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BOUNDEDNESS FOR COMMUTATORS OF LITTLEWOOD-PALEY OPERATORS ON SOME HARDY SPACES

(submitted by F. G. Avkhadiev)

ABSTRACT. In the present paper the $\left(H_b^p, L^p\right)$-type and $\left(H_b^{p,\infty}, L^{p,\infty}\right)$-type boundedness for the commutators associated with the Littlewood-Paley operators and $b \in BMO(R^n)$ are obtained, where H_b^p and $H_b^{p,\infty}$ are, respectively, variants of the standard Hardy spaces and weak Hardy spaces, and $n/(n+\varepsilon) < p \leq 1$.

1. Introduction

Let $b \in BMO(R^n)$ and T be the Calderon-Zygmund operator. The commutator $[b,T]$ generated by b and T is defined by

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x).$$

A classical result of Coifman, Rochberg and Weiss [3] proved that the commutator $[b,T]$ is bounded on $L^p(R^n)$ ($1 < p < \infty$). However, it was observed that $[b,T]$ is not bounded, in general, from $H^p(R^n)$ to $L^p(R^n)$ and from $L^1(R^n)$ to $L^{1,\infty}(R^n)$ and from $H^{p,\infty}(R^n)$ to $L^{p,\infty}(R^n)$ for $p \leq 1$. But, if $H^p(R^n)$ is replaced by a suitable atom space $H_b^p(R^n)$ and $H^{p,\infty}(R^n)$ by $H_b^{p,\infty}(R^n)$ (see [1][6]), then $[b,T]$ maps continuously $H_b^p(R^n)$ into $L^p(R^n)$ and $H_b^{p,\infty}(R^n)$ into $L^{p,\infty}(R^n)$ for $p \in (n/(n+1),1]$. In addition, we easily know that $H_b^p(R^n) \subset H^p(R^n)$ and $H_b^{p,\infty}(R^n) \subset H^{p,\infty}(R^n)$. The main purpose of this paper is to consider the boundedness of the commutators related to Littlewood-Paley operators and $BMO(R^n)$ functions from $H_b^p(R^n)$ to $L^p(R^n)$ and from $H_b^{p,\infty}(R^n)$ to $L^{p,\infty}(R^n)$ for $p \leq 1$. In fact, we prove that the commutators of Littlewood-Paley operators are bounded from $H_b^p(R^n)$ to $L^p(R^n)$ and from $H_b^{p,\infty}(R^n)$ to $L^{p,\infty}(R^n)$ for $p \in (n/(n+\varepsilon),1]$. We will work on $R^n, n > 2$. Let us first introduce some definitions (see [1] [4] [6] [9]).

Definition 1. Let b be a locally integrable function and $0 < p \leq 1$. A bounded measurable function a on R^n is said to be a (p, b) atom if

i) supp $a \subset B = B(x_0, r)$,

ii) $||a||_{L^\infty} \leq |B|^{-1/p}$,

iii) $\int a(y)dy = \int a(y)b(y)dy = 0$;

A temperate distribution f is said to belong to $H_b^p(R^n)$, if, in the Schwartz distributional sense, it can be written as

$$f(x) = \sum_{j=0}^{\infty} \lambda_j a_j(x),$$

where a_j's are (p, b) atoms, $\lambda_j \in C$ and $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$. Moreover, $||f||_{H_b^p(R^n)} \sim \left(\sum_{j=0}^{\infty} |\lambda_j|^p\right)^{1/p}$.

Definition 2. Let b be a locally integrable function and $0 < p \leq 1$. A temperate distribution f is said to belong to the space $H_b^{p,\infty}(R^n)$ if there exists a sequence of functions $\{f_k\}_{k=-\infty}^{\infty} \subset L^\infty(R^n)$ such that

a) $f(x) = \sum_{k=-\infty}^{\infty} f_k(x)$ in the Schwartz distribution sense;

b) Each f_k can be decomposed into $f_k = \sum_{j=1}^{\infty} b_j^k$ in $L^\infty(R^n) \cap H^p(R^n)$, where b_j^k satisfy the following properties:

b)$_1$ Supp $b_j^k \subset B_j^k$, B_j^k's are the balls with

$$\sup_k \sum_{j=1}^{\infty} \chi_{B_j^k}(x) < \infty, \quad \sup_k 2^{kp} \sum_{j=1}^{\infty} |B_j^k| < \infty,$$

where, and in what follows, χ_E denotes the characteristic function of the set E,

b)$_2$ there exists a constant $C = C(n, p) > 0$ such that

$$||b_j^k||_{L^\infty} \leq C2^k \text{ for every } k, j,$$

b)$_3$ $\int_{R^n} b_j^k(x)dx = \int_{R^n} b_j^k(x)b(x)dx = 0$;

The quasinorm on the space $H_b^{p,\infty}(R^n)$ is defined by

$$||f||_{H_b^{p,\infty}(R^n)}^p = \inf_{\sum_{k=-\infty}^{\infty} \sum_{j=1}^{\infty} b_j^k = f} \sup_{k \in Z} 2^{kp} \sum_{j=1}^{\infty} |B_j^k|,$$

where the infimum is taken over all decompositions of f.

Definition 3. Let $\varepsilon > 0$, fixed a function ψ satisfied the following properties:

(1) $\int \psi(x)dx = 0$,

(2) $|\psi(x)| \leq C(1 + |x|)^{-(n+\varepsilon)}$,

(3) $|\psi(x+y) - \psi(x)| \leq C|y|^\varepsilon(1 + |x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$;

Let b b a locally integrable function. The commutator of Littlewood-Paley operator is defined by

$$g_{\mu,b}^*(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_{b,t}(x,y)|^2 \frac{dydt}{t^{1+n}}\right]^{1/2}, \quad \mu > 1,$$

where

$$F_{b,t}(x, y) = \int_{R^n} \psi_t(y - z) f(z)(b(x) - b(z))dz,$$

and $\psi_t(x) = t^{-n} \psi(x/t)$ for $t > 0$. We also define

$$g_\mu^*(f)(x) = \left[\int \int_{R_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\mu} |f * \psi_t(y)|^2 \frac{dydt}{t^{1+n}} \right]^{1/2},$$

which is the Littlewood-Paley operator (see [8]).

2. Theorems and Proofs

Theorem 1.   Let $b \in BMO(R^n)$ and $1 \geq p > n/(n + \varepsilon)$, and $\mu > n/2$. Then the commutator $g_{\mu,b}^*$ is bounded from $H_b^p(R^n)$ to $L^p(R^n)$.

   Proof.   It suffices to show that there exists a constant $C > 0$ such that for every (p, b) atom a,

$$||g_{\mu,b}^*(a)||_{L^p} \leq C.$$

Let a be a (p, b) atom supported on a ball $B = B(x_0, r)$. We write

$$\int_{R^n} [g_{\mu,b}^*(a)(x)]^p dx$$

$$= \int_{|x-x_0| \leq 2r} [g_{\mu,b}^*(a)(x)]^p dx + \int_{|x-x_0| > 2r} [g_{\mu,b}^*(a)(x)]^p dx$$

$$\equiv I + II,$$

   For I, taking $q > 2$, by Hölder's inequality and the L^q-boundedness of $g_{\mu,b}^*$ when $\mu > \max(n/2, n/q)$ (see [2]), we see that

$$I \leq C||g_{\mu,b}^*(a)||_{L^q}^p \cdot |B(x_0, 2r)|^{1-p/q}$$

$$\leq C||a||_{L^q}^p |B|^{1-p/q} \leq C.$$

   For II, by the vanishing moment of a and Definition 3(3), we have

$$g_{\mu,b}^*(a)(x) \leq \left[\int \int_{R_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\mu} \left(\int_B |\psi_t(y - z) \right.\right.$$

$$\left.\left. - \psi_t(y - x_0)||a(z)||b(x) - b(z)|dz \right)^2 \frac{dydt}{t^{1+n}} \right]^{1/2}$$

$$\leq C \left[\int \int_{R_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\mu} \left(\int_B t^{-n} |a(z)||b(x) - b(z)| \right.\right.$$

$$\left.\left. \frac{(|x_0 - z|/t)^\varepsilon}{(1 + |x_0 - y|/t)^{n+1+\varepsilon}} dz \right)^2 \frac{dydt}{t^{n+1}} \right]^{1/2}$$

$$\leq C|B|^{\varepsilon/n-1/p}\left[\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{t^2}{(t+|x_0-y|)^{2(n+1+\varepsilon)}}\right.$$

$$\left.\left(\int_B|b(x)-b(z)|dz\right)^2\frac{dydt}{t^{1+n}}\right]^{1/2}$$

$$\leq C|B|^{\varepsilon/n-1/p+1}\left[\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{t^{1-n}}{(t+|x_0-y|)^{2(n+1+\varepsilon)}}\right.$$

$$\left.\left(\frac{1}{|B|}\int_B(|b(x)-b_0|+|b_0-b(z)|)dz\right)^2\right)dydt]^{1/2}$$

$$\leq C|B|^{\varepsilon/n-1/p+1}(|b(x)-b_0|+||b||_{BMO})\left[\int_0^\infty t\left(t^{-n}\int_{R^n}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\right.\right.$$

$$\left.\left.\frac{dy}{(t+|x_0-y|)^{2(n+1+\varepsilon)}}\right)dt\right]^{1/2},$$

where $b_0 = |B(x_0,r)|^{-1}\int_{B(x_0,r)}b(y)dy$, notice that

$$t^{-n}\int_{R^n}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{dy}{(t+|x_0-y|)^{2(n+1+\varepsilon)}} \leq CM\left(\frac{1}{(t+|x_0-x|)^{2(n+1+\varepsilon)}}\right)$$

$$\leq C\frac{1}{(t+|x-z|)^{2(n+1+\varepsilon)}}$$

(where Mg denotes the Hardy-Littlewood maximal function of g) and

$$\int_0^\infty\frac{tdt}{(t+|x_0-x|)^{2(n+1+\varepsilon)}} = C|x-x_0|^{-2(n+\varepsilon)},$$

Then, we deduce

$$g^*_{\mu,b}(a)(x) \leq C|B|^{\varepsilon/n-1/p+1}(|b(x)-b_0|+||b||_{BMO})|x-x_0|^{-(n+\varepsilon)},$$

thus, by Hölder's inequality,

$$II \leq C\sum_{k=1}^\infty|B(x_0,2^{k+1}r)|^{1-p}\left[\int_{2^{k+1}r\geq|x-x_0|>2^kr}g^*_{\mu,b}(a)(x)dx\right]^{1/p}$$

$$\leq C\sum_{k=1}^\infty|B(x_0,2^{k+1}r)|^{1-p}|B|^{(\varepsilon/n-1/p+1)p}||b||_{BMO}^p$$

$$\left(\int_{2^{k+1}r\geq|x-x_0|>2^kr}|x-x_0|^{-(n+\varepsilon)}dx\right)^p$$

$$+C\sum_{k=1}^\infty|B(x_0,2^{k+1}r)|^{1-p}|B|^{(\varepsilon/n-1/p+1)p}$$

$$\left(\int_{2^{k+1}r\geq|x-x_0|>2^kr}|b(x)-b_0||x-x_0|^{-(n+\varepsilon)}dx\right)^p$$

$$\equiv II_1+II_2;$$

For II_2, using the properties of $BMO(R^n)$ (see [7]), we obtain

$$II_2 \leq C \sum_{k=0}^{\infty} |B(x_0, 2^{k+1}r)|^{1-\frac{n+\varepsilon}{n}p} k^p ||b||_{BMO}^p |B|^{(1+\varepsilon/n-1/p)p}$$

$$\leq C||b||_{BMO}^p \sum_{k=0}^{\infty} k^p 2^{kn(1-(n+\varepsilon)p/n)}$$

$$\leq C||b||_{BMO}^p;$$

For II_1, similar to the estimate of II_2, we obtain $II_1 \leq C||b||_{BMO}^p$; Combining the estimates of II_1 with II_2, we gain

$$II \leq C||b||_{BMO}^p \leq C.$$

This finishes the proof of Theorem 1.

Theorem 2. Let $b \in L^\infty(R^n)$ and $p = n/(n+\varepsilon)$ and $\mu > 1$. Then $g_{\mu,b}^*$ is bounded from $H_b^p(R^n)$ to $L^{p,\infty}(R^n)$.

To prove the theorem, we recall the following lemma (see [8]):

Lemma. Let $\{f_k\}$ be a sequence of measurable functions and $p \in (0, 1)$. Assume that

$$|\{x \in R^n : |f_k(x)| > \lambda\}| \leq c\lambda^{-p} \text{ for any } k \text{ and } \lambda > 0.$$

Then, for every p-summable numerical sequence $\{C_k\}$ we have

$$\left| \left\{ x \in R^n : \left| \sum_k C_k f_k(x) \right| > \lambda \right\} \right| \leq \frac{2-p}{1-p} \frac{C}{\lambda^p} \sum_k |C_k|^p.$$

Proof of Theorem 2. It suffices to show that there exists a constant $C > 0$, such that for each (p, b) atom a and any $\lambda > 0$, we have

$$\lambda^p |\{x \in R^n : g_{\mu,b}^*(a)(x) > \lambda\}| \leq c||b||_{L^\infty}^p.$$

We write

$$\lambda^p |\{x \in R^n : g_{\mu,b}^*(a)(x) > \lambda\}|$$
$$\leq \lambda^p \left| \{x \in R^n : |b(x)| g_\mu^*(a)(x) > \lambda/2\} \right| + \lambda^p \left| \{x \in R^n : g_\mu^*(ab)(x) > \lambda/2\} \right|$$
$$\equiv I + II.$$

For I, by $a \in H^p(R^n)$ and the boundedness of g_μ^* from $H^p(R^n)$ to $L^p(R^n)$ $(0 < p \leq 1)$(see[5]), we have

$$I \leq C\lambda^p \left| \{x \in R^n : g_\mu^*(a)(x) > \lambda/(2||b||_{L^\infty})\} \right|$$
$$\leq C||b||_{L^\infty}^p;$$

For II, note that $ab/||b||_\infty$ is also a (p, ∞) atom in the space $H^p(R^n)$, thus, we gain

$$II \leq \lambda^p \left| \{x \in R^n : g_\mu^*(ab/||b||_{L^\infty})(x) > \lambda/(2||b||_{L^\infty})\} \right| \leq C||b||_{L^\infty}^p.$$

This finishes the proof of Theorem 2.

Theorem 3. Let $b \in BMO(R^n)$, $1 \geq p > n/(n+\varepsilon)$ and $\mu > n/2$. Then $g_{\mu,b}^*$ is bounded from $H_b^{p,\infty}(R^n)$ to $L^{p,\infty}(R^n)$.

Proof. Given $f \in H_b^{p,\infty}(R^n)$, let $f = \sum_{k=-\infty}^{\infty} f_k = \sum_{k=-\infty}^{\infty} \sum_{j=1}^{\infty} b_j^k$ be an atomic decomposition as in Definition 2. By a limiting argument, it suffices to show that

$$\sup_{\lambda>0} \lambda^p |\{x \in R^n : g_{\mu,b}^* \left(\sum_{k=-N}^{N} f_k \right)(x) > \lambda\}| \leq CC_1$$

fore every $N = 0, 1, 2, \cdots$, where $C_1 = \sup_{k \in Z} 2^{kp} \sum_{j=1}^{\infty} |B_j^k|$.

Given $\lambda > 0$, we taking $k_0 \in Z$ such that $2^{k_0} \leq \lambda < 2^{k_0+1}$. Let

$$\sum_{k=-N}^{N} f_k = \sum_{k=-N}^{k_0} f_k + \sum_{k=k_0+1}^{N} f_k \equiv F_1 + F_2.$$

Note that

$$|F_1(x)| \leq C \sum_{k=-N}^{k_0} 2^k \sum_{j=1}^{\infty} |b_j^k(x)| \leq C \sum_{k=-N}^{k_0} 2^k \sum_{j=1}^{\infty} \chi_{B_j^k}(x)$$

$$\leq C \sum_{k=-N}^{k_0} 2^k \chi_{\cup_{j=1}^{\infty} B_j^k}(x),$$

thus

$$\|F_1\|_{L^q} \leq C \sum_{k=-N}^{k_0} 2^k \left| \bigcup_{j=1}^{\infty} B_j^k \right|^{1/q} \leq C \sum_{k=-N}^{k_0} 2^k \left(\sum_{j=1}^{\infty} |B_j^k| \right)^{1/q}$$

$$\leq CC_1^{1/q} \sum_{k=-N}^{k_0} 2^{k-kp/q} \leq CC_1^{1/q} 2^{k_0-k_0 p/q}$$

$$\leq CC_1^{1/q} \lambda^{1-p/q}, \quad \text{for any } 1 < q < \infty.$$

Since $g_{\mu,b}^*$ is bounded on $L^q(R^n)$ for $1 < q < \infty$ (see [2]), we have

$$\lambda^p \left| \{x \in R^n : g_{\mu,b}^*(F_1)(x) > \lambda/2\} \right| \leq C\lambda^{p-q} \|F_1\|_{L^q}^q \leq CC_1;$$

For F_2, let $A_k B_j^k$ be the ball with the same center as B_j^k and A_k times the radius of B_j^k, where $A_k \geq 2^{n+1}$ is a positive number to be determined later. For brevity, let

$$B_{k_0,N} = \bigcup_{k_0+1 \leq k \leq N, j \geq 1} A_k B_j^k.$$

We write

$$\lambda^p |\{x \in R^n : g_{\mu,b}^*(F_2)(x) > \lambda/2\}|$$
$$= \lambda^p |\{x \in B_{k_0,N} : g_{\mu,b}^*\}(F_2)(x) > \lambda\}| + \lambda^p |\{x \notin B_{k_0,N} : g_{\mu,b}^*(F_2)(x) > \lambda\}|$$
$$\equiv I + II.$$

Let $C_j^k = |B_j^k|^{-1} \int_{B_j^k} b(y) dy$, first, we have

$$II \leq \lambda^p \left| \left\{ x \notin B_{k_0,N} : |b(x) - C_j^k| g_\mu^*(F_2)(x) > \lambda/2 \right\} \right|$$
$$+ \lambda^p \left| \left\{ x \notin B_{k_0,N} : g_\mu^*((b - C_j^k)F_2)(x) > \lambda/2 \right\} \right|$$
$$\equiv II_1 + II_2,$$

$$II_1 \leq C \int_{(B_{k_0,N})^c} |b(x) - C_j^k|^p (g_\mu^*(F_2)(x))^p dx$$

$$\leq C \sum_{k=k_0+1}^{N} \sum_{j=1}^{\infty} \int_{(A_k B_j^k)^c} |b(x) - C_j^k| (g_\mu^*(b_j^k)(x))^p dx,$$

let us now fix j and k, then

$$\int_{(A_k B_j^k)^c} |b(x) - C_j^k| (g_\mu^*(b_j^k)(x))^p dx$$

$$= \sum_{l=0}^{\infty} \int_{2^{l+1} A_k B_j^k \setminus 2^l A_k B_j^k} |b(x) - C_j^k|^p (g_\mu^*(b_j^k)(x))^p dx$$

$$\leq \sum_{l=0}^{\infty} \left(\int_{2^{l+1} A_k B_j^k \setminus 2^l A_k B_j^k} (g_\mu^*(b_j^k)(x))^{p/(1-p)} dx \right)^{1-p} \left(\int_{2^{l+1} A_k B_j^k \setminus 2^l A_k B_j^k} |b(x) - C_j^k| dx \right)^p$$

(where usual modification is made when $p = 1$)
$$\equiv L,$$

let $B_l = |2^{l+1} A_k B_j^k|^{-1} \int_{2^{l+1} A_k B_j^k} |b(x) - C_j^k| dx$, then

$$L \leq \sum_{l=0}^{\infty} B_l^p |2^{l+1} A_k B_j^k|^p \left(\int_{2^{l+1} A_k B_j^k \setminus 2^l A_k B_j^k} (g_\mu^*(b_j^k)(x))^{p/(1-p)} dx \right)^{1-p},$$

by $|x| > 2|y|$, similar to the proof of Theorem 1, we have

$$g_\mu^*(b_j^k)(x) \leq C 2^k |x|^{-(n+\varepsilon)} |B_j^k|^{1+\varepsilon/n},$$

thus

$$L \leq C \sum_{l=0}^{\infty} B_l^p |2^{l+1} A_k B_j^k|^p 2^{kp} |B_j^k|^{(1+\varepsilon/n)p} |2^{l+1} A_k B_j^k|^{1-p(n+\varepsilon)/n-p}$$

$$= C 2^{kp} A_k^{n(1-(n+\varepsilon)p/n)} |B_j^k| \sum_{l=0}^{\infty} B_l^p 2^{(l+1)(n-(n+\varepsilon)p)}$$

$$\leq C ||b||_{BMO}^p 2^{kp} A_k^{n-p(n+\varepsilon)} |B_j^k|,$$

now we take $A_k = A 2^{(k-k_0)/(n+\varepsilon)}$, where A is fixed and large enough, then

$$II_1 \leq C||b||_{BMO}^p \sum_{j=1}^{\infty} \sum_{k=k_0+1}^{N} 2^{kp}|B_j^k|A_k^{n-p(n+\varepsilon)}$$

$$\leq CC_1 \sum_{k=k_0+1}^{N} A_k^{n-p(n+\varepsilon)}$$

$$\leq CC_1, \text{ since } p > n/(n+\varepsilon).$$

Now, let us estimate II_2, by the estimate of g_μ^* similar to above, we reduce

$$II_2 \leq \lambda^p \left| \left\{ x \notin B_{k_0,N} : \sum_{k=k_0+1}^{N} \sum_{j=1}^{\infty} g_\mu^*((b-C_j^k)b_j^k)(x) > \lambda/2 \right\} \right|$$

$$\leq \lambda^p \left| \left\{ x \notin B_{k_0,N} : \sum_{k=k_0+1}^{N} \sum_{j=1}^{\infty} C|x|^{-(n+\varepsilon)} 2^k |B_j^k|^{1+\varepsilon/n} \left(|B_j^k|^{-1} \int_{B_j^k} |b(y) - c_j^k| dy \right) \right. \right.$$

$$\left. \left. > \lambda/2 \right\} \right|$$

$$\leq C\lambda^p \sum_{k=k_0+1}^{N} \sum_{j=1}^{\infty} \left[2^k |B_j^k|^{1+\varepsilon/k} ||b||_{BMO} \right]^{n/(n+\varepsilon)} \lambda^{-n/(n+\varepsilon)}$$

$$\leq C\lambda^{p-n/(n+\varepsilon)} ||b||_{BMO}^{n/(n+\varepsilon)} \cdot C_1 \sum_{k=k_0+1}^{N} 2^{k(n/(n+\varepsilon)-p)}$$

$$\leq CC_1 \lambda^{p-n/(n+\varepsilon)} 2^{k_0(n/(n+\varepsilon)-p)}$$

$$\leq CC_1, \text{ since } \lambda \leq 2^{k_0+1},$$

combining the estimates of II_1 with II_2, we obtain

$$II \leq CC_1.$$

Finally, let us estimate I. We have

$$I \leq \lambda^p \sum_{k=k_0+1}^{N} \sum_{j=1}^{\infty} A_k^n |B_j^k| \leq C\lambda^p C_1 \sum_{k=k_0+1}^{N} 2^{n(k-k_0)/(n+\varepsilon)} 2^{-kp}$$

$$\leq CC_1 \lambda^p 2^{-k_0 p} \sum_{k=k_0+1}^{N} 2^{(k-k_0)(n/(n+\varepsilon)-p)} \leq CC_1.$$

Now, let us combine the estimates of I with II, and let $N \to \infty$, then, we have obtained the conclusion of Theorem 3.

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