Lobachevskii Journal of Mathematics
Vol. 12, 2003, 11 – 39

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CONVERGENCE FOR STEP LINE PROCESSES UNDER SUMMATION OF RANDOM INDICATORS AND MODELS OF MARKET PRICING

(submitted by D. Kh. Mustari)

Abstract. Functional limit theorems for random step lines and random broken lines defined by sums of iid random variables with replacements are obtained and discussed. Also we obtained functional limit theorems for integrals of such random processes. We use our results to study a number of models of the financial market.

1. Introduction

The paper deals with random step lines processes and random broken lines processes defined by sums of independent identically distributed random variables multiplied by values of indicators defined on another probability space. These processes describe some models in which random variables are replaced with other one. We prove our theorems in the Skorokhod space $D_b(\mathbf{R}_+)$ of bounded functions defined on $\mathbf{R}_+ = [0, \infty)$ and in the space $C_b(\mathbf{R}_+)$ of bounded continuous functions defined on $\mathbf{R}_+$.

For the indicators we will assume that one of the following properties is valid:

1) in each line the indicators are independent;

2) the indicators are defined by occupations of series of balls.

And our limit theorems are founded on strong laws of large numbers for the indicators (Lemma 1, Lemma 2 and there Corollaries). Observe, that in the case 2) weak law of large numbers for the indicators follows from Theorem 1 on p. 22 in (Kolchin, Sevast'ianov, and Chistiakov, [1]). So Lemma 2 is a straightness of this result.

We have different types of replacements:

a) summands are replaced randomly by zeros;

b) summand are replaced randomly by nonzero random variables.

Also we consider various examples which show that the class of such indicators is sufficiently large. In the case of the convergence to gaussian process we prove the convergence for integrals of our processes in the space $C[0, T]$, $0 < T < \infty$, of bounded continuous functions defined on $[0, T]$.

As a corollaries we obtain functional limit theorems in the case then indicators and random variables are defined on the same probability space and are independent. In this case our limit theorems can be considered as functional limit theorems for random sums (see about limit theorems for random sums, for example Korolev and Kruglov [2]).

We give some applications of our results to some models of stochastic financial mathematics. We construct three models of a financial market: the market with constant number of agents, the market with increasing number of agents and the market with decreasing number of agents. We suppose that the trading policy is changing during the time and we consider various types of this changing. For such models we construct the random processes of stock market price with respect to market probability and we construct the random processes of stock market price with respect to risk-neutral probability.

The limit theorems for such models for almost all sequences of the trade policy changing are given. Using this limit theorems we obtained analogs of Black-Scholes formula for there models.

For the space $C_b(\mathbf{R}_+)$ the method of the proofs of our theorems is the same as those of Prokhorov theorem (see Prokhorov, [3]) and for the space $D_b(\mathbf{R}_+)$ the method of the proofs of our theorems is the same as those of Skorokhod theorem (see Gihman and Skorokhod, [4]). Our results generalizes functional limit theorems from Fazekas and Chuprunov [5]. We mention that in Rusakov [6], another type of replacement was considered and a functional limit theorem with an Ornstein-Uhlenbeck limit process was proved.

2. Preliminary results: strong laws of large numbers.

We will use the following denotation:
$\xrightarrow{d}$ – the convergence in distribution;
$\xrightarrow{P}$ – the convergence in probability;
$\stackrel{d}{=}$ – the equality in distribution (for random processes $\stackrel{d}{=}$ is the equality of finite dimensional distributions);
$\mathbb{N}$ – the set of positive integers;
$\mathbb{Q}$ – the set of rational numbers;
$\mathbb{Q}_+ = \mathbb{R}_+ \cap \mathbb{Q}$;
$\mathbf{Q}_T$ – the set of numbers yT where $y \in [0,1] \cap \mathbf{Q}$;
$[c]$ – integer part of a real number c;
$\{c\}$ – fractional part of a real number c;
$\mathcal{L}(X)$ – the law of distribution of a random element or a random processes X;
$\mathbb{1}_A$ – the indicator function of an event A;
$\gamma(v)$ – a gaussian random variable with the expectation zero and the variance v^2;
Φ – distribution function of $\gamma(1)$;
$\sigma^2(\xi)$ – the variance of the random variable ξ;
$W(t)$, $W'(t)$ – the independent standard Brownian motions (Bm);
$W_D(t)$, $W_{D'}(t)$, $t \in \mathbb{R}_+$ – the independent homogeneous infinitely divisible

random processes with independent increments such that $W_D(1) \stackrel{d}{=} \gamma_D$ and $W_{D'}(1) \stackrel{d}{=} \gamma_{D'}$ where γ_D and $\gamma_{D'}$ is an infinitely divisible random variables.

For any (a_i) we will suppose that $\sum\limits_{i\in\emptyset} a_i = 0$ and $\prod\limits_{i\in\emptyset} a_i = 1$.

We will consider two probability spaces $\{\Omega, \mathfrak{A}, \mathbf{P}\}$ and $\{\Omega_1, \mathfrak{A}_1, \mathbf{P}_1\}$. The first probability space will play a role of a space of noise, and the second one will play a role a space of information. Next we denote by $\mathbf{E}$ and $\mathbf{E}_1$ the mathematical expectations respectively to the probabilities $\mathbf{P}$ and $\mathbf{P}_1$.

We will consider a scheme of series parameterized by a natural n. Throughout the paper we will use a previously fixed natural parameter m playing a role of viscosity for an information. It shows a rate of changing of informational events along the discrete time. We will examine an asymptotic under a sequence (k_n) of natural numbers such that $k_n \to \infty$ as $n \to \infty$ and $k_n < k_{n+1}$, $n \in \mathbf{N}$.

We will denote informational events from $\mathfrak{A}_1$ as $A_{ij} \equiv A_{ij}(n)$ for $i, j \in \mathbb{N}$, and corresponding indicators will denote as $\mathbb{I}_{A_{ij}(n)}(\omega_1) \stackrel{\triangle}{=} \mathbb{I}_{ij} \equiv \mathbb{I}_{ij}(n)$, $\omega_1 \in \Omega_1$. We will suppose that probabilities of these events independent of the time index i, i.e. the sequence (informational flow) $(A_{ij})_{i=1}^{\infty}$ is homogeneous on the time,

$$\mathbf{P}_1\{A_{ij}(n)\} \stackrel{\triangle}{=} p_j(n) \equiv p_j, \qquad \forall i \in \mathbf{N}$$

In connect with some ways an interpretation and a formalization an information flow we will suppose that the informational events $A_{ij}(n)$ satisfy to one of the following properties:

(A) $A_{ij}(n)$ are mutually independent over all $n \in \mathbb{N}$ and such that for all $x \in \mathbb{R}_+$ for a continues function f the following limit relation holds and the corresponding limits exist

$$f(x) \stackrel{\triangle}{=} \lim_{n\to\infty} \mathbf{E}_1\{\mathbb{I}_{i1}(n)\mathbb{I}_{i2}(n)\cdots\mathbb{I}_{i[k_n x]}(n)\} = \lim_{n\to\infty} p_1(n)p_2(n)\cdots p_{[k_n x]}(n)$$

In particular, (A) implies that $f(0) = 1$ and f is decreasing.

(A1) Let ξ, ξ_j, $j \in \mathbf{N}$ be independent identically uniformly distributed on $[0,1)$ random variables defined on $(\Omega_1, \mathfrak{A}_1, \mathbf{P}_1)$, $\Delta_i = \Delta_{ni} = [\frac{i-1}{k_n}, \frac{i}{k_n})$, $1 \le i \le k_n$ and $m \in \mathbf{N}$.

Define the informational events with connection to the split Δ of the interval $[0,1)$

$$A_{ij} \equiv A_{ij}(n,m) = \cap_{k=m(j-1)+1}^{mj} \{\omega_1 \in \Omega_1 : \ \xi_k(\omega_1) \notin \Delta_i\}, \ 1 \le i \le k_n, \ j, n \in \mathbf{N}.$$

For defined here information events there exists a natural interpretation. The intervals Δ_j, $j = 1, \dots, k_n$ we treat as a row of boxes. Random variables ξ_j are independent copies of rv ξ. Each realization of rv ξ we treat as random allocation one ball onto the k_n boxes. We consider series containing m balls. And the equality $\mathbb{I}_{ij}$ to one means that for a random occupation all balls from jth series do not allocate the ith box. We will use the following simple corollaries of well-known results.

Lemma 1. *Let* $g_{ni} : \Omega_1 \to \mathbf{R}$, $1 \le i \le k_n$, $n \in \mathbf{N}$ *be independent identically distributed random variables in each series such that for all* $1 \le i \le k_n$, $n \in \mathbf{N}$,

$\mathbf{E}_1(g_{ni} - \mathbf{E}_1 g_{ni})^4 \le C$ *for some* $C > 0$ *and* $\mathbf{E}_1 g_{in} \to p$ *as* $n \to \infty$. *Then we have*

$$\frac{1}{k_n}\sum_{i=1}^{k_n} g_{ni} \to p \;\; as \;\; n \to \infty$$

almost surely.

Proof. Observe that $\mathbf{E_1}\left|\frac{1}{k_n}\sum\limits_{i=1}^{k_n}(g_{ni} - \mathbf{E}_1 g_{ni})\right|^4 \le \frac{7C}{k_n^2}$, $n \in \mathbf{N}$. Therefore for all $\varepsilon > 0$ it holds $\sum\limits_{n=1}^{\infty} \mathbf{P}_1\{\left|\frac{1}{k_n}\sum\limits_{i=1}^{k_n}(g_{ni} - \mathbf{E}_1 g_{ni})\right| > \varepsilon\} \le \frac{1}{\varepsilon^4}\sum\limits_{n=1}^{\infty}\frac{7C}{k_n^2} < \infty$. Consequently, $\frac{1}{k_n}\sum\limits_{i=1}^{k_n}(g_{ni} - \mathbf{E}_1 g_{ni}) \to 0$ as $n \to \infty$ almost sure. Then we have

$$\frac{1}{k_n}\sum_{i=1}^{k_n} g_{ni} - p = \frac{1}{k_n}\sum_{i=1}^{k_n}(g_{ni} - \mathbf{E}_1 g_{ni}) + \mathbf{E}_1 g_{ni} - p \to 0$$

as $n \to \infty$ almost sure. The proof is complete.

We will set

$$f_{1n}(t) = f_{1n}(t)(\omega_1) = \sum_{i=1}^{k_n} 1\!\!1_{[k_n t]i}(n) 1\!\!1_{([k_n x]-1)i}(n) \cdots 1\!\!1_{1i}(n), t \in \mathbf{R}_+, \omega_1 \in \Omega_1$$

and

$$f_{2n}(t) = f_{2n}(t)(\omega_1) = \sum_{i=1}^{k_n}(1 - 1\!\!1_{[k_n t]i}(n) 1\!\!1_{([k_n x]-1)i}(n) \cdots 1\!\!1_{1i}(n)), t \in \mathbf{R}_+, \omega_1 \in \Omega_1,$$

where $n \in \mathbf{N}$.

Corollary 1. *Let (A) be valid. Then there exists* $\Omega' \subset \Omega_1$ *such that* $\mathbf{P}_1(\Omega') = 1$ *and for all* $\omega_1 \in \Omega'$ *uniformly by* $t \in \mathbf{R}_+$ *one has*

$$\frac{f_{1n}(t)(\omega_1)}{k_n} \to f(t) \;\; as \;\; n \to \infty. \tag{1}$$

Proof. By Theorem A there exists $\Omega' \subset \Omega_1$ such that $\mathbf{P}_1(\Omega') = 1$ and for all $\omega_1 \in \Omega'$ for all $t \in \mathbf{Q}_+$ (1) is valid.

Let $\omega_1 \in \Omega'$ and $t \in \mathbf{R}_+$. Choose $t_1, t_2 \in \mathbf{Q}_+$ such that $0 \le t_1 \le t \le t_2$. Since f_{1n} is a decreasing function we have

$$\frac{f_{1n}(t_2)(\omega_1)}{k_n} \le \frac{f_{1n}(t)(\omega_1)}{k_n} \le \frac{f_{1n}(t_1)(\omega_1)}{k_n}.$$

Therefore, one has

$$f^*(t_2) \le \liminf_{n\to\infty}\frac{f_{1n}(t)(\omega_1)}{k_n} \le \limsup_{n\to\infty}\frac{f_{1n}(t)(\omega_1)}{k_n} \le f(t_1).$$

By the continuity of $f(t)$ this follows (1) uniformly by $t \in \mathbf{R}_+$. The corollary is proved.

Corollary 2. *Let (A) be valid. There exists* $\Omega' \subset \Omega_1$ *such that* $\mathbf{P}_1(\Omega') = 1$ *and for all* $\omega_1 \in \Omega'$ *uniformly by* $t \in \mathbf{R}_+$ *one has*

$$\frac{f_{2n}(t)(\omega_1)}{k_n} \to 1 - f(t) \ \ as \ \ n \to \infty.$$

Since $f_{2n}(t) = k_n - f_{1n}(t)$, $t \in \mathbf{R}_+$, Corollary 2 follows from Corollary 1.

From Theorem 1 on p. 22 in (Kolchin, Sevast'ianov, and Chistiakov, [1]) it follows that for all $t > 0$ $\frac{1}{k_n} \sum_{i=1}^{k_n} \prod_{j=1}^{[tk_n]} I_{pji} \to e^{-mt}$, as $n \to \infty$ in probability. Consider more strong version of this result.

LEMMA 2. *Let (A1) be valid. For all $t \in \mathbf{R}_+$ we have*

$$\frac{1}{k_n} \sum_{i=1}^{k_n} \prod_{j=1}^{[tk_n]} \mathbb{1}_{pji} \to e^{-mt}, \quad \text{as} \quad n \to \infty.$$

almost sure.

PROOF. Let $t \in \mathbf{R}_+$. Denote $g_{ni} = \prod_{j=1}^{[tk_n]} \mathbb{1}_{pji}$,

$$g_i = g_{ni} - \mathbf{E_1} g_{ni} = g_{ni} - \left(1 - \frac{1}{k_n}\right)^{m[tk_n]}.$$

We will estimate

$$\begin{aligned} I = \mathbf{E_1} \left(\sum_{i=1}^{k_n} g_i \right)^4 &= \sum_{j_1=1}^{k_n} \sum_{j_2=1}^{k_n} \sum_{j_3=1}^{k_n} \sum_{j_4=1}^{k_n} \mathbf{E} g_{j_1} g_{j_2} g_{j_3} g_{j_4} \\ &= k_n \mathbf{E}(g_1)^4 + 6k_n(k_n-1)\mathbf{E}(g_1)^2(g_2)^2 + 4k_n(k_n-1)\mathbf{E}(g_1)^3 g_2 \\ &+ 12k_n(k_n-1)(k_n-2)\mathbf{E}(g_1)^2 g_2 g_3 + 24k_n(k_n-1)(k_n-2)(k_n-3)\mathbf{E} g_1 g_2 g_3 g_4 \\ &= I_1 + 6I_2 + 4I_3 + 12I_4 + 24I_5. \end{aligned}$$

Since $|g_i| \le 1$, we have

$$I_1 + 6I_2 + 4I_3 \le k_n + 6k_n(k_n-1) + 4k_n(k_n-1) \le 10k_n^2. \tag{2}$$

Using the inequality $a^k - b^k \le k(a-b)$, $0 \le b \le a \le 1$, $k \in \mathbf{N}$ and the inequality

$$\left(1 - \frac{1}{k_n}\right)^{2m[tk_n]} \left(1 - \frac{2}{k_n}\right)^{m[tk_n]} - \left(1 - \frac{1}{k_n}\right)^{4[tk_n]} \le 0$$

we obtain

$$
\begin{aligned}
I_4 &\le k_n^3 \mathbf{E}\Bigg(\prod_{i=1}^{[tk_n]} \mathbb{I}_{\{\xi_i \notin \Delta_1\}} - \Big(1-\frac{1}{k_n}\Big)^{m[tk_n]}\Bigg)^2 \\
&\quad \times \Bigg(\prod_{i=1}^{[tk_n]} \mathbb{I}_{\{\xi_i \notin \Delta_2\}} - \Big(1-\frac{1}{k_n}\Big)^{m[tk_n]}\Bigg)\Bigg(\prod_{i=1}^{[tk_n]} \mathbb{I}_{\{\xi_i \notin \Delta_3\}} - \Big(1-\frac{1}{k_n}\Big)^{m[tk_n]}\Bigg) \\
&= k_n^3 \mathbf{E}\Bigg(\prod_{i=1}^{[tk_n]} \mathbb{I}_{\{\xi_i \notin \Delta_1\}} - \Big(2\prod_{i=1}^{[tk_n]} \mathbb{I}_{\{\xi_i \notin \Delta_1\}}\Big)\Big(1-\frac{1}{k_n}\Big)^{m[tk_n]} + \Big(1-\frac{1}{k_n}\Big)^{2m[tk_n]}\Bigg) \\
&\quad \times \Bigg(\prod_{i=1}^{[tk_n]} I_{\{\xi_i \notin \Delta_2 \cup \Delta_3\}} - \Big(\prod_{i=1}^{[tk_n]} I_{\{\xi_i \notin \Delta_2\}}\Big)\Big(1-\frac{1}{k_n}\Big)^{m[tk_n]} \\
&\quad - \Big(\prod_{i=1}^{[tk_n]} I_{\{\xi_i \notin \Delta_1\}}\Big)\Big(1-\frac{1}{k_n}\Big)^{m[tk_n]} + \Big(1-\frac{1}{k_n}\Big)^{2m[tk_n]}\Bigg) \\
&\le k_n^3 \mathbf{E}\Bigg(\Big(\prod_{i=1}^{[tk_n]} I_{\{\xi_i \notin \Delta_1\}}\Big)\Big(1-2\Big(1-\frac{1}{k_n}\Big)^{m[tk_n]}\Big) + \Big(1-\frac{1}{k_n}\Big)^{2m[tk_n]}\Bigg) \\
&\quad \times \Bigg(\prod_{i=1}^{[tk_n]} I_{\{\xi_i \notin \Delta_2 \cup \Delta_3\}} - \Big(\prod_{i=1}^{[tk_n]} I_{\{\xi_i \notin \Delta_2\}}\Big)\Big(1-\frac{1}{k_n}\Big)^{m[tk_n]} \\
&\quad - \prod_{i=1}^{[tk_n]} I_{\{\xi_i \notin \Delta_3\}}\Big(1-\frac{1}{k_n}\Big)^{m[tk_n]} + \Big(1-\frac{1}{k_n}\Big)^{2m[tk_n]}\Bigg) \\
&= k_n^3\Bigg(\Big(1-2\Big(1-\frac{1}{k_n}\Big)^{m[tk_n]}\Big)\Bigg(\Big(1-\frac{3}{k_n}\Big)^{m[tk_n]} \\
&\quad - 2\Big(1-\frac{2}{k_n}\Big)^{m[tk_n]}\Big(1-\frac{1}{k_n}\Big)^{m[tk_n]} + \Big(1-\frac{1}{k_n}\Big)^{3m[tk_n]}\Bigg) \\
&\quad + \Big(1-\frac{1}{k_n}\Big)^{2m[tk_n]}\Big(1-\frac{2}{k_n}\Big)^{m[tk_n]} - \Big(1-\frac{1}{k_n}\Big)^{4m[tk_n]}\Bigg) \\
&\le n^3\Bigg(\Big(1-\frac{1}{k_n}\Big)^{3m[tk_n]} - \Big(1-\frac{2}{k_n}\Big)^{m[tk_n]}\Big(1-\frac{1}{k_n}\Big)^{m[tk_n]}\Bigg) \\
&\le k_n^3\Bigg(\Big(1-\frac{2}{k_n}+\frac{1}{k_n^2}\Big)^{m[tk_n]} - \Big(1-\frac{2}{k_n}\Big)^{m[tk_n]}\Bigg) \le k_n^3\frac{1}{k_n^2}m[tk_n] \le mtk_n^2. \qquad (3)
\end{aligned}
$$

Now, using the Newton binomial, we have

$$\begin{aligned}
I_5 \le k_n^4 \mathbf{E} g_1 g_2 g_1 g_3 g_4 &= k_n^4 \Bigg(\Big(1-\frac{4}{k_n}\Big)^{m[tk_n]} - 4\Big(1-\frac{3}{k_n}\Big)^{m[tk_n]}\Big(1-\frac{1}{k_n}\Big)^{m[tk_n]} \\
&\quad + 6\Big(1-\frac{2}{k_n}\Big)^{m[tk_n]}\Big(1-\frac{1}{k_n}\Big)^{2m[tk_n]} - 4\Big(1-\frac{1}{k_n}\Big)^{4m[tk_n]} \\
&\quad + \Big(1-\frac{1}{k_n}\Big)^{4m[tk_n]}\Bigg) \\
&= k_n^4 \Bigg(\Big(1-\frac{4}{k_n}\Big)^{m[tk_n]} - 4\Big(1-\frac{4}{k_n}+\frac{3}{k_n^2}\Big)^{m[tk_n]} \\
&\quad + 6\Big(1-\frac{4}{k_n}+\frac{5}{k_n^2}-\frac{2}{k_n^3}\Big)^{m[tk_n]} - 3\Big(1-\frac{4}{k_n}+\frac{6}{k_n^2}-\frac{4}{k_n^3}+\frac{1}{k_n^4}\Big)^{m[tk_n]}\Bigg) \\
&= k_n^4 \Bigg(\Big(1-\frac{4}{k_n}\Big)^{m[tk_n]} - 4\sum_{k=0}^{m[tk_n]} C_{m[tk_n]}^k \Big(1-\frac{4}{k_n}\Big)^{m[tk_n]-k}\Big(\frac{3}{k_n^2}\Big)^k \\
&\quad + 6\sum_{k=0}^{m[tk_n]} C_{m[tk_n]}^k \Big(1-\frac{4}{k_n}\Big)^{m[tk_n]-k}\Big(\frac{5}{k_n^2}-\frac{2}{k_n^3}\Big)^k \\
&\quad - 3\sum_{k=0}^{m[tk_n]} C_{m[tk_n]}^k \Big(1-\frac{1}{k_n}\Big)^{m[tk_n]-k}\Big(\frac{6}{k_n^2}-\frac{4}{k_n^3}+\frac{1}{k_n^4}\Big)^k\Bigg) \\
&= k_n^4 \Bigg(k_n\Big(-\frac{12}{k_n^2}+\frac{30}{k_n^2}-\frac{12}{k_n^3}-\frac{18}{k_n^2}+\frac{12}{k_n^3}-\frac{3}{k_n^4}\Big)\Big(1-\frac{1}{k_n}\Big)^{m[tk_n]-1} \\
&\quad + \sum_{k=2}^{m[tk_n]} C_{m[tk_n]}^k \Big[-4\Big(\frac{3}{k_n^2}\Big)^k + 6\Big(\frac{5}{k_n^2}-\frac{2}{k_n^3}\Big)^k - 3\Big(\frac{6}{k_n^2}-\frac{4}{k_n^3}+\frac{1}{k_n^4}\Big)^k\Big]\Bigg).
\end{aligned}$$

Since

$$k_n\Big(-\frac{12}{k_n^2}+\frac{30}{k_n^2}-\frac{12}{k_n^3}-\frac{18}{k_n^2}+\frac{12}{k_n^3}-\frac{3}{k_n^4}\Big)\Big(1-\frac{1}{k_n}\Big)^{m[tk_n]-1} < 0$$

and

$$-4\Big(\frac{3}{k_n^2}\Big)^k + 6\Big(\frac{5}{k_n^2}-\frac{2}{k_n^3}\Big)^k - 3\Big(\frac{6}{k_n^2}-\frac{4}{k_n^3}+\frac{1}{k_n^4}\Big)^k < 0$$

for $k > 4$, we have

$$I_5 \le k_n^4 \sum_{k=2}^{4} C_{m[tk_n]}^k \Big[-4\Big(\frac{3}{k_n^2}\Big)^k + 6\Big(\frac{5}{k_n^2}-\frac{2}{k_n^3}\Big)^k - 3\Big(\frac{6}{k_n^2}-\frac{4}{k_n^3}+\frac{1}{k_n^4}\Big)^k\Big] \le$$

$$\le \frac{k_n^4}{k_n^2}\sum_{k=2}^{4} \frac{C_{m[tk_n]}^k}{k_n^{2(k-1)}} 6\cdot 5^k \le 6\Big(t^2 m^2 \frac{25}{2} + t^3 m^3 \frac{125}{6k_n} + t^4 m^4 \frac{625}{24 k_n^2}\Big) k_n^2. \qquad (4)$$

From (2), (3) and (4) we obtain

$$\mathbf{E}\left(\sum_{i=1}^{k_n}\left(g_{ni}-\mathbf{E_1}g_{ni}\right)\right)^4 \le C'k_n^2,$$

where C' does not depends from n. Further the proof repeats the proof of Lemma 1.

Corollary 1. *Let (A1) be valid. Then there exists $\Omega' \subset \Omega_1$ such that $\mathbf{P}_1(\Omega')=1$ and for all $\omega_1 \in \Omega'$ uniformly for $t \in \mathbf{R}_+$ one has*

$$\frac{f_{1n}(t)(\omega_1)}{k_n} \to e^{-mx} \quad as\ n \to \infty.$$

Corollary 2. *Let (A1) be valid. Then there exists $\Omega' \subset \Omega_1$ such that $\mathbf{P}_1(\Omega')=1$ and for all $\omega_1 \in \Omega'$ uniformly for $x \in \mathbf{R}^+$ one has*

$$\frac{f_{2n}(t)(\omega_1)}{k_n} \to 1-e^{-mt} \quad as\ n \to \infty.$$

The proofs of Corollary 1 and Corollary 2 is the same as ones of Lemma 1. We will denote $f_m(t)=e^{-mt}$, $t \in \mathbf{R}_+$.

3. Preliminary results: functional limit theorems

To construe our noise structure and to study the random processes we give conditions and definitions for considered random elements on $\{\Omega, \mathfrak{A}, \mathbf{P}\}$.

(Y) Let $Y_n, Y_{ni},\ n,i \in \mathbf{N}$ be an rectangle array of random variables defined on $\{\Omega, \mathfrak{A}, \mathbf{P}\}$ which are independent and identically distributed in each string.

(Y') Let Y'_n, Y'_{ni}, $i \in \mathbf{N}$ be an array of independent and identically distributed in each string random variables defined on $(\Omega, \mathfrak{A}, \mathbf{P})$ such that Y'_{ni}, Y_{ni}, $i \in \mathbf{N}$ are independent random variables for all $n \in \mathbf{N}$.

And we will assume that some of the following conditions are satisfied:

(C) $\displaystyle\sum_{j=1}^{k_n} Y_{nj} \xrightarrow{d} \gamma_D$, as $n \to \infty$; (C') $\displaystyle\sum_{j=1}^{k_n} Y'_{nj} \xrightarrow{d} \gamma_{D'}$, as $n \to \infty$;

(S) $\displaystyle\sum_{i=1}^{k_n} Y_{ni} \xrightarrow{d} \gamma(v)$, as $n \to \infty$; (S') $\displaystyle\sum_{i=1}^{k_n} Y'_{ni} \xrightarrow{d} \gamma(v')$, as $n \to \infty$.

The following theorem is a simple corollary of Lindeberg-Feller Theorem.
Theorem A. *Let (Y) be fulfilled.*
(1) The condition (S) is valid if and only if

(1b) for all $\varepsilon > 0$ $k_n\mathbf{P}\{|Y_n| > \varepsilon\} \to 0$ as $n \to \infty$;

(2b) $k_n\mathbf{E}Y_n\mathbb{1}_{\{|Y_n|\le 1\}} \to 0$ as $n \to \infty$;
and
(3b) $k_n\sigma^2(Y_n\mathbb{1}_{\{|V_n|\le 1\}}) \to v^2$, as $n \to \infty$.

(2) *Let (S) be valid and* $b_{ni} \in \mathbf{R}$ *be such that* $\sup_{\{1\le i\le k_n, n\in\mathbf{N}\}} |b_{ni}| < \infty$. *Denote* $U_n = \sum_{i=1}^{k_n} b_{ni} Y_{ni}$, $n \in \mathbf{N}$. *Then* $U_n \xrightarrow{d} \gamma(s)$ *as* $n \to \infty$ *if and only if*

(4b) $\sigma^2(Y_n 1\!\!\mathrm{I}_{\{|Y_n|\le 1\}}) \sum_{i=1}^{k_n} (b_{ni})^2 \to s^2$, *as* $n \to \infty$.

(D) Let $f_n(x),\ f^*(x) : \mathbb{R}_+ \mapsto \mathbb{R}_+,\ n \in \mathbb{N}$ be integer valued functions, $f^*(x)$ is a continue bounded function plays a role of the following limit

$$\frac{f_n(x)}{k_n} \to f^*(x), \qquad \text{as} \quad n \to \infty$$

for all $x \in [0, \infty]$ and one of the following conditions is true:
a) f_n are increasing functions;
b) f_n are decreasing functions.
In the paper we will examine a sequence of the random processes with a time scale managing by the functions $(f_n(x))$,

$$(Z0') \qquad X'_{*n}(x) = X'_{*n}(Y)(x) = \sum_{i=1}^{f_n(x)} Y_{ni}, \quad x \in \mathbb{R}_+, \qquad n \in \mathbb{N}$$

and their modifications formed by piece-wise linear continuous broken lines,

$$(Z0) \quad X_{0n}(x) = X_{0n}(Y)(x) = X'_{*n}(x_k) + \frac{x - x_k}{x_{k+1} - x_k} \left(X'_{*n}(x_{k+1}) - X'_{*n}(x_k) \right),$$

$n \in \mathbb{N}$, $x \in [x_k, x_{k+1})$, where $0 = x_0 < x_1 < x_2 \ldots$ are the points of jumps of f_n.

Theorem B. *Let the conditions (C) and (D) be valid. Then for the processes, defined by (Z0') one has*

$$X'_{*n} \xrightarrow{d} W_{f^*D}, \qquad \text{as} \quad n \to \infty$$

in $D_b(\mathbf{R}_+)$ *where* $W_{f^*D}(x) = W_D(f^*(x))$, $x \in \mathbf{R}_+$.

The proof of Theorem B is the same as one of Skorokhod Theorem (see Theorem 1 on p. 547, [4]).

Theorem C. *Let the conditions (S) and (D) be valid. Then for the processes, defined by (Z0) one has*

$$X_{0n} \xrightarrow{d} W_{f^*v}, \qquad \text{as} \quad n \to \infty$$

in $C_b(\mathbf{R}_+)$, *where* $W_{f^*v}(x) = W(v^2 f^*(x))$, $x \in \mathbf{R}_+$.

Proof. We represent X_{0n} as follows

$$X_{0n} = X_{0n}(Y^{(1)}) + X_{0n}(Y^{(2)}) + X_{0n}(Y^{(3)}),$$

where $Y_n^{(3)} = Y_n I_{\{|Y_n|>1\}}$, $Y_n^{(2)} = \mathbf{E} Y_n 1\!\!\mathrm{I}_{\{|Y_n|\le 1\}}$, and $Y_n^{(1)} = Y_n 1\!\!\mathrm{I}_{\{|Y_n|\le 1\}} - \mathbf{E} Y_n 1\!\!\mathrm{I}_{\{|Y_n|\le 1\}}$. Since for all $\varepsilon > 0$ it holds

$$\mathbf{P}\{\sup_{t\in\mathbf{R}_+} |X_{0n}(Y^{(3)})| > \varepsilon\} < k_n \mathbf{P}\{|Y_n| > 1\},$$

by (1b) we have $X_{0n}(Y^{(3)}) \to 0$, as $n \to \infty$ in probability in $C_b(\mathbf{R}_+)$. If $\varepsilon > k_n|\mathbf{E}Y_n \mathbb{I}_{\{|Y_n|\le 1\}}|$, then

$$\mathbf{P}\{\sup_{t\in\mathbf{R}_+} |X_{0n}(Y^{(2)})| > \varepsilon\} = 0.$$

Therefore by (2b) one has $X_{0n}(Y^{(2)}) \to 0$, as $n \to \infty$ in probability in $C_b(\mathbf{R}_+)$. The random variables $Y_{ni}^{(1)}$ satisfy to the conditions of Lindeberg-Feller theorem. Consequently, repeating the proof of Prokhorov Theorem [3] we can show that $X_n^{(1)} \xrightarrow{d} W_{fv}$ as $n \to \infty$ in $C_b(\mathbf{R}_+)$. This implies Theorem D.

It is easy to see that the correlation function of W_{f^*v} in the case of a) is $B_0(x,y) = v^2 f^{\#}(x,y)$, where $f^{\#}(x,y) = f^*(\min(x,y))$ and the correlation function of W_{f^*v} in the case of b) is $B_0(x,y) = v^2 f^{\#}(x,y)$, where $f^{\#}(x,y) = f^*(\max(x,y))$

Formulate the analogy of (D):

(D') Let $f_n^{**}(x)$, $f^{**}(x) : \mathbb{R}_+ \mapsto \mathbb{R}_+$, $n \in \mathbb{N}$ be integer valued functions, $f^{**}(x)$ is a continue bounded function plays a role of the following limit

$$\frac{f_n^{**}(x)}{k_n} \to f^{**}(x), \qquad \text{as} \quad n \to \infty$$

for all $x \in [0,\infty]$ $x \in \mathbb{R}_+$ and one of the following conditions is true:

a) f_n^{**} are increasing functions;

b) f_n^{**} are decreasing functions.

We will consider the random processes

$$(Z0") \qquad X'_{**n}(x) = X'_{**n}(Y')(x) = \sum_{i=1}^{f_n^{**}(x)} Y'_{ni}, \qquad n \in \mathbb{N} \qquad x \in \mathbb{R}_+.$$

and their modifications formed by piece-wise linear continuous broken lines, $(Z0^*)$

$$X_{**n}(x) = X_{**n}(Y')(x) = X'_{**n}(x_k) + \frac{x - x_k^*}{x_{k+1}^* - x_k^*}\left(X'_{**n}(x_{k+1}^*) - X'_{**n}(x_k^*)\right),$$

$n \in \mathbb{N}$, $x \in [x_k^*, x_{k+1}^*)$, where $0 = x_0^* < x_1^* < x_2^* \ldots$ are the points of jumps of f_n.

Since X_{0n} and X_{**n} are independent random processes from Theorem C it follows

Theorem D. *Let the conditions (S), (S'), (D) and (D') be valid. Then for the processes, defined by* $(Z0)$ *and* $(Z0^*)$ *one has*

$$X_{0n} + X_{**n} \xrightarrow{d} W_{f^*v} + W_{f^{**}v'}, \qquad \text{as} \quad n \to \infty$$

in $C_b(\mathbf{R}_+)$.

Let $-\infty \le a < b \le \infty$, For a function $x : [a,b] \to \mathbf{R}_+$ we define

$$\Delta_{a,b}(x) = \sup_{a\le t'\le t\le t"\le b} [\min\{|x(t') - x(t)|, |x(t) - x(t")|\}].$$

In order to prove a version of Theorem D for Skorokhod space we need the following lemma.

Lemma 3. *Let $V'(t)$ and $V"(t)$, $t \in [a,b]$ be separable independent random processes. Denote $V = V' + V"$. Then for all $\varepsilon > 0$ we have*

$$\begin{aligned}
&\mathbf{P}\{\Delta_{a,b}(V) > \varepsilon\} \\
&\quad\le 2\mathbf{P}\left\{\sup_{a\le t\le b}|V'(b)-V'(t)| > \frac{\varepsilon}{4}\right\}\mathbf{P}\left\{\sup_{a\le t\le b}|V"(b)-V"(t)| > \frac{\varepsilon}{4}\right\} \\
&\qquad + \mathbf{P}\left\{\Delta_{a,b}(V') > \frac{\varepsilon}{2}\right\} + \mathbf{P}\left\{\Delta_{a,b}(V") > \frac{\varepsilon}{2}\right\}.
\end{aligned}$$

Proof. Using the inequality $|a+b| \le 2\max(|a|,|b|)$, $a, b \in \mathbf{R}_+$, we obtain

$$\begin{aligned}
\mathbf{P}\{\Delta_{a,b}(V) > \varepsilon\} &\le \mathbf{P}\Big\{2\sup_{a\le t'\le t\le t"\le b}[\min\{\max(|V'(t')-V'(t)|, |V"(t')-V"(t)|), \\
&\quad \max(|V'(t)-V'(t")|, |V"(t)-V"(t")|)\}] > \varepsilon\Big\} \\
&\le \mathbf{P}\left\{\Delta_{a,b}(V') > \frac{\varepsilon}{2}\right\} + \mathbf{P}\left\{\Delta_{a,b}(V") > \frac{\varepsilon}{2}\right\} \\
&\quad \times \mathbf{P}\Big\{\sup_{a\le t'\le t\le t"\le b}[\min\{|V'(t')-V'(t)|, |V"(t)-V"(t")|\}] > \frac{\varepsilon}{2}\Big\} \\
&\quad \times \mathbf{P}\Big\{\sup_{a\le t'\le t\le t"\le b}[\min\{|V"(t')-V"(t)|, |V'(t)-V'(t")|\}] > \frac{\varepsilon}{2}\Big\} \\
&\le 2\mathbf{P}\left\{\sup_{a\le t\le b}|V'(b)-V'(t)| > \frac{\varepsilon}{4}\right\}\mathbf{P}\left\{\sup_{a\le t\le b}|V"(b)-V"(t)| > \frac{\varepsilon}{4}\right\} \\
&\quad + \mathbf{P}\left\{\Delta_{a,b}(V') > \frac{\varepsilon}{2}\right\} + \mathbf{P}\left\{\Delta_{a,b}(V") > \frac{\varepsilon}{2}\right\}.
\end{aligned}$$

The proof is complete.

The addition in Skorokhod space is not continuous. Therefore in general the sum of convergence sequences of random processes does not appear a convergence sequence. However we have

Theorem E. *Let the conditions (C), (C'), (D) and (D') be valid. Then for the processes, defined by (Z0') and (Z0") one has*

$$X'_{*n} + X'_{**n} \xrightarrow{d} W_{f^*D} + W_{f^{**}D'}, \qquad \text{as} \quad n \to \infty$$

in $D_b(\mathbf{R}_+)$.

Proof. Since by Theorem C $X'_{*n} \xrightarrow{d} W_{f^*D}$, as $n \to \infty$ and $X'_{**n} \xrightarrow{d} W_{f^{**}D'}$, as $n \to \infty$ and for all $n \in \mathbf{N}$ X'_{*n} and X'_{**n} are independent in $D_b(\mathbf{R}_+)$ and for all $n \in \mathbf{N}$ X'_{*n} and X'_{**n} are independent, finite dimensional distributions of $X'_{*n} + X'_{**n}$ converge to finite dimensional distributions of $W_{f^*D} + W_{f^{**}D'}$.

The proof of the tightness of $\{\mathcal{L}(X'_{*n} + X'_{**n}), n \in \mathbf{N}\}$ in $D_b(\mathbf{R}_+)$ is the same as Skorokhod Theorem (see Theorem 1 on p. 548, [3]). Only with the Skorokhod inequality (see Lemma 1 on p. 548, [4]) we use Lemma 3. By Theorem 2 (see p. 545, [4]) this implies Theorem E.

Now we will prove a functional limit theorem for integral of X'_{0n}. More precisely we will consider the random processes:

(Z0)

$$Z_{0n}(t) \equiv Z_{0n}(t)(Y) = \int_0^t X'_{*n}(x)dx = \sum_{k=1}^{k_t} \left(\sum_{i=1}^{f_n(x_{k-1})} Y_{ni} \right) \Delta_k + \left(\sum_{i=1}^{f_n(x_{k_t})} Y_{ni} \right) \Delta_t$$

$t \in \mathbb{R}_+$, $n \in \mathbb{N}$ and the random processes

$$\text{(Z0')} \quad Z'_{0n}(x) \equiv Z'_{0n}(x)(Y) = \sum_{k=1}^{k_t} \left(\sum_{i=1}^{f_n(x_{k-1})} Y_{ni} \right) \Delta_k, \qquad t \in \mathbb{R}_+, \quad n \in \mathbb{N}.$$

Here $\Delta_k = x_k - x_{k-1}$, $k \in \mathbf{N}$, $k_t = \sup\{k : x_k \le t\}$, $x_t = x_{k_t}$, and $\Delta_t = t - x_t$.

We will use the condition:

$$(E) \qquad \max_{1 \le k \le k_T + 1} \Delta_k \to 0 \qquad \text{as} \qquad n \to \infty.$$

Consider the function:

c) $f_n^{\#}(x, y) = \frac{f_n(\min(x,y))}{k_n}$ in the case of a);

d) $f_n^{\#}(x, y) = \frac{f_n(\max(x,y))}{k_n}$ in the case of b);

e) $g_n(x, y) = \sum_{k_1=1}^{k_x} \sum_{k_2=1}^{k_y} f_n^{\#}(x_{k_1-1}, x_{k_2-1}) \Delta_{k_1} \Delta_{k_2}$ and

f) $g(x, y) = \int_0^x \int_0^y f^{\#}(t_1, t_2) dt_1 dt_2$, $x, y \in \mathbf{R}_+$.

Lemma 4. *Let (D) and (E) be valid. Then for all $x, y \in \mathbf{R}_+$ one has*

$$g_n(x, y) \to g(x, y), \ \ \textit{as} \ \ n \to \infty,$$

Proof. Since f_n, f^* are monotone bounded functions, $f_n(x) \to f^*(x)$, as $n \to \infty$ uniformly by $x \in \mathbf{R}_+$. Therefore

$$g_n(x, y) - \sum_{k_1=1}^{k_x} \sum_{k_2=1}^{k_y} f^{\#}(x_{k_1-1}, x_{k_2-1}) \Delta_{k_1} \Delta_{k_2} \to 0, \ \ \textit{as} \ \ n \to \infty.$$

Observe that

$$\sum_{k_1=1}^{k_x} \sum_{k_2=1}^{k_y} f^{\#}(x_{k_1-1}, x_{k_2-1}) \Delta_{k_1} \Delta_{k_2}$$

is an integral sum for the integral $\int_0^{t_x} \int_0^{t_y} f^{\#}(t_1, t_2) dt_1 dt_2$. Consequently,

$$\sum_{k_1=1}^{k_x} \sum_{k_2=1}^{k_y} f^{\#}(x_{k_1-1}, x_{k_2-1}) \Delta_{k_1} \Delta_{k_2} \to \int_0^x \int_0^y f^{\#}(t_1, t_2) dt_1 dt_2, \ \ \textit{as} \ \ n \to \infty.$$

This follows the lemma.

We will use the following version of Levy maximal inequality (see. p. 262, [7]). Let U_i, $1 \le i \le n$, be independent random variables. Denote $V_k = \sum_{i=1}^{k} U_i$. Then for all $\varepsilon > 0$ one has

$$\mathbf{P}\{\max_{k\le n}|V_k|>\varepsilon\}\le 2\mathbf{P}\{|V_n|>\varepsilon-\sqrt{2\sigma^2(V_n)}\}. \tag{5}$$

Observe that (5) remain true if instead $\max_{k\le n} V_k$ the expression $\max_{k\in M} V_k$, where $M\subset\{1,2,\dots n\}$. We will use this in the following lemma.

Lemma 5. *Let (D), (Y) and (S) be valid. Then for all $c,\varepsilon>0$ it holds*

$$\limsup_{n\to\infty}\mathbf{P}\{\sup_{0\le t<\infty} c|X'_{*n}(t)|>\varepsilon\}\le 2\mathbf{P}\{c|\gamma(v_1)|>\frac{\varepsilon}{2}-\sqrt{2}v_1c\},$$

where $v_1=v\sup_{t\in\mathbf{R}_+}f^*(t)$.

Proof. Denote $m_n=sup_{0\le t<\infty}f_n(x)$. By (D) $\frac{m_n}{k_n}\to\sup_{t\in\mathbf{R}_+}f^*(t)$ as $n\to\infty$. Therefore $\sum_{i=1}^{m_n}Y_{ni}\xrightarrow{d}\gamma(v_1)$ as $n\to\infty$. So, by Theorem B(1) using denotation of Theorem D we obtain

$$\limsup_{n\to\infty}\mathbf{P}\{\sup_{0\le t<\infty}c|X'_{*n}(t)|>\varepsilon\}\le\limsup_{n\to\infty}\mathbf{P}\left\{\sup_{1\le k\le m_n}c|\sum_{i=1}^{k}Y_{ni}|>\varepsilon\right\}$$

$$\le\limsup_{n\to\infty}\left(\mathbf{P}\left\{\sup_{1\le k\le m_n}c|\sum_{i=1}^{k}Y_{ni}^{(1)}|>\frac{\varepsilon}{2}\right\}+\mathbf{P}\left\{\sup_{1\le k\le m_n}c|\sum_{i=1}^{k}(Y_{ni}^{(2)}+Y_{ni}^{(3)})|>\frac{\varepsilon}{2}\right\}\right)$$

$$\le\limsup_{n\to\infty}\left(\mathbf{P}\left\{\sup_{1\le k\le m_n}c|\sum_{i=1}^{k}Y_{ni}^{(1)})|>\frac{\varepsilon}{2}\right\}\right.$$

$$\left.+\mathbf{P}\left\{c\sum_{i=1}^{m_n}|Y_{ni}|\mathbb{I}_{\{|Y_{ni}|>1\}}>\frac{\varepsilon}{2}-ck_n|\mathbf{E}Y_n\mathbb{I}_{\{|Y_n|\le1\}}|\right\}\right)$$

$$\le 2\limsup_{n\to\infty}\mathbf{P}\left\{c|\sum_{i=1}^{k}Y_{ni}^{(1)})|>\frac{\varepsilon}{2}-c\sqrt{2\sigma^2(\sum_{i=1}^{k}Y_{ni}^{(1)})}\right\}$$

$$=2\mathbf{P}\{c|\gamma(v_1)|>\frac{\varepsilon}{2}-\sqrt{2}v_1c\}.$$

The lemma is proved.

Corollary. *Let (D), (E), (Y) and (S) be valid. For all $c,\varepsilon>0$ and $\omega_1\in\Omega'$ it holds*

$$\mathbf{P}\{\sup_{t\in[0,T]}|Z_{0n}(t)-Z'_{0n}(t)|>\varepsilon\}\to 0,\quad as\ \ n\to\infty.$$

Proof. Observe that

$$\mathbf{P}\{\sup_{t\in\mathbf{R}_+}|Z_{0n}(t)-Z'_{0n}(t)|>\varepsilon\}\le\mathbf{P}\{2\left(\sup_{1\le k\le k_T+1}\Delta_k\right)\left(\sup_{0\le k\le m_n}|\sum_{i=1}^{m_n}Y_{ni}|\right)>\varepsilon\}.$$

By Lemma 5 this implies Corollary.

Theorem F. *Assume that the conditions (Y), (S), (D) and (E) are fulfilled. Then the processes defined by (Z0) weakly converge in $C[0,T]$ as follows*

$$Z_{0n}\xrightarrow{d}W_0,\qquad \text{as } n\to\infty,$$

where W_0 is a centered gaussian random process with the covariance $B_1(t_1,t_2) = v_1^2 g(t_1,t_2)$, $t_1,t_2 \in [0,T]$.

PROOF. Let $\varepsilon > 0$, $M \in \mathbf{N}$, $0 < \delta < \frac{T}{M}$ Denote $y_k = k\frac{T}{M}$, $0 \le k \le M$. In the same way as in the proof of Theorem 1 in p. 358 [7], we have

$$\mathbf{P}\left\{\sup_{|t'-t"|\le\delta} |Z_{0n}(t') - Z_{0n}(t")| > \varepsilon\right\}$$
$$\le 4\sum_{k=1}^{M} \mathbf{P}\left\{\sup_{y_{k-1}\le t\le y_k} |Z_{0n}(t) - Z_{0n}(y_k)| > \frac{\varepsilon}{8}\right\}$$
$$\le 4\sum_{k=1}^{M} \mathbf{P}\left\{\sup_{y_{k-1}\le t\le y_k} \left|\int_{y_{k-1}}^{t} X_{*n}(t)\right| > \frac{\varepsilon}{8}\right\}$$
$$\le 4\sum_{k=1}^{M} \mathbf{P}\left\{\frac{T}{m}\sup_{1\le l\le m_n} |\sum_{i=i}^{l} Y_{ni}| > \frac{\varepsilon}{8}\right\}.$$

Thus by Lemma 5 we obtain

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \mathbf{P}\left\{\sup_{|t'-t"|\le\delta} |Z_{0n}(t') - Z_{0n}(t")| > \varepsilon\right\}$$
$$\le \lim_{M\to\infty}\limsup_{n\to\infty} \le 4\sum_{k=1}^{M} \mathbf{P}\left\{\frac{T}{M}\sup_{1\le l\le m_n} \left|\sum_{i=i}^{l} Y_{ni}\right| > \frac{\varepsilon}{8}\right\}$$
$$\le 8\lim_{M\to\infty} M\mathbf{P}\left\{\frac{T}{M}|\gamma(v_1)| > \frac{\varepsilon}{16} - \sqrt{2}v_1\frac{T}{M}\right\} = 0. \quad (6)$$

Let $M \in \mathbf{N}$, $0 \le t_1 < t_2 \dots t_M \le T$ and $a_k \in \mathbf{R}$, $1 \le k \le M$. We will show

$$\sum_{k=1}^{M} a_k Z_{0n}(t_k) \xrightarrow{d} \sum_{k=1}^{M} a_k W_0(t_k), \text{ as } n \to \infty.$$

By Corollary of Lemma 5 it is sufficient to prove that

$$\sum_{k=1}^{M} a_k Z'_{0n}(t_k) \xrightarrow{d} \sum_{k=1}^{M} a_k W_0(t_k), \text{ as } n \to \infty.$$

Consider $U_n = \sum\limits_{k=1}^{M} a_k Z'_{0n}(t_k) = \sum\limits_{i=1}^{m_n} b_{ni}Y_{ni}$. Observe that $|b_{ni}| \le C$, where $C = T\sum\limits_{k=1}^{M} |a_k|$, and

$$\sum_{i=1}^{m_n} \mathbf{E}(b_{ni}Y_{ni}^{(1)})^2 = \mathbf{E}\left(\sum_{k=1}^{M} a_k Z'_{0n}(Y^{(1)})(t_k)\right)^2 = k_n v_n^2 \sum_{l_1,l_2=1}^{M,M} a_{l_1}a_{l_2}g_n(t_{l_1},t_{l_2}),$$

where $v_n^2 = \sigma^2(Y_n^{(1)})$. Therefore by Lemma 3 we have as $n \to \infty$,

$$\sum_{i=1}^{m_n} \mathbf{E}(b_{ni}Y_{ni}^{(1)})^2 \to v^2 \left(\sum_{l_1,l_2=1}^{M,M} a_{l_1}a_{l_2}g(t_{l_1}, t_{l_2}) \right) = \mathbf{E} \left(\sum_{k=1}^{M} a_k W_1(t_k) \right)^2 ,$$

Consequently, by Theorem B(2) $\sum\limits_{k=1}^{M} a_k Z_{0n}(t_k) \xrightarrow{d} \sum\limits_{k=1}^{M} a_k W_0(t_k)$, as $n \to \infty$.

So, by Theorem 7.7 p. 76 in [8] finite dimensional distributions of Z_{0n} converge to finite dimensional distributions of W_0. By Theorem 1 on p. 522 in [4] this and (6) imply $Z_{0n} \xrightarrow{d} W_0$ as $n \to \infty$ in $C[0,T]$. The proof is complete.

Observe that the random processes Z'_{0n} have not continuous trajectories. Therefore it need to consider the convergence of Z'_{0n} in more wide spaces then $C[0,T]$.

Theorem G. *Assume that the conditions (D), (E), (Y), and (S) are fulfilled. Then the processes defined by (Z0') weakly converge in $L^\infty[0,T]$ as follows*

$$Z'_{0n} \xrightarrow{d} W_0, \qquad \text{as } n \to \infty.$$

Proof. By Corollary of Lemma 5 $Z_{0n}(t) - Z'_{0n}(t) \xrightarrow{P} 0$ as $n \to \infty$ in $L^\infty[0,T]$. The space $C[0,T]$ is a subspace of $L^\infty[0,T]$. Trajectories of $Z'_{0n}(t)$ belongs to $L^\infty[0,T]$. Therefore Theorem G follows from Theorem F. The proof is complete.

4. Main results.

At first we consider the case of the replacement by zeros.

Fix n and describe a recurrent procedure of operations on rows.

(0) *We start with initial string $(Y_{ni})_{i=1}^\infty$.*

(1) *At each row i on each term the indicator $\mathbb{I}_{ij}$ impacts. It means that in the case a term is a rv Y_{ni} remains to ith step non-zero valued (as rv) and $\mathbb{I}_{ij}$ equals to zero than the term Y_{ni} is disappear at ith step or is replacing with 0, and the term Y_{nj} remains the same if $\mathbb{I}_{ij} = 1$.*

Define the sequence $\left(S_k^{(1n)}\right)_{k=0}^\infty$ of row sums,

(X1)
$$S_0^{(1n)} = \sum_{j=1}^{k_n} Y_{nj},$$

$$S_1^{(1n)} = \sum_{j=1}^{k_n} \mathbb{I}_{1j}(n) Y_{nj},$$

$$S_2^{(1n)} = \sum_{j=1}^{k_n} \mathbb{I}_{2j}(n)\mathbb{I}_{1j}(n) Y_{nj},$$

$$\vdots$$

$$S_k^{(1n)} = \sum_{j=1}^{k_n} \mathbb{I}_{1j}(n)\mathbb{I}_{2j}(n)\cdots\mathbb{I}_{kj}(n) Y_{nj},$$

$\vdots$

We will consider sequences of step lines random processes depending from $\omega_1 \in \Omega_1$. As we shell use the same type of construction in several cases, (Zi') will denote that the process X'_{in} is defined by the random variables $S_k^{(in)}$, $i \in \mathbf{N}$. For $i \in \mathbf{N}$ let

(Zi') $\quad X'_{in}(x) \equiv X'_{in}(x\,;\omega_1)(Y) \stackrel{\triangle}{=} S^{(in)}_{[k_n x]}, \qquad x \in \mathbb{R}_+\,, \quad n \in \mathbb{N}\,, \quad \omega_1 \in \Omega_1\,.$

Theorem 1. *Assume that the conditions (Y) and (C) are satisfied.*

(1) Let (A) be valid. Then the processes defined by (X1) and (Z1') for almost all $\omega_1 \in \Omega_1$ weakly converge in $D_b(\mathbf{R}_+)$ as follows

$$X'_{1n}(\omega_1) \xrightarrow{d} W_{fD}, \ \ \textit{as} \ \ n \to \infty.$$

(2) Let (A2) be valid. Then the processes defined by (X1) and (Z1') for almost all $\omega_1 \in \Omega_1$ weakly converge in $D_b(\mathbf{R}_+)$ as follows

$$X'_{1n}(\omega_1) \xrightarrow{d} W_{f_m D}, \ \ \textit{as} \ \ n \to \infty.$$

Observe that $X'_{1n} \stackrel{d}{=} X'_{*n}$ with $f_n(x) = f_{1n}(x)$. So Theorem 1(1) follows from Theorem B and Corollary 1 of Lemma 1, Theorem 1(2) follows from Theorem B and Corollary 1 of Lemma 2.

We will consider the following random broken line sequence:

(Zi) $\qquad X_{in}(x) \equiv X_{in}(x\,;\omega_1)(Y) \stackrel{\triangle}{=} S^{(in)}_{[k_n x]} + \{k_n x\}(S^{(in)}_{[k_n x]+1} - S^{(in)}_{[k_n x]}),$

$x \in \mathbb{R}_+$, $n \in \mathbb{N}$, $\omega_1 \in \Omega_1$, $i \in \mathbf{N}$.

Since $X_{1n} \stackrel{d}{=} X_{0n}$ with $f_n = f_{1n}$, from Theorem C and Corollary 1 of Lemma 1 or Corollary 1 of Lemma 2, respectively, it follows

Theorem 2. *Assume that the conditions (Y) and (S) are fulfilled.*

(1) Let (A) be valid. Then the processes defined by (X1) and (Z1) for almost all $\omega_1 \in \Omega_1$ weakly converge in $C_b(\mathbf{R}_+)$ as follows

$$X_{1n}(\omega_1) \xrightarrow{d} W_{fv}, \qquad \text{as } n \to \infty.$$

(2) Let (A1) be valid. Then the processes defined by (X1) and (Z1) for almost all $\omega_1 \in \Omega_1$ weakly converge in $C_b(\mathbf{R}_+)$ as follows

$$X_{1n}(\omega_1) \xrightarrow{d} W_{f_m v}, \qquad \text{as } n \to \infty.$$

In Theorem 1 and Theorem 2 where considered the case of sums in which the number of summands is decreasing. Now we will consider the case in which the number of summands in sums is increasing. Let

(X2) $$S_0^{(2n)} = 0,$$

$$S_1^{(2n)} = \sum_{i=1}^{k_n} (1 - \mathbb{I}_{1i}(n)) Y'_{ni},$$

$$S_2^{(2n)} = \sum_{i=1}^{k_n} (1 - \mathbb{I}_{1i}(n) \mathbb{I}_{2i}(n)) Y'_{ni},$$

$$\vdots$$

$$X_k^{(2n)} = \sum_{i=1}^{k_n} (1 - 1\!\!1_{1i}(n) 1\!\!1_{2i}(n) \cdots 1\!\!1_{ki}(n)) Y'_{ni},$$

$$\vdots$$

Observe that $X'_{2n} \stackrel{d}{=} X'_{*n}$ and $X_{2n} \stackrel{d}{=} X_{0n}$ with $f_n(x) = f_{2n}(x)$. Therefore by Corollary 2 of Lemma 1 or Corollary 2 of Lemma 2, Theorem D or Theorem E respectively we obtain

Theorem 3. *Assume that the conditions (Y) and (C) are fulfilled.*

(1) Let (A) be valid. Then the processes defined by (X2) and (Z2') weakly converge in $D_b(\mathbf{R}_+)$ as follows

$$X'_{2n}(\omega_1) \xrightarrow{d} W_{(1-f)D'}, \qquad \text{as } n \to \infty$$

for almost all $\omega_1 \in \Omega_1$.

(2) Let (A1) be valid. Then the processes defined by (X2) and (Z2') weakly converge in $D_b(\mathbf{R}_+)$ as follows

$$X'_{2n}(\omega_1) \xrightarrow{d} W_{(1-f_m)D'}, \qquad \text{as } n \to \infty$$

for almost all $\omega_1 \in \Omega_1$.

Theorem 4. *Assume that the conditions (Y) and (S) are valid.*

(1) Let (A) be valid. Then the processes defined by (X2) and (Z2) weakly converge in $C_b(\mathbf{R}_+)$ as follows

$$X_{2n}(\omega_1) \xrightarrow{d} W_{(1-f)v'}, \qquad \text{as } n \to \infty$$

for almost all $\omega_1 \in \Omega_1$.

(2) *Let (A1) be valid. Then the processes defined by (X2) and (Z2) weakly converge in $C_b(\mathbf{R}_+)$ as follows*

$$X_{2n}(\omega_1) \xrightarrow{d} W_{(1-f_m)v'}, \qquad \text{as } n \to \infty$$

for almost all $\omega_1 \in \Omega_1$.

Now we consider the case of the replacement by non-zero random variables. We start with the summands Y_{ni}, $1 \le i \le k_n$ and in each step, in the random way, we change Y_{ni} to Y'_{ni}, $1 \le i \le k_n$. More precisely, let

(X3) $$S_0^{(3n)} = \sum_{i=1}^{k_n} Y_{ni},$$

$$S_1^{(3n)} = \sum_{i=1}^{k_n} 1\!\!1_{1i}(n) Y_{ni} + \sum_{i=1}^{k_n} (1 - 1\!\!1_{1i}(n)) Y'_{ni},$$

$$S_2^{(3n)} = \sum_{i=1}^{k_n} 1\!\!1_{1i}(n) 1\!\!1_{2i}(n) Y_{ni} + \sum_{i=1}^{k_n} (1 - 1\!\!1_{1i}(n) 1\!\!1_{2i}(n)) Y'_{ni},$$

$$\vdots$$

$$S_k^{(3n)} = \sum_{i=1}^{k_n} \mathbb{I}_{1i}(n)\mathbb{I}_{2i}(n)\cdots\mathbb{I}_{ki}(n)Y_{ni} + \sum_{i=1}^{k_n}(1-\mathbb{I}_{1i}(n)\mathbb{I}_{2i}(n)\cdots\mathbb{I}_{ki}(n))Y'_{ni},$$

$$\vdots$$

Since $X_{3n} = X_{1n}(Y) + X_{2n}(Y')$ and $X_{1n}(Y)$, $X_{2n}(Y')$ are independent random processes, from Theorem D and Theorem E, respectively, it follows

Theorem 5. *Suppose that (Y), (S), (Y') and (S') are satisfied.*

(1) Let (A) be valid. Then for the processes defined by (X3) and (Z3) one has

$$X_{3n}(\omega_1) \xrightarrow{d} W_{fvv'}, \quad n \to \infty \ \ in \quad C_b(\mathbf{R}_+)$$

for almost all $\omega_1 \in \Omega_1$, *where* $W_{fvv'}(x) = W(v^2 f(x)) + W'(v'^2(1-f(x)))$, $x \in \mathbf{R}^+$.

(2) *Let (A1) be valid. Then for the processes defined by (X3) and (Z3) one has*

$$X_{3n}(\omega_1) \xrightarrow{d} W_{f_m vv'}, \quad n \to \infty \ \ in \quad C_b(\mathbf{R}_+)$$

for almost all $\omega_1 \in \Omega_1$, *where* $W_{f_m vv'}(x) = W(v^2 f_m(x)) + W'(v'^2(1-f_m(x)))$, $x \in \mathbf{R}^+$.

Theorem 6. *Suppose that (Y), (C), (Y') and (C') are satisfied.*

(1) Let (A) be valid. Then for the processes defined by (X3) and (Z3') one has

$$X'_{3n}(\omega_1) \xrightarrow{d} W_{fDD'}, \quad n \to \infty \ \ in \quad D_b(\mathbf{R}_+)$$

for almost all $\omega_1 \in \Omega_1$, *where* $W_{fDD'}(x) = W_D(f(x)) + W_{D'}((1-f(x)))$, $x \in \mathbf{R}^+$.

(2) Let (A1) be valid. Then for the processes defined by (X3) and (Z3') one has

$$X'_{3n}(\omega_1) \xrightarrow{d} W_{f_m DD'}, \quad n \to \infty \ \ in \quad D_b(\mathbf{R}_+)$$

for almost all $\omega_1 \in \Omega_1$, *where* $W_{f_m DD'}(x) = W_D(f(x)) + W_{D'}((1-f_m(x)))$, $x \in \mathbf{R}^+$.

We will illustrate our theorems by following examples. Let $0 < \alpha, m < \infty$.

Example 1. Let $\mathbf{P}_1(A_{ij}(n)) = p_j(n) = \left(1 - \frac{m}{k_n}\right)^{\alpha}$, for all $i, j \in \mathbf{N}$. Then for all $x \in \mathbf{R}_+$ it holds

$$\frac{\mathbf{E}_1 f_{1n}(x)}{k_n} = \left(1 - \frac{m}{k_n}\right)^{[xk_n]\alpha} \to e^{-m\alpha x}, \ \text{ as } \ n \to \infty.$$

So we have:

If (Y) and (C), are satisfied, then by Theorem 1, for the processes defined by (X1) and (Z1') one has $X'_{1n}(\omega) \xrightarrow{d} W_{1D}$, $n \to \infty$ in $D(\mathbf{R}_+)$ for almost all $\omega_1 \in \Omega_1$, where $W_{1DD'}(x) = W_D(e^{-m\alpha x})$, $x \in \mathbf{R}_+$.

If (Y), (Y'), (S), and (S') are satisfied, then by Theorem 5 for the processes defined by (X3) and (Z3) one has $X_{3n}(\omega_1) \xrightarrow{d} W_{1vv'}$, $n \to \infty$ in $C_b(\mathbf{R}_+)$ for almost all $\omega_1 \in \Omega_1$, where $W_{1vv'}(x) = W(v^2 e^{-m\alpha x}) + W'(v'^2(1 - e^{-m\alpha x}))$, $x \in \mathbf{R}_+$.

Example 2. Let $\mathbf{P}_1(A_{ij}(n)) = p_j(n) = \left(1 - \frac{m}{k_n + (j-1)m}\right)^\alpha$ for all $i, j \in \mathbf{N}$ ($k_n > m$). Then for all $x \in \mathbf{R}_+$ it holds

$$\frac{\mathbf{E}_1 f_{1n}(x)}{k_n} = \left(\frac{k_n + ([k_n x] - 2)m}{k_n + ([k_n x] - 1)m}\right)^\alpha \left(\frac{k_n + ([k_n x] - 3)m}{k_n + ([k_n x] - 2)m}\right)^\alpha \cdots \left(\frac{k_n - m}{k_n}\right)^\alpha$$
$$= \left(\frac{k_n - m}{k_n + ([k_n x] - 1)m}\right)^\alpha \to \left(\frac{1}{1 + mx}\right)^\alpha, \text{ as } n \to \infty.$$

If (Y) and (C), are satisfied, then by Theorem 2 for the processes defined by (X1) and (Z1') one has $X'_{1n}(\omega_1) \xrightarrow{d} W_{2D}$, $n \to \infty$, in $D_b(\mathbf{R}_+)$ for almost all $\omega_1 \in \Omega_1$, where $W_{2D}(x) = W_D(\frac{1}{(1+mx)^\alpha})$, $x \in \mathbf{R}_+$.

If (Y), (Y'), (S), and (S') are satisfied, then by Theorem 5, for the processes defined by (X3) and (Z3) one has $X_{3n}(\omega_1) \xrightarrow{d} W_{2\sigma\sigma'}$, $n \to \infty$ in $C_b(\mathbf{R}^+)$ for almost all $\omega_1 \in \Omega_1$, where $W_{2vv'}(x) = W(\frac{v^2}{(1+mx)^\alpha}) + (W'(v'^2(1 - \frac{1}{(1+mx)^\alpha}))$, $x \in \mathbf{R}_+$.

Example 3. Let $\mathbf{P}_1(A_{ij}(n)) = p_j(n) = \left(1 - \frac{m}{k_n - (j-1)m}\right)^\alpha$ if $k_n > (j-1)m$ and $\mathbf{P}_1(A_{ij}(n)) = 0$ if $k_n \le (j-1)m$, $i, j \in \mathbf{N}$. Let $0 \le x < \frac{1}{m}$. Then we obtain

$$\frac{\mathbf{E}_1 f_{1n}(x)}{k_n} = \left(\frac{k_n - [k_n x]m}{k_n - ([k_n x] - 1)m}\right)^\alpha \left(\frac{k_n - ([k_n x] - 1)m}{k_n - ([k_n x] - 2)m}\right)^\alpha \cdots \left(\frac{k_n - m}{k_n}\right)^\alpha$$
$$= \left(\frac{k_n - [k_n x]m}{k_n}\right)^\alpha \to (1 - mx)^\alpha, \ n \to \infty, \ x < \frac{1}{m}.$$

If (Y) and (C) are satisfied, then by Theorem 1, for the processes defined by (X1) and (Z1') one has $X'_{1n}(\omega_1) \xrightarrow{d} W_{3D}$, $n \to \infty$, in $D(\mathbf{R}_+)$ for almost all $\omega_1 \in \Omega_1$, where $W_{3D}(x) = W_D((1 - mx)^\alpha I_{[0, \frac{1}{m})})$, $x \in \mathbf{R}_+$.

If (Y), (Y'), (S), and (S') are satisfied, then by Theorem 5, for the processes defined by (X3) and (Z3) one has $X_{3n}(\omega_1) \xrightarrow{d} W_{3vv'}$, $n \to \infty$ in $C_b(\mathbf{R}_+)$ for almost all $\omega_1 \in \Omega_1$, where $W_{3vv'}(x) = W(v^2(1 - mx)^\alpha I_{[0, \frac{1}{m})}) + W'(v'^2(1 - (1 - mx)^\alpha I_{[0, \frac{1}{m})}))$ $x \in \mathbf{R}_+$.

Let $\alpha = 1$. Let us speak of the sense of Example 1 and Example 3. Example 1 corresponds to the case when at the l-th step we replace m summands in the sums $S_{l-1}^{(1n)}$ by zero regardless of whether a summand equals zero or not.

Under Example 3, after each replacement, the number of Y_{ni} in $S_{l-1}^{(1n)}$ decreases by about m elements. So Example 3 corresponds to the case when at the l-th step we replace about m nonzero summands in $S_{l-1}^{(1n)}$ by zero.

The same is true for $S_l^{(3n)}$. In this case, we replace summands of $S_{l-1}^{(1n)}$ by Y'_{ni}.

Now we will prove functional limit theorems for integrals of X'_{1n}, X'_{2n} and X'_{3n} which we will use in applications to models of a financial market. More precisely we will consider the random processes:

(Wi) $$Z_{in}(x) \equiv Z_{in}(x\,;\,\omega_1)(Y) = \int_0^t X'_{in}(x)dx = \frac{1}{k_n}\sum_{j=0}^{[k_n x]-1} S_j^{(in)} + \frac{\{k_n x\}}{k_n} S_{[k_n x]}^{(in)},$$

$x \in \mathbb{R}_+$, $n \in \mathbb{N}$, $\omega_1 \in \Omega_1$ and the random processes

(Wi')

$$Z'_{in}(x) \equiv Z'_{in}(x\,;\,\omega_1)(Y) = \frac{1}{k_n}\sum_{j=0}^{[k_n x]} S_j^{(in)}, \qquad x \in \mathbb{R}_+\,, \quad n \in \mathbb{N}\,, \quad \omega_1 \in \Omega_1\,,$$

where $i \in \mathbf{N}$.

Theorem 7. *Assume that the conditions (Y) and (S) are fulfilled.*

(1) Let (A) be valid. Then the processes defined by (W1) weakly converge in $C[0,T]$ as follows

$$Z_{1n}(\omega_1) \xrightarrow{d} W_1, \qquad \text{as } n \to \infty$$

for almost all $\omega_1 \in \Omega_1$, where W_1 is a gaussian random process with the covariance $B_1(t_1,t_2) = v^2 \int_0^{t_1}\int_0^{t_2} f(\max(x,y))dydx$, $t_1,t_2 \in [0,T]$.

(2) Let (A1) be valid. Then the processes defined by (W1) weakly converge in $C[0,T]$ as follows

$$Z_{1n}(\omega_1) \xrightarrow{d} W_{m1}, \qquad \text{as } n \to \infty$$

for almost all $\omega_1 \in \Omega_1$, where W_{m1} is a gaussian random process with the covariance $B_{m1}(t_1,t_2) = v^2 \int_0^{t_1}\int_0^{t_2} e^{-m\max(x,y)}dxdy$, $t_1,t_2 \in [0,T]$.

Theorem 8. *Assume that the conditions (Y), and (S) are fulfilled.*

(1) Let (A) be valid. Then the processes defined by (W2) weakly converge in $C[0,T]$ as follows

$$Z_{2n}(\omega_1) \xrightarrow{d} W_2, \qquad \text{as } n \to \infty$$

for almost all $\omega_1 \in \Omega_1$, where W_2 is a gaussian random process with the covariance $B_2(t_1,t_2) = v'^2 \int_0^{t_1}\int_0^{t_2}(1-f(\min(x,y)))dydx$, $t_1,t_2 \in [0,T]$.

(2) Let (A1) be valid. Then the processes defined by (W2) weakly converge in $C[0,T]$ as follows

$$Z_{2n}(\omega_1) \xrightarrow{d} W_{m2}, \qquad \text{as } n \to \infty$$

for almost all $\omega_1 \in \Omega_1$, where W_{m2} is a gaussian random process with the covariance $B_{m2}(t_1,t_2) = v'^2 \int_0^{t_1}\int_0^{t_2}(1-e^{-m\min(x,y)})dydx$, $t_1,t_2 \in [0,T]$.

REMARK 1. By changing of the integration order we obtain more simple forms of the variance and the covariance function of W_i, $i \in \{1,2\}$. The variance of W_1 is equal to

$$v_1(t) = B_1(t,t) = v^2(\int_0^t f(x)xdx + \int_0^t dx \int_x^t f(y)dy) = 2v^2 \int_0^t f(x)xdx,$$

the covariance of W_1 is equal to

$$B_1(t_1,t_2) = v^2 \left(\int_0^{t_1} f(x)xdx + \int_0^{t_1} dx \int_x^{t_2} f(y)dy \right) =$$

$$v^2 \left(t_1 \int_{t_1}^{t_2} f(x)dx + 2 \int_0^{t_1} f(x)xdx \right), \quad 0 \le t_1 \le t_2.$$

The variance of W_2 is equal to

$$v_2(t) = B_1(t,t) = v^2 \left(t^2 - 2t \int_0^t f(x)dx + 2 \int_0^t f(x)xdx \right),$$

the covariance of W_2 is equal to

$$B_2(t_1,t_2) = v^2 \left(t_1 t_2 - (t_1+t_2) \int_0^{t_1} f(x)dx + 2 \int_0^{t_1} f(x)xdx \right), \quad 0 \le t_1 \le t_2.$$

Theorem 9. *Assume that (Y), (Y') ,(S) and (S') are fulfilled.*

(1) Let (A) be valid. Then the processes defined by (W3) weakly converge in $C[0,T]$ as follows

$$Z_{3n}(\omega_1) \xrightarrow{d} W_3, \qquad \text{as } n \to \infty$$

for almost all $\omega_1 \in \Omega_1$, where W_3 is a gaussian random process defined on $[0,T]$ with the following covariance function under $t_1, t_2 \in [0,T]$, $B_3(t_1,t_2) = B_1(t_1,t_2) + B_2(t_1,t_2)$.

(1) Let (A1) be valid. Then the processes defined by (W3) weakly converge in $C[0,T]$ as follows

$$Z_{3n}(\omega_1) \xrightarrow{d} W_3, \qquad \text{as } n \to \infty$$

for almost all $\omega_1 \in \Omega_1$, where W_3 is a gaussian random process defined on $[0,T]$ with the following covariance function under $t_1, t_2 \in [0,T]$, $B_{m3}(t_1,t_2) = B_{m1}(t_1,t_2) + B_{m2}(t_1,t_2)$.

PROOF. Observe that

$$Z_{3n} = Z_{1n}(Y) + Z_{2n}(Y'), \quad n \in \mathbf{N},$$

and $Z_{1n}(Y)$ and $Z_{2n}(Y')$ are independent random processes. So Theorem 9 is following from Theorem 7 and Theorem 8.

Theorem 10. *Assume that the conditions (Y), and (S) are fulfilled.*

(1) Let (A) be valid. Then the processes defined by (W1') weakly converge in $L^\infty[0,T]$ as follows

$$Z'_{1n}(\omega_1) \xrightarrow{d} W_1, \qquad \text{as } n \to \infty$$

for almost all $\omega_1 \in \Omega_1$.

(2) Let (A1) be valid. Then the processes defined by (W1') weakly converge in $L^\infty[0,T]$ as follows

$$Z'_{1n}(\omega_1) \xrightarrow{d} W_{m1}, \qquad \text{as } n \to \infty$$

for almost all $\omega_1 \in \Omega_1$.

Theorem 11. *Assume that the conditions (Y), and (S) are fulfilled.*

(1) Let (A) be valid. Then the processes defined by (W2') weakly converge in $L^\infty[0,T]$ as follows

$$Z'_{2n}(\omega_1) \xrightarrow{d} W_2, \qquad \text{as } n \to \infty$$

for almost all $\omega_1 \in \Omega_1$

(2) Let (A2) be valid. Then the processes defined by (W2') weakly converge in $L^\infty[0,T]$ as follows

$$Z'_{2n}(\omega_1) \xrightarrow{d} W_{m2}, \qquad \text{as } n \to \infty$$

for almost all $\omega_1 \in \Omega_1$

Theorem 10 and Theorem 11 follow from Theorem G.

Theorem 12. *Assume that (Y), (Y') ,(S) and (S') are fulfilled.*

(1) Let (A) be valid. Then the processes defined by (W3') weakly converge in $L^\infty[0,T]$ as follows

$$Z'_{3n}(\omega_1) \xrightarrow{d} W_3, \qquad \text{as } n \to \infty$$

for almost all $\omega_1 \in \Omega_1$.

(1) Let (A1) be valid. Then the processes defined by (W3') weakly converge in $L^\infty[0,T]$ as follows

$$Z'_{3n}(\omega_1) \xrightarrow{d} W_{m3}, \qquad \text{as } n \to \infty$$

for almost all $\omega_1 \in \Omega_1$.

PROOF. Observe that $Z'_{3n} = Z'_{1n}(Y) + Z'_{2n}(Y')$, $n \in \mathbf{N}$, and $Z'_{1n}(Y)$ and $Z'_{2n}(Y')$ are independent random processes. So Theorem 12 follows from Theorem 10 and Theorem 11.

REMARK 2. Since trajectories of Z'_{in} are random step lines with jumps in rational points, in Theorem 10, Theorem 11 and Theorem 12 instead $L^\infty[0,T]$ we can consider closed subspace B of $L^\infty[0,T]$, containing such step lines. This space is separable. Therefore the class of the functionals $h : B \to \mathbf{R}_+$ such that $\mathbf{E}h(Z_{in}) \to \mathbf{E}h(W_i)$ as $n \to \infty$, is sufficiently wide.

REMARK 3. We consider the case when $\mathbb{I}_{ji}(n)$, Y_{ni} and Y'_{ni} are defined on the same probability space $(\Omega, \mathfrak{A}, \mathbf{P})$ and for all $n \in \mathbf{N}$, $\mathbb{I}_{ji}(n)$, Y_{ni}, Y'_{ni}, $i \in \mathbf{N}$ are independent random variables. In this case, the random processes X_{in}, X'_{in}, Z_{in}, and Z'_{in} will be denoted by $X_{(i+3)n}$, $X'_{(i+3)n}$, $Z_{(i+3)n}$, and $Z'_{(i+3)n}$.

Then Theorem 1 - Theorem 12 hold in the case when instead X_{in}, X'_{in}, Z_{in}, and Z'_{in}, we use $X_{(i+3)n}$, $X'_{(i+3)n}$, $Z_{(i+3)n}$, and $Z'_{(i+3)n}$, respectively. Of course, in this case the phrase "for almost all $\omega_1 \in \Omega_1$" should be omitted. This follows from the following assertion.

Let $V_n(\omega_1), V : \Omega \to B$, $n \in \mathbf{N}$ be random elements in a separable metrizable space B with the parameter $\omega_1 \in \Omega_1$ which are Bochner measurable as functions *defined on* $\Omega \times \Omega_1$ *and such that* $V_n(\omega_1) \xrightarrow{d} V$, $n \to \infty$ *for almost all* $\omega_1 \in \Omega_1$. *Define random elements* V_n' *by* $\mathcal{L}(V_n')(A) = \mathbf{E}_1\mathcal{L}(V_n)(\omega_1)(A)$, *where* A *is a Borel subset of* B. *Then* $V_n' \xrightarrow{d} V$, $n \to \infty$.

Proof. Let $f : B \to \mathbf{R}$ be a continuous bounded function. Then for almost all $\omega_1 \in \Omega_1$ it holds $\mathbf{E}f(V_n(\omega_1)) \to \mathbf{E}f(V)$, $n \to \infty$. So by Lebesgue Theorem, $\mathbf{E}_1\mathbf{E}f(V_n) \to \mathbf{E}f(V)$, as $n \to \infty$. Observe that $\mathbf{E}_1\mathbf{E}f(V_n) = \mathbf{E}f(V_n')$, $n \in \mathbf{N}$. Therefore $\mathbf{E}f(V_n') \to \mathbf{E}f(V)$, $n \to \infty$ and. $V_n' \xrightarrow{d} V$, as $n \to \infty$ in B.

Remark 4. We will suppose that $I_{0i}(n) = 1$, $i \in \mathbf{N}$. Consider the integer valued random variables defined as

$$N_{1n}(l) = \sum_{i=1}^{k_n} \mathbb{I}_{li}(n)\mathbb{I}_{(l-1)i}(n)\cdots\mathbb{I}_{1i}(n), \ \ N_{2n}(l) = k_n - N_{1n}(l), \ \ l, n \in \mathbf{N}.$$

Next,

$$S_l^{((6+i)n)} = \sum_{i=1}^{N_{jn}(l)} Y_{ni}, \ \ i \in \{1,2\}, \ \ S_l^{(9n)} = \sum_{i=1}^{N_{1n}(l)} Y_{ni} + \sum_{i=1}^{N_{2n}(l)} Y_{ni}', \ \ l \in \mathbf{N}.$$

Then for the processes defined by $(Z(i+6)')$, $(Z(i+6))$, $(W(i+6))$ and $(W(i+6)')$ for $i \in \{1,2,3\}$ we have: $\mathcal{L}(X_{(i+3)n}') = \mathcal{L}(X_{(i+6)n}')$, $\mathcal{L}(X_{(i+3)n}) = \mathcal{L}(X_{(i+6)n})$, $\mathcal{L}(Z_{(i+3)n}') = \mathcal{L}(Z_{(i+6)n}')$ and $\mathcal{L}(Z_{(i+3)n}) = \mathcal{L}(Z_{i+6)n})$. The sums $S_l^{((i+6)n)}$ are sums with random index of summing (random sums). Thus, Remark 2 may be considered as a functional limit theorem for random sums. See, for example, Korolev and Kruglov [2] for limit theorems for random sums.

5. Applications to models of market pricing

Consider the market model with n agents, where n is large enough. At any discrete time $t \in \{0, 1, \ldots M\} = T$ each agent buys or sells asset which is called below a stock (which is called agents operation).

An agent operation in the market may take place in the moments $j \in \{0, 1, 2 \ldots M\}$ only. And operations consists that an agent buys or sells an asset which is called below a stock. At the each moment j of the time i-th agent can buy a stock with the probability p and sell the stock with the probability $q = 1 - p$. If an agent buy a stock then the stock price is multiplied by the number U, $U > 1$. If an agent sell a stock then the stock price is multiplied by the number V, $0 < V < 1$. So we suppose that the trading policy of an i-th agent at the moment $j \in T$ is a random variable with the distribution

$$\xi_{ij} = \left\{ \begin{array}{lll} U & \text{with probability} & p \\ D & \text{with probability} & q \end{array} \right. , 1 \le i \le n, \ \ 0 \le j \le m.$$

Thus, the random variable ξ_{ij} is a trade policy of the i-th agent in the moment j.

Introduce the parameter a and $v > 0$ by the following equalities

$$\left. \begin{array}{rcl} v^2 & = & (U-V)^2 pq \\ av^2 - 1 & = & Up + Dq \end{array} \right\}.$$

It is easy to see that $v^2 = \sigma^2(\xi_{ij})$. By analogy to the classical limit transition we assume that $\sigma^2(\xi_{ji}) = \frac{\sigma_0^2}{n}$, where σ_0^2 is a volatility constant not depending from n. The parameter a named as relative risk characterizes difference between "market" measure (p,q) and "risk-neutral" measure $(\tilde{p},\tilde{q})$, i.e. the numbers $\tilde{p}$ and $\tilde{q}$ are define by $U\tilde{p} + D\tilde{q} = 1$, $\tilde{q} = 1 - \tilde{q}$. Correspondingly to the classical case we assume that

$$a = a(n) \to \frac{\mu - r}{\sigma_0^2} \text{ as,} \quad n \to \infty, \quad \mu - r \in \mathbf{R}_+, \ r > 0,$$

where μ is an expected rate of stock return and r is an interest rate of discount bond: $B_t = \exp\{rt\}$.

Next we assume that every agent makes equivalent contribution in to pricing of stock S by his trading policy.

(G) At any fixed moment $k \in T$, each agent's contribution in to pricing is defined as an evolution of his trading policy during the time interval $\{0, 1, \ldots k\}$

$$Z_k(i) = \prod_{j=0}^{k} \xi_{ij}, \quad k = 0, 1, \ldots, m.$$

The following assumption is more important in the construction of the mode.

(F) At any moment $k \in T$ the value of stock is equal to geometric mean of contribution in to pricing over n agents

$$S_k = S_0 \left(\prod_{i=1}^{n} Z_k(i) \right)^{1/n} = S_0 \left(\prod_{i=1}^{n} \prod_{j=0}^{k} \xi_{ij} \right)^{1/n},$$

where S_0 is a starting point.

In accordance to the assumption of time parameterizations and to the assumption (G) for the order of the model n we define the value of the stock at the time moment $t \in [0, T]$ by

$$S_n(t) = S_0 \left(\prod_{i=1}^{n} Z_{[nt]}(i) \right)^{1/n} = S_0 \left(\prod_{i=1}^{n} \prod_{j=0}^{[nt]} \xi_{ij} \right)^{1/n}.$$

So by this equality we have defined a family of step lines with jumps in rational points, depending from the random parameter $\omega_1 \in \Omega_1$ which we consider as elements in $L^\infty[0, T]$.

The market probability measure (p, q) induces the random process $S_n = S_n(p, q)$. Analogously, the risk-neutral measure $(\tilde{p}, \tilde{q})$, induces the random process $S_n = S_n(\tilde{p}, \tilde{q})$. Recall that the rational value of the Standard European Call Option C_T with the maturity T for a strike K can be defined as

$$C_T = \mathbf{E}\{e^{-rT}(e^{rT}\tilde{S}(T) - K)_+\},$$

where the random process $S(t)$, $t \in [0, T]$ is a limit of $S_n(\tilde{p}, \tilde{q})$ as $n \to \infty$.

Model 1. Changing of a trading policy. We will assume that our market satisfies to the following property:

(A3) Actions of each agent does not depends from actions of another agents and trading policy evaluates by following: if the indicates $I_{i1} = 1, I_{i2} = 1, \cdots I_{ij} = 1$, then the trade policy of i-th agent is remaining the same and is defining by the random variable ξ_i: $\xi_{ik} = \xi_i$, $k \leq j$; If the indicators $I_{i1} = 1, I_{i2} = 1, \cdots I_{i(j-1)} = 1$, but the indicator $I_{ij} = 0$, then ξ_i is replaced by independent copy ξ_i': $\xi_{ik} = \xi_i'$, $k \geq j$. So the evolution consists that in a random moment of a trade policy is changing. That is ξ_i' replaces the random variable ξ_i. This replacement can take place only one time.

Theorem 13. (The case of the market probability.)*(1) Under assumption (A), (A2), (G) and (F) the following convergence is valid as $n \to \infty$*

$$S_n(p,q) \xrightarrow{d} S$$

in $L^\infty[0,T]$ for almost all $\omega_1 \in \Omega_1$, where $S(t) = S_0 \exp\Big\{(\mu - r)t + W_3(t) - \frac{\sigma_0^2}{2}t\Big\}$, $t \in [0,T]$ and W_3 is a gaussian random process with the covariance function $B_3(t_1,t_2) = \sigma_0^2(B_1(t_1,t_2) + B_2(t_1,t_2))$, $t_1, t_2 \in [0,T]$.

(2) Under assumption (A1), (A2), (G) and (F) the following convergence is valid as $n \to \infty$

$$S_n(p,q) \xrightarrow{d} S$$

in $L^\infty[0,T]$ for almost all $\omega_1 \in \Omega_1$, where $S(t) = S_0 \exp\Big\{(\mu - r)t + W_3(t) - \frac{\sigma_0^2}{2}t\Big\}$, $t \in [0,T]$ and W_3 is a gaussian random process with the covariance function $B_{m3}(t_1,t_2) = \sigma_0^2(B_{m1}(t_1,t_2) + B_{m2}(t_1,t_2))$, $t_1, t_2 \in [0,T]$.

Theorem 14. (The case of the risk-neutral probability.) *(1) Under assumption (A), (A2), (G) and (F) the following convergence is valid as $n \to \infty$*

$$S_n(\tilde{p},\tilde{q}) \xrightarrow{d} S$$

in $L^\infty[0,T]$ for almost all $\omega_1 \in \Omega_1$, where $\tilde{S}(t) = S_0 \exp\Big\{(W_3(t) - \frac{\sigma_0^2}{2}t\Big\}$ $t \in [0,T]$.

(2) Under assumption (A1), (A2), (G) and (F) the following convergence is valid as $n \to \infty$

$$S_n(\tilde{p},\tilde{q}) \xrightarrow{d} S$$

in $L^\infty[0,T]$ for almost all $\omega_1 \in \Omega_1$, where $\tilde{S}(t) = S_0 \exp\Big\{(W_3(t) - \frac{\sigma_0^2}{2}t\Big\}$ $t \in [0,T]$.

Theorem 15. (The Black-Scholes Formula for the model.) *Let (A) or (A1), (A2), (G), (F), (X3) and (Z3) be valid. Denote the variance*

$$v^2(t) = \mathbf{E}(W_3(t))^2 = \sigma_0^2\left(\int_0^t\int_0^t (1 - f(\min(x,y)) + f(\max(x,y)))dxdy\right),$$

where $t \in [0,T]$, in the case (A), and

$$v^2(t) = \mathbf{E}(W_3(t))^2 = \sigma_0^2\left(\int_0^t\int_0^t (1 - e^{-\min(x,y)} + e^{-\max(x,y)})dxdy\right), t \in [0,T].$$

in the case (A1).

Then the rational value of the Standard European Call Option is equal to

$$C_T = S_0 \exp\Big\{\frac{v^2(T)}{2} - \frac{\sigma_0^2}{2}T\Big\}\Phi(\rho + v(T)) - K\exp\{-rT\}\Phi(\rho),$$

where

$$\rho = \frac{\ln S_0 - \ln K + rT - \sigma_0^2 T/2}{v(T)}.$$

Model 2. Decreasing market. We consider the case of a decreasing market, that is agents go out from a market. In this case the random variables ξ_i are replaced by $\xi_i' = 1$ and (an agent which gone out from market does not can to influence on a stock price) and $\ln(\xi_i)$ is replacing by zeros. We consider a leaving of agents from market as a replacement by zeros. So will assume that it is valid the condition:

(A3) Actions of each agent does not depends from actions of another agents and trading policy evaluates by following: if the indicates $I_{i1} = 1, I_{i2} = 1, \cdots I_{ij} = 1$, then the trade policy of i-th agent is remaining the same and is defining by the random variable ξ_i: $\xi_{ik} = \xi_i$, $k \leq j$; If the indicators $I_{i1} = 1, I_{i2} = 1, \cdots I_{i(j-1)} = 1$, but the indicator $I_{ij} = 0$, then ξ_i is replaced by one: $\xi_{ik} = 1$ for $k \geq j$. That is at the moment j ith agent go out from a market. And we have the following analogs of Theorem 13, Theorem 14 and Theorem 15.

Theorem 16. (The case of the market probability.) *(1) Under assumption (A), (A3), (G) and (F) the following convergence is valid as $n \to \infty$*

$$S_n(p,q) \xrightarrow{d} S$$

in $L^\infty[0,T]$ for almost all $\omega_1 \in \Omega_1$, where $S(t) = S_0 \exp\Big\{(\mu - r)b(t) + W_1(t) - \frac{\sigma_0^2}{2}b(t)\Big\}$, $t \in [0,T]$ and W_1 is a gaussian random process with the covariance function

$$B_1(t_1,t_2) = \sigma_0^2 \int_0^{t_1}\int_0^{t_2} f(\max(x,y))dxdy, \quad t_1,t_2 \in [0,T]$$

and

$$b(t) = \int_0^t f(x)dx, t \in [0,T].$$

(2) Under assumption (A1) (A3), (G) and (F) the following convergence is valid as $n \to \infty$

$$S_n(p,q) \xrightarrow{d} S$$

in $L^\infty[0,T]$ for almost all $\omega_1 \in \Omega_1$, where $S(t) = S_0 \exp\Big\{(\mu - r)b(t) + W_1(t) - \frac{\sigma_0^2}{2}b(t)\Big\}$, $t \in [0,T]$ and W_1 is a gaussian random process with the covariance function

$$B_{m1}(t_1,t_2) = \sigma_0^2 \int_0^{t_1}\int_0^{t_2} e^{-m\max(x,y)}dxdy, \quad t_1,t_2 \in [0,T]$$

and

$$b(t) = \int_0^t e^{-mx} dx, \quad t \in [0, T].$$

Theorem 17. (The case of the risk-neutral probability.) *Under assumption (A) or (A1), (A3), (G) and (F) the following convergence is valid as $n \to \infty$*

$$S_n(\tilde{p}, \tilde{q}) \xrightarrow{d} S$$

in $L^\infty[0, T]$ for almost all $\omega_1 \in \Omega_1$, where $\tilde{S}(t) = S_0 \exp\Big\{W_1(t) - \frac{\sigma_0^2}{2} b(t)\Big\}$ $t \in [0, T]$, and in the case (A) $W_1(t)$ and $b(t)$ are defined in (1) of Theorem 16 and in the case (A1) $W_1(t)$ and $b(t)$ are defined in (2) of Theorem 16.

Theorem 18. (The Black-Scholes Formula for the model.) *Let (A) or (A1), (A3), (G) and (F) be valid. Denote*

$$v^2(t) = \mathbf{E}(W_1(t))^2 = B_1(t, t) = 2\sigma_0^2 \int_0^t \int_0^t f(\max(x, y)) dx dy,$$

$$b(t) = \int_0^t f(x) dx$$

in the case (A), and

$$v^2(t) = \mathbf{E}(W_1(t))^2 = B_1(t, t) = 2\sigma_0^2 \int_0^t \int_0^t e^{-m \max(x, y)} dx dy,$$

$$b(t) = \int_0^t e^{-mx} dx$$

in the case (A1).

Then the rational value of the Standard European Call Option is equal to

$$C_T = S_0 \exp\Big\{\frac{v^2(T)}{2} - \frac{\sigma_0^2}{2} b(T)\Big\} \Phi(\rho + v(T)) - K \exp\{-r b(T)\} \Phi(\rho),$$

where

$$\rho = \frac{\ln S_0 - \ln K + r b(T) - \sigma_0^2 b(T)/2}{v(T)}.$$

Model 3. Increasing Market. We consider the case of an increasing market, that is agents come to a market. In this case the random variables $\xi_i = 1$ is replacing by ξ_i': an agents which come to a market begin to influence on a stock price. And we will use the following condition

(A4) Actions of each agent does not depends from actions of another agents and trading policy evaluates by following: if the indicates $I_{i1} = 1, I_{i2} = 1, \cdots I_{ij} = 1$, then the trade policy of i-th agent is equal to one: $\xi_{ik} = 1$, $k \le j$; If the indicators $I_{i1} = 1, I_{i2} = 1, \cdots I_{i(j-1)} = 1$, but the indicator $I_{ij} = 0$, then $\xi_{ik} = \xi_i$, $k \ge j$. This means that at the jth moment ith agent comes to a market.

As analogs of Theorem 13, Theorem 14 and Theorem 15 we have

Theorem 19. (The case of the market probability.) *(1) Under assumption (A), (A4), (G) and (F) the following convergence is valid as $n \to \infty$*

$$S_n(p,q) \xrightarrow{d} S$$

in $L^\infty[0,T]$ *for almost all* $\omega_1 \in \Omega_1$*, where* $S(t) = S_0 \exp\Big\{(\mu - r)b(t) + W_2(t) - \frac{\sigma_0^2}{2}b(t)\Big\}$*,* $t \in [0,T]$ *and* W_2 *is a gaussian random process with the covariance function*

$$B_2(t_1,t_2) = \sigma_0^2\left(t_1t_2 - \int_0^{t_1}\int_0^{t_2} f(\min(x,y))dxdy\right), \quad t_1,t_2 \in [0,T],$$

and

$$b(t) = t - \int_0^t f(x)dx.$$

(2) Under assumption (A1), (A4), (G) and (F) the following convergence is valid as $n \to \infty$

$$S_n(p,q) \xrightarrow{d} S$$

in $L^\infty[0,T]$ *for almost all* $\omega_1 \in \Omega_1$*, where* $S(t) = S_0 \exp\Big\{(\mu - r)b(t) + W_2(t) - \frac{\sigma_0^2}{2}b(t)\Big\}$*,* $t \in [0,T]$ *and* W_2 *is a gaussian random process with the covariance function*

$$B_{m2}(t_1,t_2) = \sigma_0^2\left(t_1t_2 - \int_0^{t_1}\int_0^{t_2} e^{-m\min(x,y)}dxdy\right) \quad t_1,t_2 \in [0,T]$$

and

$$b(t) = t - \int_0^t e^{-mx}dx.$$

Theorem 20. (The case of the risk-neutral probability.) *Under assumption (A) or (A1, (A4), (G) and (F) the following convergence is valid as* $n \to \infty$

$$S_n(\tilde p,\tilde q) \xrightarrow{d} S$$

in $L^\infty[0,T]$ *for almost all* $\omega_1 \in \Omega_1$*, where* $\tilde S(t) = S_0 \exp\Big\{W_2(t) - \frac{\sigma_0^2}{2}b(t)\Big\}$ $t \in [0,T]$ *and in the case (A)* $W_2(t)$ *and* $b(t)$ *are defined in (1) of Theorem 19 and in the case (A1)* $W_2(t)$ *and* $b(t)$ *are defined in (2) of Theorem 19.*

Theorem 21. (The Black-Scholes Formula for the model.) *Let (A) or (A1), (A4), (G) and (F) be valid. Denote*

$$v^2(t) = \mathbf{E}(W_2(t))^2 = \sigma_0^2\left(t^2 - \int_0^t\int_0^t f(\min(x,y))dydx\right),$$

$$b(t) = t - \int_0^t f(x)dx.$$

in the case (A), and

$$v^2(t) = \mathbf{E}(W_2(t))^2 = \sigma_0^2\left(t^2 - \int_0^t\int_0^t e^{-m\min(x,y)}dydx\right),$$

$$b(t) = t - \int_0^t e^{-mx}dx.$$

in the case (A2).

Then the rational value of the Standard European Call Option option is equal to

$$C_T = S_0 \exp\Big\{\frac{v^2(T)}{2} - \frac{\sigma_0^2}{2} b(T)\Big\}\Phi(\rho + v(T)) - K \exp\{-rb(T)\}\Phi(\rho),$$

where $\rho = \frac{\ln S_0 - \ln K + rb(T) - \sigma_0^2 b(T)/2}{v(T)}$.

The proofs of Theorem 13 – Theorem 21 are the same as the proofs of Theorem 1, Theorem 2 and Theorem 3 [9] (see, also, [10]) with the difference that is instead of Theorem 1 from [6] we use Theorem 10, Theorem 11 and Theorem 12 respectively.

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Received February 12, 2003