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ON A PROBLEM OF AVHADIEV

(submitted by F. Avhadiev)

Abstract. In this paper we consider a lower estimate for the ratio $I(\Omega)$ of the conformal moment of a simple connected domain Ω in the complex plane to the moment of inertia of this domain about its boundary. Related functionals depending on a simple connected domain Ω and two points $w_1, w_2 \in \Omega$ with fixed hyperbolical distance between them are estimated. As a consequence a nontrivial lower estimate for $I(\Omega)$ is obtained.

1. Introduction

Let Ω be a simply connected domain on the complex plane $\mathbb{C}$ and $w \in \Omega$. Let $\rho_\Omega(w)$ be the conformal radius of Ω at the point w, and $d_\Omega(w)$ be the Euclidean distance from the point w to the boundary $\partial\Omega$ of the domain Ω.

$I_c(\Omega) = \int_\Omega \rho_\Omega^2(x+iy)dxdy$ is the conformal moment of Ω, and $I(\partial\Omega) = \int_\Omega d_\Omega^2(x+iy)dxdy$ is the moment of inertia of Ω about $\partial\Omega$. This functionals were introduced by F.G. Avkhadiev [1] for solution of the classical St. Venant problem of finding two-side estimates for the torsional

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rigidity $P(\Omega)$ of the domain Ω by simple geometric characteristics of the domain [2, 3]. As a solution the following inequalities

$$I(\partial\Omega) \leqslant I_c(\Omega) \leqslant P(\Omega) \leqslant 4I_c(\Omega) \leqslant 64I(\partial\Omega) \tag{1}$$

were obtained. The first and the last inequalities in (1) is the corollary of well-known inequalities for the ratio of the conformal radius of a domain at a point and the distance from this point to the domain boundary (see for e.g.[4, 5]). The first inequality in (1) is not sharp when $I_c(\Omega)$ and $I(\partial\Omega)$ are finite. F.G. Avkhadiev set a problem to find lower and upper sharp bounds for the ratio

$$I(\Omega) = \frac{I_c(\Omega)}{I(\partial\Omega)}, \tag{2}$$

when $I_c(\Omega)$ and $I(\partial\Omega)$ are finite. Note that $I(\Omega)$ is invariant under linear transforms of Ω.

The first and the last inequalities in (1) imply that the region of values $I(\Omega)$ is a subset of $[1, 16]$. In this paper we give a better lower estimate for $I(\Omega)$.

Let $w_1, w_2 \in \Omega$, and let a holomorphic univalent function $f(z)$ maps the unit disk $\mathbb{U} = \{z : |z| < 1\}$ onto Ω so that $f(0) = w_1$ and $f(r) = w_2$, where $r \in (0, 1)$ is a given constant. The normalization of f means, that the hyperbolical distance between w_1 and w_2 is constant. Let us consider the functional

$$G(f, \alpha) = \frac{|f'(0)|^\alpha + |f'(r)|^\alpha (1 - |r|^2)^\alpha}{d_\Omega(f(0))^\alpha + d_\Omega(f(r))^\alpha}, \quad \alpha > 0, \tag{3}$$

on the class S^0 of univalent holomorphic functions $f(z), z \in \mathbb{U}$.

Obviously $G(f, \alpha) = G(\beta + \gamma f, \alpha), \gamma \neq 0$. It is possible to consider the functional $G(f, \alpha)$ as a function of Ω and two points $w_1, w_2 \in \Omega$ with fixed hyperbolical distance between them. Namely,

$$G(f, \alpha) = \frac{\rho_\Omega(w_1)^\alpha + \rho_\Omega(w_2)^\alpha}{d_\Omega(w_1)^\alpha + d_\Omega(w_2)^\alpha}.$$

Let r_α be the unique solution of the equation

$$1 - \frac{4r}{(1-r)^2} + \frac{(1-r)^{2\alpha}}{(1+r)^{2\alpha}} + \frac{(1+r)^{2\alpha}}{(1-r)^{2\alpha}} \Big(\min_{\gamma,\theta \in \mathbb{R}} \Re e \Big[\frac{1 - 2re^{i\gamma} - r^2 e^{2i\gamma}}{(1 + re^{i\gamma})^2} + \\ + \frac{2r^2}{1-r^2} \frac{1 - re^{i\gamma}}{1 + re^{i\gamma}} + \frac{4re^{i\theta}}{1-r^2} \frac{1 - re^{i\gamma}}{1 + re^{i\gamma}} \Big] - \frac{4r}{(1-r)^2} \Big) = 0,$$

on the interval $(0, 1)$. Then we have

Theorem 1. *If $r < r_\alpha$, then every function $f(z)$ minimizing the functional $G(f,\alpha)$ on the class S^0 maps $\mathbb{U}$ onto an arc biangle bounded by circle arcs centered at the points $f(0)$ and $f(r)$.*

To improve the lower estimate of $I(\Omega)$ we need to find the sharp bound of the functional

$$F(f,r,c) = \frac{|f'(0)|^4 c + |f'(r)|^4(1-r^2)^4}{d_{f(\mathbb{U})}(f(0))^2|f'(0)|^2 c + d_{f(\mathbb{U})}(f(r))^2|f'(r)|^2(1-r^2)^2}, \qquad c>0, 0<r<1, \tag{4}$$

on class S^0. We have

Proposition 1. *If $r < r_0$, then every function $f(z)$ minimizing the functional $F(f,r,c)$ on the class S^0 maps $\mathbb{U}$ onto an arc biangle, bounded by circle arcs centered at the points $f(0)$ and $f(r)$.*

The value of r_0 will be given in the proof of Proposition 1.

By $f_{\theta,d}$ denote a function mapping $\mathbb{U}$ onto the arc biangle satisfying the following requirement: one of bounding circles has center at the origin and unit radius, the other bounding circle has center at the point $d > 0$ and θ is the argument of the intersection point of this circle lying in the upper half-plane. Taking into account that the functional $F(f,r,c)$ is invariant under linear transforms of the complex plane, we conclude that there are $d > 0, \theta \in [0,\pi]$ such that $F(f_{\theta,d}, r, c) = \sup\limits_{f\in S^0} F(f,r,c)$.

Now we can give a lower estimate for $I(\Omega)$ in terms of the functional $F(f,r,c)$.

Theorem 2. *For any simply connected domain Ω, for which $I_c(\Omega)$ and $I(\partial\Omega)$ are finite, the estimate*

$$I(\Omega) \geqslant \min_{r_1 \leqslant f_{\theta,d}^{-1}(d) \leqslant r_0} \{F(f_{\theta,d}, f_{\theta,d}^{-1}(d), \frac{1}{1-r^{*2}}), F(f_{\theta,d}, f_{\theta,d}^{-1}(d), 1-r^{*2})\} = 1.031021...$$

holds.

The values of r^*, r_1 will be given in the proof of Theorem 2.

2. Proof of theorem 1 and proposition 1

Proof of Theorem 1. It is possible to assume that $f \in S$, where S is the class of all holomorphic univalent functions $f(z)$ in $\mathbb{U}$ normalized by $f(0) = f'(0) - 1 = 0$. Let $k(z) = \frac{z}{(1-z)^2}$, $k_\varepsilon(z) = k^{-1}((1-\varepsilon)k(z))$ and $k_\varepsilon^\gamma(z) = e^{-i\gamma}k_\varepsilon(e^{i\gamma}z)$. For ε small enough we have:

$$k_\varepsilon^\gamma(z) = z - \varepsilon z \frac{1 - ze^{i\gamma}}{1 + ze^{i\gamma}} + O(\varepsilon^2). \tag{5}$$

The function $k_\varepsilon^\gamma(z)$ maps $\mathbb{U}$ onto $\mathbb{U}$ minus radial slit with endpoint at $-e^{-i\gamma}$. Note that the slit length tends to 0 as $\varepsilon \to 0$.

Assume the opposite. Then there are two open disks $D_1, D_2 \subset \mathbb{U}, \mathbb{U} \not\subset D_1 \cup D_2$, with centers at points 0 and $f(r)$ and radii $d(0)$ and $d(f(r))$ respectively. Therefore, there are $\gamma \in \mathbb{R}$ and $\varepsilon_0 > 0$ such that $D_1, D_2 \subset f_\varepsilon(\mathbb{U})$, where $f_\varepsilon(z) = f(k_\varepsilon^\gamma(z)), 0 \leqslant \varepsilon \leqslant \varepsilon_0$.

By construction $d(f_\varepsilon(0)) = d(f(0)) = d(0)$, and $|f_\varepsilon'(0)| = (1-\varepsilon)|f'(0)| = (1-\varepsilon)$. Moreover taking into account (5), we have

$$f_\varepsilon(z) = f(z) - \varepsilon f'(z) z \frac{1 - ze^{i\gamma}}{1 + ze^{i\gamma}} + O(\varepsilon^2). \tag{6}$$

Using (6) and $D_m \subset f_\varepsilon(\mathbb{U}), m = 1, 2$ we get

$$\begin{aligned} d(f_\varepsilon(r)) &\geqslant d(f(r)) - \varepsilon \Big| f'(r) r \frac{1 - re^{i\gamma}}{1 + re^{i\gamma}} \Big| + O(\varepsilon^2) \geqslant \\ &\geqslant d(f(r)) \Big(1 - \varepsilon \frac{|f'(r)|}{d(f(r))} r \frac{1+r}{1-r} \Big) + O(\varepsilon^2). \end{aligned} \tag{7}$$

Note that

$$\frac{|f'(r)|}{d(r)} = \frac{(1-r^2)|f'(r)|}{(1-r^2) d(f(r))},$$

and $|f'(r)|(1-r^2)$ is the conformal radius of the domain. Thus $|f'(r)|/d(r) \leqslant \frac{4}{1-r^2}$. Therefore

$$\begin{aligned} d(f_\varepsilon(r)) \geqslant d(f(r)) \Big(1 - \varepsilon 4r \frac{1+r}{(1-r^2)(1-r)} \Big) + O(\varepsilon^2) = \\ d(f(r)) \Big(1 - \varepsilon \frac{4r}{(1-r)^2} \Big) + O(\varepsilon^2). \end{aligned}$$

Let $p(r) = \frac{4r}{(1-r)^2}$.

Using (6) we have

$$f_\varepsilon'(z) = f'(z) - \varepsilon \Big(z f'(z) \frac{1 - ze^{i\gamma}}{1 + ze^{i\gamma}} \Big)' + O(\varepsilon^2).$$

So

$$|f_\varepsilon'(z)| = |f'(z)| \Big(1 - \varepsilon \Re e \Big[\frac{1 - 2ze^{i\gamma} - z^2 e^{2i\gamma}}{(1 + ze^{i\gamma})^2} + \frac{z f''(z)}{f'(z)} \frac{1 - ze^{i\gamma}}{1 + ze^{i\gamma}} \Big] \Big) + O(\varepsilon^2).$$

Let $q(r) = \Re e \Big[\frac{1 - 2re^{i\gamma} - r^2 e^{2i\gamma}}{(1 + re^{i\gamma})^2} + \frac{r f''(r)}{f'(r)} \frac{1 - re^{i\gamma}}{1 + re^{i\gamma}} \Big]$. Taking into account (7), we obtain

$$G(f_\varepsilon,\alpha)\leqslant\frac{|f'(0)|^\alpha(1-\alpha\varepsilon)+|f'(r)|^\alpha(1-|r|^2)^\alpha(1-q(r)\alpha\varepsilon)}{d(f(0))^\alpha+d(f(r))^\alpha(1-p(r)\alpha\varepsilon)}+O(\varepsilon^2). \tag{8}$$

Let

$$A=|f'(0)|^\alpha,$$
$$B=(1-r^2)^\alpha|f'(r)|^\alpha,$$
$$C=d(f(0))^\alpha,$$
$$D=d(f(r))^\alpha.$$

Then inequality (8) takes the form

$$G(f_\varepsilon,\alpha)\leqslant\frac{A(1-\varepsilon\alpha)+B(1-q(r)\alpha\varepsilon)}{C+D(1-p(r)\alpha\varepsilon)}+O(\varepsilon^2),$$

or

$$G(f_\varepsilon,\alpha)\leqslant\frac{A+B}{C+D}-\frac{(C+D)(A+Bq(r))-(A+B)Dp(r)}{(C+D)^2}\alpha\varepsilon+O(\varepsilon^2).$$

Therefore, in order to prove that every function $f(z)$ minimizing the functional $G(f,\alpha)$ on class S map $\mathbb{U}$ onto an arc biangle, it is sufficient to show that

$$(C+D)(A+Bq(r))-(A+B)Dp(r)>0,$$

or

$$1-p(r)+\frac{C}{D}+\frac{B}{A}(q(r)-p(r))>0. \tag{9}$$

Let us find a lower bound of $\frac{C}{D}$. Using the obvious inequality $d(f(r))\leqslant d(f(0))+|f(r)|$, we have

$$\frac{C}{D}\geqslant\Big(\frac{d(f(0))}{d(f(0))+|f(r)|}\Big)^\alpha.$$

Since $f\in S$,

$$\frac{C}{D}\geqslant\Big(\frac{1}{1+4|f(r)|}\Big)^\alpha.$$

Using the growth theorem in the class S, we have

$$\frac{C}{D}\geqslant\Big(\frac{(1-r)^2}{(1-r)^2+4r}\Big)^\alpha=\frac{(1-r)^{2\alpha}}{(1+r)^{2\alpha}}.$$

By the distortion theorem for the class S

$$\frac{(1-r)^{2\alpha}}{(1+r)^{2\alpha}} \leqslant \frac{A}{B} \leqslant \frac{(1+r)^{2\alpha}}{(1-r)^{2\alpha}}.$$

Tacking into account that the left part of (9) is monotonic on $\frac{B}{A}$, we conclude that the following inequalities

$$1-p(r)+\frac{(1-r)^{2\alpha}}{(1+r)^{2\alpha}}+\frac{(1-r)^{2\alpha}}{(1+r)^{2\alpha}}(q(r)-p(r))>0$$

and

$$1-p(r)+\frac{(1-r)^{2\alpha}}{(1+r)^{2\alpha}}+\frac{(1+r)^{2\alpha}}{(1-r)^{2\alpha}}(q(r)-p(r))>0,$$

imply inequality (9). Substituting into this inequality p, q, using the estimate for function $f \in S$ (see foe e.g. [6, p. 32])

$$\Big|\frac{rf''(r)}{f'(r)}-\frac{2r^2}{1-r^2}\Big| \leqslant \frac{4r}{1-r^2} \tag{10}$$

and the maximum principle for harmonic functions we have, that inequalities

$$\begin{gathered} M_1(r)=1-\frac{4r}{(1-r)^2}+\frac{(1-r)^{2\alpha}}{(1+r)^{2\alpha}}+ \\ +\frac{(1-r)^{2\alpha}}{(1+r)^{2\alpha}}\Big(\min_{\gamma,\theta\in\mathbb{R}} R(r,\gamma,\theta)-\frac{4r}{(1-r)^2}\Big)>0, \end{gathered} \tag{11}$$

$$\begin{gathered} M_2(r)=1-\frac{4r}{(1-r)^2}+\frac{(1-r)^{2\alpha}}{(1+r)^{2\alpha}}+ \\ +\frac{(1+r)^{2\alpha}}{(1-r)^{2\alpha}}\Big(\min_{\gamma,\theta\in\mathbb{R}} R(r,\gamma,\theta)-\frac{4r}{(1-r)^2}\Big)>0, \end{gathered} \tag{12}$$

where

$$R(r,\gamma,\theta)=\Re e\Big[\frac{1-2re^{i\gamma}-r^2e^{2i\gamma}}{(1+re^{i\gamma})^2}+\frac{2r^2}{1-r^2}\frac{1-re^{i\gamma}}{1+re^{i\gamma}}+\frac{4re^{i\theta}}{1-r^2}\frac{1-re^{i\gamma}}{1+re^{i\gamma}}\Big]$$

imply (9).

Consider the functions

$$P(r)=1-\frac{4r}{(1-r)^2}$$

and

$$Q(r)=\min_{\gamma,\theta\in\mathbb{R}} R(r,\gamma,\theta)-\frac{4r}{(1-r)^2}.$$

They are monotonic on the interval $(0,1)$. So they may have only one zero on this interval. Computations show, that $P(r)$ equals zero at the point

$3-2\sqrt{2}=0.1715...$, and $Q(r)$ equals zero at the point $r'=0.086427....$ Therefore we have that the roots of equations $M_1(r)=0$ and $M_2(r)=0$ lie in the interval $(r',1)$. It is easy to see that, for $r\in(r',1)$, $M_2(r)>M_1(r)$ and so if inequality (12) holds, then inequality (11) holds too. Using the inequality $Q(r)<0$ and the fact that $Q(r)$ is decreasing on the interval $(r',1)$ we have, that the function $M_2(r)$ is decreasing on this interval too. Thus, we conclude that if $r<r_\alpha$, where r_α is the single root of the equation

$$1-\frac{4r}{(1-r)^2}+\frac{(1-r)^{2\alpha}}{(1+r)^{2\alpha}}+\frac{(1+r)^{2\alpha}}{(1-r)^{2\alpha}}\Big(\min_{\gamma,\theta\in\mathbb{R}}R(r,\gamma,\theta)-\frac{4r}{(1-r)^2}\Big)=0$$

on the interval $(0,1)$, then every function minimizing functional (3) maps $\mathbb{U}$ onto an arc biangle, bounded by two circle arcs centered at the points $f(0)$ and $f(r)$. Note that $r_\alpha>r'$.

Proof of Proposition 1. Using the same argument as in the proof of theorem 1 and following notation

$$A=c|f'(0)|^4,$$

$$B=|f'(r)|^4(1-r^2)^4,$$

$$C=d(f(0))^2|f'(0)|^2c,$$

$$D=d(g(r))^2|f'(r)|^2(1-r^2)^2,$$

we have

$$F(f_\varepsilon,c,r)\leqslant\frac{A+B}{C+D}-$$

$$-\varepsilon\frac{4(A+q(r)B)(C+D)-2(A+B)(C+(p+q)D)}{(C+D)^2}+O(\varepsilon^2),$$

where $f_\varepsilon(z)=f(k_\varepsilon^\gamma(z))$. Thus the inequality

$$4(A+q(r)B)(C+D)-2(A+B)(C+(p(r)+q(r))D)>0$$

or

$$A(C+D(2-p(r)-q(r)))+B(D(q(r)-p(r))+C(-1+2q(r))))>0 \quad (13)$$

implies that any function minimizing functional (4), maps $\mathbb{U}$ onto an arc biangle. Using inequalities $A,B,C,D>0$ we have that inequalities

$$\begin{aligned} q(r)&>p(r),\\ p(r)+q(r)&<2, \qquad (14)\\ q(r)&>0.5 \end{aligned}$$

guarantee inequality (13). Substituting in (14) the functions $p(r)$ and $q(r)$, using estimate (10) and the maximum principle for harmonic functions, we have that the following inequalities

$$\begin{gathered} \min_{\gamma,\theta\in\mathbb{R}} R(r,\gamma,\theta) - \frac{4r}{(1-r)^2} > 0, \\ \frac{4r}{(1-r)^2} + \max_{\gamma,\theta\in\mathbb{R}} R(r,\gamma,\theta) < 2, \\ \min_{\gamma,\theta\in\mathbb{R}} R(r,\gamma,\theta) > 0.5 \end{gathered} \tag{15}$$

imply (13).

Functions in the left-side of (15) are monotonic on the interval (0,1). Computations show that, the first was violated at $r_{01} = 0.086427...$, the second at $r_{02} = 0.071796...$, the third at $r_{03} = 0.071796...$. thus we conclude that if

$$r < r_0 = \max\{r_{01}, r_{02}, r_{03}\},$$

then every function every minimizing functional (4) maps $\mathbb{U}$ onto an arc biangle, bounded by circle arcs centered at the points $f(0)$ and $f(r)$.

3. Proof of Theorem 2

Proof of Theorem 2. First we divide $\mathbb{U}$ into two subsets $\mathbb{U}_1, \mathbb{U}_2$ and construct a map $\xi(z) : \mathbb{U}_1 \to \mathbb{U}_2$ such that Jacobian of $\xi(z)$ equals the unity almost everywhere and the inequality

$$s(\xi(z), z) = \left|\frac{z - \xi(z)}{1 - \overline{z}\xi(z)}\right| < r_0, \text{a.e.}$$

holds.

First, we consider a disk $\mathbb{U}^*$ of radius r^* centered at the origin.

Let us consider the square s inscribed in $\mathbb{U}^*$ with sides parallel to the axes of coordinates. Let s_1, s_2, s_3, s_4 be the intersection of the square s with the first, the second, the third and the fourth quarters of the complex plane. Let c_1, c_3 be the sectors of $\mathbb{U}^*$, cutting off the sides of the square s, parallel to the axis OX, and by c_2, c_4 sectors, cutting off the sides of the square s, paralleling to the axis OY.

Let $s_1, s_2, c_1, c_2 \subset \mathbb{U}_1$, $s_3, s_4, c_2, c_4 \subset \mathbb{U}_2$ and

$$\xi(z) = \begin{cases} z - \frac{r^*}{\sqrt{2}} - i\frac{r^*}{\sqrt{2}}, & z \in s_1 \\ z + \frac{r^*}{\sqrt{2}} - i\frac{r^*}{\sqrt{2}}, & z \in s_2 \\ e^{i\frac{\pi}{2}} z, & z \in c_1, c_3. \end{cases}$$

Let us find the lower and upper sharp bounds of $s(\xi(z), z)$, for $z \in c_1, c_3$. It is easy to see that

$$s(\xi(z), z) = \frac{|z||1-i|}{|1+|z|^2 i|} = \frac{\sqrt{2}|z|}{\sqrt{1+|z|^4}}.$$

The function $\frac{r}{\sqrt{1+r^4}}$ increases on the interval $(0,1)$ and so we have

$$\frac{2r^*}{\sqrt{4+r^{*4}}} \leqslant s(\xi(z), z) \leqslant \frac{\sqrt{2}r^*}{\sqrt{1+r^{*4}}}, z \in c_1, c_3.$$

By construction, $s(\xi(z), z) < r_0$ a.e., so

$$r^* = \frac{\sqrt{1-\sqrt{1-r_0^4}}}{r_0} = 0.0507682....$$

Let us find the lower and upper sharp bounds of $s(\xi(z), z)$, for $z \in s_1$. Let $z = x + iy$, then we have

$$s(\xi(z), z) = \frac{r^*}{\left|1-(x+iy)\left(x - \frac{r^*}{\sqrt{2}} - i\left(y - \frac{r^*}{\sqrt{2}}\right)\right)\right|}.$$

So

$$s(\xi(z), z)^2 = \frac{r^{*2}}{\frac{r^{*2}(x-y)^2}{2} + \left(1 - x^2 - y^2 + \frac{r^*(x+y)}{\sqrt{2}}\right)^2}.$$

Let us consider the function

$$q(x,y) = 1 - x^2 - y^2 + \frac{r^*(x+y)}{\sqrt{2}}.$$

When $z \in s_1$, the function $q(x,y)$ is nonnegative, and attains its minimal value at points $\frac{r^*}{\sqrt{2}}$, $i\frac{r^*}{\sqrt{2}}$, 0, $\frac{r^*}{\sqrt{2}} + i\frac{r^*}{\sqrt{2}}$. Thus the function $\frac{r^{*2}}{2}(x-y)^2$ is minimal at $x = y$, then

$$\frac{r^{*2}(x-y)^2}{2} + \left(1 - x^2 - y^2 + \frac{r^*(x+y)}{\sqrt{2}}\right)^2$$

attains minimum at the point 0, so we have $s(\xi(z), z) \leqslant r^*$, for $z \in s_1$.

Let find the upper sharp bound for the function

$$p(x,y) = \frac{r^{*2}(x-y)^2}{2} + \left(1 - x^2 - y^2 + \frac{r^*(x+y)}{\sqrt{2}}\right)^2.$$

Calculations show that the stationary points of $p(x,y)$ are the following

$$\frac{r^*}{2\sqrt{2}} + i\frac{r^*}{2\sqrt{2}},$$

$$\frac{r^* - \sqrt{4-r^{*2}}}{2\sqrt{2}} + i\frac{r^* + \sqrt{4-r^{*2}}}{2\sqrt{2}},$$

$$\frac{r^*+\sqrt{4-r^{*2}}}{2\sqrt{2}}+i\frac{r^*-\sqrt{4-r^{*2}}}{2\sqrt{2}},$$
$$\frac{r^*-\sqrt{4+r^{*2}}}{2\sqrt{2}}+i\frac{r^*-\sqrt{4+r^{*2}}}{2\sqrt{2}},$$
$$\frac{r^*+\sqrt{4+r^{*2}}}{2\sqrt{2}}+i\frac{r^*+\sqrt{4+r^{*2}}}{2\sqrt{2}}.$$

Note that only the first point lies in s_1 provided $r^* = r_0^{-1}\sqrt{1-\sqrt{1-r_0^4}} =$ 0.0507....

The second partial derivative at the point $\frac{r^*}{2\sqrt{2}} + i\frac{r^*}{2\sqrt{2}}$ equals

$$p_{xx}\Big(\frac{r^*}{2\sqrt{2}},\frac{r^*}{2\sqrt{2}}\Big) = p_{yy}\Big(\frac{r^*}{2\sqrt{2}},\frac{r^*}{2\sqrt{2}}\Big) = \frac{(16+24r^{*2}+r^{*4})}{16},$$

$$p_{xy}\Big(\frac{r^*}{2\sqrt{2}},\frac{r^*}{2\sqrt{2}}\Big) = \frac{(4-r^{*2})^2}{16}.$$

So we have

$$p_{xx}p_{yy} - p_{xy}^2 = \frac{r^{*2}(4+r^{*2})^2}{4} > 0.$$

Therefore the function $p(x,y)$ may attain its maximum only in this point or on the boundary of the square s_1 or at the point $\frac{r^*}{2\sqrt{2}} + i\frac{r^*}{2\sqrt{2}}$. Let us show that $p(x,y)$ attains maximum at the point $\frac{r^*}{2\sqrt{2}} + i\frac{r^*}{2\sqrt{2}}$. Simple calculations show that

$$p_x\Big(\frac{r^*}{\sqrt{2}},y\Big) = -\sqrt{2}r^*\Big(1-\Big(\frac{r^*}{\sqrt{2}}-y\Big)^2\Big) < 0, \quad y \in \Big[0,\frac{r^*}{\sqrt{2}}\Big]$$

$$p_x(0,y) = \sqrt{2}r^*(1-y^2) > 0, \quad y \in \Big[0,\frac{r^*}{\sqrt{2}}\Big]$$

$$p_y\Big(x,\frac{r^*}{\sqrt{2}}\Big) = -\sqrt{2}r^*\Big(1-\Big(\frac{r^*}{\sqrt{2}}-x\Big)^2\Big) < 0, \quad x \in \Big[0,\frac{r^*}{\sqrt{2}}\Big]$$

$$p_y(x,0) = \sqrt{2}r^*(1-x^2) > 0. \quad x \in \Big[0,\frac{r^*}{\sqrt{2}}\Big]$$

Thus we have

$$\frac{4r^*}{4+r^{*2}} \leqslant s(\xi(z),z) \leqslant \frac{\sqrt{2}r^*}{\sqrt{1+r^{*4}}} = r_0, z \in s_1, s_2, c_1, c_3.$$

Let us consider the ring $1 > |z| > r^*$. Divide the circle $|z| = r$, $1 > r > r^*$ into two disjoint subsets, consisting of pairwise disjoint arc, contracting angle $\frac{\pi}{n}$ and translating into each other after rotation on angle $\frac{\pi}{n}$. Let us find n such that

$$s(\xi(z), z) = \frac{r|1 - e^{i\frac{\pi}{n}}|}{|1 - r^2 e^{i\frac{\pi}{n}}|} \leqslant r_0,$$

or

$$\cos\frac{\pi}{n} \geqslant \frac{2r^2 - r_0^2(1 + r^4)}{2r^2(1 - r_0^2)}.$$

Thus

$$n \geqslant A(r, r_0) = \frac{\pi}{\arccos\frac{2r^2 - r_0^2(1+r^4)}{2r^2(1-r_0^2)}}.$$

Let

$$n = [A(r, r_0)] + 1.$$

It is easy to check

$$s(z, \xi(z)) = \frac{r|1 - e^{i\frac{\pi}{n}}|}{|1 - r^2 e^{i\frac{\pi}{n}}|} = r\sqrt{\frac{2 - 2\cos\frac{\pi}{[A(r,r_0)]+1}}{1 + r^4 - 2r^2\cos\frac{\pi}{[A(r,r_0)]+1}}} \geqslant$$
$$r\sqrt{\frac{2 - 2\cos\frac{\pi}{A(r,r_0)+1}}{1 + r^4 - 2r^2\cos\frac{\pi}{A(r,r_0)+1}}}.$$

Calculations show that it attains minimum on the set $[r^*, 1)$ at the point $r = r^*$, and minimal value equals 0.0508335....

Thus, we have

$$r_1 = \frac{4r^*}{4 + r^{*2}} = 0.0507355... \leqslant s(\xi(z), z) \leqslant r_0 = 0.0717968...., z \in \mathbb{U}_1.$$

By construction we have that the measure of the set $\{z : s(z, \xi(z)) = r_0\}$ is zero.

Let us find the lower and upper sharp bounds for

$$c(z) = \frac{1 - |z|^2}{1 - |\xi(z)|^2} \tag{16}$$

in $z \in \mathbb{U}_1$. By construction we have $c(z) = 1$, if $z \in c_1, c_2$ or $|z| > r^*$. Let us consider the case of $z \in s_1$. Then it is easy to see that (16) attains minimum at the point $\frac{\sqrt{2}}{2}r + i\frac{\sqrt{2}}{2}r$ and maximum at the point 0. So we have that (16) lies between $1 - r^{*2}$ and $(1 - r^{*2})^{-1}$.

Now we can give an improved lower estimate of (2). Let $f(z)$ map $\mathbb{U}$ onto Ω, then we have

$$I_c(\Omega) = \int_{\mathbb{U}} \left[|f'(z)|^4(1-|z|^2)^2\right]dxdy =$$

$$\int_{\mathbb{U}_1} \left[|f'(z)|^4(1-|z|^2)^2\right]dxdy + \int_{\mathbb{U}_2} \left[|f'(z)|^4(1-|z|^2)^2\right]dxdy =$$

$$\int_{\mathbb{U}_1} \left[|f'(z)|^4(1-|z|^2)^2 + |f'(\xi(z))|^4(1-|\xi(z)|^2)^2\right]dxdy.$$

Similarly,

$$I(\partial\Omega) = \int_{\mathbb{U}} \left[|f'(z)|^2 d(f(z))^2\right]dxdy =$$

$$\int_{\mathbb{U}_1} \left[|f'(z)|^2 d(f(z))^2 + |f'(\xi(z))|^2 d(f(\xi(z)))^2\right]dxdy.$$

So we have that functional (2) is not greater then

$$\frac{|f'(z)|^4(1-|z|^2)^2 + |f'(\xi(z))|^4(1-|\xi(z)|^2)^2}{|f'(z)|^2 d(f(z))^2 + |f'(\xi(z))|^2 d(f(\xi(z)))^2}. \tag{17}$$

Making change of variable $\psi(w) = \frac{w-z}{1-\bar{z}w}$, we have that (17) is equal to

$$\frac{|g'(0)|^4 c + |g'(r)|^4(1-r^2)^4}{d(0)^2|g'(0)|^2 c + d(g(r))^2|g'(r)|^2(1-r^2)^2}, \tag{18}$$

where $g(w) = f \circ \psi(w)$, $r = s(\xi(z), z)$ and $c = \frac{(1-|z|^2)^2}{(1-|\xi(z)|^2)^2}$. Using

$$r_1 < r < r_0, \quad (1-r^{*2})^2 \leqslant \frac{(1-|\xi(z)|^2)^2}{(1-|z|^2)^2} \leqslant \frac{1}{(1-r^{*2})^2},$$

proposition 1 and that (18) is monotonic on c, we have

$$I(\Omega) \geqslant \min_{r_1 \leqslant f_{\theta,d}^{-1}(d) \leqslant r_0} \{F(f_{\theta,d}, f_{\theta,d}^{-1}(d), \frac{1}{1-r^{*2}}), F(f_{\theta,d}, f_{\theta,d}^{-1}(d), 1-r^{*2})\}.$$

Numerical calculations show that the right hand side of above inequality is not less then 1.031021....

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