infty} \frac{h(2^n e)}{2^n} = e'$. Then the mapping $h : A \to B$ is a Lie JC^*-algebra homomorphism.

Proof. Define $\varphi(x, y, z, w) = \theta(||x||^p + ||y||^p + ||z||^p + ||w||^p)$, and apply Theorem 2.1. ■

Theorem 2.3. *Let $h : A \to B$ be a mapping satisfying $h(0) = 0$ and $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $y \in A$, all $u \in U(A)$ and $n = 0, 1, 2, \cdots$, for which there exists a function $\varphi : A^4 \to [0, \infty)$ satisfying* (2.i), (2.iii) *and* (2.iv) *such that*

$$\|h(\mu x + \mu y + [z, w]) - \mu h(x) - \mu h(y) - [h(z), h(w)]\| \leq \varphi(x, y, z, w) \qquad (2.v)$$

for $\mu = 1, i$, and all $x, y, z, w \in A$. If $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then the mapping $h : A \to B$ is a Lie JC^*-algebra homomorphism.

Proof. Put $z = w = 0$ and $\mu = 1$ in (2.v). By the same reasoning as in the proof of Theorem 2.1, there exists a unique additive mapping $H : A \to B$ satisfying (2.0). The additive mapping $H : A \to B$ is given by

$$H(x) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n x)$$

for all $x \in A$. By the same reasoning as in the proof of [11, Theorem], the additive mapping $H : A \to B$ is $\mathbb{R}$-linear.

Put $y = z = w = 0$ and $\mu = i$ in (2.v). By the same method as in the proof of Theorem 2.1, one can obtain that

$$H(ix) = \lim_{n\to\infty} \frac{h(2^n ix)}{2^n} = \lim_{n\to\infty} \frac{ih(2^n x)}{2^n} = iH(x)$$

for all $x \in A$. For each element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. So

$$\begin{aligned} H(\lambda x) &= H(sx + itx) = sH(x) + tH(ix) = sH(x) + itH(x) = (s + it)H(x) \\ &= \lambda H(x) \end{aligned}$$

for all $\lambda \in \mathbb{C}$ and all $x \in A$. So

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in \mathbb{C}$, and all $x, y \in A$. Hence the additive mapping $H : A \to B$ is $\mathbb{C}$-linear.

The rest of the proof is the same as in the proof of Theorem 2.1. ■

Theorem 2.4. *Let $h : A \to B$ be a mapping satisfying $h(2x) = 2h(x)$ for all $x \in A$ for which there exists a function $\varphi : A^4 \to [0, \infty)$ satisfying* (2.i), (2.ii), (2.iii) *and* (2.iv) *such that*

$$\|h(2^n u \circ y) - h(2^n u) \circ h(y)\| \le \varphi(u, y, 0, 0) \tag{2.vi}$$

for all $y \in A$, all $u \in U(A)$ and $n = 0, 1, 2, \cdots$. Then the mapping $h : A \to B$ is a Lie JC^*-algebra homomorphism.

Proof. By the same reasoning as in the proof of Theorem 2.1, there exists a unique $\mathbb{C}$-linear mapping $H : A \to B$ satisfying (2.0).

By (2.vi) and the assumption that $h(2x) = 2h(x)$ for all $x \in A$,

$$\begin{aligned}\|h(2^n u \circ y) - h(2^n u) \circ h(y)\| &= \frac{1}{4^m}\|h(2^m 2^n u \circ 2^m y) - h(2^m 2^n u) \circ h(2^m y)\| \\ &\le \frac{1}{4^m}\varphi(2^m u, 2^m y, 0, 0) \le \frac{1}{2^m}\varphi(2^m u, 2^m y, 0, 0),\end{aligned}$$

which tends to zero as $m \to \infty$ by (2.i). So

$$h(2^n u \circ y) = h(2^n u) \circ h(y)$$

for all $y \in A$, all $u \in U(A)$ and $n = 0, 1, 2, \cdots$. But by (2.1),

$$H(x) = \lim_{n \to \infty} \frac{1}{2^n} h(2^n x) = h(x)$$

for all $x \in A$.

The rest of the proof is the same as in the proof of Theorem 2.1. ■

Now we are going to investigate Lie JC^*-algebra homomorphisms between Lie JC^*-algebras associated with the Jensen functional equation.

Theorem 2.5. *Let $h : A \to B$ be a mapping satisfying $h(0) = 0$ and $h(3^n u \circ y) = h(3^n u) \circ h(y)$ for all $y \in A$, all $u \in U(A)$ and $n = 0, 1, 2, \cdots$, for which there exists a function $\varphi : (A \setminus \{0\})^4 \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y, z, w) := \sum_{j=0}^{\infty} 3^{-j} \varphi(3^j x, 3^j y, 3^j z, 3^j w) < \infty, \tag{2.vii}$$

$$\begin{aligned}\|2h(\frac{\mu x + \mu y + [z, w]}{2}) - \mu h(x) - \mu h(y) - &[h(z), h(w)]\| \\ &\le \varphi(x, y, z, w), \qquad (2.viii)\\ \|h(3^n u^*) - h(3^n u)^*\| \le &\varphi(3^n u, 3^n u, 0, 0) \qquad (2.ix)\end{aligned}$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(A)$, all $x, y, z, w \in A$ and $n = 0, 1, 2, \cdots$. Assume that $\lim_{n\to\infty} \frac{h(3^n e)}{3^n} = e'$. Then the mapping $h : A \to B$ is a Lie JC^*-algebra homomorphism.

Proof. Put $z = w = 0$ and $\mu = 1 \in \mathbb{T}^1$ in (2.viii). It follows from Jun and Lee's Theorem [2, Theorem 1] that there exists a unique additive mapping $H : A \to B$ such that

$$\|h(x) - H(x)\| \leq \frac{1}{3}(\widetilde{\varphi}(x, -x, 0, 0) + \widetilde{\varphi}(-x, 3x, 0, 0))$$

for all $x \in A \setminus \{0\}$. The additive mapping $H : A \to B$ is given by

$$H(x) = \lim_{n\to\infty} \frac{1}{3^n} h(3^n x)$$

for all $x \in A$.

By the assumption, for each $\mu \in \mathbb{T}^1$,

$$\|2h(3^n \mu x) - \mu h(2 \cdot 3^{n-1} x) - \mu h(4 \cdot 3^{n-1} x)\| \leq \varphi(2 \cdot 3^{n-1} x, 4 \cdot 3^{n-1} x, 0, 0)$$

for all $x \in A \setminus \{0\}$. And one can show that

$$\begin{aligned}
&\|\mu h(2 \cdot 3^{n-1} x) + \mu h(4 \cdot 3^{n-1} x) - 2\mu h(3^n x)\| \\
&\qquad \leq |\mu| \cdot \|h(2 \cdot 3^{n-1} x) + h(4 \cdot 3^{n-1} x) - 2h(3^n x)\| \\
&\qquad \leq \varphi(2 \cdot 3^{n-1} x, 4 \cdot 3^{n-1} x, 0, 0)
\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A \setminus \{0\}$. So

$$\begin{aligned}
\|h(3^n \mu x) - \mu h(3^n x)\| =& \|h(3^n \mu x) - \frac{1}{2}\mu h(2 \cdot 3^{n-1} x) - \frac{1}{2}\mu h(4 \cdot 3^{n-1} x) \\
&+ \frac{1}{2}\mu h(2 \cdot 3^{n-1} x) + \frac{1}{2}\mu h(4 \cdot 3^{n-1} x) - \mu h(3^n x)\| \\
\leq& \frac{1}{2}\|2h(3^n \mu x) - \mu h(2 \cdot 3^{n-1} x) - \mu h(4 \cdot 3^{n-1} x)\| \\
&+ \frac{1}{2}\|\mu h(2 \cdot 3^{n-1} x) + \mu h(4 \cdot 3^{n-1} x) - 2\mu h(3^n x)\| \\
\leq& \frac{2}{2}\varphi(2 \cdot 3^{n-1} x, 4 \cdot 3^{n-1} x, 0, 0)
\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A \setminus \{0\}$. Thus $3^{-n}\|h(3^n \mu x) - \mu h(3^n x)\| \to 0$ as $n \to \infty$ for all $\mu \in \mathbb{T}^1$ and all $x \in A \setminus \{0\}$. Hence

$$H(\mu x) = \lim_{n\to\infty} \frac{h(3^n \mu x)}{3^n} = \lim_{n\to\infty} \frac{\mu h(3^n x)}{3^n} = \mu H(x)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A \setminus \{0\}$.

By the same reasoning as in the proof of Theorem 2.1, the unique additive mapping $H : A \to B$ is a $\mathbb{C}$-linear mapping.

By a similar method to the proof of Theorem 2.1, one can show that the mapping $h : A \to B$ is a Lie JC^*-algebra homomorphism. ∎

Corollary 2.6. *Let $h : A \to B$ be a mapping satisfying $h(0) = 0$ and $h(3^n u \circ y) = h(3^n u) \circ h(y)$ for all $y \in A$, all $u \in U(A)$ and $n = 0, 1, 2, \cdots$, for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\|2h(\frac{\mu x + \mu y + [z, w]}{2}) - \mu h(x) - \mu h(y) - [h(z), h(w)]\| \leq \theta(||x||^p + ||y||^p + ||z||^p + ||w||^p),$$

$$\|h(3^n u^*) - h(3^n u)^*\| \leq 2 \cdot 3^{np}\theta$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(A)$, all $x, y, z, w \in A \setminus \{0\}$ and $n = 0, 1, 2, \cdots$. Assume that $lim_{n\to\infty} \frac{h(3^n e)}{3^n} = e'$. Then the mapping $h : A \to B$ is a Lie JC^*-algebra homomorphism.

Proof. Define $\varphi(x, y, z, w) = \theta(||x||^p + ||y||^p + ||z||^p + ||w||^p)$, and apply Theorem 2.5. ∎

One can obtain similar results to Theorems 2.3 and 2.4 for the Jensen functional equation.

Finally, we are going to investigate Lie JC^*-algebra homomorphisms between Lie JC^*-algebras associated with the Trif functional equation.

Theorem 2.7. *Let $h : A \to B$ be a mapping satisfying $h(0) = 0$ and $h(q^n u \circ y) = h(q^n u) \circ h(y)$ for all $y \in A$, all $u \in U(A)$ and $n = 0, 1, 2, \cdots$, for which there exists a function $\varphi : A^{d+2} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x_1, \cdots, x_d, z, w) := \sum_{j=0}^{\infty} q^{-j}\varphi(q^j x_1, \cdots, q^j x_d, q^j z, q^j w) < \infty, \quad (2.x)$$

$$\|d\ {}_{d-2}C_{l-2}h(\frac{\mu x_1 + \cdots + \mu x_d}{d} + \frac{[z, w]}{d\ {}_{d-2}C_{l-2}}) + {}_{d-2}C_{l-1}\sum_{j=1}^{d} \mu h(x_j) - l \sum_{1\leq j_1<\cdots<j_l\leq d} \mu h(\frac{x_{j_1} + \cdots + x_{j_l}}{l}) - [h(z), h(w)]\| \quad (2.xi)$$

$$\leq \varphi(x_1, \cdots, x_d, z, w),$$

$$\|h(q^n u^*) - h(q^n u)^*\| \leq \varphi(\underbrace{q^n u, \cdots, q^n u}_{d \text{ times}}, 0, 0) \quad (2.xii)$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(A)$, all $x_1, \cdots, x_d, z, w \in A$ and $n = 0, 1, 2, \cdots$. Assume that $\lim_{n\to\infty} \frac{h(q^n e)}{q^n} = e'$. Then the mapping $h : A \to B$ is a Lie JC^*-algebra homomorphism.

Proof. Put $z = w = 0$ and $\mu = 1 \in \mathbb{T}^1$ in (2.xi). It follows from Trif's Theorem [17, Theorem 3.1] that there exists a unique additive mapping $H : A \to B$ such that

$$\|h(x) - H(x)\| \leq \frac{1}{l \cdot {}_{d-1}C_{l-1}}\widetilde{\varphi}(qx, \underbrace{rx, \cdots, rx}_{d-1 \text{ times}}, 0, 0)$$

for all $x \in A$. The additive mapping $H : A \to B$ is given by

$$H(x) = \lim_{n\to\infty} \frac{1}{q^n} h(q^n x)$$

for all $x \in A$. ∎

Put $x_1 = \cdots = x_d = x$ and $z = w = 0$ in (2.xi). For each $\mu \in \mathbb{T}^1$,

$$\|d\ {}_{d-2}C_{l-2}(h(\mu x) - \mu h(x))\| \le \varphi(\underbrace{x, \cdots, x}_{d \text{ times}}, 0, 0)$$

for all $x \in A$. So

$$q^{-n}\|d\ {}_{d-2}C_{l-2}(h(\mu q^n x) - \mu h(q^n x))\| \le q^{-n}\varphi(\underbrace{q^n x, \cdots, q^n x}_{d \text{ times}}, 0, 0)$$

for all $x \in A$. By (2.x),

$$q^{-n}\|d\ {}_{d-2}C_{l-2}(h(\mu q^n x) - \mu h(q^n x))\| \to 0$$

as $n \to \infty$ for all $\mu \in \mathbb{T}^1$ and all $x \in A$. Thus

$$q^{-n}\|h(\mu q^n x) - \mu h(q^n x)\| \to 0$$

as $n \to \infty$ for all $\mu \in \mathbb{T}^1$ and all $x \in A$. Hence

$$H(\mu x) = \lim_{n\to\infty} \frac{h(q^n \mu x)}{q^n} = \lim_{n\to\infty} \frac{\mu h(q^n x)}{q^n} = \mu H(x)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$.

By the same reasoning as in the proof of Theorem 2.1, the unique additive mapping $H : A \to B$ is a $\mathbb{C}$-linear mapping.

By a similar method to the proof of Theorem 2.1, one can show that the mapping $h : A \to B$ is a Lie JC^*-algebra homomorphism. ■

Corollary 2.8. *Let $h : A \to B$ be a mapping satisfying $h(0) = 0$ and $h(q^n u \circ y) = h(q^n u) \circ h(y)$ for all $y \in A$, all $u \in U(A)$ and $n = 0, 1, 2, \cdots$, for which there exist constants $\theta \ge 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned}\|d\ {}_{d-2}C_{l-2}h(\frac{\mu x_1 + \cdots + \mu x_d}{d} + \frac{[z, w]}{d\ {}_{d-2}C_{l-2}}) &+ {}_{d-2}C_{l-1}\sum_{j=1}^{d} \mu h(x_j) \\ -l \sum_{1 \le j_1 < \cdots < j_l \le d} \mu h(\frac{x_{j_1} + \cdots + x_{j_l}}{l}) &- [h(z), h(w)]\| \\ &\le \theta(\sum_{j=1}^{d} ||x_j||^p + ||z||^p + ||w||^p),\end{aligned}$$

$$\|h(q^n u^*) - h(q^n u)^*\| \le d q^{np}\theta$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(A)$, all $x_1, \cdots, x_d, z, w \in A$ and $n = 0, 1, 2, \cdots$. Assume that $\lim_{n\to\infty} \frac{h(q^n e)}{q^n} = e'$. Then the mapping $h : A \to B$ is a Lie JC^*-algebra homomorphism.

Proof. Define $\varphi(x_1, \cdots, x_d, z, w) = \theta(\sum_{j=1}^{d} ||x_j||^p + ||z||^p + ||w||^p)$, and apply Theorem 2.7. ■

One can obtain similar results to Theorems 2.3 and 2.4 for the Trif functional equation.

3. Stability of Lie JC^*-algebra homomorphisms in Lie JC^*-algebras

We are going to show the Cauchy–Rassias stability of Lie JC^*-algebra homomorphisms in Lie JC^*-algebras associated with the Cauchy functional equation.

Theorem 3.1. *Let $h : A \to B$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : A^6 \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y, z, w, a, b) := \sum_{j=0}^{\infty} 2^{-j} \varphi(2^j x, 2^j y, 2^j z, 2^j w, 2^j a, 2^j b) < \infty, \qquad (3.i)$$

$$\|h(\mu x + \mu y + [z, w] + a \circ b) - \mu h(x) - \mu h(y) - [h(z), h(w)] - h(a) \circ h(b)\| \\ \leq \varphi(x, y, z, w, a, b), \qquad (3.ii)$$

$$\|h(2^n u^*) - h(2^n u)^*\| \leq \varphi(2^n u, 2^n u, 0, 0, 0, 0) \qquad (3.iii)$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(A)$, $n = 0, 1, 2, \cdots$, and all $x, y, z, w, a, b \in A$. Then there exists a unique Lie JC^*-algebra homomorphism $H : A \to B$ such that

$$\|h(x) - H(x)\| \leq \frac{1}{2} \widetilde{\varphi}(x, x, 0, 0, 0, 0) \qquad (3.iv)$$

for all $x \in A$.

Proof. Put $z = w = a = b = 0$ and $\mu = 1 \in \mathbb{T}^1$ in (3.ii). It follows from Găvruta's Theorem [1] that there exists a unique additive mapping $H : A \to B$ satisfying (3.iv). The additive mapping $H : A \to B$ is given by

$$H(x) = \lim_{n \to \infty} \frac{1}{2^n} h(2^n x)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 2.1. ∎

Corollary 3.2. *Let $h : A \to B$ be a mapping with $h(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\|h(\mu x + \mu y + [z, w] + a \circ b) - \mu h(x) - \mu h(y) - [h(z), h(w)] - h(a) \circ h(b)\| \\ \leq \theta(||x||^p + ||y||^p + ||z||^p + ||w||^p + ||a||^p + ||b||^p),$$

$$\|h(2^n u^*) - h(2^n u)^*\| \leq 2 \cdot 2^{np} \theta$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(A)$, $n = 0, 1, 2, \cdots$, and all $x, y, z, w, a, b \in A$. Then there exists a unique Lie JC^*-algebra homomorphism $H : A \to B$ such that

$$\|h(x) - H(x)\| \leq \frac{2\theta}{2 - 2^p} ||x||^p$$

for all $x \in A$.

Proof. Define $\varphi(x, y, z, w, a, b) = \theta(||x||^p + ||y||^p + ||z||^p + ||w||^p + ||a||^p + ||b||^p)$, and apply Theorem 3.1. ∎

Theorem 3.3. *Let $h : A \to B$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : A^6 \to [0, \infty)$ satisfying* (3.i) *and* (3.iii) *such that*

$$\|h(\mu x + \mu y + [z,w] + a \circ b) - \mu h(x) - \mu h(y) - [h(z), h(w)] - h(a) \circ h(b)\| \\ \leq \varphi(x, y, z, w, a, b)$$

for $\mu = 1, i$, and all $x, y, z, w, a, b \in A$. If $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique Lie JC^*-algebra homomorphism $H : A \to B$ satisfying (3.iv).

Proof. The proof is similar to the proof of Theorem 2.3. ■

We are going to show the Cauchy–Rassias stability of Lie JC^*-algebra homomorphisms in Lie JC^*-algebras associated with the Jensen functional equation.

Theorem 3.4. *Let $h : A \to B$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : (A \setminus \{0\})^6 \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y, z, w, a, b) = \sum_{j=0}^{\infty} 3^{-j} \varphi(3^j x, 3^j y, 3^j z, 3^j w, 3^j a, 3^j b) < \infty, \quad (3.v)$$

$$\|2h(\frac{\mu x + \mu y + [z,w] + a \circ b}{2}) - \mu h(x) - \mu h(y) - [h(z), h(w)] - h(a) \circ h(b)\| \\ \leq \varphi(x, y, z, w, a, b), \quad (3.vi)$$

$$\|h(3^n u^*) - h(3^n u)^*\| \leq \varphi(3^n u, 3^n u, 0, 0, 0, 0) \quad (3.vii)$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(A)$, $n = 0, 1, 2, \cdots$, and all $x, y, z, w, a, b \in A \setminus \{0\}$. Then there exists a unique Lie JC^*-algebra homomorphism $H : A \to B$ such that

$$\|h(x) - H(x)\| \leq \frac{1}{3}(\widetilde{\varphi}(x, -x, 0, 0, 0, 0) + \widetilde{\varphi}(-x, 3x, 0, 0, 0, 0)) \quad (3.viii)$$

for all $x \in A \setminus \{0\}$.

Proof. Put $z = w = a = b = 0$ and $\mu = 1 \in \mathbb{T}^1$ in (3.vi). It follows from Jun and Lee's Theorem [2, Theorem 1] that there exists a unique additive mapping $H : A \to B$ satisfying (3.viii). The additive mapping $H : A \to B$ is given by

$$H(x) = \lim_{n \to \infty} \frac{1}{3^n} h(3^n x)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 2.5. ■

Corollary 3.5. *Let $h : A \to B$ be a mapping with $h(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\|2h(\frac{\mu x + \mu y + [z,w] + a \circ b}{2}) - \mu h(x) - \mu h(y) - [h(z), h(w)] - h(a) \circ h(b)\| \\ \leq \theta(||x||^p + ||y||^p + ||z||^p + ||w||^p + ||a||^p + ||b||^p),$$

$$\|h(3^n u^*) - h(3^n u)^*\| \leq 2 \cdot 3^{np} \theta$$

for all $\mu \in \mathbb{T}^1$, *all* $u \in U(A)$, $n = 0, 1, 2, \cdots$, *and all* $x, y, z, w, a, b \in A \setminus \{0\}$. *Then there exists a unique Lie* JC^**-algebra homomorphism* $H : A \to B$ *such that*

$$\|h(x) - H(x)\| \le \frac{3 + 3^p}{3 - 3^p}\theta||x||^p$$

for all $x \in A \setminus \{0\}$.

Proof. Define $\varphi(x, y, z, w, a, b) = \theta(||x||^p + ||y||^p + ||z||^p + ||w||^p + ||a||^p + ||b||^p)$, and apply Theorem 3.4. ∎

One can obtain a similar result to Theorem 3.3 for the Jensen functional equation.

Now we are going to show the Cauchy–Rassias stability of Lie JC^*-algebra homomorphisms in Lie JC^*-algebras associated with the Trif functional equation.

Theorem 3.6. *Let* $h : A \to B$ *be a mapping with* $h(0) = 0$ *for which there exists a function* $\varphi : A^{d+4} \to [0, \infty)$ *such that*

$$\widetilde{\varphi}(x_1, \cdots, x_d, z, w, a, b) := \sum_{j=0}^{\infty} q^{-j}\varphi(q^j x_1, \cdots, q^j x_d, q^j z, q^j w, q^j a, q^j b) < \infty, \tag{3.ix}$$

$$\begin{aligned}\|d\ {}_{d-2}C_{l-2}h(\frac{\mu x_1 + \cdots + \mu x_d}{d} + \frac{[z, w] + a \circ b}{d\ {}_{d-2}C_{l-2}}) + {}_{d-2}C_{l-1}\sum_{j=1}^{d}\mu h(x_j) \\ -l \sum_{1 \le j_1 < \cdots < j_l \le d} \mu h(\frac{x_{j_1} + \cdots + x_{j_l}}{l}) - [h(z), h(w)] - h(a) \circ h(b)\| \\ \le \varphi(x_1, \cdots, x_d, z, w, a, b),\end{aligned} \tag{3.x}$$

$$\|h(q^n u^*) - h(q^n u)^*\| \le \varphi(\underbrace{q^n u, \cdots, q^n u}_{d \text{ times}}, 0, 0, 0, 0) \tag{3.xi}$$

for all $\mu \in \mathbb{T}^1$, *all* $u \in U(A)$, $n = 0, 1, 2, \cdots$, *and all* $x_1, \cdots, x_d, z, w, a, b \in A$. *Then there exists a unique Lie* JC^**-algebra homomorphism* $H : A \to B$ *such that*

$$\|h(x) - H(x)\| \le \frac{1}{l \cdot {}_{d-1}C_{l-1}}\widetilde{\varphi}(qx, \underbrace{rx, \cdots, rx}_{d-1 \text{ times}}, 0, 0, 0, 0) \tag{3.xii}$$

for all $x \in A$.

Proof. Put $z = w = a = b = 0$ and $\mu = 1 \in \mathbb{T}^1$ in (3.x). It follows from Trif's Theorem [17, Theorem 3.1] that there exists a unique additive mapping $H : A \to B$ satisfying (3.xii). The additive mapping $H : A \to B$ is given by

$$H(x) = \lim_{n\to\infty} \frac{1}{q^n}h(q^n x)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 2.7. ∎

Corollary 3.7. *Let $h : A \to B$ be a mapping with $h(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0,1)$ such that*

$$\|d\ {}_{d-2}C_{l-2}h(\frac{\mu x_1 + \cdots + \mu x_d}{d} + \frac{[z,w] + a \circ b}{d\ {}_{d-2}C_{l-2}}) + {}_{d-2}C_{l-1}\sum_{j=1}^{d} \mu h(x_j)$$
$$-l \sum_{1 \leq j_1 < \cdots < j_l \leq d} \mu h(\frac{x_{j_1} + \cdots + x_{j_l}}{l}) - [h(z), h(w)] - h(a) \circ h(b)\|$$
$$\leq \theta(\sum_{j=1}^{d} ||x_j||^p + ||z||^p + ||w||^p + ||a||^p + ||b||^p),$$
$$\|h(q^n u^*) - h(q^n u)^*\| \leq d q^{np}\theta$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(A)$, $n = 0, 1, 2, \cdots$, and all $x_1, \cdots, x_d, z, w, a, b \in A$. Then there exists a unique Lie JC^*-algebra homomorphism $H : A \to B$ such that

$$\|h(x) - H(x)\| \leq \frac{q^{1-p}(q^p + (d-1)r^p)\theta}{l\ {}_{d-1}C_{l-1}(q^{1-p} - 1)}||x||^p$$

for all $x \in A$.

Proof. Define $\varphi(x_1, \cdots, x_d, z, w, a, b) = \theta(\sum_{j=1}^{d} ||x_j||^p + ||z||^p + ||w||^p + ||a||^p + ||b||^p)$, and apply Theorem 3.6. ∎

One can obtain a similar result to Theorem 3.3 for the Trif functional equation.

4. Stability of Lie JC^*-algebra derivations in Lie JC^*-algebras

Definition 4.1. A $\mathbb{C}$-linear mapping $D : A \to A$ is called a *Lie JC^*-algebra derivation* if $D : A \to A$ satisfies

$$D(x \circ y) = (Dx) \circ y + x \circ (Dy),$$
$$D([x, y]) = [Dx, y] + [x, Dy],$$
$$D(x^*) = D(x)^*$$

for all $x, y \in A$.

Remark 4.1. A $\mathbb{C}$-linear mapping $D : A \to A$ is a C^*-algebra derivation if and only if the mapping $D : A \to A$ is a Lie JC^*-algebra derivation.

Assume that D is a Lie JC^*-algebra derivation. Then

$$D(xy) = D([x, y] + x \circ y) = D([x, y]) + D(x \circ y)$$
$$= [Dx, y] + [x, Dy] + (Dx) \circ y + x \circ (Dy) = (Dx)y + x(Dy)$$

for all $x, y \in A$. So D is a C^*-algebra derivation.

Assume that D is a C^*-algebra derivation. Then

$$\begin{aligned} D([x,y]) &= D(\frac{xy-yx}{2}) = \frac{(Dx)y + x(Dy) - (Dy)x - y(Dx)}{2} \\ &= [Dx,y] + [x,Dy], \\ D(x \circ y) &= D(\frac{xy+yx}{2}) = \frac{(Dx)y + x(Dy) + (Dy)x + y(Dx)}{2} \\ &= (Dx) \circ y + x \circ (Dy) \end{aligned}$$

for all $x, y \in A$. So H is a Lie JC^*-algebra derivation.

We are going to show the Cauchy–Rassias stability of Lie JC^*-algebra derivations in Lie JC^*-algebras associated with the Cauchy functional equation.

Theorem 4.1. *Let $h : A \to A$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : A^6 \to [0,\infty)$ satisfying* (3.i) *and* (3.iii) *such that*

$$\begin{aligned} \|h(\mu x + \mu y + [z,w] + a \circ b) - \mu h(x) - \mu h(y) - [h(z),w] - [z,h(w)] \\ - h(a) \circ b - a \circ h(b)\| \le \varphi(x,y,z,w,a,b) \quad (4.i) \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z, w, a, b \in A$. Then there exists a unique Lie JC^*-algebra derivation $D : A \to A$ such that

$$\|h(x) - D(x)\| \le \frac{1}{2}\widetilde{\varphi}(x,x,0,0,0,0) \tag{4.ii}$$

for all $x \in A$.

Proof. Put $z = w = a = b = 0$ in (4.i). By the same reasoning as in the proof of Theorem 2.1, there exists a unique $\mathbb{C}$-linear involutive mapping $D : A \to A$ satisfying (4.ii). The $\mathbb{C}$-linear mapping $D : A \to A$ is given by

$$D(x) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n x) \tag{4.1}$$

for all $x \in A$.

It follows from (4.1) that

$$D(x) = \lim_{n\to\infty} \frac{h(2^{2n}x)}{2^{2n}} \tag{4.2}$$

for all $x \in A$. Let $x = y = a = b = 0$ in (4.i). Then we get

$$\|h([z,w]) - [h(z),w] - [z,h(w)]\| \le \varphi(0,0,z,w,0,0)$$

for all $z, w \in A$. Since

$$\frac{1}{2^{2n}}\varphi(0,0,2^n z, 2^n w,0,0) \le \frac{1}{2^n}\varphi(0,0,2^n z, 2^n w,0,0),$$

$$\frac{1}{2^{2n}}\|h([2^n z, 2^n w]) - [h(2^n z), 2^n w] - [2^n z, h(2^n w)]\| \le \frac{1}{2^{2n}}\varphi(0,0,2^n z, 2^n w, 0,0)$$
$$\le \frac{1}{2^n}\varphi(0,0,2^n z, 2^n w, 0, 0) \quad (4.3)$$

for all $z, w \in A$. By (3.i), (4.2), and (4.3),

$$\begin{aligned} D([z,w]) &= \lim_{n\to\infty} \frac{h(2^{2n}[z,w])}{2^{2n}} = \lim_{n\to\infty} \frac{h([2^n z, 2^n w])}{2^{2n}} \\ &= \lim_{n\to\infty}([\frac{h(2^n z)}{2^n}, \frac{2^n w}{2^n}] + [\frac{2^n z}{2^n}, \frac{h(2^n w)}{2^n}]) \\ &= [D(z), w] + [z, D(w)] \end{aligned}$$

for all $z, w \in A$.

Similarly, one can obtain that

$$\begin{aligned} D(a\circ b) &= \lim_{n\to\infty} \frac{h(2^{2n} a \circ b)}{2^{2n}} = \lim_{n\to\infty} \frac{h((2^n a)\circ(2^n b))}{2^{2n}} \\ &= \lim_{n\to\infty}\left((\frac{h(2^n a)}{2^n}) \circ (\frac{2^n b}{2^n}) + (\frac{2^n a}{2^n} \circ (\frac{h(2^n b)}{2^n})\right) \\ &= (Da)\circ b + a \circ (Db) \end{aligned}$$

for all $a, b \in A$. Hence the $\mathbb{C}$-linear mapping $D : A \to A$ is a Lie JC^*-algebra derivation satisfying (4.ii), as desired. ■

Corollary 4.2. *Let $h : A \to A$ be a mapping with $h(0) = 0$ for which there exist constants $\theta \ge 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned} \|h(\mu x + \mu y + [z,w] + a\circ b) &- \mu h(x) - \mu h(y) - [h(z), w] - [z, h(w)] \\ &- h(a)\circ b - a \circ h(B)\| \\ &\le \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p + \|a\|^p + \|b\|^p), \\ \|h(2^n u^*) - h(2^n u)^*\| &\le 2 \cdot 2^{np}\theta \end{aligned}$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(A)$, $n = 0, 1, 2, \cdots$, and all $x, y, z, w, a, b \in A$. Then there exists a unique Lie JC^*-algebra derivation $D : A \to A$ such that

$$\|h(x) - D(x)\| \le \frac{2\theta}{2 - 2^p}\|x\|^p$$

for all $x \in A$.

Proof. Define $\varphi(x,y,z,w,a,b) = \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p + \|a\|^p + \|b\|^p)$, and apply Theorem 4.1. ■

Theorem 4.3. *Let $h : A \to A$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : A^6 \to [0, \infty)$ satisfying (3.i) and (3.iii) such that*

$$\begin{aligned} \|h(\mu x + \mu y + [z,w] + a\circ b) &- \mu h(x) - \mu h(y) - [h(z), w] - [z, h(w)] \\ &- h(a)\circ b - a\circ h(b)\| \le \varphi(x,y,z,w,a,b) \end{aligned}$$

for $\mu = 1, i$*, and all* $x, y, z, w, a, b \in A$*. If* $h(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in A$*, then there exists a unique Lie* JC^**-algebra derivation* $D : A \to A$ *satisfying* (4.ii)*.*

Proof. By the same reasoning as in the proof of Theorem 2.3, there exists a unique $\mathbb{C}$-linear mapping $D : A \to A$ satisfying (4.ii).

The rest of the proof is the same as in the proofs of Theorems 2.1, 3.1 and 4.1. ■

We are going to show the Cauchy–Rassias stability of Lie JC^*-algebra derivations in Lie JC^*-algebras associated with the Jensen functional equation.

Theorem 4.4. *Let* $h : A \to A$ *be a mapping with* $h(0) = 0$ *for which there exists a function* $\varphi : (A \setminus \{0\})^6 \to [0, \infty)$ *satisfying* (3.v) *and* (3.vii) *such that*

$$\|2h(\frac{\mu x + \mu y + [z, w] + a \circ b}{2}) - \mu h(x) - \mu h(y) - [h(z), w] - [z, h(w)] - h(a) \circ b - a \circ h(b)\| \le \varphi(x, y, z, w, a, b) \tag{4.iii}$$

for all $\mu \in \mathbb{T}^1$ *and all* $x, y, z, w, a, b \in A \setminus \{0\}$*. Then there exists a unique Lie* JC^**-algebra derivation* $D : A \to A$ *such that*

$$\|h(x) - D(x)\| \le \frac{1}{3}(\widetilde{\varphi}(x, -x, 0, 0, 0, 0) + \widetilde{\varphi}(-x, 3x, 0, 0, 0, 0)) \tag{4.iv}$$

for all $x \in A \setminus \{0\}$*.*

Proof. Put $z = w = a = b = 0$ in (4.iii). By the same reasoning as in the proof of Theorem 2.5, there exists a unique $\mathbb{C}$-linear involutive mapping $D : A \to A$ satisfying (4.iv). The $\mathbb{C}$-linear mapping $D : A \to A$ is given by

$$D(x) = \lim_{n\to\infty} \frac{1}{3^n} h(3^n x) \tag{4.4}$$

for all $x \in A$.

It follows from (4.4) that

$$D(x) = \lim_{n\to\infty} \frac{h(3^{2n} x)}{3^{2n}} \tag{4.5}$$

for all $x \in A$. Let $x = y = a = b = 0$ in (4.iii). Then we get

$$\|2h(\frac{[z, w]}{2}) - [h(z), w] - [z, h(w)]\| \le \varphi(0, 0, z, w, 0, 0)$$

for all $z, w \in A$. Since

$$\frac{1}{3^{2n}} \varphi(0, 0, 3^n z, 3^n w, 0, 0) \le \frac{1}{3^n} \varphi(0, 0, 3^n z, 3^n w, 0, 0),$$

$$\begin{aligned} \frac{1}{3^{2n}} \|2h(\frac{1}{2}[3^n z, 3^n w]) - [h(3^n z), 3^n w] - [3^n z, h(3^n w)]\| &\le \frac{1}{3^{2n}} \varphi(0, 0, 3^n z, 3^n w, 0, 0) \\ &\le \frac{1}{3^n} \varphi(0, 0, 3^n z, 3^n w, 0, 0) \end{aligned} \tag{4.6}$$

for all $z, w \in A$. By (3.v), (4.5), and (4.6),

$$\begin{aligned}
2D(\frac{[z,w]}{2}) &= \lim_{n\to\infty} \frac{2h(\frac{3^{2n}}{2}[z,w])}{3^{2n}} = \lim_{n\to\infty} \frac{2h(\frac{1}{2}[3^n z, 3^n w])}{3^{2n}} \\
&= \lim_{n\to\infty} ([\frac{h(3^n z)}{3^n}, \frac{3^n w}{3^n}] + [\frac{3^n z}{3^n}, \frac{h(3^n w)}{3^n}]) \\
&= [D(z), w] + [z, D(w)]
\end{aligned}$$

for all $z, w \in A$. But since D is $\mathbb{C}$-linear,

$$D([z,w]) = 2D(\frac{[z,w]}{2}) = [D(z), w] + [z, D(w)]$$

for all $z, w \in A$.

Similarly, one can obtain that

$$\begin{aligned}
2D(\frac{a\circ b}{2}) &= \lim_{n\to\infty} \frac{2h(\frac{3^{2n}}{2} a\circ b)}{3^{2n}} = \lim_{n\to\infty} \frac{2h(\frac{1}{2}(3^n a)\circ(3^n b))}{3^{2n}} \\
&= \lim_{n\to\infty} \left((\frac{h(3^n a)}{3^n}) \circ (\frac{3^n b}{3^n}) + (\frac{3^n a}{3^n} \circ (\frac{h(3^n b)}{3^n}) \right) \\
&= (Da)\circ b + a \circ (Db)
\end{aligned}$$

for all $a, b \in A$. So

$$D(a\circ b) = 2D(\frac{a\circ b}{2}) = (Da)\circ b + a\circ(Db)$$

for all $a, b \in A$. Hence the $\mathbb{C}$-linear mapping $D : A \to A$ is a Lie JC^*-algebra derivation satisfying (4.iv), as desired. ∎

Corollary 4.5. *Let $h : A \to A$ be a mapping with $h(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned}
&\|2h(\frac{\mu x + \mu y + [z,w] + a\circ b}{2}) - \mu h(x) - \mu h(y) - [h(z), w] - [z, h(w)] \\
&\qquad -h(a)\circ b - a\circ h(b)\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p + \|a\|^p + \|b\|^p), \\
&\qquad \|h(3^n u^*) - h(3^n u)^*\| \leq 2\cdot 3^{np}\theta
\end{aligned}$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(A)$, $n = 0, 1, 2, \cdots$, and all $x, y, z, w, a, b \in A \setminus \{0\}$. Then there exists a unique Lie JC^*-algebra derivation $D : A \to A$ such that

$$\|h(x) - D(x)\| \leq \frac{3 + 3^p}{3 - 3^p}\theta\|x\|^p$$

for all $x \in A \setminus \{0\}$.

Proof. Define $\varphi(x, y, z, w, a, b) = \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p + \|a\|^p + \|b\|^p)$, and apply Theorem 4.4. ∎

One can obtain a similar result to Theorem 4.3 for the Jensen functional equation.

Finally, we are going to show the Cauchy–Rassias stability of Lie JC^*-algebra derivations in Lie JC^*-algebras associated with the Trif functional equation.

Theorem 4.6. *Let $h : A \to A$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : A^{d+4} \to [0, \infty)$ satisfying* (3.ix) *and* (3.xi) *such that*

$$\begin{aligned}\|d\ {}_{d-2}C_{l-2}h(\frac{\mu x_1 + \cdots + \mu x_d}{d} + \frac{[z,w] + a\circ b}{d\ {}_{d-2}C_{l-2}}) + {}_{d-2}C_{l-1}\sum_{j=1}^{d}\mu h(x_j) \\ -l\sum_{1\le j_1<\cdots<j_l\le d}\mu h(\frac{x_{j_1}+\cdots+x_{j_l}}{l}) - [h(z),w] - [z,h(w)] \quad (4.v) \\ -h(a)\circ b - a\circ h(b)\| \le \varphi(x_1,\cdots,x_d,z,w,a,b)\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \cdots, x_d, z, w, a, b \in A$. Then there exists a unique Lie JC^*-algebra derivation $D : A \to A$ such that

$$\|h(x) - D(x)\| \le \frac{1}{l\cdot {}_{d-1}C_{l-1}}\widetilde{\varphi}(qx, \underbrace{rx, \cdots, rx}_{d-1 \text{ times}}, 0, 0, 0, 0) \qquad (4.vi)$$

for all $x \in A$.

Proof. Put $z = w = a = b = 0$ in (4.v). By the same reasoning as in the proof of Theorem 2.7, there exists a unique $\mathbb{C}$-linear involutive mapping $D : A \to A$ satisfying (4.vi). The $\mathbb{C}$-linear mapping $D : A \to A$ is given by

$$D(x) = \lim_{n\to\infty}\frac{1}{q^n}h(q^n x) \qquad (4.7)$$

for all $x \in A$.

It follows from (4.7) that

$$D(x) = \lim_{n\to\infty}\frac{h(q^{2n}x)}{q^{2n}} \qquad (4.8)$$

for all $x \in A$. Let $x_1 = \cdots = x_d = a = b = 0$ in (4.v). Then we get

$$\|d\ {}_{d-2}C_{l-2}h(\frac{[z,w]}{d\ {}_{d-2}C_{l-2}}) - [h(z),w] - [z,h(w)]\| \le \varphi(\underbrace{0,\cdots,0}_{d \text{ times}}, z, w, 0, 0)$$

for all $z, w \in A$. Since

$$\frac{1}{q^{2n}}\varphi(\underbrace{0,\cdots,0}_{d \text{ times}}, q^n z, q^n w, 0, 0) \le \frac{1}{q^n}\varphi(\underbrace{0,\cdots,0}_{d \text{ times}}, q^n z, q^n w, 0, 0),$$

$$\begin{aligned}\frac{1}{q^{2n}}\|d_{d-2}C_{l-2}h(\frac{1}{d_{d-2}C_{l-2}}[q^n z, q^n w]) - [h(q^n z), q^n w] - [q^n z, h(q^n w)]\| \\ \le \frac{1}{q^{2n}}\varphi(\underbrace{0,\cdots,0}_{d \text{ times}}, q^n z, q^n w, 0, 0) \le \frac{1}{q^n}\varphi(\underbrace{0,\cdots,0}_{d \text{ times}}, q^n z, q^n w, 0, 0) \qquad (4.9)\end{aligned}$$

for all $z, w \in A$. By (3.ix), (4.8), and (4.9),

$$d_{d-2}C_{l-2}D(\frac{[z,w]}{d_{d-2}C_{l-2}}) = \lim_{n\to\infty} \frac{d_{d-2}C_{l-2}h(\frac{q^{2n}}{d_{d-2}C_{l-2}}[z,w])}{q^{2n}}$$

$$= \lim_{n\to\infty} \frac{d_{d-2}C_{l-2}h(\frac{1}{d_{d-2}C_{l-2}}[q^n z, q^n w])}{q^{2n}} = \lim_{n\to\infty}([\frac{h(q^n z)}{q^n}, \frac{q^n w}{q^n}] + [\frac{q^n z}{q^n}, \frac{h(q^n w)}{q^n}])$$
$$= [D(z), w] + [z, D(w)] \text{for all } z, w \in A.$$

But since D is $\mathbb{C}$-linear, vglue-8pt

$$\frac{D([z,w]) = d\ {}_{d-2}C_{l-2}D([z,w]}{d\ {}_{d-2}C_{l-2}) = [D(z), w] + [z, D(w)] \quad \text{for all } z, w \in A.}$$

Similarly, one can obtain that $D(a \circ b) = (Da) \circ b + a \circ (Db)$ for all $a, b \in A$. Hence the $\mathbb{C}$-linear mapping $D\colon A \to A$ is a Lie JC^*-algebra derivation satisfying (4.vi), as desired. ■

Corollary 4.7. *Let $h : A \to A$ be a mapping with $h(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\|d\ {}_{d-2}C_{l-2}h(\frac{\mu x_1 + \cdots + \mu x_d}{d} + \frac{[z,w] + a \circ b}{d\ {}_{d-2}C_{l-2}}) + {}_{d-2}C_{l-1}\sum_{j=1}^{d} \mu h(x_j)$$
$$-l \sum_{1 \leq j_1 < \cdots < j_l \leq d} \mu h(\frac{x_{j_1} + \cdots + x_{j_l}}{l}) - [h(z), w] - [z, h(w)] - h(a) \circ b$$
$$-a \circ h(b)\| \leq \theta(\sum_{j=1}^{d} \|x_j\|^p + \|z\|^p + \|w\|^p + \|a\|^p + \|b\|^p),$$
$$\|h(q^n u^*) - h(q^n u)^*\| \leq d q^{np}\theta$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(A)$, $n = 0, 1, \cdots,$ and all $x_1, \cdots, x_d, z, w, a, b \in A$. Then there exists a unique Lie JC^*-algebra derivation $D : A \to A$ such that

$$\|h(x) - D(x)\| \leq \frac{q^{1-p}(q^p + (d-1)r^p)\theta}{l\ {}_{d-1}C_{l-1}(q^{1-p} - 1)} \|x\|^p$$

for all $x \in A$.

Proof. Define $\varphi(x_1, \cdots, x_d, z, w, a, b) = \theta(\sum_{j=1}^{d} \|x_j\|^p + \|z\|^p + \|w\|^p + \|a\|^p + \|b\|^p)$, and apply Theorem 4.6. ■

One can obtain a similar result to Theorem 4.3 for the Trif functional equation.

References

[1] Găvruta, P., *A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[2] Jun, K., and Y. Lee, *A generalization of the Hyers–Ulam–Rassias stability of Jensen's equation*, J. Math. Anal. Appl. **238** (1999), 305–315.

[3] Kadison, R. V., and G. Pedersen, *Means and convex combinations of unitary operators*, Math. Scand. **57** (1985), 249–266.

[4] Kadison, R. V., and J.R. Ringrose, "Fundamentals of the Theory of Operator Algebras: Elementary Theory," Academic Press, 1983.

[5] Oh, S., C. Park, and Y. Shin, *Quantum n-space and Poisson n-space*, Comm. Algebra **30** (2002), 4197–4209.

[6] —, *A Poincaré-Birkhoff-Witt theorem for Poisson enveloping algebras*, Comm. Algebra **30** (2002), 4867–4887.

[7] Park, C., *On the stability of the linear mapping in Banach modules*, J. Math. Anal. Appl. **275** (2002), 711–720.

[8] —, *Modified Trif's functional equations in Banach modules over a C^*-algebra and approximate algebra homomorphisms*, J. Math. Anal. Appl. **278** (2003), 93–108.

[9] Park, C., and W. Park, *On the Jensen's equation in Banach modules*, Taiwanese J. Math. **6** (2002), 523–531.

[10] Rassias, J. M., *On approximation of approximately linear mappings by linear mappings*, J. Funct. Anal. **46** (1982), 126–130.

[11] Rassias, Th. M. *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[12] —, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math. **62** (2000), 23–130.

[13] —, *The problem of S.M. Ulam for approximately multiplicative mappings*, J. Math. Anal. Appl. **246** (2000), 352–378.

[14] —, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. **251** (2000), 264–284.

[15] Rassias, Th.M., and P. Šemrl, *On the behavior of mappings which do not satisfy Hyers–Ulam stability*, Proc. Amer. Math. Soc. **114** (1992), 989–993.

[16] —, *On the Hyers-Ulam stability of linear mappings*, J. Math. Anal. Appl. **173** (1993), 325–338.

[17] Trif, T., *On the stability of a functional equation deriving from an inequality of Popoviciu for convex functions*, J. Math. Anal. Appl. **272** (2002), 604–616.

[18] Upmeier, H. "Jordan Algebras in Analysis, Operator Theory, and Quantum Mechanics," Amer. Math. Soc., 1987.

Chun-Gil Park
Department of Mathematics
Chungnam National University
Daejeon 305–764
South Korea
cgpark@cnu.ac.kr

Received March 13, 2004
and in final form December 7, 2004