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DEGREE ONE COHOMOLOGY FOR THE LIE ALGEBRAS OF DERIVATIONS

(submitted by M. Arslanov)

ABSTRACT. Let R be a commutative ring and W a Lie algebra of its derivations which is an R-submodule in the full derivation algebra Der R. We consider a class of W-modules generalizing the natural representations of the Lie algebras of vector fields in tensor fields of arbitrary type. The main result consists in the determination of the cohomology of those modules in degree 1. Its applications include a description of derivations and the universal central extension for the Lie algebra W.

INTRODUCTION

This paper is concerned with the Lie algebras of derivations of a commutative ring R which are closed under the natural R-module structure on all the derivations. The basic motivating examples are the Lie algebras of all vector fields of respective smoothness class on C^∞ manifolds, real analytic ones or Stein spaces. Lie algebras of this type appear also in the classification of simple Lie algebras of finite dimension over fields with nonzero characteristic [15, 25]. Whereas the Lie algebras in the classes just mentioned were studied often separately, a natural generality of results describing their properties can be obtained in the settings of an arbitrary commutative ring R. This point of view in regard to the Lie algebra isomorphisms and the structure of ideals was emphasized in Grabowski's paper [7]. The same questions received a further treatment in [9, 13, 22].

Another group of problems centers around the representation theory. When dealing with representations it is reasonable to make certain assumptions on the ring R and an R-module Lie algebra of derivations

$W \subset \operatorname{Der} R$ which express algebraically the idea of absence of singularities (although Lie algebras associated with singular analytic spaces or algebraic varieties were a subject of specific interest in some recent work [e.g. 2, 10, 21]). Under these assumptions there is a class of W-modules which are glued, in a sense, from a family of finite dimensional irreducible modules of any possible type over the general linear Lie algebras $\mathfrak{gl}_n(R/\mathfrak{m})$, where the parameter $\mathfrak{m}$ runs through the maximal ideals of R. They generalize the natural representations of the Lie algebras of vector fields by Lie derivatives in the sections of vector bundles associated with the representations of the general linear group GL_n.

Earlier I gave a description of submodules and intertwining operators for those modules [24]. The purpose of the present paper is to determine the Chevalley-Eilenberg cohomology groups in degree 1. By theorem 5.5 these groups vanish except for modules of several exceptional types. Theorem 5.6 gives the solution in exceptional cases.

Two traditional applications of cohomology are derivations and central extensions. We show that, under our assumptions, the derivation algebra $\operatorname{Der} W$ is isomorphic with the normalizer of W in $\operatorname{Der} R$. This generalizes the description of derivations obtained by Takens [26] for the Lie algebra of vector fields on a smooth manifold, by Grabowski [8] in the real analytic and Stein cases, and by Jacobson [12] and Ree [18] for the Lie algebras of Witt type in positive characteristic.

All central extensions of W are trivial, according to theorem 7.1, when the rank of W as a projective R-module is greater than 1. If the rank equals 1 then the kernel of the universal central extension is canonically isomorphic with the cohomology group $H^1(\Omega)$ of the de Rham complex relative to W. This generalizes the classical construction of the Virasoro algebra as the universal central extension of the Lie algebra of C^∞ vector fields on a circle. In this case Ω is the ordinary de Rham complex, and $\dim_{\mathbb{R}} H^1(\Omega) = 1$. Another interesting example considered by Wagemann recently [27] is the Lie algebra $\operatorname{Vect}_{1,0} \Sigma$ of complexified C^∞ vector fields of type $(1,0)$ on a Riemann surface Σ. The relative de Rham complex is here the complex $0 \to \Omega^{0,0} \to \Omega^{1,0} \to 0$ of C^∞ differential forms of type $(*,0)$ with respect to the ∂-differential. As it comes by taking the global sections from a fine resolution of the sheaf $\overline{\mathcal{O}}$ of antiholomorphic functions on Σ, its cohomology in degree 1 is isomorphic with $H^1(\Sigma, \overline{\mathcal{O}})$. Therefore the kernel of the universal central extension has dimension equal to the genus of Σ. A nontrivial central extension is known also for the Zassenhaus algebras in positive characteristic [5].

By now the most comprehensive results on cohomology of the Lie algebras we consider have been achieved in the cases of formal and C^∞ vector fields (see [6]). The cohomology with coefficients in tensor fields of certain types was computed in all dimensions. However, the technique of glueing used here is not available in other situations. It should be mentioned that the Lie algebras and modules in those results were understood as topological objects, and the cohomology computed was that of continuous cochains. On the contrary, we deal with arbitrary linear cocycles. In fact, we don't even need a ground field and work over the ring of integers. Nevertheless, we show that every 1-cocycle in our settings is a differential operator of order at most 3. Therefore the continuous and algebraic cohomologies for the Lie algebras of vector fields coincide in degree 1. Another feature of our approach is that there is no big difference between local and global aspects. All constructions are done without the resort to glueing of cocycles defined locally.

1. The category of representations

At the beginning we fix notations and recall the definition of a certain representation category from [24, section 6]. Let R be a commutative, associative and unital ring, $W \subset \operatorname{Der} R$ a Lie $\mathbb{Z}$-algebra of derivations such that $RW = W$, that is, W is an R-submodule in $\operatorname{Der} R$. Put $\Omega^1 = \operatorname{Hom}_R(W, R)$ and define $df \in \Omega^1$ for each $f \in R$ by the rule $df(D) = Df$, $D \in W$. The assumptions throughout the whole paper are as follows:

(1.1) 2 is invertible in R,

(1.2) W is R-projective of constant finite rank $n > 0$,

(1.3) $\Omega^1 = R \cdot dR$.

We furnish the R-module $\mathfrak{g} = \Omega^1 \otimes_R W$ with an R-bilinear Lie product setting

$$[\theta \otimes D, \, \theta' \otimes D'] = \langle \theta', D \rangle \, \theta \otimes D' - \langle \theta, D' \rangle \, \theta' \otimes D$$

for elements $\theta, \theta' \in \Omega^1$ and $D, D' \in W$, where $\langle \cdot, \cdot \rangle$ stands for the natural pairing $\Omega^1 \times W \to R$. There is an isomorphism σ_W of $\mathfrak{g}$ onto the Lie algebra $\mathfrak{gl}_R W$ of all R-linear transformations of W defined by the rule

$$\sigma_W(\theta \otimes D)(D') = -\langle \theta, D' \rangle D.$$

Definition 1.1. Denote by $\mathcal{C}_1$ the category whose objects are additive groups M together with a system of operators f_M, $\rho_M(D)$, $\sigma_M(T)$ defined for each $f \in R$, $D \in W$, $T \in \mathfrak{g}$ so that the following properties are satisfied:

(1.4) the operators f_M give M the structure of an R-module,

(1.5)   the operators $\rho_M(D)$ give M the structure of a W-module,

(1.6) $[\rho_M(D),\, f_M] = (Df)_M,$

(1.7) $[\sigma_M(T),\, f_M] = 0,$

(1.8) $\rho_M(fD) = f_M \circ \rho_M(D) + \sigma_M(df \otimes D),$

(1.9) $\sigma_M(fT) = f_M \circ \sigma_M(T).$

The morphisms in $\mathcal{C}_1$ are the maps that commute with the actions of R and W. Denote by $\mathcal{C}_0$ the full subcategory of $\mathcal{C}_1$ consisting of objects M with $\sigma_M = 0$.

The category $\mathcal{C}_1$ is closed under several operations. If M, N are its objects then the R-modules $M \otimes_R N$ and $\mathrm{Hom}_R(M, N)$ are in a natural way objects of $\mathcal{C}_1$ too. We agree to write $M \otimes N$ suppressing the subscript in the tensor product. The corresponding operators are given by

$$
\begin{aligned}
\rho_{M \otimes N}(D)(u \otimes v) &= \rho_M(D)u \otimes v + u \otimes \rho_N(D)v, \\
\sigma_{M \otimes N}(T)(u \otimes v) &= \sigma_M(T)u \otimes v + u \otimes \sigma_N(T)v, \\
\rho_{\mathrm{Hom}_R(M,N)}(D)\xi &= \rho_M(D) \circ \xi - \xi \circ \rho_N(D) \\
\sigma_{\mathrm{Hom}_R(M,N)}(T)\xi &= \sigma_M(T) \circ \xi - \xi \circ \sigma_N(T),
\end{aligned}
$$

where $u \in M$, $v \in N$, $\xi \in \mathrm{Hom}_R(M, N)$. These are well defined in view of the compatibility conditions (1.6), (1.7). In particular, the r-fold tensor power $\bigotimes^r M$ of the underlying R-module of M is an object of $\mathcal{C}_1$. The same is valid for the symmetric power $S^r M$ and the exterior power $\bigwedge^r M$ of the R-module M as these are factors of $\bigotimes^r M$ by subgroups stable under all operators involved. Both W and $\mathfrak{g}$ operate in the tensor, symmetric and exterior algebras of the R-module M via derivations.

We regard R as an object of $\mathcal{C}_0$ letting ρ_R be the natural action of W on R and $\sigma_R = 0$. Similarly, W together with the adjoint representation ρ_W and σ_W defined earlier is an object of $\mathcal{C}_1$. Hence Ω^1 and $\mathfrak{g}$ are objects of $\mathcal{C}_1$ too. Most of the natural R-linear maps that we will happen to deal with are in fact morphisms in $\mathcal{C}_1$. For instance, so is $\sigma_M : \mathfrak{g} \to \mathrm{End}_R M$ for any $M \in \mathcal{C}_1$. That σ_M is a W-equivariant map is asserted in Lemma 1.1 below. Another example is the contraction $\gamma : \Omega^1 \otimes \mathfrak{g} \to R$ defined by the rule $\gamma(\theta \otimes D) = \langle \theta, D \rangle$ for $\theta \in \Omega^1$ and $D \in W$.

Note that (1.2) enables one to define the trace function $\mathrm{tr} : \mathrm{End}_R W \to R$ as follows. The R-module $\bigwedge^n W$ is projective of rank 1. Therefore $\mathrm{End}_R \bigwedge^n W \cong R$. There is a natural representation of the Lie algebra $\mathfrak{gl}_R W$ in $\bigwedge^n W$ via R-linear transformations. Each element $T_W \in \mathfrak{gl}_R W$ acts as a multiplication by a certain element of R called the trace of T_W. Now $\gamma(T) = -\mathrm{tr}\, \sigma_W(T)$ for $T \in \mathfrak{g}$. Similarly, $\gamma(T)$ is the trace of the R-linear endomorphism $\sigma_{\Omega^1}(T)$.

It may be helpful to think of a particular example in which R is the ring of C^∞ functions and W the Lie algebra of C^∞ vector fields on a smooth manifold. In this case Ω^1 is the module of linear differential forms. Tensor fields of any possible type constitute an object of the category $\mathcal{C}_1$ according to the constructions above. The representation ρ is given by Lie derivatives, whereas σ involves certain contractions of tensors. This generalizes to real analytic manifolds and Stein spaces.

Lemma 1.1. *Let M be an object of $\mathcal{C}_1$. Then the operators $\sigma_M(T)$ with $T \in \mathfrak{g}$ define a representation of $\mathfrak{g}$ and $[\rho_M(D), \sigma_M(T)] = \sigma_M(\rho_\mathfrak{g}(D)T)$ for all $D \in W$, $T \in \mathfrak{g}$. Every morphism in $\mathcal{C}_1$ is a $\mathfrak{g}$-module homomorphism.*

Proof. In view of (1.3) every element of $\mathfrak{g}$ is a sum of certain $dg \otimes D'$ with $g \in R$ and $D' \in W$, so it suffices to consider only such elements. If $T' = dg \otimes D'$ then $\sigma_M(T') = \rho_M(gD') - g_M \circ \rho_M(D')$, and

$$\begin{aligned}
[\rho_M(D), &\sigma_M(T')] \\
&= \rho_M([D,\, gD']) - g_M \circ \rho_M([D, D']) - (Dg)_M \circ \rho_M(D') \\
&\qquad\qquad\qquad\qquad\qquad\qquad = \sigma_M(\rho_\mathfrak{g}(D)T')
\end{aligned}$$

since $\rho_\mathfrak{g}(D)T' = dg \otimes [D, D'] + d(Dg) \otimes D'$. Now for $T = df \otimes D$ with $f \in R$ and $D \in W$ we express $\sigma_M(T)$ from (1.8) and get

$$[\sigma_M(T), \sigma_M(T')] = \sigma_M(\rho_\mathfrak{g}(fD)T' - f \cdot \rho_\mathfrak{g}(D)T') = \sigma_M(\sigma_\mathfrak{g}(T)T').$$

One checks that

$$\begin{aligned}
\sigma_\mathfrak{g}(T)T' &= \sigma_{\Omega^1}(T)(dg) \otimes T' + dg \otimes \sigma_W(T)D' \\
&= \langle dg,\, D \rangle\, df \otimes D' - dg \otimes \langle df,\, D' \rangle\, D = [T, T'].
\end{aligned}$$

The final conclusion of the Lemma follows again from (1.8). $\square$

If $M \in \mathcal{C}_1$ and $\mathfrak{m}$ is a maximal ideal of R then $M/\mathfrak{m}M$ is a module for the Lie algebra $\mathfrak{g}/\mathfrak{m}\mathfrak{g}$ over the field $R/\mathfrak{m}$. We may identify $\mathfrak{g}/\mathfrak{m}\mathfrak{g}$ with the Lie algebra $\mathfrak{gl}(W/\mathfrak{m}W)$ of all linear transformations of the vector space $W/\mathfrak{m}W$. In a sense M can be regarded as being glued from a family of representations of general linear Lie algebras parametrized by the maximal ideals of R. This is indeed a correct point of view provided M is R-projective. Of particular importance are the following assumptions on an object $Q \in \mathcal{C}_1$:

(1.10) Q is a finitely generated projective R-module,

(1.11)   for each maximal ideal $\mathfrak{m}$ of R the quotient $Q/\mathfrak{m}Q$ is an absolutely irreducible module for the Lie algebra $\mathfrak{g}/\mathfrak{m}\mathfrak{g}$.

Lemma 1.2. *Suppose that $Q \in \mathcal{C}_1$ satisfies (1.10), (1.11). Then the associative R-algebra $\operatorname{End}_R Q$ is generated by the endomorphisms $\sigma_Q(T)$ with $T \in \mathfrak{g}$.*

Proof. Put $A = \operatorname{End}_R Q$, and let B be its subalgebra (containing the identity endomorphism) generated by all $\sigma_Q(T)$. For each maximal ideal $\mathfrak{m}$ of R the image of B in $A/\mathfrak{m}A \cong \operatorname{End}_{R/\mathfrak{m}}(Q/\mathfrak{m}Q)$ is the associative subalgebra generated by all endomorphisms of an absolutely irreducible representation in $Q/\mathfrak{m}Q$. It is therefore the whole $A/\mathfrak{m}A$, i.e., $B + \mathfrak{m}A = A$. Since Q is a finitely generated projective R-module, A is finitely generated over R too. The global version of Nakayama's Lemma [1, II, §3, Proposition 11] yields $B = A$. $\square$

It turns out that for application to the central extensions in section 7 more general objects of $\mathcal{C}_1$ have to be dealt with. Hence we are led to the following

Definition 1.2. Suppose that $Q \in \mathcal{C}_1$ is an object satisfying (1.10), (1.11). We say that an object $M \in \mathcal{C}_1$ is *of type Q* if there is a homomorphism of associative R-algebras $\operatorname{End}_R Q \to \operatorname{End}_R M$ which takes $\sigma_Q(T)$ to $\sigma_M(T)$ for each $T \in \mathfrak{g}$.

If M is of type Q, then so is every subobject $M' \subset M$ as well. In fact, the image of $\operatorname{End}_R Q$ in $\operatorname{End}_R M$ is the subalgebra, say A_M, generated by the endomorphisms $\sigma_M(T)$ with $T \in \mathfrak{g}$. Each $\sigma_{M'}(T)$ is the restriction of $\sigma_M(T)$. Therefore M' is stable under A_M, and the resulting homomorphism of R-algebras $A_M \to \operatorname{End}_R M'$ takes $\sigma_M(T)$ to $\sigma_{M'}(T)$. Similarly, along with M, every its factor object is of type Q. A particular example of an object satisfying (1.10), (1.11) is R itself. An object $M \in \mathcal{C}_1$ is of type R if and only if $\sigma_M(T) = 0$ for all $T \in \mathfrak{g}$, that is, $M \in \mathcal{C}_0$.

Lemma 1.3. *Suppose that Q is an object of $\mathcal{C}_1$ satisfying (1.10), (1.11). Then the functor $M_0 \mapsto M_0 \otimes Q$ is an equivalence between $\mathcal{C}_0$ and the full subcategory of $\mathcal{C}_1$ consisting of objects of type Q.*

Proof. Put $A = \operatorname{End}_R Q$. Note that Q is a projective generator in the category of R-modules. By Morita theory the functor $M_0 \mapsto M_0 \otimes Q$ is an equivalence between the categories of R-modules and A-modules with the inverse equivalence $M \mapsto \operatorname{Hom}_A(Q, M)$. We will check that it induces an equivalence between the categories in question.

Suppose that $M \in \mathcal{C}_1$ is of type Q. Then the homomorphism $A \to \operatorname{End}_R M$ afforded by the definition 1.2 makes M into an A-module. Let $M_0 = \operatorname{Hom}_A(Q, M)$. Then $M_0 \subset H$ where $H = \operatorname{Hom}_R(Q, M)$ is an object of $\mathcal{C}_1$. In view of Lemma 1.2 $\eta \in H$ is in M_0 if and only if η commutes with the action of $\mathfrak{g}$. Hence M_0 is the kernel of the morphism

$\varphi : H \to \mathrm{Hom}_R(\mathfrak{g}, H)$ in $\mathcal{C}_1$ defined by the rule

$$\varphi(\eta)(T) = \sigma_H(T)\eta = \sigma_M(T) \circ \eta - \eta \circ \sigma_Q(T)$$

for $\eta \in H$, $T \in \mathfrak{g}$. Thus $M_0 \in \mathcal{C}_1$ and, since the induced action of $\mathfrak{g}$ in M_0 is trivial, in fact $M_0 \in \mathcal{C}_0$. The canonical map $M_0 \otimes Q \to M$ is a morphism in $\mathcal{C}_1$. It is bijective by Morita theory.

Conversely, suppose that $M_0 \in \mathcal{C}_0$. Then $M = M_0 \otimes Q$ is an object of $\mathcal{C}_1$ and $\sigma_M(T) = \mathrm{id} \otimes \sigma_Q(T)$ for all $T \in \mathfrak{g}$. The assignment $\xi \mapsto \mathrm{id} \otimes \xi$ defines a homomorphism of R-algebras $A \to \mathrm{End}_R M$ which takes $\sigma_Q(T)$ to $\sigma_M(T)$. Thus M is of type Q. The canonical map $M_0 \to \mathrm{Hom}_A(Q, M)$ is a bijective morphism in $\mathcal{C}_0$. $\square$

Lemma 1.4. *Let Q, Q' be two objects of $\mathcal{C}_1$ satisfying* (1.10), (1.11). *Suppose that $M, M' \in \mathcal{C}_1$ are objects of type Q and Q' respectively, so that $M \cong M_0 \otimes Q$ and $M' \cong M'_0 \otimes Q'$ for some $M_0, M'_0 \in \mathcal{C}_0$. If the $\mathfrak{g}/\mathfrak{m}\mathfrak{g}$-modules $Q/\mathfrak{m}Q$ and $Q'/\mathfrak{m}Q'$ are not isomorphic for every maximal ideal $\mathfrak{m}$ of R then $\mathrm{Mor}_{\mathcal{C}_1}(M, M') = 0$. If $Q = Q'$ then $\mathrm{Mor}_{\mathcal{C}_1}(M, M') \cong \mathrm{Mor}_{\mathcal{C}_0}(M_0, M'_0)$.*

Proof. Suppose there is a nonzero morphism $M \to M'$ in $\mathcal{C}_1$. Its image N is a factor object of M and a subobject of M' in $\mathcal{C}_1$. It is therefore of type Q and Q' simultaneously. Put $A = \mathrm{End}_R Q$, $A' = \mathrm{End}_R Q'$, and let B be the subalgebra of the associative R-algebra $\mathrm{End}_R N$ generated by all endomorphisms $\sigma_N(T)$ with $T \in \mathfrak{g}$. There is a homomorphism of R-algebras $A \to B$ which takes $\sigma_Q(T)$ to $\sigma_N(T)$ for each $T \in \mathfrak{g}$. Clearly it is surjective. Since A is finitely generated over R, so is B as well. Furthermore, $B \neq 0$ because $N \neq 0$. By Nakayama's Lemma there exists a maximal ideal $\mathfrak{m}$ of R such that $B \neq \mathfrak{m}B$. Fix such an ideal.

The factor algebra $B/\mathfrak{m}B$ is a homomorphic image of a simple associative algebra $A/\mathfrak{m}A \cong \mathrm{End}_{R/\mathfrak{m}}(Q/\mathfrak{m}Q)$. It follows that $B/\mathfrak{m}B \cong A/\mathfrak{m}A$. By symmetry we have $B/\mathfrak{m}B \cong A'/\mathfrak{m}A'$ as well. Up to isomorphism, $Q/\mathfrak{m}Q$ is a unique simple module for $A/\mathfrak{m}A$. Similarly, $A'/\mathfrak{m}A'$ has a unique simple module $Q'/\mathfrak{m}Q'$. If we let $A/\mathfrak{m}A$ operate in $Q'/\mathfrak{m}Q'$ via the algebra isomorphism $\varphi : A/\mathfrak{m}A \xrightarrow{\sim} A'/\mathfrak{m}A'$ constructed above, there has to be an isomorphism of $A/\mathfrak{m}A$-modules $\iota : Q/\mathfrak{m}Q \xrightarrow{\sim} Q'/\mathfrak{m}Q'$. Denote by $\sigma_{\mathfrak{m}} : \mathfrak{g}/\mathfrak{m}\mathfrak{g} \to A/\mathfrak{m}A$ the reduction modulo $\mathfrak{m}$ of the map $\sigma_Q : \mathfrak{g} \to A$ and by $\sigma'_{\mathfrak{m}} : \mathfrak{g}/\mathfrak{m}\mathfrak{g} \to A'/\mathfrak{m}A'$ the reduction of $\sigma_{Q'} : \mathfrak{g} \to A'$. Then $\sigma'_{\mathfrak{m}} = \varphi \circ \sigma_{\mathfrak{m}}$ by the construction. Hence ι is an isomorphism of $\mathfrak{g}/\mathfrak{m}\mathfrak{g}$-modules. This proves the first statement of the Lemma. The second one is a general fact that a category equivalence is bijective on morphisms. $\square$

Next we are going to introduce certain operators on objects $M \in \mathcal{C}_1$. Whereas by the definition these operators are R-linear endomorphisms of M, Lemma 1.5 shows that they can be expressed in terms of the endomorphisms $\rho_M(D)$, $D \in W$, solely. It is worth keeping in mind that the χ_M defined below is an R-multilinear function of its arguments. For $\theta, \theta' \in \Omega^1$ and $D, D' \in W$ put

$$\chi_M(\theta, \theta', D, D') = \big(\sigma_M(\theta \otimes D) - \langle \theta, D \rangle_M\big) \sigma_M(\theta' \otimes D')$$
$$+ \big(\sigma_M(\theta' \otimes D) - \langle \theta', D \rangle_M\big) \sigma_M(\theta \otimes D') .$$

Lemma 1.5. *Let $M \in \mathcal{C}_1$. Then for all $f, g \in R$ and $D, D' \in W$ we have*

$$- \rho_M(fgD) \rho_M(D') + \rho_M(fD) \rho_M(gD')$$
$$+ \rho_M(gD) \rho_M(fD') - \rho_M(D) \rho_M(fgD') = \chi_M(df, dg, D, D') .$$

This is checked straightforwardly using relations (1.6)–(1.9) [24, Lemma 6.1].

We need a modification of the category $\mathcal{C}_1$ in which the role of Ω^1 is transferred to the R-module of Kähler differentials which we denote as $\widetilde{\Omega}^1$. Recall that it is defined together with a derivation $d : R \to \widetilde{\Omega}^1$, universal in the class of derivations with values in R-modules (a $\mathbb{Z}$-linear map $\Delta : R \to M$ with M an arbitrary R-module is *a derivation* if $\Delta(fg) = f \Delta g + g \Delta f$ for all $f, g \in R$). The universality property gives a unique R-linear map $\widetilde{\Omega}^1 \to \Omega^1$ rendering commutative the diagram

$$\begin{array}{ccc} & R & \\ {\scriptstyle d} \swarrow & & \searrow {\scriptstyle d} \\ \widetilde{\Omega}^1 & \longrightarrow & \Omega^1 \end{array}$$

This map is surjective in view of (1.3). It induces an R-bilinear pairing $\widetilde{\Omega}^1 \times W \to R$. The same formula as in the case of $\mathfrak{g}$ defines now a Lie multiplication on $\widetilde{\mathfrak{g}} = \widetilde{\Omega}^1 \otimes W$, the tensor product being over R.

Definition 1.3. Denote by $\widetilde{\mathcal{C}}_1$ the category whose objects are additive groups M together with a system of operators $f_M, \rho_M(D), \sigma_M(T)$ defined for each $f \in R$, $D \in W$, $T \in \widetilde{\mathfrak{g}}$ subject to the conditions (1.4)–(1.9). The morphisms in $\widetilde{\mathcal{C}}_1$ are the maps that commute with the actions of R and W.

The epimorphism of R-modules $\widetilde{\Omega}^1 \to \Omega^1$ induces a surjective homomorphism of Lie algebras $\widetilde{\mathfrak{g}} \to \mathfrak{g}$ over R. The category $\mathcal{C}_1$ can be identified therefore with the full subcategory in $\widetilde{\mathcal{C}}_1$ whose objects M satisfy $\sigma_M(T) = 0$ for all T in the kernel of $\widetilde{\mathfrak{g}} \to \mathfrak{g}$. All natural operations in $\mathcal{C}_1$ have their analogues in $\widetilde{\mathcal{C}}_1$. If $M \in \widetilde{\mathcal{C}}_1$ then its R-linear transformations $\chi_M(\theta, \theta', D, D')$ make sense for elements $\theta, \theta' \in \widetilde{\Omega}^1$ and $D, D' \in W$. Lemmas 1.1 and 1.5 generalize to $\widetilde{\mathcal{C}}_1$ as they are formal consequences of (1.4)–(1.9).

Lemma 1.6. *The R-module $\widetilde{\Omega}^1$ can be in a unique way furnished with additional operators which make it an object of $\widetilde{\mathcal{C}}_1$ and the universal derivation $d : R \to \widetilde{\Omega}^1$ a W-equivariant map. Furthermore,*

$$\sigma_{\widetilde{\Omega}^1}(\theta \otimes D)\,\theta' = \langle \theta', D \rangle\, \theta \qquad \text{for } \theta, \theta' \in \widetilde{\Omega}^1 \text{ and } D \in W.$$

The canonical map $\widetilde{\Omega}^1 \to \Omega^1$ is an epimorphism in $\widetilde{\mathcal{C}}_1$ and its kernel J an object of the subcategory $\mathcal{C}_0$.

Proof. Put $M = \widetilde{\Omega}^1$. Given $D \in W$, make the direct sum of additive groups $E = M \oplus M$ into an R-module setting

$$f \cdot (\theta', \theta) = \big(f\theta' + (Df)\theta,\ f\theta\big) \qquad \text{for } f \in R \text{ and } \theta, \theta' \in M.$$

The projection $\pi : E \to M$ onto the second summand is a homomorphism of R-modules. One checks that the assignment $f \mapsto (dDf,\, df)$ defines a derivation $R \to E$. By the universality property of Kähler differentials there is an R-linear map $\varphi : M \to E$ sending df to $(dDf,\, df)$. The composite $\pi\varphi$ has to be the identity endomorphism of M. Therefore $\varphi(\theta) = \big(\rho(D)\theta,\, \theta\big)$ for some operator $\rho(D)$ on M. One has $\rho(D)(df) = dDf$ and

$$\rho(D)(f\theta) = f \cdot \rho(D)\theta + (Df)\,\theta$$

for all $f \in R$, $D \in W$, $\theta \in M$. These two identities determine $\rho(D)$ uniquely as $M = R \cdot dR$. It follows $\rho([D, D']) = [\rho(D), \rho(D')]$ for $D, D' \in W$, i.e., ρ is a Lie algebra representation. Next, $\rho(fD) - f_M \circ \rho(D)$ is an R-linear transformation of M. Indeed, this is a consequence of the identity (1.6) which we have verified above. One computes

$$\big(\rho(fD) - f_M \circ \rho(D)\big)(dg)$$
$$= d(f \cdot Dg) - f \cdot d(Dg) = (Dg) \cdot df = \langle dg, D \rangle\, df = \sigma(df \otimes D)(dg)$$

where $f, g \in R$, $D \in W$ and $\sigma = \sigma_M$ is defined in the statement of the Lemma. Clearly σ satisfies (1.7) and (1.9). Since $M = R \cdot dR$, (1.8) holds too. The map $\widetilde{\Omega}^1 \to \Omega^1$ is a W-equivariant homomorphism of R-modules, i.e., a morphism in the category $\widetilde{\mathcal{C}}_1$. If $\theta' \in J$ then $\langle \theta', D \rangle = 0$ for all

$D \in W$. It follows from the definition of σ that $\sigma(T)\theta' = 0$ for all $T \in \widetilde{\mathfrak{g}}$. Thus $J \in \mathcal{C}_0$. $\square$

2. THE DIFFERENTIAL ORDER OF A COCYCLE

A 1-*cocycle* $\varphi : W \to M$ with coefficients in a W-module M is a $\mathbb{Z}$-linear map satisfying

$$\varphi([D, D']) = \rho_M(D)\varphi(D') - \rho_M(D')\varphi(D)$$

for all $D, D' \in W$. In this section we will prove that, when $M \in \mathcal{C}_1$, every 1-cocycle is a differential operator of order at most 3. In general, given two R-modules M and N, a $\mathbb{Z}$-linear map $\xi : N \to M$ and an element $f \in R$, define

$$\delta_f \xi = \xi \circ f_N - f_M \circ \xi$$

where f_M and f_N are multiplication operators on M and N. We call ξ a *differential operator of order $\leq r$* if $\delta_{f_1} \cdots \delta_{f_{r+1}} \xi = 0$ for all $f_1, \ldots, f_{r+1} \in R$. Denote by $\mathrm{Diff}_r(N, M)$ the group of all differential operators of order $\leq r$. For $f, g \in R$ one has

$$\delta_f \delta_g \xi = \delta_g \delta_f \xi \qquad \text{and} \qquad \delta_{fg} \xi = (\delta_f \xi) \circ g_N + f_M \circ (\delta_g \xi).$$

It follows that the map $R \times \cdots \times R$ (r times) $\to \mathrm{Hom}_R(N, M)$ given by the rule $(f_1, \ldots, f_r) \mapsto \delta_{f_1} \cdots \delta_{f_r} \xi$ is symmetric and is a derivation in each of its arguments whenever $\xi \in \mathrm{Diff}_r(N, M)$. By the universality property of Kähler differentials it induces a symmetric R-multilinear map $\widetilde{\Omega}^1 \times \cdots \times \widetilde{\Omega}^1$ (r times) $\to \mathrm{Hom}_R(N, M)$, hence also an R-linear map from the r-th symmetric power $S^r \widetilde{\Omega}^1$ to $\mathrm{Hom}_R(N, M)$. There is therefore an R-linear map

$$\xi^\flat : S^r \widetilde{\Omega}^1 \otimes N \to M$$

such that

$$\xi^\flat(df_1 \cdots df_r \otimes u) = (\delta_{f_1} \cdots \delta_{f_r} \xi)(u)$$

for $f_1, \ldots, f_r \in R$ and $u \in N$. It is called *the r-th order symbol of ξ*. The notation $\xi^\flat$ that we use is somewhat ambiguous as every differential operator of order $\leq r$ is also a differential operator of order $\leq r + 1$. If P is a third R-module and $\eta : P \to N$ a $\mathbb{Z}$-linear map then $\delta_f(\xi \circ \eta) = \delta_f \xi \circ \eta + \xi \circ \delta_f \eta$. It follows by induction that $\xi \circ \eta$ is a differential operator of order $\leq r + s$ when $\xi \in \mathrm{Diff}_r(N, M)$ and $\eta \in \mathrm{Diff}_s(P, N)$. Its symbol can be computed as

$$(\xi \circ \eta)^\flat(\theta_1 \cdots \theta_{r+s} \otimes u) = \sum \xi^\flat\left(\theta_{i_1} \cdots \theta_{i_r} \otimes \eta^\flat(\theta_{i_{r+1}} \cdots \theta_{i_{r+s}} \otimes u)\right)$$

for $\theta_1, \ldots, \theta_{r+s} \in \widetilde{\Omega}^1$ and $u \in P$, where the sum ranges over all permutations of indices $1, \ldots, r + s$ such that $i_1 < \ldots < i_r$ and $i_{r+1} < \ldots < i_{r+s}$.

The exact sequence $0 \to J \to \widetilde{\Omega}^1 \to \Omega^1 \to 0$ in our settings induces a surjective homomorphism of symmetric algebras $S\widetilde{\Omega}^1 \to S\Omega^1$ whose kernel is the ideal of $S\widetilde{\Omega}^1$ generated by J. If the symbol of a differential operator $\xi \in \mathrm{Diff}_r(N, M)$ vanishes on all elements in the image of the canonical map $(J \cdot S^{r-1}\widetilde{\Omega}^1) \otimes N \to S^r\widetilde{\Omega}^1 \otimes N$ then it induces an R-linear map $S^r\Omega^1 \otimes N \to M$ which we still call the symbol of ξ and denote as $\xi^\flat$.

Lemma 2.1. *Let $M \in \mathcal{C}_1$. Then every 1-coboundary $\varphi : W \to M$ is a differential operator of order ≤ 1.*

Proof. There is $m \in M$ such that $\varphi(D) = \rho_M(D)m$ for all $D \in W$. By (1.8) $(\delta_f\varphi)(D) = \sigma_M(df \otimes D)m$ for $f \in R$, $D \in W$, whence $\delta_f\varphi$ is an R-linear map according to (1.9). $\square$

Lemma 2.2. *Suppose that $\varphi : W \to M$ is a 1-cocycle where M is an object of $\mathcal{C}_1$. If φ is a differential operator of order ≤ 2 then its symbol $\varphi^\flat$ induces a morphism $S^2\Omega^1 \otimes W \to M$ in $\mathcal{C}_1$. If φ is a differential operator of order ≤ 3 and either $\mathrm{rk}_R W > 1$ or 3 is invertible in R then $\varphi^\flat$ induces a morphism $S^3\Omega^1 \otimes W \to M$.*

Proof. Assume $\varphi \in \mathrm{Diff}_3(W, M)$. Its symbol $\varphi^\flat : S^3\widetilde{\Omega}^1 \otimes W \to M$ is an R-module homomorphism. We will show that $\varphi^\flat$ is a W-module homomorphism as well, hence a morphism in $\widetilde{\mathcal{C}}_1$. The Lie algebra W operates on the $\mathbb{Z}$-linear maps $\xi : W \to M$ in a natural way, and $D \cdot \delta_f\xi = \delta_{Df}\xi + \delta_f D\xi$ for $f \in R$, $D \in W$. Now $D\varphi$ is the coboundary of $\varphi(D) \in M$, whence $\delta_f\delta_g(D\varphi) = 0$ for all $f, g \in R$ by Lemma 2.1. We deduce

$$D \cdot \delta_{f_1}\delta_{f_2}\delta_{f_3}\varphi = \delta_{Df_1}\delta_{f_2}\delta_{f_3}\varphi + \delta_{f_1}\delta_{Df_2}\delta_{f_3}\varphi + \delta_{f_1}\delta_{f_2}\delta_{Df_3}\varphi$$

for $f_1, f_2, f_3 \in R$ and $D \in W$. Evaluating at $D' \in W$ yields

$$\rho_M(D)\varphi^\flat(\omega \otimes D') - \varphi^\flat(\omega \otimes [D, D']) = \varphi^\flat(D\omega \otimes D')$$

where $\omega = df_1 \cdot df_2 \cdot df_3$. Since $\widetilde{\Omega}^1 = R \cdot dR$, the equality holds actually for all $\omega \in S^3\widetilde{\Omega}^1$ and gives the W-invariance of $\varphi^\flat$.

Next we want to show that $\varphi^\flat$ factors through $S^3\Omega^1 \otimes W$. The kernel $\mathfrak{k}$ of the canonical homomorphism $\widetilde{\mathfrak{g}} \to \mathfrak{g}$ annihilates M and W since both modules are in $\mathcal{C}_1$. As $\varphi^\flat$ is $\widetilde{\mathfrak{g}}$-equivariant by the $\widetilde{\mathcal{C}}_1$ version of Lemma 1.1, it must vanish on the R-submodule $K \subset S^3\widetilde{\Omega}^1 \otimes W$ spanned by the tensors $\omega \otimes D$ with $\omega \in \mathfrak{k} \cdot S^3\widetilde{\Omega}^1$ and $D \in W$. We will show that $\mathfrak{k} \cdot S^3\widetilde{\Omega}^1 = J \cdot S^2\widetilde{\Omega}^1$, which means that K is the kernel of the canonical epimorphism $S^3\widetilde{\Omega}^1 \otimes W \to S^3\Omega^1 \otimes W$, as required. Recall that $\widetilde{\mathfrak{g}}$ operates in the symmetric algebra $S\widetilde{\Omega}^1$ via derivations. Given $D \in W$, let i_D

denote the derivation of $S\widetilde{\Omega}^1$ such that $i_D f = 0$ for $f \in R$ and $i_D \theta' = \langle \theta', D \rangle$ for $\theta' \in \widetilde{\Omega}^1$. The composite θi_D of i_D with the multiplication by $\theta \in \widetilde{\Omega}^1$ is again a derivation. Hence $\theta \otimes D$ acts in the symmetric algebra as θi_D, both derivations having the same values on elements of R and $\widetilde{\Omega}^1$. Now $\mathfrak{k}$ is spanned by the tensors $\theta \otimes D$ with $\theta \in J$, $D \in W$. Therefore $\mathfrak{k} \cdot S^3 \widetilde{\Omega}^1 = JN$, where $N \subset S^2 \widetilde{\Omega}^1$ is the R-submodule spanned by the elements $i_D \omega$ with $D \in W$, $\omega \in S^3 \widetilde{\Omega}^1$. It remains to verify that $N = S^2 \widetilde{\Omega}^1$.

If $P \subset \widetilde{\Omega}^1$ is a finitely generated R-submodule then so is the span $P^2 \subset S^2 \widetilde{\Omega}^1$ of all $\theta \theta'$ with $\theta, \theta' \in P$. By Nakayama's Lemma the inclusion $P^2 \subset N$ holds if and only if $P^2 \subset N + \mathfrak{m}P^2$ for all maximal ideals $\mathfrak{m}$ of R. It suffices to prove these inclusions only for those P that project onto the whole Ω^1 since the latter is finitely generated over R. Fix P and $\mathfrak{m}$. Let $\theta \in P$. If $\mathrm{rk}_R W > 1$ then $\Omega^1 / \mathfrak{m}\Omega^1 \cong (W/\mathfrak{m}W)^*$ has dimension > 1 over $R/\mathfrak{m}$. We can find $D' \in W$ and $\theta' \in P$ such that $\langle \theta, D' \rangle \equiv 0$, $\langle \theta', D' \rangle \equiv 1$ modulo $\mathfrak{m}$. Then $\theta^2 \equiv i_{D'}(\theta^2 \theta')$ modulo $\mathfrak{m}P^2$, so that $\theta^2 \in N + \mathfrak{m}P^2$. In view of (1.1) the squares θ^2 span the whole P^2. Suppose now $\mathrm{rk}_R W = 1$ but $3R = R$. If $\theta \notin \mathfrak{m}P$ we can find $D \in W$ such that $\langle \theta, D \rangle \equiv 1$ modulo $\mathfrak{m}$. Then $3\theta^2 \equiv i_D(\theta^3)$ modulo $\mathfrak{m}P^2$, whence again $\theta^2 \in N + \mathfrak{m}P^2$. If $\theta \in \mathfrak{m}P$, take $\theta' \in P$ such that $\theta' \notin \mathfrak{m}P$. Since the squares of θ', $\theta' + \theta$, $\theta' - \theta$ are all in $N + \mathfrak{m}P^2$, so is θ^2 as well. That completes the proof. The case when φ is a differential operator of order ≤ 2 is treated similarly. $\square$

Lemma 2.3. *The $\mathbb{Z}$-linear span X of all endomorphisms $\chi_W(\theta, \theta', D, D')$ with $\theta, \theta' \in \Omega^1$ and $D, D' \in W$ coincides with $A = \mathrm{End}_R W$.*

Proof. Obviously, X is an R-submodule in A. Since A is finitely generated over R, it suffices to show that $A = X + \mathfrak{m}A$ for every maximal ideal $\mathfrak{m}$ of R. Note that

$$\sigma_W(\theta \otimes D) \circ \sigma_W(\theta' \otimes D') = -\langle \theta, D' \rangle \, \sigma_W(\theta' \otimes D)$$

and therefore $\chi_W(\theta, \theta, D, D) = -4\langle \theta, D \rangle \, \sigma_W(\theta \otimes D)$ where we take $\theta, \theta' \in \Omega^1$ and $D, D' \in W$. If $\langle \theta, D \rangle \notin \mathfrak{m}$ then we can find $f \in R$ such that $-4\langle \theta, D \rangle f \equiv 1$ modulo $\mathfrak{m}$. Multiplying the previous equality by f, we deduce immediately that $\sigma_W(\theta \otimes D) \in X + \mathfrak{m}A$. Suppose that $\langle \theta, D \rangle \in \mathfrak{m}$ but $D \notin \mathfrak{m}W$. Then $\langle \theta', D \rangle \notin \mathfrak{m}$ for some $\theta' \in \Omega^1$. Since $\sigma_W(\theta' \otimes D)$ and $\sigma_W((\theta + \theta') \otimes D)$ are both in $X + \mathfrak{m}A$, so is $\sigma_W(\theta \otimes D)$ as well. Suppose finally that $D \in \mathfrak{m}W$. Pick out $D' \in W$ such that $D' \notin \mathfrak{m}W$. Then $\sigma_W(\theta \otimes D')$ and $\sigma_W(\theta \otimes (D + D'))$ are both in $X + \mathfrak{m}A$, whence so is $\sigma_W(\theta \otimes D)$. We proved that $\sigma_W(\mathfrak{g}) \subset X + \mathfrak{m}A$, which gives the assertion because $A = \sigma_W(\mathfrak{g})$. $\square$

Proposition 2.4. *Let $M \in \mathcal{C}_1$. Then every 1-cocycle $\varphi : W \to M$ is a differential operator of order ≤ 3. If $\mathrm{rk}_R W > 1$ then every 1-cocycle is actually a differential operator of order ≤ 2.*

Proof. If $D \in W$ then $D\varphi$ is a coboundary. By Lemma 2.1

$$D\varphi = \rho_M(D) \circ \varphi - \varphi \circ \rho_W(D) \in \mathrm{Diff}_1(W, M) \,.$$

In view of (1.6) $\rho_M(D) \in \mathrm{Diff}_1(M, M)$ and similarly $\rho_W(D) \in \mathrm{Diff}_1(W, W)$. Hence

$$\rho_M(D)\,\rho_M(D')\,\varphi - \varphi\,\rho_W(D)\,\rho_W(D') = \rho_M(D)\left(\rho_M(D')\,\varphi - \varphi\,\rho_W(D')\right)$$
$$+ \left(\rho_M(D)\,\varphi - \varphi\,\rho_W(D)\right)\rho_W(D')$$

is a differential operator of order ≤ 2 for all $D, D' \in W$. The linear combination given in Lemma 1.5 yields

$$\chi_M(\theta, \theta', D, D') \circ \varphi - \varphi \circ \chi_W(\theta, \theta', D, D') \in \mathrm{Diff}_2(W, M) \qquad (*)$$

for all $\theta, \theta' \in dR \subset \Omega^1$ and $D, D' \in W$. The inclusion holds actually for all $\theta, \theta' \in \Omega^1$ since $\chi_M(f\theta, \theta', D, D') = \chi_M(\theta, \theta', D, fD')$, where $f \in R$, and similarly for χ_W. We get

$$(\delta_f\varphi)\circ\chi_W(\theta, \theta', D, D') = \varphi\circ f_W\circ\chi_W(\theta, \theta', D, D')-f_M\circ\varphi\circ\chi_W(\theta, \theta', D, D')$$
$$= f_M \circ \left(\chi_M(\theta, \theta', D, D') \circ \varphi - \varphi \circ \chi_W(\theta, \theta', D, D')\right)$$
$$- \chi_M(f\theta, \theta', D, D') \circ \varphi + \varphi \circ \chi_W(f\theta, \theta', D, D') \in \mathrm{Diff}_2(W, M) \,.$$

It follows $(\delta_{f_1} \cdots \delta_{f_4}\varphi) \circ \chi = 0$ for all $f_1, \ldots, f_4 \in R$ and all χ in the subgroup $X \subset \mathrm{End}_R W$ described in Lemma 2.3. Since $1_W \in X$, we deduce $\delta_{f_1} \cdots \delta_{f_4}\varphi = 0$, i.e., φ is a differential operator of order ≤ 3.

Suppose $\mathrm{rk}_R W > 1$ further on. Consider the symbol $\varphi^\flat : S^3\Omega^1 \otimes W \to M$. The map φ is a differential operator of order ≤ 2 if and only if $\varphi^\flat = 0$. Put

$$N = \left\{ \omega \in S^3\Omega^1 \mid \omega \otimes D \in \mathrm{Ker}\,\varphi^\flat \text{ for all } D \in W \right\}.$$

Applying $\delta_{f_1}\delta_{f_2}\delta_{f_3}$ to $(*)$, we get

$$\chi_M(\theta, \theta', D, D') \circ \delta_{f_1}\delta_{f_2}\delta_{f_3}\varphi = \delta_{f_1}\delta_{f_2}\delta_{f_3}\varphi \circ \chi_W(\theta, \theta', D, D'), \quad \text{i.e.,}$$
$$\chi_M(\theta, \theta', D, D')\left(\varphi^\flat(\theta_1\theta_2\theta_3 \otimes D'')\right) = \varphi^\flat\left(\theta_1\theta_2\theta_3 \otimes \chi_W(\theta, \theta', D, D')D''\right)$$

for all $\theta, \theta', \theta_1, \theta_2, \theta_3 \in dR$ and $D, D', D'' \in W$. In view of (1.3) this holds actually for all θ's in Ω^1. In other words,

$$\chi_M(\theta, \theta', D, D') \circ \varphi^\flat = \varphi^\flat \circ \left(\mathrm{id} \otimes \chi_W(\theta, \theta', D, D')\right). \qquad (**)$$

It follows that all endomorphisms $\mathrm{id} \otimes \chi_W(\theta, \theta', D, D')$, hence by Lemma 2.3 all endomorphisms in $\mathrm{id} \otimes \mathrm{End}_R W$, leave the kernel of $\varphi^\flat$ stable. The endomorphism $\mathrm{id} \otimes \sigma_W(\theta \otimes D)$ decomposes as

$$S^3\Omega^1 \otimes W \xrightarrow{\ -i_\theta\ } S^3\Omega^1 \xrightarrow{\ j_D\ } S^3\Omega^1 \otimes W$$

where the maps i_θ, j_D are given by

$$i_\theta(\omega \otimes D') = \langle \theta, D' \rangle \omega, \qquad j_D(\omega) = \omega \otimes D, \qquad \omega \in S^3\Omega^1, \ D' \in W.$$

As $\mathrm{Ker}\,\varphi^\flat$ is stable under $j_D \circ i_\theta$ for every $D \in W$ and $\theta \in \Omega^1$ by the above, we see that $i_\theta(\mathrm{Ker}\,\varphi^\flat) \subset N$ for every θ. On the other hand, $\varphi^\flat$ is a morphism in $\mathcal{C}_1$ by Lemma 2.2. In particular, $\varphi^\flat$ commutes with the actions of $\mathfrak{g}$. Therefore

$$\chi_M(\theta, \theta', D, D') \circ \varphi^\flat = \varphi^\flat \circ \chi_{S^3\Omega^1 \otimes W}(\theta, \theta', D, D').$$

Comparing this with $(\ast\ast)$, we see that the endomorphisms

$$\chi_{S^3\Omega^1 \otimes W}(\theta, \theta', D, D') - \mathrm{id} \otimes \chi_W(\theta, \theta', D, D')$$

have images in $\mathrm{Ker}\,\varphi^\flat$.

Now we are ready to prove that $N = S^3\Omega^1$, and so $\varphi^\flat = 0$. By Nakayama's Lemma it suffices to show that $S^3\Omega^1 = N + \mathfrak{m} \cdot S^3\Omega^1$ for all maximal ideals $\mathfrak{m}$ of R. Fix $\mathfrak{m}$. Let $\theta, \theta' \in \Omega^1$. We will check that $\theta^2\theta' \in N + \mathfrak{m} \cdot S^3\Omega^1$, whence our assertion. Since the vector space $W/\mathfrak{m}W$ and its dual $\Omega^1/\mathfrak{m}\Omega^1$ have dimension > 1 over $R/\mathfrak{m}$, we can find $D_1 \in W$, $D_1 \notin \mathfrak{m}W$, such that $\langle \theta', D_1 \rangle \equiv 0$, and then find $D_2 \in W$, $\theta_1, \theta_2 \in \Omega^1$ such that $\langle \theta_i, D_j \rangle \equiv \delta_{ij}$ modulo $\mathfrak{m}$ for $i, j = 1, 2$. Put

$$a = \big(\chi_{S^3\Omega^1 \otimes W}(\theta, \theta, D_1, D_1) - \mathrm{id} \otimes \chi_W(\theta, \theta, D_1, D_1)\big)(\theta_1^2\theta' \otimes D_2) \in S^3\Omega^1 \otimes W.$$

As we noted above, $a \in \mathrm{Ker}\,\varphi^\flat$. Now compute a. Using Lemma 2.5 below, we get $a = \omega_1 \otimes D_1 + \omega_2 \otimes D_2$ where ω_1 turns out to be irrelevant, while

$$\omega_2 = \chi_{S^3\Omega^1}(\theta, \theta, D_1, D_1)(\theta_1^2\theta') \equiv 4\theta^2\theta' \pmod{\mathfrak{m} \cdot S^3\Omega^1}.$$

It follows $i_{\theta_2}(a) \equiv \omega_2 \equiv 4\theta^2\theta'$ modulo $\mathfrak{m} \cdot S^3\Omega^1$. Since $i_{\theta_2}(a) \in N$, we get the conclusion about $\theta^2\theta'$. $\square$

Lemma 2.5. *Suppose that* $\theta \in \Omega^1$, $D \in W$ *and* $A, B \in \mathcal{C}_1$. *Then the endomorphism* $\chi_{A \otimes B}(\theta, \theta, D, D)$ *is equal to*

$$\chi_A(\theta, \theta, D, D) \otimes 1_B + 4\sigma_A(\theta, D) \otimes \sigma_B(\theta, D) + 1_A \otimes \chi_B(\theta, \theta, D, D).$$

Denote by i_D *the derivation of the symmetric algebra* $S\Omega^1$ *such that* $i_D f = 0$ *for* $f \in R$ *and* $i_D \theta' = \langle \theta', D \rangle$ *for* $\theta' \in \Omega^1$. *If* $A = S\Omega^1$ *and* $\omega \in A$ *then*

$$\chi_A(\theta, \theta, D, D)\,\omega = 2\theta^2 \cdot i_D^2\omega.$$

Proof. One checks (i) straightforwardly using the definitions of operators. Under hypotheses of (ii), $\sigma_A(\theta, D)\omega = \theta \cdot i_D\omega$. The conclusion of (ii) follows from the computation

$$\sigma_A(\theta, D)^2\omega = \theta^2 \cdot i_D^2\omega + \langle\theta, D\rangle\theta \cdot i_D\omega = \theta^2 \cdot i_D^2\omega + \langle\theta, D\rangle\,\sigma_A(\theta, D)\omega\,.$$

$\square$

3. First order prolongations

If M is an object of the category $\mathcal{C}_0$ then the standard cochain complex $C^\bullet(W, M)$ of $\mathbb{Z}$-multilinear alternating maps $W \times \cdots \times W \to M$ contains a subcomplex $C_R^\bullet(W, M)$ whose elements are R-multilinear maps. The cohomology $H_R^\bullet(W, M)$ of the latter is one of the ingredients in the cohomology $H^\bullet(W, M)$ of the ambient complex and can not be simplified any further in the general settings. Since W is a finitely generated projective R-module, $C_R^\bullet(W, M) \cong M \otimes \Omega^\bullet$ where $\Omega^\bullet$ is the exterior algebra of the R-module Ω^1. In particular, $C_R^\bullet(W, R) \cong \Omega^\bullet$ generalizes the classical de Rham complex. If now $M \in \mathcal{C}_1$ then a part of $H^\bullet(W, M)$ is related to the cohomology of R-multilinear cochains for a certain extension of W which we describe below.

Consider, more generally, a pair $\widetilde{W}, \pi$ where $\widetilde{W}$ is an additive group endowed with structures of a Lie algebra over $\mathbb{Z}$ and a module over R, and $\pi : \widetilde{W} \to W$ a map which is surjective and is a homomorphism of both structures simultaneously. Assume, moreover, that

(3.1) $[\widetilde{D}, f\widetilde{D}'] = f[\widetilde{D}, \widetilde{D}'] + \big(\pi(\widetilde{D})f\big)\widetilde{D}'$, for all $f \in R$ and $\widetilde{D}, \widetilde{D}' \in \widetilde{W}$.

Such algebraic structures under different names were considered by many people [11, 14, 17, 19, 23]. In particular, the pair W, id_W satisfies these conditions. Suppose M is an additive group endowed with an R-module and a $\widetilde{W}$-module structures. Denote by $\widetilde{\rho}_M(\widetilde{D})$ and f_M the operators on M corresponding to elements $\widetilde{D} \in \widetilde{W}$ and $f \in R$. If the identities

(3.2) $[\widetilde{\rho}_M(\widetilde{D}), f_M] = \big(\pi(\widetilde{D})f\big)_M\,,$

(3.3) $\widetilde{\rho}_M(f\widetilde{D}) = f_M \circ \widetilde{\rho}_M(\widetilde{D})$

are fulfilled for M then the R-multilinear alternating maps $\widetilde{W} \times \cdots \times \widetilde{W} \to M$ form a cochain complex with respect to the standard differential (see the references above). Denote by $H_R^\bullet(\widetilde{W}, M)$ its cohomology.

The kernel $\mathfrak{k}$ of π is an ideal of $\widetilde{W}$ on which the Lie multiplication is R-bilinear in view of (3.1). Similarly, (3.2), (3.3) show that the induced action $\mathfrak{k} \times M \to M$ is an R-bilinear operation. Denote by $H_R^\bullet(\mathfrak{k}, M)$ the cohomology of the standard complex of R-multilinear alternating

maps $\mathfrak{k} \times \cdots \times \mathfrak{k} \to M$. The Lie algebra $\widetilde{W}$ operates naturally in that cohomology group. As the action of $\mathfrak{k}$ is trivial, there is the induced action of the factor algebra $W \cong \widetilde{W}/\mathfrak{k}$. In particular, the $\mathfrak{k}$-invariants in M form an R-submodule $M^{\mathfrak{k}}$ which has the induced structure of a W-module. As (3.2), (3.3) carry over to the induced representation of W in $M^{\mathfrak{k}}$, the R-linear cohomology $H_R^{\bullet}(W, M^{\mathfrak{k}})$ is defined too.

There is an analogue of the Hochschild-Serre spectral sequence relating the cohomology groups just described (see [16] for the construction in the settings of Lie algebroids). We will need only an exact sequence associated with the initial terms of this spectral sequence.

Lemma 3.1. *There is an exact sequence*

$$0 \to H_R^1(W, M^{\mathfrak{k}}) \to H_R^1(\widetilde{W}, M) \to H_R^1(\mathfrak{k}, M)^W \to H_R^2(W, M^{\mathfrak{k}}).$$

Proof. The map on the left is obtained by taking the composites of the R-linear 1-cocycles $W \to M^{\mathfrak{k}}$ with π. It is injective since a 1-coboundary $\widetilde{W} \to M$ factors through W only if it is the differential of a 0-cochain lying in $M^{\mathfrak{k}}$. Restricting the R-linear 1-cocycles $\widetilde{W} \to M$ to $\mathfrak{k}$ gives the next map. Its images are W-invariant classes in $H_R^1(\mathfrak{k}, M)$ since $\widetilde{W}$ acts in $H_R^1(\widetilde{W}, M)$ trivially. If a cocycle $\widetilde{W} \to M$ vanishes on $\mathfrak{k}$ then it has values in $M^{\mathfrak{k}}$ and factors through W, which shows the exactness at the second term.

Suppose now we are given an R-linear 1-cocycle $\varphi' : \mathfrak{k} \to M$ whose cohomology class is W-invariant. Then $\eta(\widetilde{D}) = \widetilde{D}\varphi'$ is a coboundary for every $\widetilde{D} \in \widetilde{W}$. We get thus a map $\eta : \widetilde{W} \to B_R^1(\mathfrak{k}, M)$ with values in the group of coboundaries $\mathfrak{k} \to M$. If $f \in R$, $\widetilde{D} \in \widetilde{W}$, $T \in \mathfrak{k}$ then

$$\eta(f\widetilde{D})(T) = \widetilde{\rho}_M(f\widetilde{D})\varphi'(T) - \varphi'\big([f\widetilde{D}, T]\big)$$
$$= f \cdot \rho_M(\widetilde{D})\varphi'(T) - \varphi'\big(f[\widetilde{D}, T]\big) = f \cdot \eta(\widetilde{D})(T).$$

Hence η is R-linear. Obviously $\eta(T)$ is the coboundary of $\varphi'(T)$ for every $T \in \mathfrak{k}$. We can extend φ' to an R-linear map $\widetilde{\varphi} : \widetilde{W} \to M$ with the property that $\eta(\widetilde{D})$ is the coboundary of $\widetilde{\varphi}(\widetilde{D})$ for every $\widetilde{D} \in \widetilde{W}$. In fact, the exact sequence

$$0 \to \mathfrak{k} \to \widetilde{W} \to W \to 0$$

splits as a sequence of R-modules in view of (1.2). In other words, $\widetilde{W} = \mathfrak{k} \oplus \mathfrak{c}$ with $\mathfrak{c}$ an R-submodule. Since $\mathfrak{c}$ is R-projective and the differential $M \to B_R^1(\mathfrak{k}, M)$ is an epimorphism of R-modules, the restriction $\mathfrak{c} \to B_R^1(\mathfrak{k}, M)$ of η can be lifted to an R-linear map $\varphi'' : \mathfrak{c} \to M$. Taking $\widetilde{\varphi}$ to be φ' on $\mathfrak{k}$ and φ'' on $\mathfrak{c}$ fulfills our requirement. Let now $\psi : \widetilde{W} \times \widetilde{W} \to M$

be the coboundary of $\widetilde{\varphi}$. Then ψ vanishes when one of its arguments is in $\mathfrak{k}$. It induces therefore a 2-cocycle $\overline{\psi} : W \times W \to M$ which takes values in $M^{\mathfrak{k}}$. The last map in the statement of the Lemma takes the cohomology class of φ' to that of $\overline{\psi}$. One can check that it is well defined and gives the exactness in the same way as for ordinary Lie algebras. $\square$

Now take $\widetilde{W} = \mathfrak{g} \times W$ to be the direct product of underlying additive groups. Note that W operates on $\mathfrak{g}$ by means of $\rho_{\mathfrak{g}}$ as a Lie algebra of derivations. Furnish $\widetilde{W}$ with the semidirect product of Lie algebra structures and a certain R-module structure. Explicit formulas are

$$\left[(T, D), (T', D')\right] = \left([T, T'] + \rho_{\mathfrak{g}}(D)T' - \rho_{\mathfrak{g}}(D')T, [D, D']\right),$$
$$f \cdot (T, D) = (fT - df \otimes D, fD),$$

where $f \in R$, $T, T' \in \mathfrak{g}$ and $D, D' \in W$. The projection π onto the second factor is clearly a homomorphism of both structures. The same definitions are in effect when $\mathfrak{g}$ is replaced by $\widetilde{\mathfrak{g}}$. The kernel of π is $\mathfrak{g}$ in the former case and $\widetilde{\mathfrak{g}}$ in the latter. Identity (3.1) is a bit cumbersome, but we propose a more sophisticated argument in just a moment.

Definition 3.1. We call $\mathfrak{g} \times W$ (respectively $\widetilde{\mathfrak{g}} \times W$) with the Lie algebra and R-module structures just described the first order prolongation of W with kernel $\mathfrak{g}$ (respectively $\widetilde{\mathfrak{g}}$).

Lemma 3.2. *Suppose that $\widetilde{W}$ is the first order prolongation of W with kernel $\mathfrak{g}$ or $\widetilde{\mathfrak{g}}$. Then $\mathcal{C}_1$ (respectively $\widetilde{\mathcal{C}}_1$) is isomorphic to the category whose objects are additive groups M together with an R-module and a $\widetilde{W}$-module structures satisfying identities (3.2), (3.3) and morphisms are the maps compatible with both structures.*

Proof. If we fix an R-module structure on an additive group M then the rule

$$\widetilde{\rho}_M(T, D) = \sigma_M(T) + \rho_M(D), \qquad T \in \mathfrak{g}, \quad D \in W,$$

establishes a one-to-one correspondence between the families of operators $\widetilde{\rho}_M(\widetilde{D})$ on M defined for each $\widetilde{D} \in \widetilde{W}$ and the families of operators $\rho_M(D)$, $\sigma_M(T)$ defined for each $D \in W$ and $T \in \mathfrak{g}$. Property (3.2) translates to the pair of (1.6) and (1.7), property (3.3) to (1.8) and (1.9). If $\widetilde{\rho}_M$ is a Lie algebra representation then so is its restriction ρ_M to W. Conversely, if $M \in \widetilde{\mathcal{C}}_1$ then it is immediate from Lemma 1.1 that $\widetilde{\rho}_M$ is a Lie algebra representation. The same Lemma shows also that the morphisms in $\widetilde{\mathcal{C}}_1$ are precisely the maps that are homomorphisms of R-module and $\widetilde{W}$-module structures. $\square$

Now we can verify (3.1). Suppose that M is an object of $\mathcal{C}_1$ (respectively $\widetilde{\mathcal{C}}_1$) such that the induced representation $\widetilde{\rho}_M$ is faithful. For

instance, we can take $M = W$ in case when the kernel is $\mathfrak{g}$. Since W is a faithful R-module by (1.2), property (3.2) implies that $\ker \widetilde{\rho}_M \subset \ker \pi = \mathfrak{g}$. However $\mathfrak{g}$ acts in W faithfully. In case of kernel $\widetilde{\mathfrak{g}}$ we can take $M = \widetilde{\Omega}^1$. Again (1.2) ensures the faithfulness of σ_M and $\widetilde{\rho}_M$. It remains to observe that, in view of (3.2) and (3.3), the elements at both sides of (3.1) act in M as

$$f_M \circ \left[\widetilde{\rho}_M(\widetilde{D}), \widetilde{\rho}_M(\widetilde{D}') \right] + \left(\pi(\widetilde{D})f \right)_M \circ \widetilde{\rho}_M(\widetilde{D}') .$$

Thus we can apply Lemma 3.1 to the situation where M is an object of $\mathcal{C}_1$ or even $\widetilde{\mathcal{C}}_1$ and $\widetilde{W}$ is the first order prolongation of W with kernel $\widetilde{\mathfrak{g}}$. In this case $\mathfrak{k} = \widetilde{\mathfrak{g}}$. Consider now the canonical embedding $\iota : W \to \widetilde{W}$ such that $\iota(D) = (0, D)$ for $D \in W$. Then ι is a differential operator of order 1. In fact $(\delta_f \iota)(D) = (df \otimes D, 0)$ for $f \in R$ and $D \in W$. Hence the symbol of ι is given by the formula $\iota^\flat(T) = (T, 0)$ for $T \in \widetilde{\mathfrak{g}}$. $\square$

Lemma 3.3. *Suppose that $M \in \mathcal{C}_1$ and $\widetilde{W}$ is the first order prolongation of W with kernel $\widetilde{\mathfrak{g}}$. Given a differential operator $\varphi : W \to M$ of order ≤ 1, there exists a unique R-linear map $\widetilde{\varphi} : \widetilde{W} \to M$ such that $\varphi = \widetilde{\varphi} \circ \iota$. Moreover, if φ is a cocycle then so is $\widetilde{\varphi}$ as well. In this case the symbol $\varphi^\flat : \widetilde{\mathfrak{g}} \to M$ of φ is also an R-linear cocycle. In order that a cocycle φ be R-linear, it is necessary and sufficient that $\widetilde{\varphi}$ vanish on $\widetilde{\mathfrak{g}}$.*
Proof. Every $\mathbb{Z}$-linear map $\widetilde{\varphi} : \widetilde{W} \to M$ satisfying $\varphi = \widetilde{\varphi} \circ \iota$ can be written as

$$\widetilde{\varphi}(T, D) = \varphi'(T) + \varphi(D) \qquad \text{for } T \in \widetilde{\mathfrak{g}} \text{ and } D \in W.$$

where φ' is a $\mathbb{Z}$-linear map $\widetilde{\mathfrak{g}} \to M$. In order that $\widetilde{\varphi}$ be R-linear, it is necessary and sufficient that φ' be R-linear and the equality $\varphi(fD) - f\varphi(D) = \varphi'(df \otimes D)$ hold for all $f \in R$ and $D \in W$. Thus $\varphi' = \varphi^\flat$ is the only choice which gives the desired property. Suppose that φ is a cocycle. Let $N = M \oplus R$ be the direct sum of two R-modules. Define operators

$$\rho_N(D)(m, h) = \left(\rho_M(D)m + h\,\varphi(D),\ Dh \right)$$
$$\sigma_N(T)(m, h) = \left(\sigma_M(T)m + h\,\varphi^\flat(T),\ 0 \right)$$

where $D \in W$, $T \in \widetilde{\mathfrak{g}}$, $m \in M$, $h \in R$. One checks straightforwardly that N is now an object of the category $\widetilde{\mathcal{C}}_1$. By Lemma 3.2 ρ_N extends to a representation $\widetilde{\rho}_N$ of $\widetilde{W}$ satisfying (3.2), (3.3). In fact we have an exact sequence $0 \to M \to N \to R \to 0$ in $\widetilde{\mathcal{C}}_1$ which is also an exact sequence of $\widetilde{W}$-modules. Furthermore,

$$\widetilde{\rho}_N(\widetilde{D})(0, 1) = \sigma_N(T)(0, 1) + \rho_N(D)(0, 1) = \left(\varphi^\flat(T) + \varphi(D),\ 0 \right) = \left(\widetilde{\varphi}(\widetilde{D}),\ 0 \right)$$

for any $\widetilde{D} = (T, D) \in \widetilde{W}$. Applying the operator

$$\widetilde{\rho}_N([\widetilde{D}, \widetilde{D}']) = [\widetilde{\rho}_N(\widetilde{D}), \widetilde{\rho}_N(\widetilde{D}')]$$

to $(0, 1)$, where $\widetilde{D}' \in \widetilde{W}$ is a second element, we get the cocycle condition for $\widetilde{\varphi}$. Since the symbol $\iota^{\flat} : \widetilde{\mathfrak{g}} \to \widetilde{W}$ is the canonical embedding which is a homomorphism of Lie algebras, $\varphi^{\flat} = \widetilde{\varphi} \circ \iota^{\flat}$ is a cocycle as well. Finally, φ is R-linear if and only if $\varphi^{\flat} = 0$, i.e., if $\widetilde{\varphi} \circ \iota^{\flat} = 0$. $\square$

4. CONSTRUCTION OF UNIVERSAL COCYCLES

Among the 1-cocycles $\varphi : W \to M$ of differential order ≤ 2 with values in the objects of $\mathcal{C}_1$ one can look for a one which satisfies the following universality property: for every object $M' \in \mathcal{C}_1$ and a 1-cocycle $\varphi' : W \to M'$ of differential order ≤ 2 there exists a unique morphism $\xi : M \to M'$ in $\mathcal{C}_1$ such that the 1-cocycle $\varphi' - \xi \circ \varphi$ is a differential operator of order ≤ 1. In fact proposition 4.1 gives such a cocycle $\varphi : W \to S^2\Omega^1 \otimes W$ which will be called *the universal differential order* 2 *cocycle*. When $\mathrm{rk}_R W = 1$, proposition 4.5 describes a 1-cocycle $W \to \Omega^1 \otimes \Omega^1$ satisfying a similar universality property with respect to 1-cocycles of differential order ≤ 3. We call it *the universal differential order* 3 *cocycle*.

Definition 4.1. A $\mathbb{Z}$-bilinear map $\nabla : W \times W \to W$ is *a torsion-free connection on W* if

(4.1) $\quad \nabla(fD', D'') - f \nabla(D', D'') = 0$,

(4.2) $\quad \nabla(D', fD'') - f \nabla(D', D'') = (D'f) D''$,

(4.3) $\quad \nabla(D', D'') - \nabla(D'', D') = [D', D'']$

for all $f \in R$ and $D', D'' \in W$. If only the first two identities are fulfilled then ∇ is *a connection on W*. A 1-cocycle $\psi : W \to R$ is *a divergence on W* if it satisfies the identity

(4.4) $\quad \psi(fD) - f \psi(D) = Df$, $\qquad f \in R$, $\quad D \in W$.

Proposition 4.1. *There is a 1-cocycle $\varphi : W \to S^2\Omega^1 \otimes W$ which is a differential operator of order 2 and whose symbol $\varphi^{\flat}$ is the identity endomorphism of $S^2\Omega^1 \otimes W$.*

Proof. The Lie algebra W operates on the $\mathbb{Z}$-bilinear maps $W \times W \to W$ in a natural way. If ∇ is a torsion-free connection on W, put

$$\varphi(D) = D \cdot \nabla, \qquad D \in W.$$

One checks straightforwardly that $D \cdot \nabla$ is a symmetric R-bilinear map for every element D. By the assumption (1.2) we may identify $S^2\Omega^1 \otimes W$ with the group of such maps. Namely, given $\theta_1, \theta_2 \in \Omega^1$ and $D \in W$, the

tensor $\theta_1\theta_2 \otimes D$ determines the map

$$(D', D'') \mapsto \big(\langle\theta_1, D'\rangle\langle\theta_2, D''\rangle + \langle\theta_1, D''\rangle\langle\theta_2, D'\rangle\big)D\,, \qquad D', D'' \in W.$$

Thus φ takes values in $S^2\Omega^1 \otimes W$. It is a cocycle since it comes from a coboundary in a larger module. Let $f, g \in R$. We have

$$
\begin{aligned}
(\delta_f\varphi)&(D)(D', D'') \\
&= [fD,\, \nabla(D', D'')] - f\,[D,\, \nabla(D', D'')] - \nabla([fD,\, D'],\, D'') \\
&\quad + f\,\nabla([D,\, D'],\, D'') - \nabla(D',\, [fD,\, D'']) + f\,\nabla(D',\, [D,\, D'']) \\
&= -\big(\nabla(D', D'')f\big)\,D + (D'f)\,\nabla(D, D'') - (D'f)\,[D,\, D''] \\
&\qquad\qquad\qquad + (D''f)\,\nabla(D', D) + (D'D''f)\,D\,,
\end{aligned}
$$

and

$$(\delta_g\delta_f\varphi)(D)(D', D'') = \big((D'f)(D''g) + (D''f)(D'g)\big)D\,.$$

In other words, $(\delta_g\delta_f\varphi)(D) = (df \cdot dg) \otimes D$. We see that φ is a differential operator of order 2 with symbol $\varphi^\flat = \mathrm{id}$. To complete the proof we need the Lemma below.

Lemma 4.2. *A torsion-free connection on W does exist.*

Proof. Let $\widetilde{W}$ be the first order prolongation of W with kernel $\mathfrak{g}$, as described in section 3. The projection $\widetilde{W} \to W$ is an epimorphism of R-modules. It splits by the projectivity of W over R. Thus there exists a map $\xi : W \to \mathfrak{g}$ such that the assignment $D \mapsto \big(\xi(D),\, D\big)$ defines an R-module homomorphism $W \to \widetilde{W}$. That means that $\xi(fD) = f\,\xi(D) - df \otimes D$ for all $f \in R$, $D \in W$. Now $\mathfrak{g}$ operates on W by means of σ_W, and we have

$$\xi(fD)D' = f \cdot \xi(D)D' + (D'f)D$$

for all $f \in R$, $D, D' \in W$. Setting $\nabla(D', D'') = \xi(D'')D'$, we get a connection on W. Let

$$\tau(D', D'') = \nabla(D', D'') - \nabla(D'', D') - [D',\, D'']\,.$$

Then $\tau : W \times W \to W$ is a skewsymmetric R-bilinear map, the torsion of the connection. By a well known characterization of finitely generated projective modules, there exists a finite number of elements $D_1, \ldots, D_s \in W$ and $\theta_1, \ldots, \theta_s \in \Omega^1$ such that every $D \in W$ is expressed as $\sum \langle\theta_i, D\rangle D_i$. We get

$$\tau(D', D'') = \sum_{i,j=1}^{s} \langle\theta_i, D'\rangle\langle\theta_j, D''\rangle\,\tau(D_i, D_j) = \nu(D', D'') - \nu(D'', D')$$

where $\nu(D', D'') = \sum_{i<j} \langle \theta_i, D' \rangle \langle \theta_j, D'' \rangle \tau(D_i, D_j)$. Thus $\nu : W \times W \to W$ is an R-bilinear map and $\nabla' = \nabla - \nu$ a torsion-free connection on W. $\square$

Lemma 4.3. *There exists a $\mathbb{Z}$-linear map $\psi : W \to R$ satisfying identity (4.4). If $\mathrm{rk}_R W = 1$ then any such ψ is a divergence.*

Proof. Let $\gamma : \mathfrak{g} \to R$ be the contraction and $\xi : W \to \mathfrak{g}$ the map considered in the proof of Lemma 4.2. Then $\psi = -\gamma \circ \xi$ satisfies (4.4). Assume that ψ is an arbitrary $\mathbb{Z}$-linear map satisfying (4.4). One checks straightforwardly that its coboundary $\omega : W \times W \to R$,

$$\omega(D, D') = D\big(\psi(D')\big) - D'\big(\psi(D)\big) - \psi\big([D, D']\big) \qquad \text{for } D, D' \in W,$$

is R-bilinear and skewsymmetric. It corresponds therefore to a homomorphism of R-modules $\bigwedge^2 W \to R$. If $\mathrm{rk}_R W = 1$, then $\bigwedge^2 W = 0$, and $\omega = 0$. In other words, ψ is a cocycle. $\square$

Lemma 4.4. *The composite $\varphi : W \to R \to \Omega^1$ of a divergence ψ with the differential d is a 1-cocycle. Furthermore, φ is a differential operator of order 2 whose symbol $\varphi^\flat$ is the epimorphism $\pi : S^2\Omega^1 \otimes W \to \Omega^1$ in $\mathcal{C}_1$ defined by the rule*

$$\theta\theta' \otimes D \mapsto \langle \theta, D \rangle \theta' + \langle \theta', D \rangle \theta \quad \text{for } \theta, \theta' \in \Omega^1 \text{ and } D \in W.$$

Proof. Clearly π is a morphism in $\mathcal{C}_1$. Its surjectivity can be verified by passing to the reductions modulo the maximal ideals of R, where it becomes immediate. Since ψ is a cocycle and d a W-equivariant map, φ is a cocycle. Both ψ and d are differential operators of order 1 whose symbols are given by

$$\psi^\flat(\theta \otimes D) = \langle \theta, D \rangle \qquad \text{and} \qquad d^\flat(\theta \otimes h) = h\,\theta,$$

where $\theta \in \Omega^1$, $D \in W$, $h \in R$. Hence φ is a differential operator of order 2, and the computation of its symbol gives π, as required. $\square$

Proposition 4.5. *Suppose that $\mathrm{rk}_R W = 1$ and 3 is invertible in R. Then there is a 1-cocycle $\varphi : W \to \Omega^1 \otimes \Omega^1$ which is a differential operator of order 3 with symbol $\varphi^\flat : S^3\Omega^1 \otimes W \to \Omega^1 \otimes \Omega^1$ an isomorphism in $\mathcal{C}_1$.*

Proof. The symmetric and the tensor powers of the R-module Ω^1 coincide because $\mathrm{rk}_R \Omega^1 = \mathrm{rk}_R W = 1$. To put it differently, every R-multilinear expression involving several arguments from Ω^1 is symmetric in these arguments. The same observation applies to W. Identify $\Omega^1 \otimes \Omega^1$ with the group of R-bilinear maps $W \times W \to R$ so that a tensor $\theta_1 \otimes \theta_2$ with $\theta_1, \theta_2 \in \Omega^1$ corresponds to the map

$$(D', D'') \mapsto \langle \theta_1, D' \rangle \langle \theta_2, D'' \rangle, \quad \text{where} \quad D', D'' \in W.$$

Let $\psi : W \to R$ be a divergence. Take φ to be the composite

$$W \xrightarrow{\ \psi\ } R \xrightarrow{\ d\ } \Omega^1 \xrightarrow{\ \psi^*\ } \Omega^1 \otimes \Omega^1,$$

where $\psi^*(\theta)(D', D'') = \psi(D'), \theta(D'') - D''(\langle \theta, D' \rangle)$ for $\theta \in \Omega^1$ and D', $D'' \in W$.

It is immediate that $\psi^*(\theta)$ is R-linear in D''. Given $f \in R$, the expression

$$\psi^*(\theta)(fD', D'') - f \cdot \psi^*(\theta)(D', D'') = (D'f)\,\langle \theta, D'' \rangle - (D''f)\,\langle \theta, D' \rangle$$

is skewsymmetric in D', D''. So it has to vanish as well. Thus ψ^* is well defined. Now

$$\psi^*(f\theta)(D', D'') - f \cdot \psi^*(\theta)(D', D'') = -(D''f)\,\langle \theta, D'' \rangle,$$

whence $\psi^*(f\theta) - f\,\psi^*(\theta) = -\theta \otimes df$. It follows that ψ^* is a differential operator of order 1 with symbol $(\psi^*)^\flat$ minus identity transformation of $S^2\Omega^1 \cong \Omega^1 \otimes \Omega^1$. We saw in the proof of Lemma 4.4 that both ψ and d are differential operators of order 1. The composite φ is therefore a differential operator of order 3. Its symbol is computed as follows

$$\varphi^\flat(\theta_1\theta_2\theta_3 \otimes D) = -\sum \langle \theta_{i_1}, D \rangle\,\theta_{i_2}\theta_{i_3} = -6\,\langle \theta_1, D \rangle\,\theta_2\theta_3$$

for $\theta_1, \theta_2, \theta_3 \in \Omega^1$ and $D \in W$, where the sum is taken over all permutations of indices $1, 2, 3$ and we use that the terms are symmetric in θ's. Since the natural pairing between Ω^1 and W induces an isomorphism of R-modules $\Omega^1 \otimes W \cong R$, again by the rank one assumption, $\varphi^\flat : S^3\Omega^1 \otimes W \to S^2\Omega^1$ is an isomorphism as well. To show that φ is a cocycle we embed $S^2\Omega^1$ into the W-module of all $\mathbb{Z}$-bilinear maps $W \times W \to R$. Define $\omega : W \times W \to R$ by the rule

$$\omega(D', D'') = \psi(D')\,\psi(D'') - D'\big(\psi(D'')\big) - D''\big(\psi(D')\big)$$

for $D', D'' \in W$. As ψ is a cocycle, $D\psi$ is the coboundary of $\psi(D)$ for every $D \in W$. Hence

$$(D\omega)(D', D'')$$
$$= (D\psi)(D')\cdot\psi(D'')+\psi(D')\cdot(D\psi)(D'')-D'\big((D\psi)(D'')\big)-D''\big((D\psi)(D')\big)$$
$$= D'\big(\psi(D)\big)\cdot\psi(D'')+\psi(D')\cdot D''\big(\psi(D)\big)-D'D''\big(\psi(D)\big)-D''D'\big(\psi(D)\big).$$

On the other hand, computing $\varphi(D)$ straightforwardly yields

$$\varphi(D)(D', D'') = \psi(D') \cdot D''\big(\psi(D)\big) - D''D'\big(\psi(D)\big).$$

Since the left hand side is symmetric in D', D'', we get $D\omega = 2\varphi(D)$. Thus φ is a coboundary in a larger module. $\square$

5. Determination of cohomology

Let $M \in \mathcal{C}_1$. We introduce a filtration on the group of 1-cocycles $Z^1(W, M)$ letting $F_i Z^1(W, M)$ for $i \geq 0$ denote its subgroup consisting of cocycles $W \to M$ which are differential operators of order $\leq i$. Set $F_{-1} Z^1(W, M) = 0$. We have seen in proposition 2.4 that $F_3 Z^1(W, M)$ (respectively $F_2 Z^1(W, M)$ when $\mathrm{rk}_R W > 1$) exhausts all the 1-cocycles. Let $F_i H^1(W, M)$ be the image of $F_i Z^1(W, M)$ in the cohomology group $H^1(W, M)$. First we are going to determine the factors

$$\mathrm{gr}_i H^1(W, M) = F_i H^1(W, M) / F_{i-1} H^1(W, M).$$

Proposition 5.1. *Let $M \in \mathcal{C}_1$. Then there are isomorphisms*
- (5.1) $\mathrm{gr}_0 H^1(W, M) \cong H_R^1(W, M^{\mathfrak{g}})$,
- (5.2) $\mathrm{gr}_1 H^1(W, M) \cong \mathrm{Ker}\big(H_R^1(\widetilde{\mathfrak{g}}, M)^W \to H_R^2(W, M^{\mathfrak{g}})\big)$,
- (5.3) $\mathrm{gr}_2 H^1(W, M) \cong \mathrm{Mor}_{\mathcal{C}_1}(S^2 \Omega^1 \otimes W, M)$,
- (5.4) $\mathrm{gr}_3 H^1(W, M) \cong \mathrm{Mor}_{\mathcal{C}_1}(\Omega^1 \otimes \Omega^1, M)$ *when* $\mathrm{rk}_R W = 1$ *and* $3R = R$.

Proof. Let $\widetilde{W}$ be the first order prolongation of W with kernel $\widetilde{\mathfrak{g}}$. The canonical embedding $W \to \widetilde{W}$ is a Lie algebra homomorphism and also a differential operator of order 1. Hence the restriction to W of every R-linear cocycle $\widetilde{W} \to M$ is in $F_1 Z^1(W, M)$. The resulting map $Z_R^1(\widetilde{W}, M) \to F_1 Z^1(W, M)$ is bijective by Lemma 3.3. Obviously the $\widetilde{W}$-coboundaries correspond to the W-coboundaries. It follows $F_1 H^1(W, M) \cong H_R^1(\widetilde{W}, M)$. Again by Lemma 3.3

$$\mathrm{gr}_0 H^1(W, M) \cong F_0 H^1(W, M) \cong \mathrm{Ker}\big(H_R^1(\widetilde{W}, M) \to H_R^1(\widetilde{\mathfrak{g}}, M)\big)$$

since the cohomology classes on the left are represented by the R-linear cocycles $W \to M$, while those on the right by the cocycles $\widetilde{W} \to M$ with zero restriction to $\widetilde{\mathfrak{g}}$. Lemma 3.1 gives now (5.1). Moreover, (5.2) also follows because

$$\mathrm{gr}_1 H^1(W, M) \cong \mathrm{Coker}\big(F_0 H^1(W, M) \to F_1 H^1(W, M)\big)$$
$$\cong \mathrm{Coker}\big(H_R^1(W, M^{\mathfrak{g}}) \to H_R^1(\widetilde{W}, M)\big).$$

Next, by Lemma 2.2 we have a map

$$F_2 Z^1(W, M) \to \mathrm{Mor}_{\mathcal{C}_1}(S^2 \Omega^1 \otimes W, M) \qquad (*)$$

which assigns to a 1-cocycle $\varphi : W \to M$ in the F_2 term of the filtration its symbol $\varphi^\flat$. The kernel of this map is clearly $F_1 Z^1(W, M)$. Since all coboundaries are in $F_1 Z^1(W, M)$ by Lemma 2.1, the map above induces an embedding of $\mathrm{gr}_2 H^1(W, M)$ into $\mathrm{Mor}_{\mathcal{C}_1}(S^2 \Omega^1 \otimes W, M)$. Suppose now that $\xi : S^2 \Omega^1 \otimes W \to M$ is a morphism in $\mathcal{C}_1$. Let $\varphi : W \to S^2 \Omega^1 \otimes W$ be

the 1-cocycle given by proposition 4.1. Then $\xi \circ \varphi$ is a 1-cocycle $W \to M$ which lies in $F_2 Z^1(W, M)$ and has symbol $\xi \circ \varphi^\flat = \xi$ as ξ is R-linear and $\varphi^\flat = \mathrm{id}$. Thus $(*)$ is surjective. The assertion about $\mathrm{gr}_3 H^1(W, M)$ is proved similarly, taken into account proposition 4.5. $\square$

Lemma 5.2. *Suppose that $0 \to C^0 \to C^1 \to \ldots$ is a cochain complex of finitely generated projective R-modules and $q > 0$ an integer such that $H^i(C^\bullet/\mathfrak{m}C^\bullet) = 0$ for all maximal ideals $\mathfrak{m}$ of R in all degrees $i < q$. Then $H^i(C^\bullet \otimes M) = 0$ for any R-module M and $i < q$.*

Proof. The differential $d : C^0 \to C^1$ induces by passing to the reductions modulo any maximal ideal $\mathfrak{m}$ an injective map $C^0/\mathfrak{m}C^0 \to C^1/\mathfrak{m}C^1$ because $H^0(C^\bullet/\mathfrak{m}C^\bullet)$ vanishes. It follows that d maps C^0 isomorphically onto a direct summand of C^1. In fact, the localizations $C^0_\mathfrak{m}$ and $C^1_\mathfrak{m}$ at $\mathfrak{m}$ are free modules of finite rank over the local ring $R_\mathfrak{m}$, and therefore $C^0_\mathfrak{m}$ is mapped isomorphically onto a direct summand of $C^1_\mathfrak{m}$ by [1, II, §3, Proposition 6]. Hence d is injective by [1, II, §3, Theorem 1]. The cokernel of d is a finitely presented R-module because C^0 and C^1 are finitely generated and projective. By [1, II, §3, Corollary 1 to Proposition 12] dC^0 is a direct summand of C^1, as required.

Now $C^\bullet$ decomposes into a direct sum $A^\bullet \oplus B^\bullet$ of two subcomplexes where $A^0 = C^0$, $A^1 = dC^0$ and $A^i = 0$ in all other degrees, while B^1 is any R-module complement of A^1 in C^1. If M is an R-module then $C^\bullet \otimes M$ is a direct sum of complexes $A^\bullet \otimes M$ and $B^\bullet \otimes M$. Therefore

$$H^i(C^\bullet \otimes M) \cong H^i(A^\bullet \otimes M) \oplus H^i(B^\bullet \otimes M)$$

for all i. Since the differential $A^0 \otimes M \to A^1 \otimes M$ is an isomorphism, we have $H^i(A^\bullet \otimes M) = 0$ for all i. Taking $M = R/\mathfrak{m}$ gives $H^i(B^\bullet/\mathfrak{m}B^\bullet) = 0$ for all maximal ideals $\mathfrak{m}$ in degrees $i < q$. Since nonzero terms in $B^\bullet$ start from degree 1, we complete the proof reindexing $B^\bullet$ and applying induction on q. $\square$

Lemma 5.3. *Suppose that $Q \in \mathcal{C}_1$ is an object satisfying (1.10), (1.11) and $M \in \mathcal{C}_1$ an object of type Q. If $H^1(\mathfrak{g}/\mathfrak{m}\mathfrak{g}, Q/\mathfrak{m}Q) = 0$ for all maximal ideals $\mathfrak{m}$ of R then $H^1_R(\mathfrak{g}, M) = 0$ as well.*

Proof. Consider the standard cochain complex $C^\bullet_R(\mathfrak{g}, Q)$ of R-multilinear alternating maps $\mathfrak{g} \times \ldots \times \mathfrak{g} \to Q$. Its components are finitely generated projective R-modules since so are Q and the exterior powers of $\mathfrak{g}$. Furthermore, if M_0 is an R-module on which $\mathfrak{g}$ operates trivially, then

$$C^\bullet_R(\mathfrak{g}, Q \otimes M_0) \cong C^\bullet_R(\mathfrak{g}, Q) \otimes M_0 \,.$$

In particular, $C^\bullet(\mathfrak{g}/\mathfrak{m}\mathfrak{g}, Q/\mathfrak{m}Q) \cong C^\bullet_R(\mathfrak{g}, Q/\mathfrak{m}Q) \cong C^\bullet_R(\mathfrak{g}, Q) \otimes R/\mathfrak{m}$ for every $\mathfrak{m}$. The reduced modulo $\mathfrak{m}$ complexes have zero cohomology

in degree 1 according to the assumptions. Since $\mathfrak{g}/\mathfrak{m}\mathfrak{g}$ has a proper commutant, the first cohomology group of the trivial $\mathfrak{g}/\mathfrak{m}\mathfrak{g}$-module is nonzero. Hence $Q/\mathfrak{m}Q$ is not trivial, and therefore $H^0(\mathfrak{g}/\mathfrak{m}\mathfrak{g}, Q/\mathfrak{m}Q) = 0$ as well. By Lemma 1.3 $M \cong Q \otimes M_0$ for a suitable M_0. We complete the proof applying Lemma 5.2 with $q = 2$. $\square$

Lemma 5.4. *Let V be a vector space of dimension n over a field k of characteristic $p \neq 2$. Denote by $\overline{K}$ the kernel of the linear map $S^2V^* \otimes V \to V^*$ defined by the rule*

$$\theta\theta' \otimes v \mapsto \langle\theta, v\rangle\, \theta' + \langle\theta', v\rangle\, \theta \qquad \text{for } \theta, \theta' \in V^* \text{ and } v \in V.$$

The $\mathfrak{gl}\,V$-modules V, V^*, $V^* \otimes \bigwedge^n V^*$, and $\overline{K}$ when $n > 1$ and $n + 1 \not\equiv 0 \pmod{p}$ are nontrivial, absolutely irreducible, pairwise nonisomorphic and have zero cohomology in degree 1 (with the exceptions only for $p = 3$, $n \leq 2$).

The $\mathfrak{gl}\,V$-module $S^2V^* \otimes V$ is isomorphic with the direct sum $\overline{K} \oplus V^*$ of two irreducible modules when $n > 1$ and $n + 1 \not\equiv 0 \pmod{p}$. It has a single irreducible factor module isomorphic to V^* otherwise. The $\mathfrak{gl}\,V$-module $\overline{K}$ has never V^* as its factor module.

Proof. The assertion about irreducibility is immediate for all modules stated, except for $\overline{K}$. In fact, $S^2V^* \otimes V$ is the component of degree 1 in the graded Lie algebra of general Cartan type $G_{-1} \oplus G_0 \oplus G_1 \oplus \cdots$ having $G_{-1} = V$ and $G_0 = \mathfrak{gl}\,V$. The structure of G_1 as a G_0-module was investigated in [15, I, §10]. If $n > 1$ and p does not divide $n + 1$ then $\overline{K}$ is irreducible and $G_1 \cong \overline{K} \oplus V^*$. This is clearly not affected by field extensions, so that we get the absolute irreducibility. Note that

$$\dim \overline{K} = \dim S^2V^* \otimes V - \dim V^* = \frac{1}{2}n^2(n+1) - n = \frac{1}{2}n(n-1)(n+2) > n.$$

However, the first three modules under consideration all have dimension n. It is easy to check that these three are nonetheless pairwise nonisomorphic with an exception for $p = 3$, $n = 1$.

The vanishing of cohomology is a general fact when $p = 0$. If $p > 0$ it is achieved by inspection of weights with respect to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{gl}\,V$. The weights $\varepsilon_1, \ldots, \varepsilon_n$ of V constitute a basis for the dual space $\mathfrak{h}^*$. The weights of V^* are now $-\varepsilon_i$, the weights of $V^* \otimes \bigwedge^n V^*$ are $-\varepsilon_i - \delta$, and those of $\overline{K}$ are $\varepsilon_l - \varepsilon_i - \varepsilon_j$, where $\delta = \varepsilon_1 + \cdots + \varepsilon_n$ and $1 \leq i, j, l \leq n$. None of them is among the roots of $\mathfrak{gl}\,V$, which are $\varepsilon_l - \varepsilon_i$ (except when $p = 3$, $n = 2$).

Look now at the other possibilities for G_1 and $\overline{K}$. If $n = 1$ then $\overline{K} = 0$ and $G_1 \cong V^*$. Suppose that $n > 1$ but p divides $n + 1$. By [15] $\overline{K}$ is a single maximal submodule of G_1. Moreover, G_1 contains a single

irreducible submodule, say V', which satisfies $V' \subset \overline{K}$ and $V' \cong V^*$. The factor module $\overline{K}/V'$ is irreducible of dimension greater than n. The only exception is the case $n = 2$, $p = 3$ when $\mathfrak{gl}\,V$ acts trivially in $\overline{K}/V'$. In any case V^* is not a factor module of $\overline{K}$, as asserted. $\square$

The category $\mathcal{C}_1$ contains objects Q satisfying (1.10) and (1.11) which give a certain canonical glueing of the $\mathfrak{g}/\mathfrak{mg}$-modules of each type considered in Lemma 5.4. Three of these are W, Ω^1, and $\Omega^1 \otimes \Omega^n$ where $n = \mathrm{rk}_R W$ and $\Omega^n = \bigwedge^n \Omega^1$. Denote by K the kernel of the epimorphism $\pi : S^2\Omega^1 \otimes W \to \Omega^1$ defined in Lemma 4.4. Since both Ω^1 and $S^2\Omega^1 \otimes W$ are finitely generated projective R-modules, so is K as well. If $\mathfrak{m}$ is a maximal ideal of R then $K/\mathfrak{m}K$ is the kernel of the induced linear map $S^2V^* \otimes V \to V^*$ of vector spaces over $R/\mathfrak{m}$ where $V = W/\mathfrak{m}W$ and V^* is its dual. Suppose that $(n+1)R = R$. Then it follows from Lemma 5.4 that $K/\mathfrak{m}K$ is an absolutely irreducible $\mathfrak{g}/\mathfrak{mg}$-module, and so K satisfies (1.10), (1.11). Moreover, there is a canonical decomposition

$$S^2\Omega^1 \otimes W \cong K \oplus \Omega^1$$

in $\mathcal{C}_1$. To see this consider the morhism $\mu : \Omega^1 \otimes \mathfrak{g} \to S^2\Omega^1 \otimes W$ defined by the rule $\theta \otimes (\theta' \otimes D) \mapsto \theta\theta' \otimes D$ for $\theta, \theta' \in \Omega^1$ and $D \in W$. Then

$$\pi \circ \mu(\theta \otimes T) = \sigma_{\Omega^1}(T)\theta + \gamma(T)\theta \qquad \text{for all } \theta \in \Omega^1 \text{ and } T \in \mathfrak{g},$$

where $\gamma : \mathfrak{g} \to R$ is the contraction. Since $\sigma_{\Omega^1} : \mathfrak{g} \to \mathrm{End}_R\,\Omega^1$ is an isomorphism in $\mathcal{C}_1$, there is a W-invariant element $1 \in \mathfrak{g}$ which corresponds to the identity endomorphism of Ω^1. The assignment $\theta \mapsto \mu(\theta \otimes 1)$ defines obviously a morphism $\nu : \Omega^1 \to S^2\Omega^1 \otimes W$ in $\mathcal{C}_1$. Since $\gamma(1) = \mathrm{tr}\,1_{\Omega^1} = \mathrm{rk}_R\,\Omega^1 = n$, we deduce that $\pi \circ \nu = (n+1) \cdot 1_{\Omega^1}$. If $n+1$ is invertible in R then the restriction of π gives an isomorphism $\mathrm{Im}\,\nu \to \Omega^1$. In this case $\mathrm{Im}\,\nu$ is a subobject complementary to K in $S^2\Omega^1 \otimes W$.

Now we have come to the final results on cohomology. Let $Q \in \mathcal{C}_1$ be an object satisfying conditions (1.10), (1.11) and $M \in \mathcal{C}_1$ an object of type Q. Recall that $n = \mathrm{rk}_R W$. In addition to (1.1) assume that 3 is invertible in R as well, at least when $n = 1$.

Theorem 5.5. *If for every maximal ideal $\mathfrak{m}$ of R the $\mathfrak{g}/\mathfrak{mg}$-module $Q/\mathfrak{m}Q$ is isomorphic to neither the trivial module, nor $W/\mathfrak{m}W$, nor $(W/\mathfrak{m}W)^*$, nor $K/\mathfrak{m}K$ when $n > 1$ and $\mathrm{char}\,R/\mathfrak{m}$ does not divide $n+1$, nor $\bigotimes^2(W/\mathfrak{m}W)^*$ when $n = 1$, then*

$$H^1(W, M) \cong H^1_R(\mathfrak{g}, M)^W.$$

If, moreover, $H^1(\mathfrak{g}/\mathfrak{mg},\,Q/\mathfrak{m}Q) = 0$ for all $\mathfrak{m}$ then $H^1(W, M) = 0$.

If R is an algebra over a field of characteristic 0 then char $R/\mathfrak{m} = 0$ for every maximal ideal $\mathfrak{m}$ of R. In this case the reductive Lie algebra $\mathfrak{g}/\mathfrak{m}\mathfrak{g}$ over the field $R/\mathfrak{m}$ has zero cohomology with coefficients in any nontrivial irreducible finite dimensional module, i.e., the condition $H^1(\mathfrak{g}/\mathfrak{m}\mathfrak{g},\, Q/\mathfrak{m}Q) = 0$ is fulfilled automatically. We will be proving theorem 5.5 simultaneously with the next result which treats exceptional modules.

Theorem 5.6. *Suppose that $M = M_0 \otimes Q$ where $M_0 \in \mathcal{C}_0$, and let M_0^W be the subgroup of W-invariants in M_0.*

i) *If $Q = R$, so that $M \in \mathcal{C}_0$, then there is an exact sequence*

$$0 \to H^1_R(W, M) \to H^1(W, M) \to M^W.$$

The final map is surjective provided that there exists a divergence $W \to R$. If $\varphi : W \to M$ is a 1-cocycle and $m \in M^W$ the image of its cohomology class then $\varphi(fD) = f\,\varphi(D) + (Df)m$ for all $f \in R$ and $D \in W$.

ii) *If $Q = W$ then M is canonically embedded into the W-module $\operatorname{Der}(R, M_0)$ of $\mathbb{Z}$-linear derivations $R \to M_0$. Denote by N the group of those $\Delta \in \operatorname{Der}(R, M_0)$ which satisfy $D \cdot \Delta \in M$ for all $D \in W$. Then $H^1(W, M) \cong N/M$.*

iii) *If $Q = \Omega^1$ then*

$$H^1(W, M) \cong \operatorname{Mor}_{\mathcal{C}_1}(\Omega^1, M) \cong M_0^W.$$

Every cohomology class is represented by a cocycle which is the composite of the universal differential order 2 cocycle $W \to S^2\Omega^1 \otimes W$, the canonical morphism $S^2\Omega^1 \otimes W \to \Omega^1$ and a morphism $\Omega^1 \to M$ in $\mathcal{C}_1$.

iv) *Suppose $n > 1$ and $(n+1)R = R$. If $Q = K$ then*

$$H^1(W, M) \cong \operatorname{Mor}_{\mathcal{C}_1}(K, M) \cong M_0^W.$$

Every cohomology class is represented by a cocycle which is the composite of the universal differential order 2 cocycle $W \to S^2\Omega^1 \otimes W$, the canonical projection $S^2\Omega^1 \otimes W \to K$ and a morphism $K \to M$ in $\mathcal{C}_1$.

v) *Suppose $n = 1$. If $Q = \Omega^1 \otimes \Omega^1$ then*

$$H^1(W, M) \cong \operatorname{Mor}_{\mathcal{C}_1}(\Omega^1 \otimes \Omega^1,\, M) \cong M_0^W.$$

Every cohomology class in this group is represented by a cocycle which is the composite of the universal differential order 3 cocycle $W \to \Omega^1 \otimes \Omega^1$ and a morphism $\Omega^1 \otimes \Omega^1 \to M$ in $\mathcal{C}_1$.

Proof. Using proposition 5.1, we examine those objects Q for which the groups $\mathrm{gr}_i H^1(W, M)$ are not all zero. If a nonzero factor occurs for $i = 0$, then $M^{\mathfrak{g}} \neq 0$. Clearly $M^{\mathfrak{g}}$ is a subobject of M, and in fact $M^{\mathfrak{g}} \in \mathcal{C}_0$. The inclusion $M^{\mathfrak{g}} \hookrightarrow M$ is a nonzero morphism in $\mathcal{C}_1$. By Lemma 1.4 $Q/\mathfrak{m}Q$ is a trivial $\mathfrak{g}/\mathfrak{m}\mathfrak{g}$-module for at least one maximal ideal $\mathfrak{m}$ of R.

Next, $\mathrm{gr}_1 H^1(W, M) \neq 0$ implies that $H_R^1(\widetilde{\mathfrak{g}}, M)^W \neq 0$. There is a surjective homomorphism $\widetilde{\mathfrak{g}} \to \mathfrak{g}$ of Lie algebras over R whose kernel is $J \otimes W$, where J is the kernel of the canonical map $\widetilde{\Omega}^1 \to \Omega^1$. Each R-linear cocycle $\varphi : \widetilde{\mathfrak{g}} \to M$ determines by restriction an R-linear map $J \otimes W \to M$. Furthermore, the coboundaries restrict to $J \otimes W$ trivially because $\widetilde{\mathfrak{g}}$ operates in M via $\mathfrak{g}$. If the cohomology class of φ is W-invariant, then $D\varphi$ is a coboundary and therefore restricts to $J \otimes W$ trivially for every $D \in W$. That means that the restriction $J \otimes W \to M$ of φ is a W-invariant map and therefore a morphism in $\mathcal{C}_1$. It is zero if and only if φ factors through $\mathfrak{g}$. Thus there is an exact sequence

$$0 \to H_R^1(\mathfrak{g}, M)^W \to H_R^1(\widetilde{\mathfrak{g}}, M)^W \to \mathrm{Mor}_{\mathcal{C}_1}(J \otimes W, M). \qquad (*)$$

If $H_R^1(\mathfrak{g}, M) \neq 0$ then $H^1(\mathfrak{g}/\mathfrak{m}\mathfrak{g}, Q/\mathfrak{m}Q) \neq 0$ for at least one $\mathfrak{m}$ by Lemma 5.3. On the other hand, $J \otimes W \in \mathcal{C}_1$ is an object of type W since $J \in \mathcal{C}_0$ according to Lemma 1.6. By Lemma 1.4 $\mathrm{Mor}_{\mathcal{C}_1}(J \otimes W, M) \neq 0$ implies $Q/\mathfrak{m}Q \cong W/\mathfrak{m}W$ as $\mathfrak{g}/\mathfrak{m}\mathfrak{g}$-modules for at least one $\mathfrak{m}$.

Consider the case $\mathrm{gr}_2 H^1(W, M) \neq 0$. By (5.3) there exists a nonzero morphism $\xi : S^2\Omega^1 \otimes W \to M$ in $\mathcal{C}_1$. Its image, say M', is a nonzero subobject of M. Then $M' \in \mathcal{C}_1$ is an object of type Q, and $M' \cong M_0' \otimes Q$ with $M_0' \in \mathcal{C}_0$ by Lemma 1.3. Since W and Ω^1 are finitely generated over R, so is M' as well. Hence $M' \neq \mathfrak{m}M'$ for at least one maximal ideal $\mathfrak{m}$. The quotient $M_0'/\mathfrak{m}M_0'$ is a trivial $\mathfrak{g}/\mathfrak{m}\mathfrak{g}$-module. The $\mathfrak{g}/\mathfrak{m}\mathfrak{g}$-module $M'/\mathfrak{m}M' \cong M_0'/\mathfrak{m}M_0' \otimes Q/\mathfrak{m}Q$ is therefore completely reducible with all irreducible submodules isomorphic to $Q/\mathfrak{m}Q$. On the other hand, $M'/\mathfrak{m}M'$ is an epimorphic image of $S^2(W/\mathfrak{m}W)^* \otimes W/\mathfrak{m}W$. According to Lemma 5.4 either $Q/\mathfrak{m}Q \cong (W/\mathfrak{m}W)^*$ or $n > 1$, $n + 1$ is invertible in $R/\mathfrak{m}$ and $Q/\mathfrak{m}Q \cong K/\mathfrak{m}K$.

Finally, $\mathrm{gr}_3 H^1(W, M) \neq 0$ implies that $n = 1$ and

$$Q/\mathfrak{m}Q \cong \bigotimes^2 (W/\mathfrak{m}W)^*$$

for at least one $\mathfrak{m}$ by similar reasons. Under the hypotheses of theorem 5.5, the groups $\mathrm{gr}_i H^1(W, M)$ vanish for all $i \neq 1$. Both $M^{\mathfrak{g}}$ and $\mathrm{Mor}_{\mathcal{C}_1}(J \otimes W, M)$ are zero by Lemma 1.4. Hence (5.2) and $(*)$ give

$$H^1(W, M) \cong \mathrm{gr}_1 H^1(W, M) \cong H_R^1(\widetilde{\mathfrak{g}}, M)^W \cong H_R^1(\mathfrak{g}, M)^W.$$

The final statement of theorem 5.5 follows from Lemma 5.3. We now check one by one all cases in theorem 5.6.

(i) By the above $\mathrm{gr}_i H^1(W, M) = 0$ for $i = 2, 3$. Hence an exact sequence

$$0 \to \mathrm{gr}_0 H^1(W, M) \to H^1(W, M) \to \mathrm{gr}_1 H^1(W, M) \to 0 \,.$$

The term on the left is retrieved from (5.1), where $M^{\mathfrak{g}} = M$. In view of (5.2) and ($*$) there is an embedding $\mathrm{gr}_1 H^1(W, M) \hookrightarrow H^1_R(\widetilde{\mathfrak{g}}, M)^W \cong H^1_R(\mathfrak{g}, M)^W$. Since $\mathfrak{g}$ annihilates M, we have

$$H^1_R(\mathfrak{g}, M) \cong \mathrm{Hom}_R(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}], \, M).$$

The commutant $[\mathfrak{g}, \mathfrak{g}]$ consists of all elements of $\mathfrak{g}$ that act in W with trace zero. Hence $[\mathfrak{g}, \mathfrak{g}]$ is the kernel of the contraction $\gamma : \mathfrak{g} \to R$. It follows that γ induces an isomorphism $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \cong R$, and $H^1_R(\mathfrak{g}, M) \cong \mathrm{Hom}_R(R, M) \cong M$. We get thus the required exact sequence. Every 1-cocycle $\varphi : W \to M$ is a differential operator of order ≤ 1 and its symbol $\varphi^\flat : \mathfrak{g} \to M$ is an R-linear cocycle. By the above $\varphi^\flat(\theta \otimes D) = \langle \theta, D \rangle m$ for $\theta \in \Omega^1$, $D \in W$, where $m \in M^W$ is the element corresponding to the cohomology class of φ. If $f \in R$, $D \in W$ then

$$\varphi(fD) - f\,\varphi(D) = \varphi^\flat(df \otimes D) = \langle df, D \rangle m = (Df)m \,.$$

Suppose now that $\psi : W \to R$ is a divergence. Given a W-invariant $m \in M^W$, the assignment $f \mapsto fm$ defines a morphism $\iota_m : R \to M$ in $\mathcal{C}_1$. The composite $\varphi = \iota_m \circ \psi$ is then a cocycle $W \to M$. Furthermore, $\varphi^\flat = \iota_m \circ \psi^\flat$. Since $\psi^\flat = \gamma$, the map $H^1(W, R) \to R^W$ takes the cohomology class of ψ to 1. As the map $H^1(W, M) \to M^W$ is natural in M, it takes the cohomology class of φ to $\iota_m(1) = m$.

(ii) Here $\mathrm{gr}_i H^1(W, M)$ vanishes for $i \neq 1$. By lemmas 5.3 and 5.4 $H^1_R(\mathfrak{g}, M) = 0$. From (5.2) and ($*$) we deduce now an embedding

$$H^1(W, M) \cong \mathrm{gr}_1 H^1(W, M) \hookrightarrow H^1_R(\widetilde{\mathfrak{g}}, M)^W \hookrightarrow \mathrm{Mor}_{\mathcal{C}_1}(J \otimes W, \, M) \,.$$

The exact sequence $0 \to J \to \widetilde{\Omega}^1 \to \Omega^1 \to 0$ in $\widetilde{\mathcal{C}}_1$ splits as a sequence of R-modules since Ω^1 is R-projective. It gives rise therefore to an exact sequence in $\widetilde{\mathcal{C}}_1$

$$0 \to \mathrm{Hom}_R(\Omega^1, \, M_0) \to \mathrm{Hom}_R(\widetilde{\Omega}^1, \, M_0) \to \mathrm{Hom}_R(J, M_0) \to 0 \,.$$

Here $\mathrm{Hom}_R(\Omega^1, \, M_0) \cong M_0 \otimes \mathrm{Hom}_R(\Omega^1, \, R) \cong M_0 \otimes W = M$. On the other hand, $\mathrm{Hom}_R(\widetilde{\Omega}^1, M_0) \cong \mathrm{Der}(R, M_0)$ by the universality property of Kähler differentials. Thus M is embedded into $\mathrm{Der}(R, M_0)$ and

$$N/M = \big(\mathrm{Der}(R, M_0)/M\big)^W \cong \mathrm{Hom}_R(J, M_0)^W = \mathrm{Mor}_{\mathcal{C}_0}(J, M_0) \,.$$

Consider the diagram

$$\begin{array}{ccc} \big(\mathrm{Der}(R, M_0)/M\big)^W & \cong & \mathrm{Mor}_{\mathcal{C}_0}(J, M_0) \\ \downarrow & & \downarrow \wr \\ H^1(W, M) & \longrightarrow & \mathrm{Mor}_{\mathcal{C}_1}(J \otimes W, M) \end{array}$$

where the left vertical arrow is the connecting map in cohomology arising from the short exact sequence of W-modules $0 \to M \to \mathrm{Der}(R, M_0) \to \mathrm{Der}(R, M_0)/M \to 0$ and the right one is the isomorphism of Lemma 1.4. Now take a W-invariant coset $\Delta + M$ where $\Delta \in N$. Pushing it right in the diagram gives a morphism $J \to M_0$ in $\mathcal{C}_0$ which is the restriction of the R-linear map $\xi : \widetilde{\Omega}^1 \to M_0$ defined by the rule $\xi(df) = \Delta f$ for $f \in R$. Pushing $\Delta + M$ down gives the cohomology class of the cocycle $\varphi : W \to M$ such that $\varphi(D) = D \cdot \Delta$ for $D \in W$. Pushing φ further right, we come to a morphism $J \otimes W \to M$ in $\mathcal{C}_1$ which is the restriction of the symbol $\varphi^\flat : \widetilde{\Omega}^1 \otimes W \to M$ of φ. If $f \in R$, $D \in W$ then the derivation $(fD) \cdot \Delta - f(D \cdot \Delta)$ takes $g \in R$ to

$$\rho_{M_0}(fD)(\Delta g) - f \cdot \rho_{M_0}(D)(\Delta g) - \Delta(fDg) + f \cdot \Delta(Dg) \in M_0 \,.$$

The first two terms cancel as $\sigma_{M_0} = 0$. Hence the result is $-(Dg) \cdot (\Delta f)$, which is the value at g of the element $-\Delta f \otimes D \in M_0 \otimes W$ regarded as a derivation $R \to M_0$. In other words, $\varphi^\flat(df \otimes D) = -\Delta f \otimes D$. Since $\varphi^\flat = -\xi \otimes 1_W$, the diagram is anticommutative. It follows that the left arrow is an isomorphism.

(iii) In this case $\mathrm{gr}_i H^1(W, M) = 0$ for $i \neq 2$. If $\xi : K \to M$ is a morphism in $\mathcal{C}_1$ then its image K' is a subobject of M. Hence K' is of type Ω^1, and so $K' \cong K_0' \otimes \Omega^1$ with $K_0' \in \mathcal{C}_0$ by Lemma 1.3. For every maximal ideal $\mathfrak{m}$ of R the $\mathfrak{g}/\mathfrak{m}\mathfrak{g}$-module $K'/\mathfrak{m}K' \cong K_0'/\mathfrak{m}K_0' \otimes (W/\mathfrak{m}W)^*$ is completely reducible with all irreducible submodules isomorphic to $(W/\mathfrak{m}W)^*$. On the other hand, $K'/\mathfrak{m}K'$ is an epimorphic image of $K/\mathfrak{m}K$ which does not have $(W/\mathfrak{m}W)^*$ as its factor module according to Lemma 5.4. Hence $K' = \mathfrak{m}K'$. Since K' is finitely generated over R, it follows $K' = 0$ by Nakayama's Lemma. Thus every morphism $S^2\Omega^1 \otimes W \to M$ vanishes on K and therefore factors through Ω^1. By (5.3)

$$H^1(W, M) \cong \mathrm{gr}_2 H^1(W, M) \cong \mathrm{Mor}_{\mathcal{C}_1}(S^2\Omega^1 \otimes W, M) \cong \mathrm{Mor}_{\mathcal{C}_1}(\Omega^1, M) \,.$$

(iv) Again $\mathrm{gr}_i H^1(W, M)$ is nonzero for $i = 2$ only. Since $\mathrm{Mor}_{\mathcal{C}_1}(\Omega^1, M) = 0$ by Lemma 1.4, we have

$$H^1(W, M) \cong \mathrm{gr}_2 H^1(W, M) \cong \mathrm{Mor}_{\mathcal{C}_1}(K \oplus \Omega^1, M) \cong \mathrm{Mor}_{\mathcal{C}_1}(K, M) \,.$$

(v) Here $\mathrm{gr}_i H^1(W, M) = 0$ for $i \neq 3$, and

$$H^1(W, M) \cong \mathrm{gr}_3 H^1(W, M) \cong \mathrm{Mor}_{\mathcal{C}_1}(\Omega^1 \otimes \Omega^1, M).$$

The description of cocycles representing cohomology classes is immediate from the construction of isomorphisms. Lemma 1.4 shows also that in any of the cases

$$\mathrm{Mor}_{\mathcal{C}_1}(Q, M) \cong \mathrm{Mor}_{\mathcal{C}_0}(R, M_0) = \mathrm{Hom}_R(R, M_0)^W \cong M_0^W.$$

$\square$

Corollary 5.7. *The Lie algebra* $\mathrm{Der}\, W$ *of all* $\mathbb{Z}$*-linear derivations of* W *is isomorphic with the normalizer* N *of* W *in* $\mathrm{Der}\, R$.
Proof. The adjoint representation of N in its ideal W induces a homomorphism of Lie algebras $N \to \mathrm{Der}\, W$. It is an isomorphism because so is the induced map $N/W \to \mathrm{Der}\, W/\mathrm{ad}\, W \cong H^1(W, W)$ by (ii). $\square$

This result was obtained earlier [22] under assumptions weaker than (1.2), (1.3). If $k \subset R^W$ is a subring, then a derivation $\Delta \in N$ is k-linear if and only if so is the induced derivation $\mathrm{ad}\, \Delta \in \mathrm{Der}\, W$. Therefore the subalgebra $\mathrm{Der}_k W$ of k-linear derivations of W is isomorphic with $N \cap \mathrm{Der}_k R$, where $\mathrm{Der}_k R$ are the k-linear derivations of R. If R is the ring of functions on X, a smooth manifold, real analytic one or a Stein space, and k the field of real or complex numbers then $\mathrm{Der}_k R$ is isomorphic with the Lie algebra of vector fields $\mathrm{Vect}\, X$ of respective class on X. It follows that all k-linear derivations of $\mathrm{Vect}\, X$ are inner.

6. The case of commuting derivations

We will specialize our assumptions on R, W. Suppose that R is an algebra over a field k and that W is a free R-module generated by a system of pairwise commuting k-linear derivations $\partial_1, \ldots, \partial_n$. We still keep our basic assumptions (1.1)–(1.3). There is a very explicit construction of certain representations of W in this case. I would like to thank Naihong Hu who drew my attention to Shen's paper [20] where this construction appeared under the name of mixed products. Accordingly, we are able to write down the 1-cocycles quite explicitly. Commuting derivations appear in many interesting situations. For instance, the Lie algebras of Witt type in positive characteristic fit into our present settings. Degree one cohomology in that special case was considered by Chiu and Shen [3] and Dzhumadil'daev [4].

Denote by $\mathfrak{D}$ the k-linear span of $\partial_1, \ldots, \partial_n$. This is an abelian subalgebra of $W = R\mathfrak{D} \cong R \otimes_k \mathfrak{D}$. We have

$$\mathfrak{g} \cong \mathfrak{gl}_R W \cong R \otimes_k \mathfrak{gl}_k \mathfrak{D} \, .$$

Let E_{ij} be the linear transformation of $\mathfrak{D}$ such that $E_{ij}\partial_l = \delta_{jl}\partial_i$ where indices i, j, l are taken among $1, \ldots, n$. If $\theta \in \Omega^1$ then the element $\theta \otimes \partial_i \in \mathfrak{g}$ corresponds to $-\sum_j \langle \theta, \partial_j \rangle \otimes E_{ij}$ under the isomorphism above. Given a representation σ_V of $\mathfrak{gl}_k \mathfrak{D}$ in a vector space V over k, put

$$\mathcal{R}V = R \otimes_k V$$

and define a representation of W in this space setting

$$\rho_{\mathcal{R}V}(g\partial_i) = g\partial_i \otimes \mathrm{id}_V - \sum_j (\partial_j g)_R \otimes \sigma_V(E_{ij}) \qquad \text{for } g \in R, \ i = 1, \ldots, n,$$

where f_R stands for the multiplication operator on R corresponding to an element $f \in R$. The operators $f_R \otimes \mathrm{id}_V$ give $\mathcal{R}V$ a compatible R-module structure, so that (1.6) is fulfilled. Extend σ_V by R-linearity to a representation of $\mathfrak{g}$ in $\mathcal{R}V$. If $\theta \in \Omega^1$ then

$$\sigma_{\mathcal{R}V}(\theta \otimes \partial_i) = -\sum_j \langle \theta, \partial_j \rangle_R \otimes \sigma_V(E_{ij}) \, .$$

Take $f, g \in R$. As

$$\rho_{\mathcal{R}V}(g\partial_i) - (g_R \otimes \mathrm{id}_V) \circ \rho_{\mathcal{R}V}(\partial_i) = -\sum_j (\partial_j g)_R \otimes \sigma_V(E_{ij}) = \sigma_{\mathcal{R}V}(dg \otimes \partial_i) \, ,$$

we have for $D = g\partial_i \in W$

$$\rho_{\mathcal{R}V}(fD) - (f_R \otimes \mathrm{id}_V) \circ \rho_{\mathcal{R}V}(D)$$
$$= \sigma_{\mathcal{R}V}(d(fg) \otimes \partial_i) - \sigma_{\mathcal{R}V}(f \cdot dg \otimes \partial_i) = \sigma_{\mathcal{R}V}(df \otimes D) \, .$$

Thus (1.8) is fulfilled too. We see that $\mathcal{R}V$ together with the three module structures we have described is an object of $\mathcal{C}_1$. In fact $\mathcal{R}$ is a functor from the category of $\mathfrak{gl}_k \mathfrak{D}$-modules to $\mathcal{C}_1$. It takes the tensor product of two $\mathfrak{gl}_k \mathfrak{D}$-modules to the tensor product of the corresponding objects in $\mathcal{C}_1$, the symmetric and exterior powers of a $\mathfrak{gl}_k \mathfrak{D}$-module to the symmetric and exterior powers in $\mathcal{C}_1$. If $\dim V < \infty$ and V^* is the contragredient $\mathfrak{gl}_k \mathfrak{D}$-module then $\mathcal{R}V^* \cong \mathrm{Hom}_R(\mathcal{R}V, R)$ in $\mathcal{C}_1$.

Clearly $\mathcal{R}V$ is free over R. It has finite rank provided $\dim V < \infty$. If V is an absolutely irreducible $\mathfrak{gl}_k \mathfrak{D}$-module, then so is $\mathcal{R}V/(\mathfrak{m} \cdot \mathcal{R}V) \cong R/\mathfrak{m} \otimes_k V$ as a module over $\mathfrak{g}/\mathfrak{m}\mathfrak{g} \cong R/\mathfrak{m} \otimes_k \mathfrak{gl}_k \mathfrak{D}$ where $\mathfrak{m}$ is any maximal ideal of R. In this case $\mathcal{R}V$ satisfies (1.10) and (1.11). Therefore we can determine its cohomology applying theorems 5.5, 5.6 where we take $M = Q = \mathcal{R}V$. Note that Q corresponds to $M_0 = R$ under the equivalence

of Lemma 1.3. Denote by R^W the ring of W-invariant elements in R. In case char $k = 3$ assume $n > 1$. There are several cases.

(i) If $V = k$ is the trivial $\mathfrak{gl}_k \mathfrak{D}$-module then $\mathcal{R}V \cong R$ and

$$H^1(W, R) = H^1(\Omega) \oplus R^W \cdot \mathrm{cls}(\psi)$$

where the first summand is the subgroup of cohomology classes represented by the R-linear cocycles and the second summand is a free cyclic module over the ring R^W whose generator is the cohomology class of a divergence $\psi : W \to R$. One can take ψ with zero values on $\mathfrak{D}$. Then $\psi(f\partial_i) = \partial_i f$ for $f \in R$ and $i = 1, \ldots, n$.

(ii) If $V = \mathfrak{D}$ is the natural $\mathfrak{gl}_k \mathfrak{D}$-module, then we have $\mathcal{R}V \cong W$ and $H^1(W, W) \cong N/W$, where N is the normalizer of W in $\mathrm{Der}\, R$.

(iii) If $V = \mathfrak{D}^*$ then $\mathcal{R}V \cong \Omega^1$ and $H^1(W, \Omega^1)$ is a free cyclic module over R^W generated by the cohomology class of any cocycle $\varphi : W \to \Omega^1$ whose symbol is the canonical epimorphism $S^2\Omega^1 \otimes W \to \Omega^1$ in $\mathcal{C}_1$. By Lemma 4.4 one can take $\varphi = d \circ \psi$ where $\psi : W \to R$ is a divergence. If ψ is the same as in (i), then

$$\varphi(f\partial_i)(\partial_j) = \partial_j\big(\psi(f\partial_i)\big) = \partial_j\partial_i f \qquad \text{for } f \in R \text{ and } 1 \le i, j \le n.$$

(iv) Suppose $n > 1$ and char k does not divide $n + 1$. If V is the kernel of the canonical linear map $S^2\mathfrak{D}^* \otimes \mathfrak{D} \to \mathfrak{D}^*$ then $\mathcal{R}V \cong K$ is the kernel of the canonical epimorphism $\pi : S^2\Omega^1 \otimes W \to \Omega^1$ in $\mathcal{C}_1$. The group $H^1(W, K)$ is a free cyclic module over R^W generated by the cohomology class of the cocycle $\varphi = \pi_K \circ \varphi_u$ where $\pi_K : S^2\Omega^1 \otimes W \to K$ is the projection and $\varphi_u : W \to S^2\Omega^1 \otimes W$ the universal differential order 2 cocycle of proposition 4.1. We can take a torsion free connection $\nabla : W \times W \to R$ with zero restriction to $\mathfrak{D} \times \mathfrak{D}$. Let $f, g \in R$ and $1 \le i, j, j' \le n$. Then $\nabla(f\partial_j, g\partial_{j'}) = (f\partial_j g) \cdot \partial_{j'}$, and so

$$\varphi_u(f\partial_i)(\partial_j, \partial_{j'}) = -\nabla([f\partial_i, \partial_j], \partial_{j'}) - \nabla(\partial_j, [f\partial_i, \partial_{j'}]) = (\partial_j\partial_{j'} f) \cdot \partial_i$$

We have $\pi_K = \mathrm{id} - \frac{1}{n+1} \nu \circ \pi$ where $\nu : \Omega^1 \to S^2\Omega^1 \otimes W$ is the canonical morphism in $\mathcal{C}_1$ such that $\pi \circ \nu = (n+1) \cdot 1_{\Omega^1}$. Let $\varepsilon_1, \ldots, \varepsilon_n$ be the dual basis for the free R-module Ω^1, so that $\langle \varepsilon_i, \partial_j \rangle = \delta_{ij}$. Since $\sum_r \varepsilon_r \otimes \partial_r \in \mathfrak{g}$ corresponds to the identity endomorphism of Ω^1 under σ_{Ω^1}, the morphism ν takes $\theta \in \Omega^1$ to $\sum_r \theta\varepsilon_r \otimes \partial_r$. Now $\varphi_u(f\partial_i) = \frac{1}{2} \sum_{r,s} (\partial_r\partial_s f) \cdot \varepsilon_r\varepsilon_s \otimes \partial_i$. Applying π to this, we get $\sum_s (\partial_i\partial_s f) \cdot \varepsilon_s$ in Ω^1. Applying next ν, we get $\sum_{r,s} (\partial_i\partial_s f) \cdot \varepsilon_s\varepsilon_r \otimes \partial_r$ in $S^2\Omega^1 \otimes W$. It follows

$$\varphi(f\partial_i)(\partial_j, \partial_{j'}) = (\partial_j\partial_{j'} f) \cdot \partial_i - \frac{1}{n+1}(\partial_i\partial_j f) \cdot \partial_{j'} - \frac{1}{n+1}(\partial_i\partial_{j'} f) \cdot \partial_j.$$

(v) Suppose that $n = 1$. If $V = \mathfrak{D}^* \otimes \mathfrak{D}^*$ then $\mathcal{R}V \cong \Omega^1 \otimes \Omega^1$ and $H^1(W, \Omega^1 \otimes \Omega^1)$ is a free cyclic module over R^W generated by the cohomology class of the cocycle φ described in proposition 4.5. Recall that $\varphi = \psi^* \circ d \circ \psi$ where ψ is a divergence. As an R-module, $\Omega^1 \otimes \Omega^1$ is free with one generator $\varepsilon^2 = \varepsilon \otimes \varepsilon$, where $\varepsilon \in \Omega^1$ is specified by the relation $\langle \varepsilon, \partial_1 \rangle = 1$. We take ψ with zero value on ∂_1. Then $\psi^*(f\varepsilon)(\partial_1, \partial_1) = -\partial_1 f$, and so $\psi^*(f\varepsilon) = -(\partial_1 f) \cdot \varepsilon^2$, for $f \in R$. As $d\psi(f\partial_1) = (\partial_1^2 f) \cdot \varepsilon$, we get

$$\varphi(f\partial_1) = -(\partial_1^3 f) \cdot \varepsilon^2.$$

(vi) Suppose that V is an absolutely irreducible finite dimensional $\mathfrak{gl}_k \mathfrak{D}$-module other than those considered in (i)–(v). Then $H^1(W, \mathcal{R}V) \cong H_R^1(\mathfrak{g}, \mathcal{R}V)^W$. The standard cochain complex for $\mathfrak{g}$ admits the following identification:

$$C_R^\bullet(\mathfrak{g}, \mathcal{R}V) \cong \mathrm{Hom}_R(\textstyle\bigwedge \mathfrak{g}, \mathcal{R}V) \cong R \otimes_k \mathrm{Hom}_k(\textstyle\bigwedge \mathfrak{gl}_k \mathfrak{D}, V) = \mathcal{R}C^\bullet(\mathfrak{gl}_k \mathfrak{D}, V)$$

with the differential $1_R \otimes d_V$ where d_V is the differential of the standard cochain complex $C^\bullet(\mathfrak{gl}_k \mathfrak{D}, V)$ of k-multilinear alternating maps $\mathfrak{gl}_k \mathfrak{D} \times \cdots \times \mathfrak{gl}_k \mathfrak{D} \to V$. What we get above is, moreover, an isomorphism in the category $\mathcal{C}_1$. Since d_V is a $\mathfrak{gl}_k \mathfrak{D}$-equivariant map, the differential $1_R \otimes d_V$ is a morphism in $\mathcal{C}_1$. By passing to the cohomology we still get objects of $\mathcal{C}_1$ and an isomorphism in this category

$$H_R^\bullet(\mathfrak{g}, \mathcal{R}V) \cong R \otimes_k H^\bullet(\mathfrak{gl}_k \mathfrak{D}, V) = \mathcal{R}H^\bullet(\mathfrak{gl}_k \mathfrak{D}, V).$$

As $\mathfrak{gl}_k \mathfrak{D}$ acts in the cohomology group of its module V trivially, the action of W in the corresponding object of $\mathcal{C}_1$ is given by the rule $D \cdot (f \otimes \zeta) = Df \otimes \zeta$ for $D \in W$, $f \in R$ and $\zeta \in H^\bullet(\mathfrak{gl}_k \mathfrak{D}, V)$. Taking the W-invariants, we conclude

$$H^1(W, \mathcal{R}V) \cong R^W \otimes_k H^1(\mathfrak{gl}_k \mathfrak{D}, V).$$

We want to describe this isomorphism on the level of cocycles. Given a 1-cocycle $\varphi' : \mathfrak{gl}_k \mathfrak{D} \to V$, define a map $\varphi : W \to \mathcal{R}V$ by the rule

$$\varphi(f\partial_i) = -\sum_j \partial_j f \otimes \varphi'(E_{ij}) \qquad \text{for } f \in R \text{ and } i = 1, \ldots, n.$$

Clearly, φ is a differential operator of order 1. Since $\varphi(\partial_i) = 0$, we have

$$\varphi^\flat(df \otimes \partial_i) = \varphi(f\partial_i) = -\sum_j (1_R \otimes \varphi')(\partial_j f \otimes E_{ij}) = (1_R \otimes \varphi')\big(\sigma_W(df \otimes \partial_i)\big).$$

Thus $\varphi^\flat = (1_R \otimes \varphi') \circ \sigma_W$, that is, $\varphi^\flat : \mathfrak{g} \to \mathcal{R}V$ is an R-linear cocycle and $\mathrm{cls}(\varphi^\flat)$ corresponds to $1 \otimes \mathrm{cls}(\varphi')$, where cls stands for the cohomology class of a cocycle. Let us check that φ is itself a cocycle. Consider the first order prolongation $\widetilde{W}$ of W with kernel $\widetilde{\mathfrak{g}}$. By Lemma 3.3 there

is an R-linear map $\widetilde{\varphi} : \widetilde{W} \to \mathcal{R}V$ whose composite with the canonical embedding $\iota : W \to \widetilde{W}$ gives φ. Denote by $\eta : \widetilde{W} \times \widetilde{W} \to \mathcal{R}V$ its coboundary. Then $\eta(T, T') = 0$ for all $T, T' \in \widetilde{\mathfrak{g}}$ since $\widetilde{\varphi} \circ \iota^\flat = \varphi^\flat$ is a cocycle. Observe now that φ is a $\mathfrak{D}$-equivariant map and vanishes on $\mathfrak{D}$. Hence $\widetilde{\varphi}$ is also $\mathfrak{D}$-equivariant and vanishes on $\iota(\mathfrak{D})$. It follows that η has zero value whenever one of its arguments is in $\iota(\mathfrak{D})$. Since η is R-bilinear and $\widetilde{\mathfrak{g}} + \iota(\mathfrak{D})$ generates $\widetilde{W}$ as an R-module, η is identically zero. Thus $\widetilde{\varphi}$ is a cocycle, and so is φ as well. We see that $\mathrm{cls}(\varphi)$ corresponds to $1 \otimes \mathrm{cls}(\varphi')$.

7. Universal central extensions

Recall that *a $\mathbb{Z}$-split central extension* of W by a $\mathbb{Z}$-module V is an exact sequence $0 \to V \to L \to W \to 0$ where $L \to W$ is a homomorphism of Lie algebras over $\mathbb{Z}$ whose kernel is a central ideal and also a $\mathbb{Z}$-module direct summand of L. We call V *the kernel* of the central extension. An extension is said to be *split* if L contains a subalgebra mapped isomorphically onto W. The equivalence classes of $\mathbb{Z}$-split central extensions of W by V are in a one-to-one correspondence with the cohomology classes of $\mathbb{Z}$-bilinear 2-cocycles $W \times W \to V$, the coefficients being a trivial W-module. A $\mathbb{Z}$-split central extension L^u of W by a $\mathbb{Z}$-module U is *universal* if for every other $\mathbb{Z}$-module V and a $\mathbb{Z}$-split central extension L of W by V there is a unique $\mathbb{Z}$-linear map $U \to V$ which extends to a morphism between the two extensions:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & U & \longrightarrow & L^u & \longrightarrow & W & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & V & \longrightarrow & L & \longrightarrow & W & \longrightarrow & 0.
\end{array}
$$

In other words, a universal central extension corresponds to an isomorphism of functors $H^2(W, V) \cong \mathrm{Hom}_{\mathbb{Z}}(U, V)$ in V. Recall that $\Omega^\bullet = \bigwedge \Omega^1$ is the de Rham complex relative to W.

Theorem 7.1. *Assume that $3R = R$. If $\mathrm{rk}_R W > 1$ then every $\mathbb{Z}$-split central extension of W splits. If $\mathrm{rk}_R W = 1$ then the universal central extension of W has kernel $H^1(\Omega)$ and is determined by the 2-cocycle $\varphi : W \times W \to H^1(\Omega)$ such that $\varphi(D, D')$ for $D, D' \in W$ is the cohomology class of the 1-form $\psi(D) \cdot d\psi(D')$, where $\psi : W \to R$ is a divergence.*

Without assumption $3R = R$ the same proof shows that W has no nontrivial central extensions provided $\mathrm{rk}_R W > 2$.

Proof. Put $n = \operatorname{rk}_R W$. There is an obvious embedding

$$H^2(W, V) \hookrightarrow H^1\big(W, \operatorname{Hom}_{\mathbb{Z}}(W, V)\big) \tag{$*$}$$

obtained by separating the two arguments of the cocycles $W \times W \to V$. The image of $H^2(W, V)$ consists precisely of the cohomology classes represented by the 1-cocycles $\varphi : W \to \operatorname{Hom}_{\mathbb{Z}}(W, V)$ which satisfy $\varphi(D)(D) = 0$ for all $D \in W$. Consider the W-module $M = \operatorname{Hom}_{\mathbb{Z}}(W, V)$ and denote by ρ_M the corresponding representation of W. In a natural way M is a right module for the ring $\operatorname{End}_R W$, hence a left module for the opposite ring $(\operatorname{End}_R W)^{\mathrm{op}}$. In particular, M is also an R-module. Let $f \in R$, $D \in W$, $\xi \in M$. As $\rho_M(D)\xi = -\xi \circ \rho_W(D)$ and $f_M \xi = \xi \circ f_W$, we get

$$[\rho_M(D), f_M]\,\xi = \xi \circ \big(-f_W \circ \rho_W(D) + \rho_W(D) \circ f_W\big) = \xi \circ (Df)_W = (Df)_M \xi,$$

which verifies (1.6). For $T \in \mathfrak{g}$ define an endomorphism $\sigma_M(T) \in \operatorname{End}_R M$ by the rule

$$\sigma_M(T)\,\xi = \xi \circ \big(\gamma(T) \cdot 1_W - \sigma_W(T)\big), \qquad \xi \in M,$$

where $\gamma : \mathfrak{g} \to R$ is the contraction. As

$$\big(\rho_M(fD)\xi - f \cdot \rho_M(D)\xi\big)(D') = \xi\big(-[fD, D'] + [D, fD']\big)$$
$$= \xi\big((Df)D' - \sigma_W(df \otimes D)D'\big)$$

for $f \in R$ and $D, D' \in W$, we get (1.8). The other identities in the definition 1.1 are immediate. Thus M is now an object of $\mathcal{C}_1$. Put $Q = \Omega^1 \otimes \Omega^n$. This is an object of $\mathcal{C}_1$ satisfying (1.10), (1.11). Since Ω^n is a projective R-module of rank 1, its endomorphism algebra can be identified with R, which yields isomorphisms of R-algebras

$$\operatorname{End}_R Q \cong \operatorname{End}_R \Omega^1 \otimes \operatorname{End}_R \Omega^n \cong \operatorname{End}_R \Omega^1 \cong (\operatorname{End}_R W)^{\mathrm{op}}.$$

Let $T \in \mathfrak{g}$. The endomorphism $\sigma_{\Omega^n}(T)$ is just the multiplication by the trace of $\sigma_{\Omega^1}(T)$, that is, by $\gamma(T)$. Hence

$$\sigma_Q(T) = \sigma_{\Omega^1}(T) \otimes 1_{\Omega^n} + 1_{\Omega^1} \otimes \sigma_{\Omega^n}(T) = \sigma_{\Omega^1}(T) \otimes 1_{\Omega^n} + \gamma(T) \cdot 1_{\Omega^1} \otimes 1_{\Omega^n}.$$

Under the isomorphisms above it is sent to $\sigma_{\Omega^1}(T) + \gamma(T) \cdot 1_{\Omega^1}$ in $\operatorname{End}_R \Omega^1$, and then to $\gamma(T) \cdot 1_W - \sigma_W(T)$ in $(\operatorname{End}_R W)^{\mathrm{op}}$. Thus M is of type Q.

We apply theorems 5.5, 5.6 to compute $H^1(W, M)$. If $\mathfrak{m}$ is a maximal ideal of R then $Q/\mathfrak{m}Q \cong (W/\mathfrak{m}W)^* \otimes \bigwedge^n (W/\mathfrak{m}W)^*$. By Lemma 5.4 this $\mathfrak{g}/\mathfrak{m}\mathfrak{g}$-module is isomorphic to neither the trivial module, nor $W/\mathfrak{m}W$, nor $(W/\mathfrak{m}W)^*$, nor $K/\mathfrak{m}K$. Furthermore, $H^1(\mathfrak{g}/\mathfrak{m}\mathfrak{g}, Q/\mathfrak{m}Q) = 0$. It follows $H^1(W, M) = 0$ provided $n > 1$. If $n = 1$ then

$$H^1(W, M) \cong \operatorname{Mor}_{\mathcal{C}_1}(\Omega^1 \otimes \Omega^1, M).$$

The right hand side can be described as the W-invariant elements in

$$\mathrm{Hom}_R(\Omega^1 \otimes \Omega^1, M) \cong \mathrm{Hom}_{\mathbb{Z}}(\Omega^1 \otimes \Omega^1 \otimes W, V) \cong \mathrm{Hom}_{\mathbb{Z}}(\Omega^1, V)$$

because $\Omega^1 \otimes W \cong R$. Taking the W-invariants, we get

$$\mathrm{Mor}_{\mathcal{C}_1}(\Omega^1 \otimes \Omega^1, M) \cong \mathrm{Hom}_W(\Omega^1, V) \cong \mathrm{Hom}_{\mathbb{Z}}(\Omega^1/\rho_{\Omega^1}(W)(\Omega^1), V).$$

Since $\Omega^2 = 0$, the classical formula relating the action of W and the interior product on differential forms gives $\rho_{\Omega^1}(D)\,\theta = d(\langle \theta, D \rangle)$ for $\theta \in \Omega^1$, $D \in W$. Since the elements $\langle \theta, D \rangle$ span the whole R, again by the projectivity assumption, the group $\rho_{\Omega^1}(W)(\Omega^1)$ consists of all exact 1-forms. Hence $\Omega^1/\rho_{\Omega^1}(W)(\Omega^1) \cong H^1(\Omega)$.

Now take a $\mathbb{Z}$-linear map $\eta : H^1(\Omega) \to V$ and write out the corresponding 1-cocycle $\varphi : W \to M$. Tracing back the isomorphisms above, we first find the morphism $\zeta : \Omega^1 \otimes \Omega^1 \to M$ in $\mathcal{C}_1$. It is related to η as follows:

$$\zeta(\omega)(D) = \eta\big(\mathrm{cls}(i_D\omega)\big) \qquad \text{for} \quad \omega \in \Omega^1 \otimes \Omega^1, \ \ D \in W,$$

where $i_D\omega \in \Omega^1$ is given by $(i_D\omega)(D') = \omega(D, D')$ for $D' \in W$ (recall that $\Omega^1 \otimes \Omega^1$ can be identified with the group of R-bilinear maps $W \times W \to R$) and cls refers to the cohomology class of a 1-form. Then φ is the composite of ζ and the cocycle of proposition 4.5, that is, $\varphi = \zeta \circ \psi^* \circ d \circ \psi$ where $\psi : W \to R$ is a divergence. Note that

$$i_{D'}\big(\psi^*(\theta)\big) = \psi(D')\,\theta - d(\langle \theta, D' \rangle) \qquad \text{for} \quad \theta \in \Omega^1, \ \ D' \in W.$$

We have $\varphi(D) = \zeta(\omega)$ where $\omega = \psi^*(\theta)$ and $\theta = d\psi(D)$. As $i_{D'}\omega \equiv \psi(D')\theta$ modulo the exact 1-forms, it follows

$$\varphi(D)(D') = \eta\big(\mathrm{cls}(\psi(D') \cdot d\psi(D))\big) \qquad \text{for} \quad D, D' \in W.$$

Since $\psi(D) \cdot d\psi(D) = \frac{1}{2}d\psi(D)^2$ is an exact form, we have $\varphi(D)(D) = 0$. This shows that $(*)$ is an isomorphism and so

$$H^2(W, V) \cong \mathrm{Hom}_{\mathbb{Z}}\big(H^1(\Omega), V\big).$$

Thus the central extension corresponding to $V = H^1(\Omega)$ and the identity map $\eta : H^1(\Omega) \to V$ is universal. The corresponding cocycle is determined as well. $\square$

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