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On the use of explicit bounds on residues of Dedekind zeta functions taking into account the behavior of small primes

par Stéphane R. LOUBOUTIN

Résumé. Nous donnons des majorants explicites des résidus au point $s = 1$ des fonctions zêta $\zeta_K(s)$ des corps de nombres tenant compte du comportement des petits nombres premiers dans K. Dans le cas où K est abélien, de telles majorations sont déduites de majorations de $|L(1,\chi)|$ tenant compte du comportement de χ sur les petits nombres premiers, pour χ un caractère de Dirichlet primitif. De nombreuses applications sont données pour illustrer l'utilité de tels majorants.

Abstract. Lately, explicit upper bounds on $|L(1,\chi)|$ (for primitive Dirichlet characters χ) taking into account the behaviors of χ on a given finite set of primes have been obtained. This yields explicit upper bounds on residues of Dedekind zeta functions of abelian number fields taking into account the behavior of small primes, and it as been explained how such bounds yield improvements on lower bounds of relative class numbers of CM-fields whose maximal totally real subfields are abelian. We present here some other applications of such bounds together with new bounds for non-abelian number fields.

1. Introduction

Let K be a number field of degree $n = r_1 + 2r_2 > 1$. Let d_K, w_K, Reg_K, h_K and $\mathrm{Res}_{s=1}(\zeta_K(s))$ be the absolute value of its discriminant, the number of complex roots of unity in K, the regulator, class number, and residue at $s = 1$ of the Dedekind zeta function $\zeta_K(s)$ of K. Recall the class number formula:

$$h_K = \frac{w_K\sqrt{d_K}}{2^{r_1}(2\pi)^{r_2}\mathrm{Reg}_K}\mathrm{Res}_{s=1}(\zeta_K(s)).$$

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In order to obtain bounds on h_K we need bounds on $\operatorname{Res}_{s=1}(\zeta_K(s))$. The best general upper bound is (see [Lou01, Theorem 1]):

$$\operatorname{Res}_{s=1}(\zeta_K(s)) \leq \left(\frac{e\log d_K}{2(n-1)}\right)^{n-1}.$$

If K is totally real cubic, then we have the better upper bound (see [Lou01, Theorem 2]):

$$\operatorname{Res}_{s=1}(\zeta_K(s)) \leq \frac{1}{8}\log^2 d_K.$$

Finally, if K is abelian, then we have the even better general upper bound

$$\operatorname{Res}_{s=1}(\zeta_K(s)) \leq \left(\frac{\log d_K}{2(n-1)}+\kappa\right)^{n-1},$$

where $\kappa = (5-2\log 6)/2 = 0.70824\cdots$, by [Ram01, Corollary 1].

However, from the Euler product of $\zeta_K(s)$ we expect to have better upper bounds for $\operatorname{Res}_{s=1}(\zeta_K(s))$, provided that the small primes do not split in K. For any prime $p \geq 1$, we set

$$\Pi_K(p) := \prod_{\mathcal{P}|p}(1-(N(\mathcal{P}))^{-1})^{-1} \geq 1,$$

where $\mathcal{P}$ runs over all the primes ideals of K above p. A careful analysis of the proofs of all the previous bounds suggests that we should expect that there exists some $\kappa' > 0$ such that

$$\operatorname{Res}_{s=1}(\zeta_K(s)) \leq \begin{cases} \frac{\Pi_K(2)}{\Pi_{\mathbf{Q}}^n(2)}\left(\frac{e\log d_K}{2(n-1)}+\kappa'\right)^{n-1} \text{in general,}\\ \frac{1}{8}\frac{\Pi_K(2)}{\Pi_{\mathbf{Q}}^n(2)}(\log d_K+\kappa')^2 \quad \text{if } K \text{ is a totally real cubic field,}\\ \frac{\Pi_K(2)}{\Pi_{\mathbf{Q}}^n(2)}\left(\frac{\log d_K}{2(n-1)}+\kappa'\right)^{n-1} \text{if } K \text{ is abelian.}\end{cases}$$

Notice that the factor $\Pi_K(2)/\Pi_{\mathbf{Q}}^n(2)$ is always less than or equal to 1, but is equal to $1/(2^n-1)$, hence small, if the prime $p=2$ is inert in K. Combined with lower bounds for $\operatorname{Res}_{s=1}(\zeta_K(s))$ depending on the behavior of the small primes in K (see [Lou03, Theorem 1]), we would as a consequence obtain better lower bounds for relative class number of CM-fields. The aim of this paper is to illustrate on various examples the use of such better bounds on $\operatorname{Res}_{s=1}(\zeta_K(s))$.

To begin with, we recall:

Theorem 1. *(See* [Ram04] [1]*; see also* [Lou04a] *and* [Lou04d]*). Let S be a given finite set of pairwise distinct rational primes. Set $\kappa_S := \#S \cdot \log 4 + 2\sum_{p\in S} \frac{\log p}{p-1}$. Then, for any primitive Dirichlet character χ of conductor $q_\chi > 1$ such that $p \in S$ implies that p does not divide q_χ, we have*

$$|L(1,\chi)| \leq \frac{1}{2}\Big(\prod_{p\in S}\frac{p-1}{|p-\chi(p)|}\Big)\Big(\log q_\chi \ + \kappa_S + 5 - 2\log 6 + \frac{3\pi}{q_\chi}\prod_{p\in S}\frac{p^2-1}{4p^2}\Big)$$

if χ is odd, and

$$|L(1,\chi)| \leq \frac{1}{2}\Big(\prod_{p\in S}\frac{p-1}{|p-\chi(p)|}\Big)\Big(\log q_\chi \ + \kappa_S\Big)$$

if χ is even and $q_\chi \geq 6.2 \cdot 4^{\#S}$.

We refer the reader to [BHM], [Le], [MP], [Mos], [MR], [SSW] and [Ste] for various applications of such explicit bounds on L-functions. They are not the best possible theoretically. However, if such better bounds are made explicit, we end up with useless ones in a reasonable range for q_χ (see [Lou04a] and [Boo]). Therefore, applications of these better bounds to practical problems are not yet possible.

2. Upper bounds for relative class numbers

Corollary 2. *Let $q \equiv 5 \pmod 8$, $q \neq 5$, be a prime, let χ_q denote anyone of the two conjugate odd quartic characters of conductor q and let h_q^- denote the relative class number of the imaginary cyclic quartic field N_q of conductor q. Then,*

$$h_q^- = \frac{q}{2\pi^2}|L(1,\chi_q)|^2 \leq \frac{q}{A_\chi \pi^2}\Big(\log q + 5 - 2\log 6 + \log B_\chi + \frac{9\pi}{16q}\Big)^2,$$

which implies $h_q^- < q$ for $q \leq C_\chi$, where A_χ, B_χ and C_χ are as follows:

Values of (A_χ, B_χ, C_χ)

	$\chi_q(3) = +1$	$\chi_q(3) = -1$	$\chi_q(3) = \pm i$
$\chi_q(5) = +1$	$(40, 16, 6450000)$	$(160, 192, 2 \cdot 10^{14})$	$(100, 192, 5 \cdot 10^{10})$
$\chi_q(5) = -1$	$(90, 64\sqrt{5}, 10^{10})$	$(360, 768\sqrt{5}, 10^{22})$	$(225, 768\sqrt{5}, 4 \cdot 10^{16})$
$\chi_q(5) = \pm i$	$(65, 64\sqrt{5}, 10^{8})$	$(260, 768\sqrt{5}, 10^{18})$	$(325/2, 768\sqrt{5}, 3 \cdot 10^{13})$

[1] Note however the misprint in [Ram04, Top of page 143] where the term $\frac{3\pi\phi(hk)}{2hkq}\prod_{p|hk}\frac{p^2-1}{4p^2}$ should be $\frac{3\pi\phi(hk)}{2hk4^{\omega(h)}q}\prod_{p|hk}\frac{p^2-1}{p^2}$.

Proof. Since $q \equiv 5 \pmod 8$, we have $\chi_q(2)^2 = (\frac{2}{q}) = -1$ and $\chi_q(2) = \pm i$. Set $S = \{p \in \{2,3,5\};\ \chi(p) \neq +1\}$. Then $2 \in S$ and according to Theorem 1 we may choose

$$A_\chi = 8\prod_{p\in S}\left|\frac{p-\chi(p)}{p-1}\right|^2 = 40\prod_{2\neq p\in S}\left|\frac{p-\chi(p)}{p-1}\right|^2$$

and

$$\log B_\chi = \#S\cdot\log 4 + 2\sum_{p\in S}\frac{\log p}{p-1} = (\#S+1)\log 4 + 2\sum_{2\neq p\in S}\frac{\log p}{p-1}.$$

□

Remarks 3. *Using Corollary 2 to alleviate the amount of required relative class number computation, several authors have been trying to solve in* [JWW] *the open problem hinted at in* [Lou98]*: determine the least (or at least one) prime* $q \equiv 5 \pmod 8$ *for which* $h_q^- > q$*. Indeed, according to Corollary 2, for finding such a* q *in the range* $q < 5\cdot 10^{10}$*, we may assume that* $\chi_q(3) = +1$*, which amounts to eliminating three quarters of the primes* q *in this range. In the same way, in the range* $q < 3\cdot 10^{13}$ *we may assume that* $\chi_q(3) = +1$ *or* $\chi_q(5) = +1$*, which amounts to eliminating* $9/16$ *of the primes* q *in this range.*

3. Real cyclotomic fields of large class numbers

In [CW], G. Cornell and L. C. Washington explained how to use simplest cubic and quartic fields to produce real cyclotomic fields $\mathbf{Q}^+(\zeta_p)$ of prime conductor p and class number $h_p^+ > p$. They could find only one such real cyclotomic field. We explain how to use our bounds on L-functions to find more examples of such real cyclotomic fields. However, it is much more efficient to use simplest quintic and sextic fields to produce real cyclotomic fields of prime conductors and class numbers greater than their conductors (see [Lou02a] and [Lou04c]).

3.1. Using simplest cubic fields. The **simplest cubic fields** are the real cyclic cubic number fields associated with the $\mathbf{Q}$-irreducible cubic polynomials

$$P_m(x) = x^3 - mx^2 - (m+3)x - 1$$

of discriminants $d_m = \Delta_m^2$ where $\Delta_m := m^2+3m+9$. Since $-x^3P_m(1/x) = P_{-m-3}(x)$, we may assume that $m \geq -1$. We let

$$(1)\quad \rho_m = \frac{1}{3}\Big(2\sqrt{\Delta_m}\cos\Big(\frac{1}{3}\arctan(\frac{\sqrt{27}}{2m+3})\Big)+m\Big) = \sqrt{\Delta_m}-\frac{1}{2}+O(\frac{1}{\sqrt{\Delta_m}})$$

denote the only positive root of $P_m(x)$. Moreover, we will assume that the conductor of K_m is equal to Δ_m, which amounts to asking that (i)

$m \not\equiv 0 \pmod 3$ and Δ_m is squarefree, or (ii) $m \equiv 0,\ 6 \pmod 9$ and $\Delta_m/9$ is squarefree (see [Wa, Prop. 1 and Corollary]). In that situation, $\{-1, \rho_m, -1/(\rho_m+1)\}$ generate the full group of algebraic units of K_m and the regulator of K_m is

$$\text{Reg}_{K_m} = \log^2 \rho_m - (\log \rho_m)(\log(1+\rho_m)) + \log^2(1+\rho_m), \tag{2}$$

which in using (1) yields

$$\text{Reg}_{K_m} = \frac{1}{4}\log^2 \Delta_m - \frac{\log \Delta_m}{\sqrt{\Delta_m}} + O(\frac{\log \Delta_m}{\Delta_m}) \leq \frac{1}{4}\log^2 \Delta_m. \tag{3}$$

Lemma 4. *The polynomial $P_m(x)$ has no root mod 2, has at least one root mod 3 if and only if $m \equiv 0 \pmod 3$, and has at least one root mod 5 if and only if $m \equiv 1 \pmod 5$. Hence, if Δ_m is square-free, then 2 and 3 are inert in K_m, and if $m \not\equiv 1 \pmod 5$ then 5 is also inert in K_m.*

As in [Lou02a, Section 5.1], we let χ_{K_m} be the primitive, even, cubic Dirichlet characters modulo Δ_m associated with K_m satisfying

$$\chi_{K_m}(2) = \begin{cases} \omega^2 & \text{if } m \equiv 0 \pmod 2, \\ \omega & \text{if } m \equiv 1 \pmod 2. \end{cases}$$

Since the regulators of these K_m's are small, they should have large class numbers. In fact, we proved (see [Lou02c, (12)]):

$$h_{K_m} = \frac{\Delta_m}{4\text{Reg}_{K_m}} |L(1, \chi_{K_m})|^2 \geq \frac{\Delta_m}{e \log^3 \Delta_m} \tag{4}$$

Corollary 5. *Assume that $m \geq -1$ is such that $\Delta_m = m^2 + 3m + 9$ is squarefree. Then,*

$$h_{K_m} \leq \begin{cases} \Delta_m/60 & \text{if } m > 16, \\ \Delta_m/100 & \text{if } m \not\equiv 1 \pmod 5 \text{ and } m > 37. \end{cases}$$

Proof. If a prime $l \geq 2$ is inert in K_m then $\chi_{K_m}(l) \in \{\exp(2i\pi/3), \exp(4i\pi/3)\}$. According to Lemma 4 and to Theorem 1 (with $S = \{2,3\}$ and $S = \{2,3,5\}$), we have

$$|L(1, \chi_{K_m})|^2 \leq \begin{cases} (\log \Delta_m + \log(192))^2/91, \\ 16(\log \Delta_m + \log(768\sqrt{5}))^2/2821 & \text{if } m \not\equiv 1 \pmod 5. \end{cases}$$

Now, according (4) and (3), the desired results follow for $m \geq 95000$. The numerical computation of the class numbers of the remaining K_m provides us with the desired bounds (see [Lou02a]). □

From now on, we assume (i) that $p = \Delta_m = m^2 + 3m + 9$ is prime (hence $m \not\equiv 0 \pmod 3$) and (ii) that $p \equiv 1 \pmod{12}$, which amounts to asking that $m \equiv 0,\ 1 \pmod 4$. In that case, both K_m and $k_m := \mathbf{Q}(\sqrt{\Delta_m})$ are subfields of the real cyclotomic field $\mathbf{Q}^+(\zeta_p)$ and the product $h_2 h_3$ of the class numbers $h_2 := h_{k_m}$ and $h_3 := h_{K_m}$ of k_m and K_m divides the class number h_p^+ of $\mathbf{Q}^+(\zeta_p)$. Now, $h_3 \leq \Delta_m/60$ and $h_2h_3 \geq \Delta_m$ imply $h_2 \geq 60$, hence $h_2 \geq 61$ (for h_2 is odd), and Cohen-Lenstra heuristics predict that real quadratic number fields of prime conductors with class numbers greater than or equal to 61 are few and far between. Hence, such simplest cubic fields K_m of prime conductors $\Delta_m = m^2 + 3m + 9 \equiv 1 \pmod 4$ with $h_2h_3 > \Delta_m$ are few and far between. As we have at hand a very efficient method for computing class numbers of real quadratic fields (see [Lou02b] and [WB]), we used this explicit necessary condition $h_2 \geq 61$ to compute (using [Lou02a]) the class numbers of only 584 out of the 46825 simplest cubic fields K_m of prime conductors $\Delta_m \equiv 1 \pmod{12}$ with $-1 \leq m \leq 1066285$ to obtain the following Table. (Using the fact that $h_2 \geq 61$, the class number formula for k_m and Theorem 1 for $S = \emptyset$ imply $\mathrm{Reg}_2 \leq \sqrt{\Delta_m}(\log \Delta_m)/244$, where Reg_2 denotes the regulator of the real quadratic field $k_m = \mathbf{Q}(\sqrt{\Delta_m})$, and taking into account the fact that Reg_2 is much faster to compute than h_2, we could still improve the speed of the required computations). Notice that the authors of [CW] and [SWW] only came up with one such K_m, the one for $m = 106253$.

Least values of $m \geq -1$ for which $\Delta_m = m^2 + 3m + 9$ is prime and $h_2h_3 \geq \Delta_m$

m	$\lvert\theta(\chi_{K_m})\rvert$	$\arg W(\chi_{K_m})$	h_2	h_3	h_2h_3/Δ_m
102496	$20.268\cdots$	$\frac{1}{3}\arctan(\frac{3\sqrt{3}}{2m+3}) + \frac{\pi}{3}$	891	13152913	$1.115\cdots$
106253	$34.364\cdots$	$\frac{1}{3}\arctan(\frac{3\sqrt{3}}{2m+3})$	2685	6209212	$1.476\cdots$
319760	$202.162\cdots$	$\frac{1}{3}\arctan(\frac{3\sqrt{3}}{2m+3})$	1887	57772549	$1.066\cdots$
554869	$88.861\cdots$	$\frac{1}{3}\arctan(\frac{3\sqrt{3}}{2m+3}) + \frac{\pi}{3}$	7983	93739324	$2.430\cdots$
726845	$20.938\cdots$	$\frac{1}{3}\arctan(\frac{3\sqrt{3}}{2m+3})$	13533	176702419	$4.526\cdots$
791021	$129.812\cdots$	$\frac{1}{3}\arctan(\frac{3\sqrt{3}}{2m+3})$	1737	445142272	$1.235\cdots$
796616	$357.252\cdots$	$\frac{1}{3}\arctan(\frac{3\sqrt{3}}{2m+3})$	1155	696739264	$1.268\cdots$
839401	$293.373\cdots$	$\frac{1}{3}\arctan(\frac{3\sqrt{3}}{2m+3}) + \pi$	1575	554491633	$1.239\cdots$
906437	$93.697\cdots$	$\frac{1}{3}\arctan(\frac{3\sqrt{3}}{2m+3})$	1955	469911916	$1.118\cdots$
1066285	$140.662\cdots$	$\frac{1}{3}\arctan(\frac{3\sqrt{3}}{2m+3}) + \frac{\pi}{3}$	5389	473034223	$2.242\cdots$

3.2. Using simplest quartic fields. The so called **simplest quartic fields** (dealt with in [Laz1], [Laz2] and [Lou04b]) are the real cyclic quartic number fields associated with the quartic polynomials

$$P_m(x) = x^4 - mx^3 - 6x^2 + mx + 1$$

of discriminants $d_m = 4\Delta_m^3$ where $\Delta_m := (m^2 + 16)^3$. Since $P_m(-x) = P_{-m}(x)$, we may and we will assume that $m \geq 0$. The reader will easily check (i) that $P_m(x)$ has no rational root, (ii) that $P_m(x)$ is $\mathbf{Q}$-irreducible, except for $m = 0$ and $m = 3$, and (iii) that $P_m(x)$ has a only one root $\rho_m > 1$. Set $\beta_m = \rho_m - \rho_m^{-1} > 0$. Then, $\beta_m^2 - m\beta_m - 4 = 0$ and

$\beta_m = (m+\sqrt{\Delta_m})/2$. In particular, $k_m = \mathbf{Q}(\sqrt{\Delta_m})$ is the quadratic subfield of the cyclic quartic field K_m. It is known that h_{k_m} divides h_{K_m}, and we set $h_{K_m}^* = h_{K_m}/h_{k_m}$. Since $\rho_m > 1$ and $\rho_m^2 - \beta_m\rho_m - 1 = 0$, we obtain

$$\rho_m = \frac{1}{2}\Big(\frac{m+\sqrt{\Delta_m}}{2} + \sqrt{\frac{\Delta_m + m\sqrt{\Delta_m}}{2}}\Big) = \sqrt{\Delta_m}\Big(1 - \frac{3}{\Delta_m} + O(\frac{1}{\Delta_m^2})\Big)$$

(use $m = \sqrt{\Delta_m - 16}$),

$$\rho_m' := \frac{1}{2}\Big(\frac{m-\sqrt{\Delta_m}}{2} + \sqrt{\frac{\Delta_m - m\sqrt{\Delta_m}}{2}}\Big) = 1 - \frac{2}{\sqrt{\Delta_m}} + O(\frac{1}{\Delta_m}),$$

and
(5)
$$\text{Reg}_{\rho_m}^* = \log^2\rho_m + \log^2\rho_m' = \frac{1}{4}\log^2\Delta_m - \frac{3\log\Delta_m}{\Delta_m} + O(\frac{1}{\Delta_m}) \leq \frac{1}{4}\log^2\Delta_m$$

for $m \geq 1$.

Proposition 6. *Assume that $m \geq 1$ is odd and that $\Delta_m = m^2+16$ is prime. Then, the discriminant of the real quadratic subfield $k_m = \mathbf{Q}(\sqrt{\Delta_m})$ of K_m is equal to Δ_m, the discriminant of K_m is equal to Δ_m^3, its conductor is equal to Δ_m, the class numbers of K_m and k_m are odd,* $\text{Reg}_{K_m}/\text{Reg}_{k_m} = \text{Reg}_{\rho_m}^*$ *and (see* [Lou04b, Theorem 9]*)*

$$h_{K_m}^* = \frac{\Delta_m}{4\text{Reg}_{K_m}^*}|L(1,\chi_{K_m})|^2 \geq \frac{2\Delta_m}{3e(\log\Delta_m + 0.35)^4}, \tag{6}$$

where χ_{K_m} is anyone of the two conjugate primitive, even, quartic Dirichlet characters modulo Δ_m associated with K_m. Moreover, $\chi_{K_m}(2) = -1$, and $m \geq 5$ implies

$$h_{K_m}^* < \Delta_m/26.$$

Proof. According (6), to Theorem 1 (with $S = \{2\}$) which yields

$$|L(1,\chi_{K_m})|^2 \leq (\log\Delta_m + \log(16))^2/36,$$

and to (5), we have $h_{K_m}^* \leq \Delta_m/(36+o(1))$ and $h_{K_m}^* < \Delta_m/26$ for $m \geq 3000$. The numerical computation of the class numbers of the remaining K_m provides us with the desired bound (see [Lou02a]). □

Now, $h_{K_m} = h_{k_m}h_{K_m}^* \geq \Delta_m$ and $h_{K_m}^* < \Delta_m/26$ imply $h_{k_m} \geq 27$ (for h_{k_m} is odd), and Cohen-Lenstra heuristics predict that real quadratic number fields of prime discriminants with class numbers greater than or equal to 27 are few and far between. Hence such simplest quartic fields K_m of prime conductors $\Delta_m = m^2 + 16$ with $h_{K_m} > \Delta_m$ are few and far between. As we have at hand a very efficient method for computing rigorously class numbers of real quadratic fields (see [Lou02b] and [WB]), we used this explicit necessary condition $h_{k_m} \geq 27$ to compute only 1687 of the class numbers of the 86964 simplest quartic fields K_m of prime conductors $\Delta_m = m^2 + 16 \equiv 1 \pmod 4$ with $1 \leq m \leq 1680401$ to obtain the following Table. Notice that G. Cornell and L. C. Washington did not find any such K_m (see [CW, bottom of page 268]).

Least values of $m \geq 1$ for which $\Delta_m = m^2 + 16$ is prime and $h_{K_m} \geq \Delta_m$

m	Δ_m	h_{k_m}	$h^*_{K_m}$	h_{K_m}/Δ_m
524285	274874761241	1911	181442581	$1.261\cdots$
1680401	2823747520817	1537	1878644993	$1.022\cdots$

4. The imaginary cyclic quartic fields with ideal class groups of exponent ≤ 2

We explain how one could alleviate the determination in [Lou95] of all the non-quadratic imaginary cyclic fields of 2-power degrees $2n = 2^r \geq 4$ with ideal class groups of exponents ≤ 2 (the time consuming part bieng the computation of the relative class numbers of the fields sieved by Proposition 8 or Remark 9 below). To simplify, we will now only deal with imaginary cyclic quartic fields of odd conductors.

Theorem 7. *Let K be an imaginary cyclic quartic field of odd conductor f_K, Let k, f_k and χ_K denote the real quadratic subfield of K, the conductor of k, and anyone of the two conjugate primitive quartic Dirichlet characters modulo f_K associated with K. Then,*

$$h_K^- \geq \frac{C_K f_K}{e\pi^2(\log f_k + \kappa_k)\log(f_k f_K^2)}, \tag{7}$$

where

$$C_K = \frac{32}{|2-\chi_K(2)|^2} = \begin{cases} 32 & \text{if } \chi_K(2) = +1, \\ 32/9 & \text{if } \chi_K(2) = -1, \\ 32/5 & \text{if } \chi_K(2) = \pm i, \end{cases}$$

and

$$\kappa_k = \begin{cases} 0 & \text{if } f_k \equiv 1 \pmod 8, \\ 4\log 2 = 2.772\cdots & \text{if } f_k \equiv 5 \pmod 8). \end{cases}$$

Proof. Use [Lou03, (34)], [Lou03, (31)] with [Ram01, Corollary 1] and [Ram01, Corollary 2]'s values for $\kappa_k = \kappa_\chi$, where χ is the primitive even Dirichlet character of conductor d_k associated with k, and $\Pi_{K/k}(\{2\}) = 4/|2-\chi_K(2)|^2$. □

Proposition 8. *Assume that the exponent of the ideal class group of an imaginary cyclic quartic field K of odd conductor f_K is less than or equal to 2. Then, $f_k \leq 1889$ and $f_K \leq 10^7$ (where k is the real quadratic subfield* *of K). Moreover, whereas there are* 1 377 361 *imaginary cyclic fields K of odd conductors $f_K \leq 10^7$ and such that $f_k \leq 1889$, only* 400 *out of them may have their ideal class groups of exponents ≤ 2, the largest possible conductor being $f_K = 5619$ (for $f_k = 1873$ and $f_{K/k} := f_K/f_k = 3$).*

Proof. It is known that if the exponent of the ideal class group of K of odd conductor f_K is ≤ 2, then $f_k \equiv 1 \pmod 4$ is prime and

$$h_K^- = 2^{t_{K/k}-1}, \tag{8}$$

where $t_{K/k}$ denotes the number of prime ideals of k which are ramified in K/k (see [Lou95, Theorems 1 and 2]). Conversely, for a given real quadratic field k of prime conductor $f_k \equiv 1 \pmod 4$, the conductors f_K of the imaginary cyclic quartic fields K of odd conductors and containing k are of the form $f_K = f_k f_{K/k}$ for some positive square-free integer $f_{K/k} \geq 1$ relatively prime with f_k and such that

$$(f_k - 1)/4 + (f_{K/k} - 1)/2 \text{ is odd} \tag{9}$$

(in order to have $\chi_K(-1) = -1$, i.e. in order to guarantee that K is imaginary). Moreover, for such a given k and such a given $f_{K/k}$, there exists only one imaginary cyclic quartic field K containing k and of conductor $f_K = f_k f_{K/k}$, and for this K we have

$$t_{K/k} = 1 + \sum_{p \mid f_{K/k}} (3 + (\frac{p}{f_k}))/2, \tag{10}$$

where $(\frac{\cdot}{f_k})$ denote the Legendre's symbol. Finally, if we let ϕ_k denote anyone of the two conjugate quartic characters modulo a prime $f_k \equiv 1 \pmod 4$, then $\chi_K(n) = \phi_k(n)(\frac{n}{f_{K/k}})$, where $(\frac{\cdot}{f_{K/k}})$ denote the Jacobi's symbol, and

(11)

$$\chi_K(2) = \begin{cases} \phi_k(2) = 1 & \text{if } f_k \equiv 1 \pmod 8 \text{ and } 2^{\frac{f_k-1}{4}} \equiv 1 \pmod{f_k}, \\ \phi_k(2) = -1 & \text{if } f_k \equiv 1 \pmod 8 \text{ but } 2^{\frac{f_k-1}{4}} \not\equiv 1 \pmod{f_k}, \\ -\phi_k(2) = \pm i & \text{if } f_k \equiv 5 \pmod 8). \end{cases}$$

Hence, we may easily compute κ_k, c_K and $t_{K/k}$ from f_k and $f_{K/k}$. In particular, we easily obtain that there are 1 377 361 imaginary cyclic fields K of odd conductors $f_K \leq 10^7$ and such that $f_k \leq 1889$, and that $c_K = 32$ for 149 187 out of them, $c_K = 32/5$ for 938 253 out of them, and $c_K = 32/9$ for 289 921 out of them. Now, let P_n denote the product of the first n odd primes $3 = p_1 < 5 = p_2 < \cdots < p_n < \cdots$ (hence, $P_0 = 1$, $P_1 = 3$, $P_2 = 15$, $\cdots$). There are two cases to consider:

(1) If $\chi_k(2) = +1$. Then, $f_k \equiv 1 \pmod 8$ is prime, $\kappa_k = 0$, $c_K \geq 32/9$, $f_K = f_k f_{K/k}$ where $f_{K/k}$ is a product of $n \geq 0$ distinct odd primes.

Hence, $f_{K/k} \geq P_n$, $t_{K/k} \leq 1 + 2n$, $h_K^- = 2^{t_{K/k}-1} \leq 4^n$ and using (7) we obtain

$$F_k(n) := \frac{32 f_k P_n}{9e\pi^2 4^n (\log f_k) \log(f_k^3 P_n^2)} \leq 1.$$

Assume that $f_k \geq 36$. Then $3 f_k^{3/2} \geq 5^4$ and for $n \geq 1$ we have $p_{n+1} \geq p_2 = 5$, $P_n \geq P_1 = 3$ and

$$\frac{F_k(n+1)}{F_k(n)} = \frac{p_{n+1} \log(f_k^{3/2} P_n)}{4 \log(p_{n+1} f_k^{3/2} P_n)} \geq \frac{5 \log(f_k^{3/2} P_n)}{4 \log(5 f_k^{3/2} P_n)} \geq \frac{5 \log(3 f_k^{3/2})}{4 \log(15 f_k^{3/2})} \geq 1.$$

Since we clearly have $F_k(1) \leq F_k(0)$, we obtain $\min_{n \geq 0} F_k(n) = F_k(1)$ and

$$\frac{8 f_k}{3e\pi^2 (\log f_k) \log(9 f_k^3)} = F_k(1) \leq F_k(n) \leq 1,$$

which implies $f_k \leq 1899$, hence $f_k \leq 1889$ (for $f_k \equiv 1 \pmod 8$ must be prime). Hence, using (7), we obtain

$$h_K^- \geq \frac{32 f_K}{9e\pi^2 (\log(1889)) \log(1889 f_K^2)}.$$

Let now n denote the number of distinct prime divisors of f_K. Then $f_K \geq P_n$, $t_{K/k} \leq 2(n-1)+1$ and $h_K^- = 2^{t_{K/k}-1} \leq 4^{n-1}$. Hence, using (7), we obtain

$$4^{n-1} \geq \frac{32 P_n}{9e\pi^2 (\log(1889)) \log(1889 P_n^2)},$$

which implies $n \leq 7$, $h_K^- \leq 4^6$,

$$4^6 \geq \frac{32 f_K}{9e\pi^2 (\log(1889)) \log(1889 f_K^2)}$$

and yields $f_K \leq 10^7$.

(2) If $\chi_k(2) = -1$. Then $f_k \equiv 5 \pmod 8$ is prime, $\kappa_k \leq 2.78$, $c_K \geq 32/5$ and we follow the previous case. We obtain $f_k \leq 1329$, hence $f_k \leq 1301$ (for $f_k \equiv 5 \pmod 8$ must be prime), $n \leq 7$, $h_K^- \leq 4^6$ and $f_K \leq 7 \cdot 10^6$.

Hence, the first assertion Proposition 8 is proved. Now, for a given odd prime $f_k \leq 1889$ equal to 1 modulo 4, and for a given odd square-free integer $f_{K/k} \leq 10^7/f_k$ relatively prime with f_k, we compute κ_k, $t_{K/k}$ (using (10)), c_K (using (11)) and use (7) and (8) to deduce that if the exponent of the ideal class group of K is less than or equal to 2, then

$$2^{t_{K/k}-1} \geq \frac{c_K f_k f_{K/k}}{e\pi^2 (\log f_k + \kappa_k) \log(f_k^3 f_{K/k}^2)}. \tag{12}$$

Now, an easy calculation yields that only 400 out of 1 377 361 imaginary cyclic fields K of odd conductors and such that $f_k \leq 1889$ and $f_K \leq 10^7$ satisfy (12), and the second assertion of the Proposition is proved. □

Remarks 9. *Our present lower bound* (7) *should be compared with the bound*

$$h_K^- \geq \frac{2f_K}{e\pi^2(\log f_k + 0.05)\log(f_k f_K^2)}$$

obtained in [Lou97]. *If we used this worse lower bound for* h_K^- *then we would end up with the worse following result: If the exponent of the ideal class group of an imaginary cyclic quartic field* K *of odd conductor* f_K *is less than or equal to* 2, *then* $f_k \leq 4053$ *and* $f_K \leq 2\cdot 10^7$. *Moreover, whereas there are* 2 946 395 *imaginary cyclic fields of odd conductors* $f_K \leq 2\cdot 10^7$ *and such that* $f_k \leq 4053$, *only* 1175 *out of them may have their ideal class groups of exponents* ≤ 2, *the largest possible conductor being* $f_K = 11667$ *(for* $f_k = 3889$ *and* $f_{K/k} = 3$*).*

5. The non-abelian case

We showed in [Lou03] how taking into account the behavior of the prime 2 in CM-fields can greatly improve upon the upper bounds on the root numbers of the normal CM-fields with abelian maximal totally real subfields of a given (relative) class number. We now explain how we can improve upon previously known upper bounds for residues of Dedekind zeta functions of non-necessarily abelian number fields by taking into accound the behavior of the prime 2:

Theorem 10. *Let* K *be a number field of degree* $m \geq 3$ *and root discriminant* $\rho_K = d_K^{1/m}$. *Set* $v_m = (m/(m-1))^{m-1} \in [9/4, e)$, *and* $E(x) := (e^x - 1)/x = 1 + O(x)$ *for* $x \to 0^+$. *Then,*

(13)
$$\text{Res}_{s=1}(\zeta_K(s)) \leq (e/2)^{m-1} v_m \frac{\Pi_K(2)}{\Pi_{\mathbf{Q}}^m(2)} \left(\log\rho_K + (\log 4)E(\frac{\log 4}{\log\rho_K})\right)^{m-1}.$$

Moreover, $0 < \beta < 1$ *and* $\zeta_K(\beta) = 0$ *imply*

(14)
$$\text{Res}_{s=1}(\zeta_K(s)) \leq (1-\beta)(e/2)^m \frac{\Pi_K(2)}{\Pi_{\mathbf{Q}}^m(2)} \left(\log\rho_K + (\log 4)E(\frac{\log 4}{\log\rho_K})\right)^m.$$

Proof. We only prove (13), the proof of (14) being similar. We set

$$\Pi_K(2,s) := \prod_{\mathcal{P}|2}(1 - (N(\mathcal{P}))^{-s})^{-1}$$

(which is ≥ 1 for $s > 0$). According to [Lou01, Section 6.1] but using the bound

$$\zeta_K(s) \leq \frac{\Pi_K(2,s)}{\Pi_{\mathbf{Q}}^m(2,s)}\zeta^m(s)$$

instead of the bound $\zeta_K(s) \leq \zeta^m(s)$, we have

$$\begin{aligned}\mathrm{Res}_{s=1}(\zeta_K(s)) &\leq \frac{\Pi_K(2)}{\Pi_{\mathbf{Q}}^m(2)}\left(\frac{e\log d_K}{2(m-1)}\right)^{m-1} g(s_K)\\ &= (e/2)^{m-1}v_m\frac{\Pi_K(2)}{\Pi_{\mathbf{Q}}^m(2)}(\log\rho_K)^{m-1}g(s_K),\end{aligned}$$

where $s_K = 1 + 2(m-1)/\log d_K \in [1,6]$ and

$$g(s) := \frac{\Pi_K(2,s)/\Pi_K(2)}{\Pi_{\mathbf{Q}}^m(2,s)/\Pi_{\mathbf{Q}}^m(2)} \leq h(s) := \Pi_{\mathbf{Q}}^m(2)/\Pi_{\mathbf{Q}}^m(2,s)$$

(for $\Pi_K(2,s) \leq \Pi_K(2,1) = \Pi_K(2)$ for $s \geq 1$). Now, $\log h(1) = 0$ and $(h'/h)(s) = \frac{m\log 2}{2^s-1} \leq m\log 2$ for $s \geq 1$. Hence,

$$\begin{aligned}\log h(s_K) &\leq (s_K-1)m\log 2 = \frac{(m-1)\log 4}{\log\rho_K},\\ g(s_K) \leq h(s_K) &\leq \left(\exp(\frac{\log 4}{\log\rho_K})\right)^{m-1},\end{aligned}$$

and (13) follows. □

Corollary 11. *(Compare with* [Lou01, Theorems 12 and 14] *and* [Lou03, Theorems 9 and 22]*). Set* $c = 2(\sqrt{3}-1)^2 = 1.07\cdots$ *and* $v_m := (m/(m-1))^{m-1} \in [2,e)$*. Let* N *be a normal CM-field of degree* $2m > 2$*, relative class number* h_N^- *and root discriminant* $\rho_N = d_N^{1/2m} \geq 650$*. Assume that* N *contains no imaginary quadratic subfield (or that the Dedekind zeta functions of the imaginary quadratic subfields of* N *have no real zero in the range* $1-(c/\log d_N) \leq s < 1$*). Then,*

$$h_N^- \geq \frac{c}{2mv_me^{c/2-1}}\left(\frac{4\sqrt{\rho_N}}{3\pi e\big(\log\rho_N + (\log 4)E(\frac{\log 4}{\log\rho_N})\big)}\right)^m. \tag{15}$$

Hence, $h_N^- > 1$ *for* $m \geq 5$ *and* $\rho_N \geq 14610$*, and for* $m \geq 10$ *and* $\rho_N \geq 9150$*. Moreover,* $h_N^- \to \infty$ *as* $[N:\mathbf{Q}] = 2m \to \infty$ *for such normal CM-fields* N *of root discriminants* $\rho_N \geq 3928$*.*

Proof. To prove (15), follow the proof of [Lou01, Theorems 12 and 14] and [Lou03, Theorems 9 and 22], but now make use of Theorem 10 instead of

[Lou01, Theorem 1] and finally notice that

$$\frac{\Pi_N(2)}{\Pi_K(2)/\Pi_{\mathbf{Q}}^m(2)} = 2^m \Pi_N(2)/\Pi_K(2) = 2^m \prod_{\mathcal{P}|(2)} \big(1 - \frac{\chi(\mathcal{P})}{N(\mathcal{P})}\big)^{-1} \geq (4/3)^m$$

(χ is the quadratic character associated with the quadratic extension N/K, and $\mathcal{P}$ ranges over all the primes ideals of K lying above the rational prime 2). □

We also refer the reader to [LK] for a recent paper dealing with upper bounds on the degrees and absolute values of the discriminants of the CM-fields of class number one, under the assumption of the generalized Riemann hypothesis. The proof relies on a generalization of Odlyzko ([Odl]), Stark ([Sta]) and Bessassi's ([Bes]) upper bounds for residues of Dedekind zeta functions of totally real number fields of large degrees, this generalization taking into account the behavior of small primes. All these bounds are better than ours, but only for numbers fields of large degrees and small root discriminants, whereas ours are developped to deal with CM-fields of small degrees.

6. An open problem

Let k be a non-normal totally real cubic field of positive discriminant d_k. It is known that (see [Lou01, Theorem 2]):

$$\mathrm{Res}_{s=1}(\zeta_k(s)) \leq \frac{1}{8}\log^2 d_k.$$

This bound has been used in [BL02] to try to solve the class number one problem for the non-normal sextic CM-fields K containing no quadratic subfields. However, to date this problem is in fact not completely solved for we had a much too large bound $d_K \leq 2\cdot 10^{29}$ on the absolute values d_K of their discriminants (see [BL02, Theorem 12]). In order to greatly improve upon this upper bound, we would like to prove that there exists some explicit constant κ such that

$$\mathrm{Res}_{s=1}(\zeta_k(s)) \leq \frac{1}{8}\frac{\Pi_k(2)}{\Pi_{\mathbf{Q}}^3(2)}(\log d_k + \kappa)^2$$

holds true for any non-normal totally real cubic field k. However, adapting the proof of [Lou01, Theorem 2] is not that easy and we have not come up yet with such a result, the hardest cases to handle being the cases $(2) = \mathcal{P}$ or $\mathcal{P}_1\mathcal{P}_2$ in k, where we would expect bounds of the type

$$\mathrm{Res}_{s=1}(\zeta_k(s)) \leq \begin{cases} (\log d_k + \kappa''')^2/24 & \text{if } (2) = \mathcal{P}_1\mathcal{P}_2 \text{ in } k, \\ (\log d_k + \kappa'''')^2/56 & \text{if } (2) = \mathcal{P} \text{ in } k. \end{cases}$$

At the moment, we can only prove the following result which already yields a 1000-fold improvement on our previous bound $d_K \leq 2 \cdot 10^{29}$:

Theorem 12. *Let k be a totally real cubic number field. Then,*

$$\mathrm{Res}_{s=1}(\zeta_k(s)) \leq \begin{cases} (\log d_k - \kappa)^2/8 & \text{if } (2) = \mathcal{P}_1\mathcal{P}_2\mathcal{P}_3 \text{ in } k, \\ (\log d_k - \kappa')^2/16 & \text{if } (2) = \mathcal{P}_1\mathcal{P}_2^2,\ \mathcal{P}_1\mathcal{P}_2 \text{ or } \mathcal{P} \text{ in } k, \\ (\log d_k + \kappa'')^2/32 & \text{if } (2) = \mathcal{P}^3 \text{ in } k, \end{cases}$$

where

$$\begin{cases} \kappa = 2\log(4\pi) - 2\gamma - 2 = 1.90761\cdots, \\ \kappa' = 2\log(2\pi) - 2\gamma - 2 = 0.52132\cdots, \\ \kappa'' = 2 + 2\gamma - 2\log\pi = 0.86497\cdots. \end{cases}$$

As a consequence, if K is a non-normal sextic CM-field containing no quadratic subfield and if the class number of K is equal to one, then $d_K \leq 2 \cdot 10^{26}$.

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Stéphane R. Louboutin
Institut de Mathématiques de Luminy, UMR 6206
163, avenue de Luminy, Case 907
13288 Marseille Cedex 9, FRANCE
E-mail : `louboutiQiml.univ-mrs.fr`