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ON THE BRAUER MONOID OF S_3

(submitted by M. Arslanov)

ABSTRACT. In [HLS], the authors showed that the Brauer monoid of a finite Galois group can be written as a disjoint union of smaller pieces (groups). Each group can be computed following Stimets by defining a chain complex and checking its exactness. However, this method is not so encouraging because of the difficulty of dealing with such computations even with small groups. Unfortunately, this is the only known method so far. This paper is to apply Stimets' method to some idempotent weak 2-cocycles defined on S_3. In particular, the idempotent 2-cocycles whose associated graphs have two generators. Some nice results appear in the theory of noncommutative polynomials.

1. PRELIMINARIES

Let K/F be a finite Galois extension of fields and let G be its Galois group. A weak 2–cocycle is a function $f : G \times G \to K$ satisfying

(i) $f^\sigma(\tau,\nu)f(\sigma,\tau\nu) = f(\sigma\tau,\nu)f(\sigma,\tau)$
(ii) $f(\sigma,1) = f(1,\sigma) = 1$

for all $\sigma, \tau, \nu \in G$.

If we define an algebra A_f associated with f to be the algebra generated as a K-vector space by the indeterminates $\{x_\sigma : \sigma \in G\}$ with the relations $x_\sigma x_\tau = f(\sigma,\tau)x_{\sigma\tau}$ and $x_\sigma k = k^\sigma x_\sigma$ for $k \in K$, $x_1 = 1$, then A_f is called a *weak crossed product*. Condition (i) above guarantees the associativity of A_f.

In the classical theory f does not take the value 0, and in that case A_f is a central simple F-algebra. The set $H = \{\sigma \in G : f(\sigma,\sigma^{-1}) \neq 0\}$ is a subgroup of G called the inertial subgroup of f. We define an order "$\leq$"

on G/H by $\sigma H \leq \tau H$ if and only if $f(\sigma, \sigma^{-1}\tau) \neq 0$. This order is lower subtractive, that is if $\sigma H \leq \tau H$ then $\sigma H \leq \alpha H \leq \tau H$ if and only if $\sigma^{-1}\alpha H \leq \sigma^{-1}\tau H$. In this order, H is the unique minimal element (root) and has the property $f(H \times H) \subseteq K^*$ [HLS]. The weak crossed product A_f can be written as

$$A_f = \bigoplus_{\sigma \in H} Kx_\sigma \oplus \bigoplus_{\sigma \notin H} Kx_\sigma = B \oplus J$$

where $J = \bigoplus_{\sigma \notin H} Kx_\sigma$ is the radical of A_f and $B = \bigoplus_{\sigma \in H} Kx_\sigma$ is a classical crossed product algebra for $f|_{H \times H}$. In particular B is a central simple K^H–algebra.

Given a weak 2-cocycle f, define a function $e : G \times G \to \{0, 1\}$ by $e(\sigma, \tau) = 0$ if and only if $f(\sigma, \tau) = 0$. Then e is a weak 2-cocycle called the idempotent weak 2-cocycle associated to f.

Two weak 2-cocycles f, g are called cohomologous (or equivalent) if there is a function $\alpha : G \to K^*$ such that

$$f(\sigma, \tau) = \frac{\alpha(\sigma)\alpha^\sigma(\tau)}{\alpha(\sigma\tau)} g(\sigma, \tau) \quad \text{for all} \quad \sigma, \tau \in G.$$

Any two cohomologous weak 2-cocycles have the same associated idempotent cocycle. Under the equivalence relation introduced above the set of classes of weak 2-cocycles from $G \times G$ to K forms a monoid denoted by $M^2(G, K)$. The subgroup of invertible elements of this monoid is the usual cohomology group $H^2(G, K^*)$.

Let e be an idempotent weak 2-cocycle. If f is a weak 2-cocycle associated to e then we can define a function $g : G \times G \to K$ by

$$g(\sigma, \tau) = \begin{cases} (f(\sigma, \tau))^{-1} & \text{if } f(\sigma, \tau) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then g is a weak 2-cocycle associated to e. If $[\,\cdot\,]$ denotes the equivalence class in the relation above, let $M_e^2(G, K) = \{[f] \in M^2(G, K) \mid [f][e] = [f]$ and there is a weak 2-cocycle g such that $[f][g] = [e]\}$. Then $M_e^2(G, K)$ is a group with identity $[e]$ and $M^2(G, K) = \cup_e M_e^2(G, K)$ (disjoint) where the union is over all idempotent weak 2-cocycles, [H1]. In a similar way, we can define the group $M_e^i(G, K)$ of ith dimension.

The set of all idempotent cocycles on $G \times G$ with inertial subgroup H is in $1 - 1$ correspondence with all lower subtractive orders (graphs) on G/H with unique root H. Whenever we refer to the graph for e, we mean the graph associated to the weak 2-cocycle e. If $[f] \in M_e^2(G, K)$ then f and e have the same associated graph. The following results and definitions are from [S1] and [S2]. Each idempotent 2-cocycle e with

trivial inertial subgroup H determines a ring $R_e = \mathbb{Z}\{x_\sigma : \sigma \in G\}/I_e$ where I_e is the ideal generated by $\{x_\sigma x_\tau - x_{\sigma\tau} \mid \sigma \leq \sigma\tau\}$. The ring R_e is called the derived ring of e. To define a graded R_e–module, it is more convenient to use the notation g_i for the elements of the group G and $[g_1, g_2, \ldots, g_k]$ for the free generators of the R_e–module, where $g_i \leq g_{i+1}$ according to the relation defined above.

Define a graded R_e–module $\mathcal{M}$ by $\mathcal{M} = \bigoplus_{n \in \mathbb{Z}} M_n$ where

$$
M_n = \begin{cases}
\displaystyle\bigoplus_{g_1 < \cdots < g_n} R_e[g_1, \ldots, g_n] & n \geq 1, \ g_i \neq 1 \\
R_e & n = 0 \\
\mathbb{Z} & n = -1 \\
0 & n \leq -2
\end{cases}
$$

with differentials $d_n[g_1, \ldots, g_n] = x_{g_1}[g_1^{-1}g_2, \ldots, g_1^{-1}g_n] + \sum(-1)^i[g_1, \ldots, \hat{g}_i, \ldots, g_n]$. We call the pair $(\mathcal{M}, d)$ the chain complex of e. This definition can be given in a similar manner if $H \neq \{1\}$. Suppose f is a function from G^n to a field K which satisfies

(i) in case

$$
f(g_1, g_2, \ldots, g_{i-1}, 1, g_{i+1}, \ldots, g_n) \neq 0,
$$

we have $f(g_1, g_2, \ldots, g_{i-1}, 1, g_{i+1}, \ldots, g_n) = 1$ for all $g_1, g_2, \ldots, g_n$ in G and all $1 \leq i \leq n$,

(ii) $f(1, 1, \ldots, 1, g, 1, \ldots, 1) = 1$ for all $g \in G$, and

(iii) for each $g_1, g_2, \ldots, g_{n+1}$ in G,

$$
f^{g_1}(g_2, \ldots, g_{n+1}) \prod_{i \text{ even}} f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1})
$$
$$
= f(g_1, \ldots, g_n) \prod_{i \text{ odd}} f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1})
$$

if n is even, and

$$
f^{g_1}(g_2, \ldots, g_{n+1}) f(g_1, \ldots, g_n) \prod_{i \text{ even}} f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1})
$$
$$
= \prod_{i \text{ odd}} f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1})
$$

if n is odd. We then call f a *weak n-cocycle*. The first condition is analogous to the standard degeneracy conditions used in homological algebra, but still allows for the possibility that certain cochains may take on non-invertible values.

The second condition ensures that the cochains (cocycles) have a sufficient amount of invertibility.

If f is a weak n-cocycle and there is another weak n-cocycle h and an invertible cochain $\beta : G^{n-1} \to K^*$ such that $f = \partial\beta \cdot h$, where

$$
\begin{aligned}
&\partial\beta(g_1, \ldots, g_n) \\
&= \beta^{g_1}(g_2, \ldots, g_n)\beta^{(-1)^n}(g_1, \ldots, g_{n-1}) \prod \beta^{(-1)^i}(g_1, \ldots, g_i g_{i+1}, \ldots, g_n),
\end{aligned}
$$

then we say f is cohomologous to h and write $f \sim h$. Let $M^n(G, K)$ be the monoid of weak n-cocycles modulo the equivalence relation $\sim$. Then $M^n(G, K)$ is called the weak Galois cohomology monoid. The class of cocycles equivalent (cohomologous) to the identity are known as *weak coboundaries*.

Since the groups $M_e^i(G, K)$ are components of the required monoid, we are interested in computing these groups. Stimets ([S1],[S2]) has shown that under a sharp condition (exactness of $(\mathcal{M}, d)$) we have the following isomorphism $M_e^i(G, K) \simeq Ext_{R_e}^i(\mathbb{Z}, K^*)$ for all i, where the groups $Ext_{R_e}^i(\mathbb{Z}, K^*)$ are relatively easier to deal with because they are well known and R_e acts on K^* in the obvious way. (See [S1],[S2] for details). In this paper, we investigate exactness in some special cases. Exactness is always guaranteed at M_0 ([S1]). It is quite difficult to check exactness in general but using the "contraction process" makes the situation somewhat easier. If $g_1 < \cdots < g_n$, call $[g_1, \cdots, g_n]$ (n)–cell in M_n.

Proposition 1.1 (Contraction process) (Stimets). *Let $C_n \in M_n$, $C_{n-1} \in M_{n-1}$ be two cells such that C_n does not appear in the boundary of M_{n+1} and in the boundary of an n–cell not equal to C_n, the cell C_{n-1} does not appear. But if C_{n-1} appears in the boundary of C_n, then the chain complex $(\mathcal{M}', d)$ obtained by removing C_n and C_{n-1} is exact at M_n' if and only if $(\mathcal{M}, d)$ is exact at M_n. Moreover, $(\mathcal{M}, d)$ and $(\mathcal{M}', d)$ have the same homology groups.*

This procedure allows us to cancel cells gradually until we reach a point after which we cannot proceed any further. Then we can investigate exactness at fewer modules.

2. Contracting to a Simpler Graph

In some cases, we can simplify the given graph to a smaller one. This section is devoted to demonstrating a class of graphs that can be contracted to a specific form of Z_4 and showing that such graphs possess exact chain complex.

Lemma 2.1. *In the ring $R = \mathbb{Z}\{x, y\}/(x^2 = y^2)$, if $\alpha(x - 1) = \beta(y - 1)$, then $\alpha = h(x + 1)$, $\beta = h(y + 1)$ for some $h \in R$.*

Proof. We claim that the set $V = \{y^\varepsilon xyxy \ldots xyx^i \mid \varepsilon \in \{0, 1\}, i \geq 0\}$ forms a $\mathbb{Z}$-basis for R. Clearly V generates R. We show the independence. Let M be a free module on $T = \{Y^\varepsilon XY \cdots YX^i \mid \varepsilon \in \{0, 1\}, i \geq 0\}$. The module M can be viewed as a right R-module by defining the action:

$$(Y^\varepsilon XY_1 X \cdots XY_j X^i) \cdot x = Y^\varepsilon XY_1 X \cdots XY_j X^{i+1}$$

$$(Y^\varepsilon XY_1 X \cdots XY_j X^i) \cdot y = \begin{cases} Y^\varepsilon XY_1 X \cdots XY_{j+1} X^{i-1} & \text{if } i \text{ is odd} \\ Y^\varepsilon XY_1 X \cdots XY_{j-1} X^{i+3} & \text{if } i \text{ is even.} \end{cases}$$

This action is well-defined and notice that

$$
\begin{aligned}
(Y^\varepsilon XY_1 X \cdots XY_j X^i) \cdot y^2 &= (Y^\varepsilon XY_1 X \cdots XY_j X^i y) \cdot y \\
&= \begin{cases} Y^\varepsilon XY_1 X \cdots XY_{j+1} X^{i-1} \cdot y & \text{if } i \text{ is odd} \\ Y^\varepsilon XY_1 X \cdots XY_{j-1} X^{i+3} \cdot y & \text{if } i \text{ is even} \end{cases} \\
&= \begin{cases} (Y^\varepsilon XY_1 X \cdots XY_j XY_{j+1} \cdot y)X^{i-1} \\ Y^\varepsilon XY_1 X \cdots XY_j X^{i+2} \end{cases} \\
&= Y^\varepsilon XY_1 X \cdots XY_j X^{i+2} \\
&= (Y^\varepsilon XY_1 X \cdots XY_j X^i) \cdot x^2.
\end{aligned}
$$

So we can define a homomorphism $\varphi : R \to \mathrm{End}_\mathbb{Z}(M)$ by

$$
\begin{aligned}
\varphi(x) &= \text{right multiplication by } x. \\
\varphi(y) &= \text{right multiplication by } y.
\end{aligned}
$$

Now, for $z_i \in V$, if $\sum a_i z_i = 0$, then $\varphi(\sum a_i z_i)(1) = 0 = \sum a_i Z_i \implies a_i = 0$ for all i, and we showed the claim. Now, we define a total order on the basis elements as follows: To each $y^\varepsilon xy_1 x \cdots xy_j x^i \in V$, assign a degree $(\varepsilon, \underbrace{2, 2, \ldots, 2}_{j}, \underbrace{1, 1, \ldots, 1}_{i})$ and let $\deg m = (\frac{1}{2})$ for all $m \in \mathbb{Z}^*$, $\deg 0 = 0$.

If $u, v \in V$, we define $u \leq v$ if the length of the corresponding tuple of v is greater than the length of the corresponding tuple of u. If they have the same length, then compare the first numbers on the right, if they are the same go to the next numbers and so on. Notice that

$$\left(\varepsilon, \underbrace{2, 2, \ldots, 2}_{j}, \underbrace{1, 1, \ldots, 1}_{i}\right) \cdot x = \left(\varepsilon, \underbrace{2, 2, \ldots, 2}_{j}, \underbrace{1, 1, \ldots, 1}_{i+1}\right)$$

and

$$\left(\varepsilon, \underbrace{2, 2, \ldots, 2}_{j}, \underbrace{1, 1, \ldots, 1}_{i}\right) \cdot y = \begin{cases} \left(\varepsilon, \underbrace{2, 2, \ldots, 2}_{j-1}, \underbrace{1, 1, \ldots, 1}_{i+3}\right) & i \text{ is even} \\[3ex] \left(\varepsilon, \underbrace{2, 2, \ldots, 2}_{j+1}, \underbrace{1, 1, \ldots, 1}_{i-1}\right) & i \text{ is odd.} \end{cases}$$

This shows that multiplying an element of V by x or y increases the degree. Since any monomial in V ends from the right with either y or x^i, $i > 0$, we can write any element in V as a sum $f + gy$ where f and g are sums of monomials of V ending with x^i, $i > 0$. We say f is the component of 1 and g is the component of y. Remember that in this form any term in the sum g must end with x^1. Let $\alpha = f_1 + g_1 y$, $\beta = h_1 + k_1 y$ where $f_1, g_1 h_1$ and k_1 are sums of elements in V which end with x^i, $i > 0$. Let f, g, h and k be of the largest degrees in f_1, g_1, h_1 and k_1 respectively. The equation $\alpha(x - 1) = \beta(y - 1)$ gives that

$$fx + gyx - gy + \text{lower terms in both components}$$
$$= kx^2 - h + hy - ky + \text{lower terms in both components. (2.1)}$$

Let us denote by (g) for the degree of g, $(g, 2, 1)$ for the degree of gyx and $(k, 1, 1)$ for the degree of kx^2. From (2.1), if we assume that $g \neq 0$, and h ends with x^1, we get

Case 1. *If $(h) < (k)$ then $(g) = (k)$ and $(g, 2, 1) \leq (k, 1, 1)$ (impossible).*

Case 2. *If $(h) \geq (k)$ then $(g) = (h)$ and $(g, 2, 1) \leq \max\{(h), (k, 1, 1,)\}$ (impossible).*

Now assume $g \neq 0$ and h ends with x^i, $i > 1$. This implies that $g = k$ and $fx = hy$. Let $\sum_{\ell=0}^{t} h_\ell$ be the component of 1 in β where h_ℓ ends with x^i, $i > 1$, for all ℓ and $h_t = h$. Equation(2.1) implies that there exists h_j for some j such that $gyx = h_j y$ or $gyx = h_j$. The latter is clearly impossible since h_j ends with x^i, $i > 1$. The first is also impossible since

$$h_j y = h'_j x^i y = \begin{cases} h'_j y x^i & i \text{ is even} \\ h'_j x y x^{i-1} & i \text{ is odd} \geq 3, \end{cases}$$

and in both cases $h_j y$ ends with x^r, $r > 1$. So, $gyx \neq h_j y$ for all j.

Thus g must be 0 and either $k = 0$ or $k = h$. In the first case, $\sum_{\ell=0}^{t} h_\ell = \beta$ where h_ℓ ends with x^i, $i > 1$ for all ℓ. Equation (2.1) implies that $fx = hy$. But hy always ends with x^{2i}. so f ends with x^{2i-1} and $f = uyx^{2i-1}$ for some u. So $h = \begin{cases} ux^{2i} \\ u'yx^{2i+1} \end{cases}$.
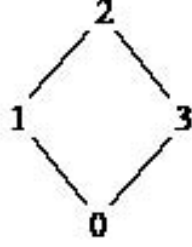
FIGURE 1

Cancel fx with hy in (2.1) we get the term of the highest degree in the left hand side is f, but $f = h_\ell y$ for any ℓ. Therefore $f = h$ which is impossible. This forces $h = k$.

Hence the original equation takes the form $f_1(x-1) = h'(1+y)(y-1)$ where h' is the sum of either components. This implies that $f_1(x-1) = h'(y^2-1) = h'(x^2-1) = h'(x+1)(x-1) \implies \alpha = h'(x+1)$ and $\beta = h'(y+1)$. $\square$

Theorem 2.2. *The chain complex of the idempotent weak 2-cocycle e whose graph is given by:*

over $\mathbb{Z}_4$ is exact.

Proof. We have $M_0 = \mathbb{Z}\{x,y\}/(x^2 = y^2) = R$, $M_1 = R[1] \oplus R[2] \oplus R[3]$. $M_2 = R[1,2] \oplus R[3,2]$, where $x = x_1$, $y = x_3$. We need to check the exactness of the following chain complex:

$$0 \longrightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \longrightarrow 0.$$

As stated earlier, we always have exactness at M_0. Now $d_2(\alpha[1,2] + \beta[3,2]) = 0 \implies \alpha(x_1[1]-[2]+[1])+\beta(x_3[3]-[2]+[3]) = 0 \implies \alpha x_1+\alpha = 0$, $-\alpha - \beta = 0$ and $\beta x_3 + \beta = 0 \implies \alpha = \beta = 0$ and d_2 is injective. To check exactness at M_1, let $z = a[1] + b[2] + c[3] \in \ker d_1$, then $z \in \mathrm{im}d_2$ if and only if $z' = z + d_2(b[1,2]) = (a + bx_1 + b)[1] + c[3] \in \mathrm{im}d_2$. Let $-s = a + bx_1 + b$. Now, $z' \in \ker d_1 \implies s(x_1 - 1) = c(x_3 - 1)$. By the lemma above, the unique solution for such an equation is $s = h(x_1 + 1)$ and $c = h(x_3+1)$ for some $h \in R$. So $z' = -h(x_1+1)[1] + h(x_3+1)[3] = d_2(-h[1,2] + h[3,2])$. Hence $z \in \mathrm{im}d_2$, so $\mathrm{im}d_2 = \ker d_1$. $\square$

Remark 2.3. *We joint each weak 2–cocycle with a chain complex and a unique graph, so excising cells in the complex is equivalent to cancelling edges in the corresponding graph, and we are free to talk about one of these two contractions instead of the other.*
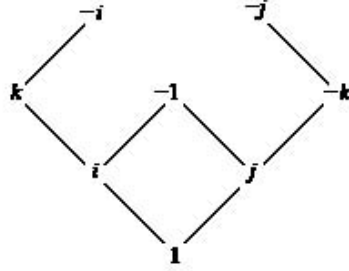
FIGURE 2

FIGURE 3

Corollary 2.4. *Let e be an idempotent weak 2-cocycle and let us denote by Γ_e its lower subtractive graph on a finite group G with a derived ring $\mathbb{Z}\{x, y\}/(x^2 = y^2)$. If Γ_e can be contracted to a graph of the form: with $\sigma^2 = \tau^2 = \nu$, then the chain complex of e is exact.*

Proof. This is a direct consequence of Theorem 2.2 and Proposition 1.1.
$\square$

Example 2.1. *Consider the idempotent e which is given by its graph over the quaternion group $\mathbf{Q} = \{\pm 1, \pm i, \pm j, \pm k\}$:*

It is easy to check that $R_e = \mathbb{Z}\{x, y\}/(x^2 = y^2)$ where $x = x_i$, $y = x_j$.

$d_3[i, k, -i] = x_i[j, -1] - [k, -i] + [i, -i] - [i, k]$

$d_3[j, -k, -j] = x_j[i, -1] - [-k, -j] + [j, -j] - [j, -k]$

The cells $[i, k, -i]$ and $[j, -k, -j]$ can be excised with $[i, k]$ and $[-k, -j]$ respectively. We now only have the relations:

$d_2[i, -1] = x_i[i] - [-1] + [i]$

$d_2[i, -i] = x_i[-1] - [-i] + [i]$

$d_2[j, -k] = x_j[i] - [-k] + [j]$

$d_2[j, -1] = x_j[j] - [-1] + [j]$

$d_2[j, -j] = x_j[-1] - [-j] + [j]$
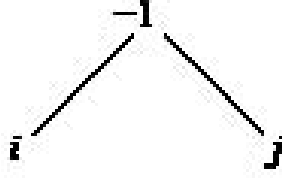
$d_2[k, -i] = x_k[j] - [-i] + [k]$

FIGURE 4

Notice that the cells $[k, -i]$, $[j, -j]$, $[j, -k]$ and $[i, -i]$ can be excised with $[k]$, $[-j]$, $[-k]$ and $[-i]$ respectively. And then we are left with a graph of the form: where $i^2 = j^2 = -1$. So, the chain complex of e is exact.

Corollary 2.5. *In the previous example $M_e^i(\mathbf{Q}, K) \simeq Ext_{R_e}^i(\mathbb{Z}, K^*)$ where $\mathbf{Q} = \{\pm 1, \pm i, \pm j, \pm k\}$ is the Galois group of the extension K/F for some base field F.*

3. S_3 WITH TWO GENERATORS

We will need some lemmas before we state and prove the main result in this work.

Lemma 3.1. *There is no non-trivial solution for the equation $\alpha(x-1) = \beta(y - 1)$ over the ring $R = \mathbb{Z}\{x, y\}/(yx = xy^2)$.*

Proof. First, we show that the set $\{x^i y^j | i, j \geq 0\}$ forms a basis for R over $\mathbb{Z}$. Let M be the free $\mathbb{Z}$–module generated by $\{X^i Y^j | i, j \geq 0\}$. Define the following action to make M a right R–module. $X^i Y^j x = X^{i+1} Y^{2j}$, $X^i Y^j y = X^i Y^{j+1}$ for all i, j. This action is well-defined as we have seen in Lemma 2.1 and we can define a homomorphism $\varphi : R \to End_{\mathbb{Z}}(M)$ by $x \mapsto$ multiplication from the right by x, $y \mapsto$ multiplication from the right by y. So, for any combination $t = \sum a_{ij} x^i y^j = 0$, we have $\varphi(t)(1) = 0$ or $\sum a_{ij} X^i Y^j = 0$ and hence $a_{ij} = 0$ for all i, j. Let $x^s y^t$ be denoted by (s, t).

We define a degree function on the basis elements in R by $(s, t) < (s', t')$ if $s < s'$. If $s = s'$, then $(s, t) < (s', t')$ if $t < t'$. Notice that multiplying by x, y from the right gives $(s, t)x \mapsto (s + 1, 2t)$, $(s, t)y \mapsto (s, t + 1)$. Let $\alpha = (s_1, t_1) + (s_2, t_2) +$ lower degrees, where $(s_1, t_1) > (s_2, t_2)$, $\beta = (s', t') +$ lower degrees. $\alpha(x - 1) = \beta(y - 1)$ gives that $(s_1 + 1, 2t_1) -$

$(s_1, t_1) + (s_2 + 1, 2t_2) +$ lower degrees $= (s', t' + 1) - (s', t') +$ lower degrees $\implies (s_1 + 1, 2t_1) = (s', t' + 1)$, that is, $s' = s_1 + 1$ and $t' = 2t_1 - 1$. We claim that $(s', t') > \max\{(s_1, t_1), (s_2 + 1, 2t_2)\}$. Clearly $s' > s_1$ since $s' = s_1 + 1$. $s_2 \leq s_1 \implies s_2 + 1 \leq s'$, if $s_2 + 1 < s'$, we are done. So, let $s_2 + 1 = s'$, thus, $s_1 = s_2$. This implies $t_1 > t_2 \ldots$ (*). Assume that $t' \leq 2t_2$, so $2t_1 - 1 \leq 2t_2$ or $t_2 \geq t_1$ which contradicts (*). $\qquad\square$

Lemma 3.2. *In the ring $R = \mathbb{Z}\{x, y\}/(xyx = y^2)$, $\alpha(x - 1) = \beta(y - 1)$ implies that $\alpha = k_0(x^{n-1} + x^{n-2} + \cdots + 1)(1 + xy)$ and $\beta = k_0(1 - x^n + (x^{n-1} + x^{n-2} + \cdots + 1)y)$, for any $k_0 \in R$ and $n \in \mathbb{N}$.*

Proof. Using an idea similar to what is in Lemma 3.1, we can show that $S = \{y^\varepsilon x^{i_1} y x^{i_2} y \cdots y x^{i_\ell}, \varepsilon \in \{0, 1\}, i_j, \ell \in \mathbb{Z}^+ \text{ for } j \neq \ell, i_\ell \in \{0, 1, \ldots\}\}$ forms a $\mathbb{Z}$-basis for R. Assign to each element $y^\varepsilon x^{i_1} y x^{i_2} y \ldots y x^{i_\ell}$ a degree $(\varepsilon, i_1, i_2, \ldots, i_\ell)$ if $i_\ell > 0$ and $(\varepsilon, i_1, i_2, \ldots, i_{\ell-1})$ for $y^\varepsilon x^{i_2} y \ldots y x^{i_\ell - 1} y$. If f ends from the right with x^i, $i > 0$, denote by (f) for degree of f, $(f + k)$ for fx^k, $(f, 1)$ for fyx. Notice also that we can express any element in R as $f_0 + g_0 y$ where f_0 and g_0 are sums of monomials from S ending with x^i, $i > 0$. Call f_0 the component of 1 and g_0 the component of y.

Let f, g, h and k be of the largest degree in α, β in both components. So

$$(f + gy + \cdots)(x - 1) = (h + ky + \cdots)(y - 1) \qquad (3.1)$$

$\implies fx + gyx - gy +$ lower terms $= kxyx - h + hy - ky +$ lower terms $\implies$

(i) $(g) = (h - k)$
(ii) $\max((f + 1), (g, 1)) = \max((k + 1, 1), (h))$.

We discuss the following cases:

Case 1. *If $g = 0$ then $h = k$ and $(f + 1) = (k + 1, 1)$ (impossible).*

Case 2. *labelcs2.6 If $g \neq 0$ and $(h) < (k)$ then $(g) = (k)$ and from (ii) $(f + 1) = (k + 1, 1)$ (impossible).*

Case 3. *If $g \neq 0$ and $(h) > (k)$ then $(g, 1) = (k + 1, 1) \implies g = mkx$ and $h = rkx$ for some $m, r \in \mathbb{Z}^*$. Substituting in the equation (3.1) yields $m = 1 = -r$ and so, $g = kx = -h$. Let g', k', h' be of the next largest degree $(g') < (g)$, $(k') < (k)$, $(h') < (h)$, then we get*

$$fx + kxyx + g'yx - kxy - g'y + \text{lower terms}$$
$$= -kxy + kx + h'y + kxyx - ky + k'xyx + \text{lower terms} \implies$$

(i)′ $(g') = (h' - k)$
(ii)′ $\max((f + 1), (g', 1)) = \max((k + 1), (k' + 1, 1))$.

Again if $g' = 0$ then since $h' = k = tfx$, $t \in \mathbb{Z}^*$ and equation (3.1) gives $fx+$ lower terms $= tfx^2+$ lower terms $\Longrightarrow f = 0$ and hence $h' = k = f = g' = g = h = 0$.

If $g' \neq 0$, then $(h') \leq (k) \Longrightarrow (g') = (k)$ and hence $(g') = (k' + 1) \Longrightarrow g' = sk = tk'x$, $s, t \in \mathbb{Z}^*$. As above we find $s = t = 1$. Repeat this step to get $g_i = k_i x$, $k_{i+1} = k_i x$ where $(k_{i+1}) > (k_i)$ and equation (3.1) becomes

$$(F+k_0(x^n+x^{n-1}+\cdots+x)y)(x-1) = (-k_0x^n+H'+k_0(x^{n-1}+\cdots+1)y)(y-1) \tag{3.2}$$

where F is the sum of components of 1 in α, H' the sum of components of 1 in β that have degrees less than h. Let $A := (x^{n-1} + \cdots + 1)$, so equation (3.2) reads

$$(F + k_0 Axy)(x - 1) = (-k_0x^n + H' + k_0 Ay)(y - 1) \Longrightarrow$$
$$Fx - F + k_0 Axyx - k_0 Axy$$
$$= H'y - H' + k_0 Axyx - k_0 x^n y - k_0 Ay + k_0 x^n \Longrightarrow$$
$$Fx - F = H'y - H' - k_0 y + k_0 x^n$$
$$(\text{since} - x^n - A = -Ax - 1) \Longrightarrow$$
$$k_0 = H' \Longrightarrow$$
$$F(x - 1) = k_0(x^n - 1) \Longrightarrow$$
$$F(x - 1) = k_0(x^{n-1} + \cdots + 1)(x - 1) = k_0 A(x - 1) \Longrightarrow$$
$$F = k_0 A.$$

Therefore

$$\alpha(x - 1) = (k_0 A + k_0 Axy)(x - 1) = k_0 A(1 + xy)(x - 1) \tag{3.3}$$

and

$$\begin{aligned}
\beta(y - 1) &= (-k_0x^n + k_0 + k_0 Ay)(y - 1) = k_0(-x^n + 1 + Ay)(y - 1) \\
&= k_0(-x^n y + x^n + y - 1 + Axyx - Ay) \\
&= k_0(-Axy - y + x^n + y - 1 + Axyx) \\
&\quad (-x^n y - Ay = -Axy - y) \\
&= k_0((x^n - 1) + Axy(x - 1)) \\
&= k_0(A(x - 1) + Axy(x - 1)) \\
&= k_0 A(1 + xy)(x - 1) = \alpha(x - 1).
\end{aligned}$$

□

Lemma 3.3. *In the ring $R = \mathbb{Z}\{x, y\}/(xyx = yxy)$, the solutions of $\alpha(x - 1) = \beta(y - 1)$ are $\alpha = h(xy - y + 1)$ and $\beta = h(yx - x + 1)$ for some $h \in R$.*

Proof. Since $xyx = yxy$ in R, any element in R can be written in a unique way as a combination of elements of the form $x^{i_1}y^{j_1}x^{i_2}y^{j_2}\ldots x^{i_\ell}y^{j_\ell}$ where $i_1, j_\ell \geq 0$, $i_2, \ldots, i_{\ell-1} \geq 2$, $j_1, \ldots j_{\ell-1} \geq 1$, $i_\ell \geq 2$ if $j_\ell > 0$ and $i_\ell \geq 1$ if $j_\ell = 0$. The form yxy is always replaced by xyx. This set of monomials forms a $\mathbb{Z}$-basis for R by applying the same trick in Lemma 3.1. Assign to each such element $x^{i_1}y^{j_1}x^{i_2}y^{j_2}\ldots x^{i_\ell}y^{j_\ell}$ a degree $(i_1, j_2, i_2, j_2, \ldots, i_\ell, j_\ell)$ and notice that

$$(i_1, j_1, i_2, j_2, \ldots, i_\ell, j_\ell)x = \begin{cases} (i_1, j_1, \ldots, i_\ell + 1, 0) & \text{if } j_\ell = 0 \\ (i_1, j_1, \ldots, i_\ell, j_\ell, 1, 0) & \text{if } j_\ell \neq 0 \end{cases}$$

$$(n_1, m_1, \ldots, m_{t-1}, n_t, m_t)y =$$

$$= \begin{cases} (n_1, m_1, \ldots, n_t, m_t + 1) & \text{if } m_t \neq 0 \\ (n_1, m_1, \ldots, m_{t-1}, n_t, 1) & \text{if } m_t = 0 \text{ and } n_t > 1 \\ (n_1, m_1, \ldots, n_{t-1} + 1, 1, m_{t-1}, 0) & \text{if } m_t = 0 \text{ and } n_t = 1 \end{cases}$$

Obviously, these two terms can not be equal except if $m_t = 0$ and $n_t = 1$. If $j_\ell = 0$ then the equality implies $(h'x^{i_\ell-1}yx^{i_\ell})x = (h'x^{i_\ell-1-1}y^{i_\ell+1}x)y$ for some $h' \in R$. If $j_\ell \neq 0$ then the equality gives $j_\ell = m_{t-1} = 1$ and $i_\ell = n_{t-1} + 1$. So, $(h''x^{i_1}y)x = (h''x^{i_\ell-1}yx)y$ for some $h'' \in R$. But in the first case we note that $h'x^{i_\ell-1}yx^{i_\ell} = h'x^{i_\ell-1-1}y^{i_\ell}xy$, so in all cases if $(i_1, j_1, i_2, j_2, \ldots, i_\ell, j_\ell)x = (n_1, m_1, \ldots, m_{t-1}, n_t, m_t)y$ then there is $h \in R$ such that $hxy = (i_1, j_1, i_2, j_2, \ldots, i_\ell, j_\ell)$, $hyx = (n_1, m_1, \ldots, m_{t-1}, n_t, m_t)$. Now, let $\alpha = \sum_{i=1}^{r} \alpha_i$, $\beta = \sum_{i=1}^{s} \beta_i$, so by a suitable rearrangement, $\alpha(x - 1) = \beta(y - 1)$ implies that $\alpha_1 x = \beta_1 y \Longrightarrow \alpha_1 = h_1 xy$ and $\beta_1 = h_1 yx$ and hence $h_1 xyx - h_1 xy + \alpha_2 x - \alpha_2 + \alpha_3 x - \alpha_3 + \cdots = h_1 yxy - h_1 yx + \beta_2 y - \beta_2 + \beta_3 y - \beta_3 + \ldots \Longrightarrow \alpha_2 x = -h_1 yx$ or $\alpha_2 = -h_1 y$. Similarly, we find $\beta_2 = -h_1 x$, $\alpha_3 = h_1$, $\beta_3 = h_1$, $\alpha_4 = h_2 xy$, $\beta_4 = h_2 yx, \ldots \Longrightarrow \alpha = \sum_{i=1}^{q} h_i(xy - y + 1)$ and $\beta = \sum_{i=1}^{q} h_i(yx - x + 1)$ for some integer q. Take $h = \sum_{i=1}^{q} h_i$. $\square$

Definition 3.1. *Let e be a weak 2–cocycle and let Γ_e be its graph. Then, we call the elements of G of level 1 generators of Γ_e. That is the elements lie right above the root of Γ_e are called generators.*

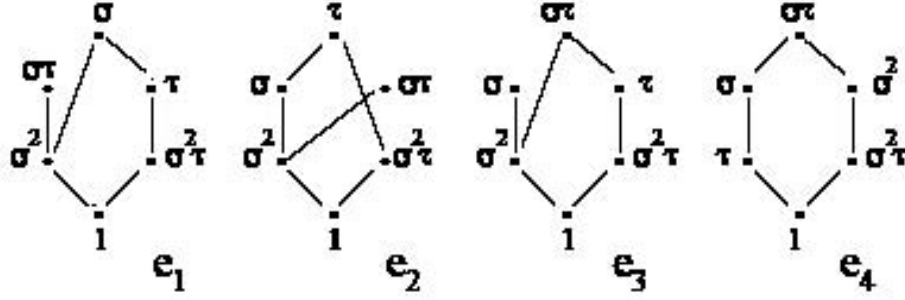Note that generators of the graph certainly generate the group itself.

FIGURE 5

Theorem 3.4. *Let e be an idempotent 2-cocycle whose graph has two generators over $S_3 = \{\sigma, \tau | \sigma^2\tau = \tau\sigma, \sigma^3 = \tau^2 = 1\}$. Then e admits an exact chain complex.*

Proof. Using a computer program, it can be shown that there are nine distinct lower subtractive graphs with two generators over S_3. Five of them are trees which admit exact graded modules (see [S2]). The remaining four are:

By excising cells, we end up with $M_j^i = 0$ for all $j > 2$, $i = 1, 2, 3, 4$. Let $0 \to M_2^i \xrightarrow{d_2^i} M_1^i \xrightarrow{d_1^i} M_0^i \to \mathbb{Z} \to 0$ be the chain complex of e_i, $i = 1, 2, 3, 4$, which we get after excision. We always have exactness at M_0^i. At M_2^i, it is easy to show that d_2^i is injective for all i. To show exactness at M_1^3, let $z = a[\sigma^2\tau] + b[\sigma^2] + c[\sigma] + d[\sigma\tau] \in \ker d_1^3$, so $z \in \mathrm{im}d_2^3$ if and only if $z' = z + d_2^3(c[\sigma^2, \sigma] + d[\sigma^2, \sigma\tau]) = \alpha[\sigma^2\tau] + \beta[\sigma^2] \in \mathrm{im}d_2^3$, for some $\alpha, \beta \in R_3$. But $z \in \ker d_1^3 \implies z' \in \ker d_1^3 \implies \alpha(x-1) = \beta(y-1) \implies$ by Lemma 3.1 $\alpha = \beta = 0$. Thus, $z \in \mathrm{im}d_2^3$ if and only if $0 \in \mathrm{im}d_2^3$ which is always true. A similar idea can be applied to show exactness at M_1^2. For M_1^4, let $u = p[\sigma^2\tau] + q[\sigma^2] + r[\tau] + s[\sigma\tau] \in \ker d_1^4$, so $u \in \mathrm{im}d_2^4$ if and only if $u' = \gamma[\sigma^2\tau] - \delta[\tau] \in \mathrm{im}d_2^4$ for some $\gamma, \delta \in R_{e_4}$. Since $u' \in \ker d_1^4 \implies$ by Lemma 3.3 $\gamma = h(xy - y + 1)$, $\delta = h(yx - x + 1)$ for some $h \in R_{e_4} \implies$ $u \in \mathrm{im}d_2^4$ if and only if $u' = h(xy - y + 1)[\sigma^2\tau] - h(yx - x + 1)[\tau] \in \mathrm{im}d_2^4$, but clearly $d_2^4\{h(1 - y)[\sigma^2\tau, \sigma^2] + h[\sigma^2, \sigma\tau] - h[\tau, \sigma\tau]\} = u'$. For M_1^1, let $z = a[\sigma^2\tau] + b[\sigma^2] + c[\sigma] + d[\sigma\tau] \in \ker d_1^1 \Rightarrow z \in \mathrm{im}d_2^1$ if and only if $z' = z + d_2^1(c[\sigma^2, \sigma] + d[\sigma^2, \sigma\tau]) = \alpha(\sigma^2\tau) - \beta[\sigma^2] \in \mathrm{im}d_2^1$ for some $\alpha, \beta \in R_{e_1}$. Since $z \in \ker d_1^1 \Rightarrow z' \in \ker d_1^1$ and $\alpha(x - 1) = \beta(x - 1)$ in the ring $\mathbb{Z}\{x, y\}/(xyx = y^2) = R_{e_1}$. This implies that $\alpha = k_0(x^{n-1} + \cdots + 1)(1 + xy)$, $\beta = k_0(1 - x^n + (x^{n-1} + \cdots + 1)y)$ (by Lemma 3.2). So $z \in \mathrm{im}d_2^1$

if and only if $z' = k_0 A(1+xy)[\sigma^2\tau] - k_0(1-x^n+Ay)[\sigma^2] \in \mathrm{im}d_2^1$. Notice that $1-x^n = A-Ax$. So $z' \in \mathrm{im}d_2^1$ if and only if $z'' = k_0 A((1+xy)[\sigma^2\tau] - (1-x+y)[\sigma^2]) \in \mathrm{im}d_2^1$. But $z'' = d_2^1(x[\sigma^2, \sigma\tau] + [\sigma^2\tau, \sigma] - [\sigma^2, \sigma])$.   □

Corollary 3.5. *For any idempotent weak 2-cocycle e over S_3 with two generators, $M_e^i(S_3, K) \approx \mathrm{Ext}_{R_e}^i(\mathbb{Z}, K^*)$.*

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