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ON THE HIGHER ORDER GEOMETRY OF WEIL BUNDLES OVER SMOOTH MANIFOLDS AND OVER PARAMETER-DEPENDENT MANIFOLDS

(submitted by B. N. Shapukov)

Abstract. The Weil bundle $T^{\mathbb{A}}M_n$ of an n-dimensional smooth manifold M_n determined by a local algebra $\mathbb{A}$ in the sense of A. Weil carries a natural structure of an n-dimensional $\mathbb{A}$-smooth manifold. This allows ones to associate with $T^{\mathbb{A}}M_n$ the series $B^r(\mathbb{A})T^{\mathbb{A}}M_n$, $r = 1, \dots, \infty$, of $\mathbb{A}$-smooth r-frame bundles. As a set, $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ consists of r-jets of $\mathbb{A}$-smooth germs of diffeomorphisms $(\mathbb{A}^n, 0) \to T^{\mathbb{A}}M_n$. We study the structure of $\mathbb{A}$-smooth r-frame bundles. In particular, we introduce the structure form of $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ and study its properties.

Next we consider some categories of m-parameter-dependent manifolds whose objects are trivial bundles $M_n \times \mathbb{R}^m \to \mathbb{R}^m$, define (generalized) Weil bundles and higher order frame bundles of m-parameter-dependent manifolds and study the structure of these bundles. We also show that product preserving bundle functors on the introduced categories of m-parameter-dependent manifolds are equivalent to generalized Weil functors.

1. Introduction.

The Weil bundle $T^{\mathbb{A}}M_n$ of a smooth manifold M_n corresponding to a local Weil algebra $\mathbb{A}$ was introduced by A. Weil [40] as a generalization

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of the bundle $T^q_m M_n$ of (m,q)-velocities of C. Ehresmann [6]. Various aspects of geometry of Weil bundles were studied by P.C. Yuen [43], [44], L.-N. Patterson [26], A. Morimoto [23], A.P. Shirokov [30], E. Okassa [25], I. Kolář [12], M. Doupovec and I. Kolář [4], A.Ya. Sultanov [35], J. Muñoz, J. Rodriguez, and F.J. Muriel [24] and other researchers.

Studying product preserving functors on the category of smooth manifolds, D.J. Eck [5], G. Kainz and P. Michor [10], and O.O. Luciano [19] proved that such functors reduce to Weil functors $T^{\mathbb{A}}$, which assign to a manifold M_n their Weil bundles $T^{\mathbb{A}}M_n$. In a series of papers [20], [15], [13] I. Kolář and W.M. Mikulski clarified the structure of product preserving and fiber product preserving bundle functors on the category of fibered manifolds. A. Kriegl and P.W. Michor [16] studied product preserving functors of infinite dimensional manifolds.

A.P. Shirokov [29], [30] discovered that the Weil bundle $T^{\mathbb{A}}M_n$ carries a natural structure of an n-dimensional manifold over the algebra $\mathbb{A}$, that is, a manifold whose local coordinates take values in $\mathbb{A}$ and coordinate transformations are $\mathbb{A}$-smooth in the sense of G. Scheffers [28]. Natural structures of manifolds over algebras arise also on semitangent bundles studied by V.V. Vishnevsky, [37], [38]. Extensive lists of references on the subject can be found in [30], [14], [37], [38], [34].

The bundle $T^{\mathbb{A}}M_n$ is naturally associated with the principal q-frame bundle B^qM_n whose structure group is the differential group G^q_n, where q is the height (order) of $\mathbb{A}$. The $\mathbb{A}$-smooth structure of $T^{\mathbb{A}}M_n$ implies that $T^{\mathbb{A}}M_n$ can be considered as a bundle with structure group $D_n(\mathbb{A})$, so-called [31], [34] $\mathbb{A}$-affine differential group. In the case of tangent bundle TM_n, this group is the group of affine transformations of $\mathbb{R}^n$. In [2], it was shown that $D_n(\mathbb{A})$ appears as a natural structure group of the (generalized) Weil bundle $\widehat{T}^{\mathbb{A}}(M_n\times\mathbb{R}^\ell)$, ℓ is the width of $\mathbb{A}$, over an object of the category of trivial bundles $M_n\times\mathbb{R}^\ell\to\mathbb{R}^\ell$ with smooth mappings $M_n\times\mathbb{R}^\ell\to M'_k\times\mathbb{R}^\ell$ projecting into the identity mapping $\mathrm{id}:\mathbb{R}^\ell\to\mathbb{R}^\ell$ as morphisms. $\widehat{T}^{\mathbb{A}}(M_n\times\mathbb{R}^\ell)\to M_n\times\mathbb{R}^\ell$ is defined as the bundle over $M_n\times\mathbb{R}^\ell$ of $\mathbb{A}$-velocities of smooth sections $\mathbb{R}^\ell\to M_n\times\mathbb{R}^\ell$. Note that Weil bundles of type $T^{\mathbb{A}}M\times\mathbb{R}$ and natural affinors on such bundles were studied by M. Doupovec and I. Kolář in [4].

This paper is devoted to the study of higher order geometry of Weil bundles considered as manifolds over algebras and to the study of geometry of (generalized) Weil bundles over m-parameter-dependent manifolds $M_n\times\mathbb{R}^m$.

In Section 2, we recall necessary notions and results from the theory of Weil bundles and smooth manifolds over Weil algebras. We also describe here the category $\mathcal{M}f^m$ of m-parameter-dependent smooth manifolds and outline some results from [3] concerning the structure of product preserving bundle functors on $\mathcal{M}f^m$.

In Section 3, we study the structure of the bundle $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ of $\mathbb{A}$-smooth r-frames over the Weil bundle $T^{\mathbb{A}}M_n$. We construct the structure form Θ^r of $B^r(\mathbb{A})T^{\mathbb{A}}M_n$, study its properties, and derive the structure equations of $B^r(\mathbb{A})T^{\mathbb{A}}M_n$. In particular, it is proved that a local diffeomorphism of $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ which maps the structure form Θ^r into itself coincides in a neighborhood of every point with the prolongation of a local $\mathbb{A}$-diffeomorphism of $T^{\mathbb{A}}M_n$.

In Section 4, we study the structure of product preserving bundle functors on the category $\mathcal{M}f^m_{\mathrm{tr}}$ whose objects are the trivial fiber bundles $M_n \times U \to U$, where M_n is a smooth manifold and U is an open subset of $\mathbb{R}^m$, and whose morphisms are the smooth mappings $M_n \times U \to M' \times U'$ which project into translations of $\mathbb{R}^m$. It is proved that each such functor is equivalent to a (generalized) Weil functor $\widehat{T}^{\mathbb{A}}_{\sigma}$ determined by a Weil algebra $\mathbb{A}$ and a constant section $\sigma : \mathbb{R}^m \to T^{\mathbb{A}}\mathbb{R}^m$.

Section 5 is devoted to the higher order geometry of manifolds from the category $\mathcal{M}f^m_{\mathrm{tr}}$. We construct the principal r-frame bundles $\widehat{B}^r(M_n \times U)$ associated with a manifold $M_n \times U$ from $\mathcal{M}f^m_{\mathrm{tr}}$, define the structure form of $\widehat{B}^r(M_n \times \mathbb{R}^m)$, prove that a local diffeomorphism of $\widehat{B}^r(M_n \times U)$ which maps the structure form into itself is the natural prolongation of a local isomorphism from the category $\mathcal{M}f^m_{\mathrm{tr}}$, and derive the structure equations of $\widehat{B}^r(M_n \times U)$. We also construct connections in $\widehat{B}^r(M_n \times U)$ and the associated connections in the Weil bundles $\widehat{T}^{\mathbb{A}}_{\sigma}(M_n \times U)$.

2. Preliminaries.

2.1. **Weil algebras.** A finite-dimensional commutative associative $\mathbb{R}$-algebra $\mathbb{A}$ with unity $1_{\mathbb{A}}$ is called a local Weil algebra or, briefly, a Weil algebra [40], [14], [34] if it has a unique maximal ideal $\mathfrak{m} = \mathfrak{m}(\mathbb{A})$ consisting of all nilpotent elements of $\mathbb{A}$ and the quotient algebra $\mathbb{A}/\mathfrak{m}$ is isomorphic to $\mathbb{R}$.

The dimension ℓ of the quotient algebra $\mathfrak{m}/\mathfrak{m}^2$ is called the *width* of $\mathbb{A}$. The natural number q defined by the relations $\mathfrak{m}^q \neq 0$, $\mathfrak{m}^{q+1} = 0$ is called the *height* or the *order* of $\mathbb{A}$. The ideal $\mathfrak{m}$ is generated by every collection of elements $\{\tau^a\}$, $a = 1, \dots, \ell$, such that the collection of residue classes $\{\tau^a + \mathfrak{m}^2\}$ is a basis of $\mathfrak{m}/\mathfrak{m}^2$. Following V.V. Wagner [39], we will call such a collection $\{\tau^a\}$, $a = 1, \dots, \ell$, a *pseudobasis* of $\mathfrak{m}$ (and of $\mathbb{A}$). Every element X of $\mathbb{A}$ can be represented in the form of a linear combination of products of powers of pseudobasis elements: $X = \sum_{|p|=0}^{q} X_p \tau^p$, where $p = (p_1, \dots, p_\ell)$ is a multiindex of length ℓ, $|p| = p_1 + \dots + p_\ell$, $\tau^p = (\tau^1)^{p_1} \dots (\tau^\ell)^{p_\ell}$, $\tau^0 = 1_{\mathbb{A}}$.

A Weil algebra $\mathbb{A}$ of width ℓ and height q is isomorphic to a quotient algebra of the algebra $\mathbb{R}[[t^1, \dots, t^\ell]] = \mathbb{R}(\ell, \infty)$ of formal power series in ℓ variables $t^1, \dots, t^\ell$ with coefficients in $\mathbb{R}$. The mapping φ which assigns to a formal power series $\sum_{|p|=0}^{\infty} \alpha_p t^p$ (as above, $p = (p_1 \dots, p_\ell)$ is a multiindex) the element $\sum_{|p|=0}^{\infty} \alpha_p \tau^p \in \mathbb{A}$ is an epimorphism of algebras, and $\mathbb{A} \cong \mathbb{R}(\ell, \infty)/\ker \varphi$. The epimorphism of algebras $\varphi : \mathbb{R}(\ell, \infty) \to \mathbb{A}$ induces the epimorphism of the modules of n-tuples $\varphi : \mathbb{R}(\ell, \infty)^n \to \mathbb{A}^n$, which, for simplicity, is denoted by the same symbol φ.

2.2. **Weil bundles.** Let M_n be a smooth (C^∞ differentiable) manifold, and let $f : (\mathbb{R}^\ell, 0) \to (M_n, x)$ be a smooth germ. In a local chart (U, h) on M_n, where $h : U \ni x \mapsto \{x^i = h^i(x)\} \in U' \subset \mathbb{R}^n$, $i = 1, \dots, n$, to the jet $j^\infty f$ of f there corresponds the jet $j^\infty(h \circ f)$ which can be considered as the collection of n jets $j^\infty(h^i \circ f)$ or formal Taylor series of the germs $h^i \circ f$. Therefore, $j^\infty(h \circ f)$ can be considered as an element of the module $\mathbb{R}(\ell, \infty)^n$. The jets $j^\infty f$ and $j^\infty g$ of germs $f, g : (\mathbb{R}^\ell, 0) \to (M_n, x)$ are said to be $\mathbb{A}$-equivalent if $\varphi(j^\infty(h \circ f)) = \varphi(j^\infty(h \circ g))$ or $\varphi(j^\infty(h \circ f - h \circ g)) = 0$. $\mathbb{A}$-equivalence of jets $j^\infty f$ and $j^\infty g$ does not depend on the choice of a chart (U, h) and is an equivalence relation on the set $J_0^\infty(\mathbb{R}^\ell, M_n)$ of ∞-jets of smooth germs from $\mathbb{R}^\ell$ to M_n at zero (see [34], [14]). The equivalence class of a jet $j^\infty f$ is called an $\mathbb{A}$-*jet* or an $\mathbb{A}$-*velocity* on M_n and denoted by $j^{\mathbb{A}} f$. On the set $T^{\mathbb{A}} M_n$ of $\mathbb{A}$-velocities on M_n, there arises a natural structure of a smooth manifold. The natural projection $\pi : T^{\mathbb{A}} M_n \to M_n$ which assigns to the $\mathbb{A}$-velocity of a germ $f : (\mathbb{R}^\ell, 0) \to (M_n, x)$ the point $x \in M_n$ turns the manifold $T^{\mathbb{A}} M_n$ into a fiber bundle over M_n called the *Weil bundle*. A smooth mapping $g : M_n \to M'_k$ induces naturally the mapping of Weil bundles $T^{\mathbb{A}} g : T^{\mathbb{A}} M_n \to T^{\mathbb{A}} M'_k$, $j^{\mathbb{A}} f \mapsto j^{\mathbb{A}}(g \circ f)$. The correspondence $T^{\mathbb{A}}$ which assigns to a smooth manifold M_n the bundle $T^{\mathbb{A}} M_n$ and to a smooth mapping g the mapping $T^{\mathbb{A}} g$ is a functor from the category of smooth manifolds to the category of fibered manifolds. The functor $T^{\mathbb{A}}$ is called a *Weil functor*. The Weil functor $T^{\mathbb{A}}$ depends on the choice of a pseudobasis in $\mathbb{A}$ (or on an epimorphism φ). The choice of another pseudobasis gives an equivalent functor.

A chart (U, h) on a manifold M_n induces the mapping $h^{\mathbb{A}} : \pi^{-1}(U) \to \mathbb{A}^n$, $j^{\mathbb{A}} f \mapsto \varphi(j^\infty(h \circ f))$, which defines an $\mathbb{A}^n$-valued chart $(\pi^{-1}(U), h^{\mathbb{A}})$ on the bundle $T^{\mathbb{A}} M_n$, and the C^∞ atlas $\{(U_\alpha, h_\alpha)\}_{\alpha \in A}$ on M_n induces the

C^∞ atlas $\{(\pi^{-1}(U_\alpha), h_\alpha^{\mathbb{A}})\}_{\alpha\in A}$ on $T^{\mathbb{A}}M_n$ defining on $T^{\mathbb{A}}M_n$ a structure of an $\mathbb{A}$-smooth manifold modeled on the $\mathbb{A}$-module $\mathbb{A}^n$ [34]. In the case of manifolds $\mathbb{R}$ and $\mathbb{R}^n$, this leads to the natural identifications [34], [33]

$$T^{\mathbb{A}}\mathbb{R} \equiv \mathbb{A}, \quad T^{\mathbb{A}}\mathbb{R}^n \equiv \mathbb{A}^n. \tag{1}$$

In what follows, as a rule, the maximal ideal $\mathfrak{m} = \mathfrak{m}(\mathbb{A})$ of a Weil algebra $\mathbb{A}$ will be denoted by $\mathring{\mathbb{A}}$. By $\mathring{\mathbb{A}}^n$ we will denote the submodule in $\mathbb{A}^n$ consisting of all elements with components belonging to $\mathring{\mathbb{A}}$. On the identifications (1), the fibers $T_0^{\mathbb{A}}\mathbb{R}$ and $T_0^{\mathbb{A}}\mathbb{R}^n$ of the bundles $T^{\mathbb{A}}\mathbb{R}$ and $T^{\mathbb{A}}\mathbb{R}^n$ are identified with the ideal $\mathring{\mathbb{A}}$ and the submodule $\mathring{\mathbb{A}}^n$ respectively.

The mapping $h^{\mathbb{A}}$ maps the domain $\pi^{-1}(U)$ onto $U' \times \mathring{\mathbb{A}}^n \subset \mathbb{A}^n$. As this takes place, the fibers of $T^{\mathbb{A}}M_n$ are mapped bijectively onto the submodule $\mathring{\mathbb{A}}^n$. Thus, $T^{\mathbb{A}}M_n$ is a locally trivial fiber bundle with standard fiber $\mathring{\mathbb{A}}^n$. The bundle $T^{\mathbb{A}}M_n$ is naturally associated with the q-frame bundle B^qM_n [42], [14], [34] of M_n whose elements are the q-jets of germs of diffeomorphisms $(\mathbb{R}^n, 0) \to M_n$. The structure group of B^qM_n is the differential group G_n^q consisting of the q-jets of germs of diffeomorphisms $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$. The group G_n^q acts on the left on $\mathring{\mathbb{A}}^n \equiv T_0^{\mathbb{A}}\mathbb{R}^n$ as follows: if $j^q\varphi \in G_n^q$ and $j^{\mathbb{A}}f \in T_0^{\mathbb{A}}\mathbb{R}^n$ are determined, respectively, by germs φ and f, then

$$G_n^q \times T_0^{\mathbb{A}}\mathbb{R}^n \ni (j^q\varphi, j^{\mathbb{A}}f) \mapsto j^q\varphi \circ j^{\mathbb{A}}f = j^{\mathbb{A}}(\varphi \circ f).$$

Thus, to the bundle $T^{\mathbb{A}}M_n$, there is naturally associated the sequence of principal bundles of higher order frames

$$\begin{aligned} M_n \xleftarrow{\pi_0^1} B^1M_n \xleftarrow{\pi_1^2} B^2M_n \xleftarrow{\pi_2^3} \\ \dots \xleftarrow{\pi_{q-1}^q} B^qM_n \xleftarrow{\pi_q^{q+1}} B^{q+1}M_n \xleftarrow{\pi_{q+1}^{q+2}} \dots \longleftarrow B^\infty M_n\,, \end{aligned} \tag{2}$$

where $B^\infty M_n$ is the bundle of infinite order frames, the limit of the projective system $M_n \leftarrow B^1M_n \leftarrow B^2M_n \leftarrow \dots$ endowed with the corresponding structure of an infinite-dimensional smooth manifold in the sense of Bernshtein–Rozenfeld [1]. The bundle $B^\infty M_n$ is formed by the infinite order jets of germs of diffeomorphisms $(\mathbb{R}^n, 0) \to M_n$. The group G_n^∞ consisting of the infinite order jets of germs of diffeomorphisms $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ acts naturally on the right on $B^\infty M_n$.

2.3. **The structure of $\mathbb{A}$-smooth mappings.** Let $U \subset \mathbb{A}^n$ be an open subset. A smooth mapping $F : U \subset \mathbb{A}^n \to \mathbb{A}^k$ is called $\mathbb{A}$-*smooth* if the tangent mapping $T_XF : T_XU \cong \mathbb{A}^n \to T_{F(X)}\mathbb{A}^k \cong \mathbb{A}^k$ is $\mathbb{A}$-linear for any

$X \in U$. Let $\{e_a\}$, $a = 0, 1, \dots, k$, $e_0 = 1_{\mathbb{A}} \equiv 1 \in \mathbb{R}$, be a basis in $\mathbb{A}$, γ^a_{bc} the structure constants of $\mathbb{A}$ with respect to the basis $\{e_a\}$, and let $X^i = x^{ia}e_a$, $F^{i'} = f^{i'b}e_b$, $i = 1, \dots, n$, $i' = 1, \dots, k$, be expansions of elements of $\mathbb{A}$ in terms of $\{e_a\}$ written in accordance with the standard summation convention. A mapping $U \ni \{X^i\} \mapsto F^{i'}(X^i) = f^{i'b}(x^{ia})e_b \in \mathbb{A}$ is $\mathbb{A}$-smooth if and only if it satisfies the Scheffers conditions ([28]; [34],(2.6); [33], (10))

$$\partial_{ia} f^{i'b} = \gamma^b_{ag} \partial_{i0} f^{i'g}, \quad \text{where} \quad \partial_{ia} f^{i'b} = \partial f^{i'b}/\partial x^{ia}. \tag{3}$$

For an $\mathbb{A}$-smooth function $F(X^i)$, the partial derivatives $\partial F/\partial X^i$ with respect to the variables X^i are defined, and (see [34], [33])

$$\partial F/\partial X^i = \partial F/\partial x^{i0}. \tag{4}$$

An arbitrary $\mathbb{A}$-smooth mapping $F : U \subset \mathbb{A}^n \to \mathbb{A}^k$ is of the form [34], [33]

$$X^{i'} = f^{i'}(x^i) + \sum_{|p|=1}^{q} \frac{1}{p!} \frac{D^p f^{i'}}{Dx^p} \overset{\circ}{X}{}^p, \tag{5}$$

where $p = (p_1 \dots, p_n)$ is a multiindex of length n, $\overset{\circ}{X}{}^p = (\overset{\circ}{X}{}^1)^{p_1} \dots (\overset{\circ}{X}{}^n)^{p_n}$, $\overset{\circ}{X} \in \overset{\circ}{\mathbb{A}}$ denotes the component of $X \in \mathbb{A}$ in accordance with the decomposition $\mathbb{A} = \mathbb{R} \oplus \overset{\circ}{\mathbb{A}}$, and $f : U \ni \{x^i\} \mapsto \{y^{i'} = f^{i'}(x^i)\} \in \mathbb{A}^k$ is a smooth mapping projectable with respect to the *canonical* $\overset{\circ}{\mathbb{A}}{}^n$*-foliation* on U generated by the projection $\pi : \mathbb{A}^n \to \mathbb{R}^n$. If U is a simple open set [22] for the canonical $\overset{\circ}{\mathbb{A}}{}^n$-foliation, i. e., the preimages of points from $\pi(U)$ under the projection $\pi : \mathbb{A}^n \to \mathbb{R}^n$ are connected, then $\mathbb{A}$-smooth mapping $F : U \to \mathbb{A}^k$ prolongs uniquely to an $\mathbb{A}$-smooth mapping

$$\tilde{F} : \pi^{-1}(\pi(U)) \to \mathbb{A}^k. \tag{6}$$

In this case, we have $f = (\widetilde{F}|\pi(U)) \circ \pi$.

This implies, in particular, that every $\mathbb{A}$-smooth germ $F : (\mathbb{A}^n, 0) \to \mathbb{A}^k$ is defined along the whole fiber $\overset{\circ}{\mathbb{A}}{}^n$, i. e., it is an equivalence class of $\mathbb{A}$-smooth mappings defined on neighborhoods of the form $\pi^{-1}(U)$, where U are neighborhoods of zero in $\mathbb{R}^n$. For any two germs of $\mathbb{A}$-diffeomorphisms $F_1 : (\mathbb{A}^n, 0) \to (\mathbb{A}^n, \overset{\circ}{X})$, $F_2 : (\mathbb{A}^n, 0) \to (\mathbb{A}^n, \overset{\circ}{Y})$, where $\overset{\circ}{X}, \overset{\circ}{Y} \in \overset{\circ}{\mathbb{A}}{}^n$, the composition $F_2 \circ F_1 : (\mathbb{A}^n, 0) \to \mathbb{A}^n$ is well-defined.

The mapping $F : U \subset \mathbb{A}^n \to \mathbb{A}^k$ defined by (5) is called the $\mathbb{A}$*-prolongation* (*analytic prolongation* [34]) of $f : U \to \mathbb{A}^k$.

Similar relations hold for $\mathbb{A}$-smooth germs of the form

$$F : (T^{\mathbb{A}} M_n, X) \to (T^{\mathbb{A}} M'_k, Y).$$

Every such a germ is defined along the whole fiber $T^{\mathbb{A}}_x M_n$, where $x = \pi(X)$, and is the $\mathbb{A}$-prolongation of a smooth germ $f : (M_n, x) \to (T^{\mathbb{A}} M'_k, Y)$. In terms of local coordinates, the germs f and F are given, respectively, by equations $y^{i'} = f^{i'}(x^i)$ and (5).

2.4. **The category of m-parameter-dependent manifolds $\mathcal{M}f^m$.** In [3], the following category $\mathcal{M}f^m$ of m-parameter-dependent manifolds was considered. The objects of $\mathcal{M}f^m$ are the trivial fiber bundles $p : M_n \times \mathbb{R}^m \to \mathbb{R}^m$, where M_n is a smooth manifold. The morphisms of $\mathcal{M}f^m$ are the commutative diagrams of the form

$$\begin{tikzcd} M_n \times \mathbb{R}^m \arrow[r, "f"] \arrow[d, "p"'] & M'_k \times \mathbb{R}^m \arrow[d, "p'"] \\ \mathbb{R}^m \arrow[r, "\mathrm{id}"] & \mathbb{R}^m \end{tikzcd} \tag{7}$$

In terms of local coordinates (x^i, t^a), $i = 1, \dots, n$, $a = 1, \dots, m$, on $M_n \times \mathbb{R}^m$ and $(x^{i'}, t^a)$, $i' = 1', \dots, k'$, on $M'_k \times \mathbb{R}^m$, a morphism (7) is given by equations $x^{i'} = f^{i'}(x^i, t^a)$, $t^{a'} = t^a$.

The category $\mathcal{FM}^m$ of m-parameter-dependent fibered manifolds is defined as follows. The objects of $\mathcal{FM}^m$ are the commutative diagrams

$$\begin{tikzcd} E \times \mathbb{R}^m \arrow[r, "p_E"] \arrow[d, "\pi"'] & \mathbb{R}^m \arrow[d, "\mathrm{id}"] \\ M_n \times \mathbb{R}^m \arrow[r, "p"] & \mathbb{R}^m \end{tikzcd}$$

where $p_E : E \times \mathbb{R}^m \to \mathbb{R}^m$ and $p : M_n \times \mathbb{R}^m \to \mathbb{R}^m$ are objects of $\mathcal{M}f^m$. The morphisms of $\mathcal{FM}^m$ are the commutative diagrams

$$\begin{tikzcd} & E \times \mathbb{R}^m \arrow[rr, "f"] \arrow[dl, "\pi"'] \arrow[dd] & & E' \times \mathbb{R}^m \arrow[dl, "\pi'"'] \arrow[dd] \\ M_n \times \mathbb{R}^m \arrow[rr, "\overline{f}"] \arrow[dd, "p"'] & & M'_k \times \mathbb{R}^m \arrow[dd, "p'"] & \\ & \mathbb{R}^m \arrow[rr, "\mathrm{id}"] \arrow[dl, "\mathrm{id}"'] & & \mathbb{R}^m \arrow[dl, "\mathrm{id}"'] \\ \mathbb{R}^m \arrow[rr, "\mathrm{id}"] & & \mathbb{R}^m & \end{tikzcd} \tag{8}$$

The base functor

$$B : \mathcal{FM}^m \to \mathcal{M}f^m$$

is defined as follows:

$$\begin{array}{ccc} E\times\mathbb{R}^m & \longrightarrow & \mathbb{R}^m \\ \big\downarrow & & \big\downarrow \\ M_n\times\mathbb{R}^m & \longrightarrow & \mathbb{R}^m \end{array} \quad \longmapsto \quad (M_n\times\mathbb{R}^m\to\mathbb{R}^m), \quad (f,\overline{f})\mapsto\overline{f}\,.$$

The functor

$$\varepsilon:\mathcal{FM}^m\to\mathcal{M}f^m$$

which erases the fibered structure is defined as follows:

$$\begin{array}{ccc} E\times\mathbb{R}^m & \longrightarrow & \mathbb{R}^m \\ \big\downarrow & & \big\downarrow \\ M_n\times\mathbb{R}^m & \longrightarrow & \mathbb{R}^m \end{array} \quad \longmapsto \quad (E\times\mathbb{R}^m\to\mathbb{R}^m), \quad (f,\overline{f})\mapsto f\,.$$

Note 1. Objects of the category $\mathcal{M}f$ of smooth manifolds can be considered as fiber bundles of the form $M_n\to\{\mathrm{pt}\}$, where $\{\mathrm{pt}\}=\mathbb{R}^0$ is a fixed one-point manifold. With this in mind, we will identify the categories $\mathcal{M}f$ and $\mathcal{M}f^0$.

Every smooth mapping $\gamma:\mathbb{R}^{m'}\to\mathbb{R}^m$, by means of the pullback construction [14], defines the functor $R_\gamma:\mathcal{M}f^m\to\mathcal{M}f^{m'}$ which acts on morphisms as follows:

$$R_\gamma(f(x,t))=f(x,\gamma(t))\,.$$

Such functors satisfy the relation

$$R_{\gamma_1}\circ R_{\gamma_2}=R_{\gamma_2\circ\gamma_1}\,. \tag{9}$$

In particular, the mapping

$$t:\{\mathrm{pt}\}\to\mathbb{R}^m,\quad \{\mathrm{pt}\}\mapsto t\in\mathbb{R}^m,$$

defines the functor $R_t:\mathcal{M}f^m\to\mathcal{M}f$ which acts on objects and morphisms, respectively, as follows: $R_t(M_n\times\mathbb{R}^m)=M_n$, $R_t(f:M_n\times\mathbb{R}^m\to M'_k\times\mathbb{R}^m)=(f|_t:M_n\to M'_k)$. The action of these functors on objects is the same for all $t\in\mathbb{R}^m$:

$$R_t|_{\mathrm{Ob}(\mathcal{M}f^m)}=R_0|_{\mathrm{Ob}(\mathcal{M}f^m)}\,,$$

where R_0 is the functor corresponding to $t=0$.

The mapping $\mathrm{pt}:\mathbb{R}^m\to\{\mathrm{pt}\}$ defines the functor $R_{\mathrm{pt}}:\mathcal{M}f\to\mathcal{M}f^m$:

$$R_{\mathrm{pt}}(M_n)=M_n\times\mathbb{R}^m,\quad R_{\mathrm{pt}}(f)=f\times\mathrm{id},$$

which embeds $\mathcal{M}f$ into $\mathcal{M}f^m$.

A covariant functor

$$F:\mathcal{M}f^m\to\mathcal{FM}^m \tag{10}$$

satisfying the *prolongation* condition $B \circ F = \mathrm{id}_{\mathcal{M}f^m}$ is called a *prolongation functor*.

To a prolongation functor (10), one can associate the functor $F_0 : \mathcal{M}f \to \mathcal{M}f$ defined by

$$F_0 = R_0 \circ \varepsilon \circ F \circ R_{\mathrm{pt}}\,.$$

A prolongation functor (10) is called a *bundle functor* if it satisfies the following *localization* condition (see [14] for the case of the category $\mathcal{M}f$): if $V \subset M_n$ is an open subset and $i : V \times \mathbb{R}^m \to M_n \times \mathbb{R}^m$ is an $\mathcal{M}f^m$-inclusion, then $F_0(V) \times \mathbb{R}^m = \pi_M^{-1}(V \times \mathbb{R}^m)$ and $F(i) : \pi_M^{-1}(V \times \mathbb{R}^m) \to F_0(M_n) \times \mathbb{R}^m$ is an $\mathcal{FM}^m$-inclusion (for brevity, we indicate only the upper rows of the diagrams).

The action of a bundle functor F on objects can be written as follows:

$$F :\ (M_n \times \mathbb{R}^m \to \mathbb{R}^m) \quad \longmapsto \quad \begin{tikzcd} F_0(M_n) \times \mathbb{R}^m \arrow[r] \arrow[d] & \mathbb{R}^m \arrow[d, "\mathrm{id}"] \\ M_n \times \mathbb{R}^m \arrow[r] & \mathbb{R}^m \end{tikzcd}$$

2.5. **Products in the category $\mathcal{M}f^m$.** The product of two objects C_1 and C_2 of a category $\mathscr{C}$ is defined [17] to be a triple $(P, \mathrm{pr}_1, \mathrm{pr}_2)$ consisting of an object P and two morphisms $\mathrm{pr}_1 : P \to C_1$, $\mathrm{pr}_2 : P \to C_2$ of $\mathscr{C}$ satisfying the following property: for any object D and any morphisms $f_1 : D \to C_1$ and $f_2 : D \to C_2$, there exists a unique morphism $f : D \to P$ such that $f_1 = \mathrm{pr}_1 \circ f$ and $f_2 = \mathrm{pr}_2 \circ f$, i. e., the diagram

$$\begin{tikzcd} C_1 & P \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] & C_2 \\ & D \arrow[ul, "f_1"] \arrow[u, dashed, "f"] \arrow[ur, "f_2"'] & \end{tikzcd} \tag{11}$$

is commutative.

In [3], it was shown that the triple $(R_{\mathrm{pt}}(M_n \times M'_k), R_{\mathrm{pt}}(\mathrm{pr}_1), R_{\mathrm{pt}}(\mathrm{pr}_2))$ is a product of the objects $p_1 : M_n \times \mathbb{R}^m \to \mathbb{R}^m$ and $p_2 : M'_k \times \mathbb{R}^m \to \mathbb{R}^m$ of the category $\mathcal{M}f^m$.

2.6. **Product preserving bundle functors on the category $\mathcal{M}f^m$.** Let $\mathscr{C}$ and $\mathscr{D}$ be arbitrary categories. A functor $F : \mathscr{C} \to \mathscr{D}$ is called [14] a *product preserving functor* if, for any product $(A, \mathrm{pr}_1, \mathrm{pr}_2)$ in the category $\mathscr{C}$, the triple $(F(A), F(\mathrm{pr}_1), F(\mathrm{pr}_2))$ is a product in the category $\mathscr{D}$.

Let $\mathcal{M}f \times \mathbb{R}^m$ denote the subcategory of $\mathcal{M}f^m$ whose class of objects coincides with that of the category $\mathcal{M}f^m$ and whose morphisms are of the form $f\times \mathrm{id} : M_n\times\mathbb{R}^m \to M'_k\times\mathbb{R}^m$, where $f : M_n\to M'_k$ is an arbitrary smooth mapping. The restriction of a bundle functor $F:\mathcal{M}f^m\to\mathcal{FM}^m$ to the subcategory $\mathcal{M}f\times\mathbb{R}^m$ is denoted by $\overline{F}:\mathcal{M}f\times\mathbb{R}^m\to\mathcal{FM}^m$. If F preserves products, then $\overline{F}$ also preserves products.

The following theorem has been proved in [3].

Theorem 1. *Let $F:\mathcal{M}f^m\to\mathcal{FM}^m$ be a product preserving bundle functor. Then the functor $\overline{F}:\mathcal{M}f\times\mathbb{R}^m\to\mathcal{FM}^m$ is naturally equivalent to an m-parameter family of Weil functors $\widetilde{T}^{\mathbb{A}(t)}$.*

A product preserving bundle functor $F:\mathcal{M}f^m\to\mathcal{FM}^m$ is uniquely determined (up to a natural equivalence) by an m-parameter family of Weil functors $\widetilde{T}^{\mathbb{A}(t)}$ and a collection of functions $\mathbb{R}^m\ni t\mapsto \overset{\circ}{S}{}^a(t)\in\overset{\circ}{\mathbb{A}}(t)$, $a=1,\dots,m$, which gives a section $\mathbb{R}^m\to\mathbb{A}(t)^m\times\mathbb{R}^m$.

For exact definitions of the notions of m-parameter families of Weil algebras and Weil functors, see Section 4.

In addition, in terms of local coordinates, the morphism $F(f)$ has equations of the form

$$Y^i=f^i(x,t)+\sum_{p+s=1}^{q}\frac{1}{p!s!}\frac{\partial^{|p+s|}f^i}{\partial x^p\partial t^s}\overset{\circ}{X}{}^p(\overset{\circ}{S}(t))^s. \qquad (12)$$

Equations (12) can be rewritten as follows:

$$Y^i=f^i(x,t)+\sum_{p=0}^{q}\frac{1}{p!}\frac{\partial^{|p|}}{\partial x^p}\left\{\sum_{s=1}^{q}\frac{1}{s!}\frac{\partial^{|s|}f^i}{\partial t^s}(\overset{\circ}{S}(t))^s\right\}\overset{\circ}{X}{}^p=$$
$$=\widehat{f^i}(x,t)+\sum_{p=1}^{q}\frac{1}{p!}\frac{\partial^{|p|}\widehat{f^i}}{\partial x^p}\overset{\circ}{X}{}^p,$$

where

$$\widehat{f^i}(x,t)=f^i(x,t)+\sum_{s=1}^{q}\frac{1}{s!}\frac{\partial^{|s|}f^i}{\partial t^s}(\overset{\circ}{S}(t))^s,$$

whence it follows that the restriction of $F(f)$ to the fiber over $t\in\mathbb{R}^m$ is an $\mathbb{A}(t)$-smooth mapping.

3. Principal fiber bundles of $\mathbb{A}$-smooth frames on $T^{\mathbb{A}}M_n$.

3.1. **The bundle $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ of $\mathbb{A}$-smooth r-frames on $T^{\mathbb{A}}M_n$.** An $\mathbb{A}$-*smooth frame of order* r (an $\mathbb{A}$-*smooth* r-*frame*) on $T^{\mathbb{A}}M_n$ is the r-jet of a germ of $\mathbb{A}$-diffeomorphism $\Phi:(\mathbb{A}^n,0)\to(T^{\mathbb{A}}M_n,X)$ [32]. The set $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ of $\mathbb{A}$-smooth r-frames on $T^{\mathbb{A}}M_n$ is an $\mathbb{A}$-smooth principal fiber bundle over $T^{\mathbb{A}}M_n$, and the canonical projection $\pi^r(\mathbb{A})$:

$B^r(\mathbb{A})T^{\mathbb{A}}M_n \to T^{\mathbb{A}}M_n$ is an $\mathbb{A}$-smooth mapping. Since every $\mathbb{A}$-smooth germ $\Phi : (\mathbb{A}^n, 0) \to (T^{\mathbb{A}}M_n, X)$ is uniquely determined by its restriction $\varphi = \Phi|\mathbb{R}^n : (\mathbb{R}^n, 0) \to (T^{\mathbb{A}}M_n, X)$, to an $\mathbb{A}$-smooth r-frame $j^r\Phi$ there corresponds bijectively the r-jet $j^r\varphi \in T^{\mathbb{R}(n,r)}T^{\mathbb{A}}M_n$, where $\mathbb{R}(n,r)$ is the algebra of truncated polynomials of degree $\leq r$ in n variables. Taking into account the natural equivalence of the functors $T^{\mathbb{R}(n,r)} \circ T^{\mathbb{A}} \cong T^{\mathbb{A}} \circ T^{\mathbb{R}(n,r)} \cong T^{\mathbb{R}(n,r)\otimes\mathbb{A}}$ [23], [14], [32], we obtain the natural identification $B^r(\mathbb{A})T^{\mathbb{A}}M_n \cong T^{\mathbb{A}}(B^rM_n)$ and the natural embedding of $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ into $T^{\mathbb{R}(n,r)\otimes\mathbb{A}}M_n$ as an open subset, whence it follows that *the bundle* $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ *carries a natural structure of a smooth manifold over the algebra* $\mathbb{A} \otimes \mathbb{R}(n,r)$. Under the identification $B^r(\mathbb{A})T^{\mathbb{A}}M_n \cong T^{\mathbb{A}}(B^rM_n)$, to the zero section $0 : B^rM_n \to T^{\mathbb{A}}(B^rM_n)$ there corresponds the natural embedding

$$\iota : B^rM_n \ni j^r_0 f \mapsto j^r_0(T^{\mathbb{A}}f) \in B^r(\mathbb{A})T^{\mathbb{A}}M_n, \tag{13}$$

where $f : (\mathbb{R}^n, 0) \to (M_n, x)$ is a germ of diffeomorphism and $T^{\mathbb{A}}f : (\mathbb{A}^n, 0) \to (T^{\mathbb{A}}M_n, x)$ is the $\mathbb{A}$-prolongation of f, a germ of $\mathbb{A}$-diffeomorphism.

The structure group of the principal bundle $\pi^r(\mathbb{A}) : B^r(\mathbb{A})T^{\mathbb{A}}M_n \to T^{\mathbb{A}}M_n$ is the Lie group $G^r_n(\mathbb{A})$ of r-jets of germs of $\mathbb{A}$-diffeomorphisms $(\mathbb{A}^n, 0) \to (\mathbb{A}^n, 0)$, which is isomorphic to the Lie group $T^{\mathbb{A}}G^r_n$ [32].

Thus, in addition to the sequence of principal bundles of higher order frames (2), there is associated to $T^{\mathbb{A}}M_n$ the sequence of principal bundles of $\mathbb{A}$-smooth frames

$$\begin{aligned} &T^{\mathbb{A}}M_n \xleftarrow{\pi^1_0(\mathbb{A})} B^1(\mathbb{A})T^{\mathbb{A}}M_n \xleftarrow{\pi^2_1(\mathbb{A})} B^2(\mathbb{A})T^{\mathbb{A}}M_n \xleftarrow{\pi^3_2(\mathbb{A})} \dots \\ &\qquad \xleftarrow{\pi^r_{r-1}(\mathbb{A})} B^r(\mathbb{A})T^{\mathbb{A}}M_n \xleftarrow{\pi^{r+1}_r(\mathbb{A})} \dots \longleftarrow B^\infty(\mathbb{A})T^{\mathbb{A}}M_n\,, \end{aligned} \tag{14}$$

where $B^\infty(\mathbb{A})T^{\mathbb{A}}M_n$ is the bundle of $\mathbb{A}$-smooth frames of infinite order, the limit of the projective system $T^{\mathbb{A}}M_n \leftarrow B^1(\mathbb{A})T^{\mathbb{A}}M_n \leftarrow \dots$ endowed with a structure of an infinite-dimensional smooth (and $\mathbb{A}$-smooth) manifold in the sense of Bernshtein–Rozenfeld [1]. Let $\pi^\infty_r(\mathbb{A}) : B^\infty(\mathbb{A})T^{\mathbb{A}}M_n \to B^r(\mathbb{A})T^{\mathbb{A}}M_n$ denote the canonical projection. The bundle $B^\infty(\mathbb{A})T^{\mathbb{A}}M_n$ consists of all infinite order jets of germs of $\mathbb{A}$-diffeomorphisms $(\mathbb{A}^n, 0) \to T^{\mathbb{A}}M_n$. The differential group $G^\infty_n(\mathbb{A})$ consisting of all infinite order jets of germs of $\mathbb{A}$-diffeomorphisms $(\mathbb{A}^n, 0) \to (\mathbb{A}^n, 0)$ acts naturally on the right on $B^\infty(\mathbb{A})T^{\mathbb{A}}M_n$.

In what follows we will need to consider simultaneously elements of the manifolds $T^{\mathbb{A}}M_n$ and $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ whose coordinates are elements of the algebras $\mathbb{A}$ and $\mathbb{A} \otimes \mathbb{R}(n,r)$ respectively. For this reason, we introduce the following notations for elements of the algebras $\mathbb{A}$ and $\mathbb{A}\otimes\mathbb{R}(n,r)$. An element $X\in\mathbb{A}$ has the representation $X = x + \overset{\circ}{X}$, where $x\in\mathbb{R}$ and $\overset{\circ}{X}\in\overset{\circ}{\mathbb{A}}$ in accordance with the decomposition $\mathbb{A} = \mathbb{R}\oplus\overset{\circ}{\mathbb{A}}$. An element $\overline{X}\in\mathbb{A}\otimes\mathbb{R}(n,r)$ can be represented in the form $\overline{X} = x + \overset{\circ}{X} + \overset{*}{X}$, where $x\in\mathbb{R}$, $\overset{\circ}{X}\in\overset{\circ}{\mathbb{A}}$, and $\overset{*}{X}\in\mathbb{A}\otimes\overset{\circ}{\mathbb{R}}(n,r)$, in accordance with the decomposition $\mathbb{A}\otimes\mathbb{R}(n,r) = \mathbb{R}\oplus\overset{\circ}{\mathbb{A}}\oplus\mathbb{A}\otimes\overset{\circ}{\mathbb{R}}(n,r)$, it can also be represented in the form $\overline{X} = x + \overset{\bullet}{X}$, where $x\in\mathbb{R}$ and $\overset{\bullet}{X} = \overset{\circ}{X}+\overset{*}{X}\in\mathfrak{m}(\mathbb{A}\otimes\mathbb{R}(n,r))$, in accordance with the decomposition $\mathbb{A}\otimes\mathbb{R}(n,r) = \mathbb{R}\oplus\mathfrak{m}(\mathbb{A}\otimes\mathbb{R}(n,r))$, $\mathfrak{m}(\mathbb{A}\otimes\mathbb{R}(n,r)) = \overset{\circ}{\mathbb{A}}\oplus\mathbb{A}\otimes\overset{\circ}{\mathbb{R}}(n,r)$. According to the above introduced notations, the coordinates $\overline{X}^i$ of an element $\overline{X}$ of $T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M_n$ (in particular, of an element of $B^r(\mathbb{A})T^{\mathbb{A}}M_n$) are represented as follows:

$$\overline{X}^i = x^i + \overset{\circ}{X}{}^i + \overset{*}{X}{}^i = x^i + \overset{\bullet}{X}{}^i. \tag{15}$$

Using the standard basis $\{\varepsilon^p\}$, $p = (p_1,\dots,p_n)$, $|p| = 0,1,\dots,r$, in the algebra of truncated polynomials $\mathbb{R}(n,r)$, where ε^p is the residue class of the monomial $t^p = (t^1)^{p_1}\dots(t^n)^{p_n}$ from the algebra $\mathbb{R}[[t^1,\dots,t^n]] = \mathbb{R}(n,\infty)$, one can also represent an element $\overline{X}\in\mathbb{A}\otimes\mathbb{R}(n,r)$ as follows:

$$\overline{X} = X + \overset{*}{X} = \sum_{|p|=0}^{r} X_p\otimes\varepsilon^p, \quad \text{where} \quad X = x + \overset{\circ}{X}, \quad X_p\in\mathbb{A}. \tag{16}$$

The expansions of the coordinates $\overline{X}^i$ of an element $\overline{X}$ of $T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M_n$ corresponding to (16) is of the form

$$\overline{X}^i = X^i + \overset{*}{X}{}^i = \sum_{|p|=0}^{r} X^i_p\otimes\varepsilon^p, \quad \text{where} \quad X^i_p\in\mathbb{A}. \tag{17}$$

3.2. **The Lie group $G^r_n(\mathbb{A})$.** By the definitions given above, the group $G^r_n(\mathbb{A})$ is the fiber $(\pi^r(\mathbb{A}))^{-1}(0)$ of the bundle $\pi^r(\mathbb{A}) : B^r(\mathbb{A})T^{\mathbb{A}}\mathbb{R}^n \to T^{\mathbb{A}}\mathbb{R}^n$. Therefore, the standard coordinates $\{x^i\}$, $i = 1,\dots,n$, on $\mathbb{R}^n$ induce the globally defined coordinates $\overset{*}{Z}{}^i\in\mathbb{A}\otimes\mathfrak{m}(\mathbb{R}(n,r))$ on $G^r_n(\mathbb{A})$. In accordance with (17), the coordinates $\overset{*}{Z}{}^i$ can be represented in the form

$$\overset{*}{Z}{}^i = \sum_{|p|=1}^{r} Z^i_p\otimes\varepsilon^p, \quad \text{where} \quad Z^i_p\in\mathbb{A},\ \det(Z^i_j)\notin\overset{\circ}{\mathbb{A}}. \tag{18}$$

In terms of these coordinates, the composition $Y = X\circ Z$ in $G^r_n(\mathbb{A})$ is written in the form

$$\overset{*}{Y}{}^i = \sum_{|p|=1}^{r} X^i_p\,\overset{*}{Z}{}^p. \tag{19}$$

The same formula (19) also gives the right action

$$R_Z : B^r(\mathbb{A})T^{\mathbb{A}}M_n \ni \overline{X} \mapsto \overline{Y} = R_Z(\overline{X}) \in B^r(\mathbb{A})T^{\mathbb{A}}M_n \qquad (20)$$

of $G_n^r(\mathbb{A})$ on $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ in terms of the local coordinates $\overline{X}^i = x^i + \overset{\circ}{X}{}^i + \overset{*}{X}{}^i$ on $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ induced by local coordinates x^i on M_n.

Proposition 1. i) *The Lie group $G_n^r(\mathbb{A})$ is isomorphic to the Lie group of $\mathbb{A}$-linear automorphisms of the algebra $\mathbb{A} \otimes \mathbb{R}(n, r)$.*

ii) *The Lie algebra $\mathfrak{g}_n^r(\mathbb{A})$ of $G_n^r(\mathbb{A})$ is isomorphic to the Lie algebra of $\mathbb{A}$-linear derivations of the algebra $\mathbb{A} \otimes \mathbb{R}(n, r)$ with bracket*

$$[D_1, D_2] = D_2 \circ D_1 - D_1 \circ D_2. \qquad (21)$$

Proof. Passing to the r-jets in the composition $F \circ G$ of $\mathbb{A}$-smooth germs $G : (\mathbb{A}^n, 0) \to (\mathbb{A}^n, 0)$ and $F : (\mathbb{A}^n, 0) \to T^{\mathbb{A}}M'_k$, where M'_k is an arbitrary k-dimensional smooth manifold, we obtain, similarly to (20), the right action of $G_n^r(\mathbb{A})$ on $T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k$:

$$T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k \times G_n^r(\mathbb{A}) \ni \{\overline{X}, Z\} \mapsto \overline{Y} = R_Z(\overline{X}) \in T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k. \qquad (22)$$

In terms of the local $\mathbb{A} \otimes \mathbb{R}(n, r)$-coordinates $\overline{X}^{i'} = x^{i'} + \overset{\circ}{X}{}^{i'} + \overset{*}{X}{}^{i'}$ on $T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k$ induced by local coordinates $x^{i'}$ on M'_k, this action is given by the equations (19): $\overset{*}{Y}{}^{i'} = \sum_{|p|=1}^{r} X_p^{i'} \overset{*}{Z}{}^p$. In particular, taking as M'_k the field of real numbers, we obtain the right action of $G_n^r(\mathbb{A})$ on $T^{\mathbb{A}\otimes\mathbb{R}(n,r)}\mathbb{R} \cong \mathbb{A} \otimes \mathbb{R}(n, r)$:

$$\mathbb{A} \otimes \mathbb{R}(n, r) \times G_n^r(\mathbb{A}) \ni \{\overline{X}, Z\} \mapsto \overline{Y} = R_Z(\overline{X}) \in \mathbb{A} \otimes \mathbb{R}(n, r), \qquad (23)$$

$$\{x + \overset{\circ}{X} + \overset{*}{X}, \overset{*}{Z}{}^i\} \mapsto \{x + \overset{\circ}{X} + \sum_{|p|=1}^{r} X_p \overset{*}{Z}{}^p\}. \qquad (23')$$

Let $\overline{X} = j^r F_1$, $\overline{Y} = j^r F_2$, and $Z = j^r G$, where $F_1 : (\mathbb{A}^n, 0) \to \mathbb{A}$, $F_2 : (\mathbb{A}^n, 0) \to \mathbb{A}$ and $G : (\mathbb{A}^n, 0) \to (\mathbb{A}^n, 0)$ are $\mathbb{A}$-smooth germs. Then $(F_1+F_2)\circ G = F_1\circ G+F_2\circ G$ and $(F_1\cdot F_2)\circ G = (F_1\circ G)\cdot(F_2\circ G)$. Passing to r-jets, we obtain $(\overline{X}+\overline{Y})\circ Z = \overline{X}\circ Z+\overline{Y}\circ Z$ and $(\overline{X}\cdot\overline{Y})\circ Z = (\overline{X}\circ Z)\cdot(\overline{Y}\circ Z)$. Thus, the action (23) of $Z \in G_n^r(\mathbb{A})$ on $\mathbb{A} \otimes \mathbb{R}(n, r)$ is an automorphism of $\mathbb{A} \otimes \mathbb{R}(n, r)$. A fixed element $X \in \mathbb{A} \subset \mathbb{R}(n, q) \otimes \mathbb{A}$ can be considered as the r-jet of the constant mapping $F_1 : (\mathbb{A}^n, 0) \to \mathbb{A}$. In this case, the relation $(F_1 \cdot F_2) \circ G = (F_1 \circ G) \cdot (F_2 \circ G)$ takes the form $(X \cdot F_2) \circ G = X \cdot (F_2 \circ G)$. Therefore, $(X \cdot \overline{Y}) \circ Z = X \cdot (\overline{Y} \circ Z)$, and so R_Z is an $\mathbb{A}$-linear automorphism. This fact also follows from (23′).

Let now $h : \mathbb{A} \otimes \mathbb{R}(n, r) \to \mathbb{A} \otimes \mathbb{R}(n, r)$ be an arbitrary $\mathbb{A}$-linear automorphism of $\mathbb{A} \otimes \mathbb{R}(n, r)$. The ideal $\overset{\circ}{\mathbb{A}}$ of $\mathbb{A}$, considered as a subset of $\mathbb{A}\otimes\mathbb{R}(n,r)$, generates the ideal $\mathring{\mathbb{A}}\otimes\mathbb{R}(n,r)$ of $\mathbb{A}\otimes\mathbb{R}(n,r)$. The quotient algebra $\mathbb{A}\otimes\mathbb{R}(n,r)/\mathring{\mathbb{A}}\otimes\mathbb{R}(n,r)$ can be identified with $\mathbb{R}(n,r)$. The ideal $\mathring{\mathbb{A}}\otimes\mathbb{R}(n,r)$ is invariant under $\mathbb{A}$-linear automorphisms of $\mathbb{A}\otimes\mathbb{R}(n,r)$. Therefore, the automorphism h induces an automorphism of $\mathbb{R}(n,r)$. Hence it follows that h acts on the generators ε^i, $i=1,\dots,n$ of the algebra $\mathbb{R}(n,r)\subset\mathbb{A}\otimes\mathbb{R}(n,r)$ being the residue classes of the monomials t^i, $i=1,\dots,n$, as follows:

$$h:\varepsilon^i\mapsto\sum_{|p|=1}^{r}\alpha^i_p\otimes\varepsilon^p+\alpha^i,\quad\text{where}\quad \alpha^i_p\in\mathbb{A},\ \det(\alpha^i_j)\notin\mathring{\mathbb{A}},\ \alpha^i\in\mathring{\mathbb{A}}.\tag{24}$$

Denote $\widetilde{\varepsilon}^i=\sum_{|p|=1}^{r}\alpha^i_p\otimes\varepsilon^p$ and $\widetilde{\alpha}^i=\widetilde{\varepsilon}^i+\alpha^i$. Since $\widetilde{\varepsilon}^p=0$ and $\widetilde{\alpha}^p=0$ for $|p|>r$, it follows that, in (24), $\alpha^i=0$. Thus, $h(\varepsilon^i)=\overset{*}{Z}{}^i\in\mathbb{A}\otimes\mathfrak{m}(\mathbb{R}(n,r))$ and, for an arbitrary $\overline{X}=\sum_{|p|=0}^{r}X_p\otimes\varepsilon^p\in\mathbb{R}(n,q)\otimes\mathbb{A}$, $X_p\in\mathbb{A}$, we have $h(\overline{X})=\sum_{|p|=0}^{r}X_ph(\varepsilon^p)=\sum_{|p|=0}^{r}X_p\overset{*}{Z}{}^p=R_Z(\overline{X})$, where $Z=\{\overset{*}{Z}{}^i\}\in G^r_n(\mathbb{A})\subset(\mathbb{A}\otimes\mathfrak{m}(\mathbb{R}(n,q))^n$. Thus, $h=R_Z$, and the first statement has been proved.

To prove the second statement, we need to find the fundamental vector fields of the action of $G^r_n(\mathbb{A})$ on $T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k$.

Consider first a more general situation. Let $F:T^{\mathbb{A}}M_n\to T^{\mathbb{A}}M'_k$ be an $\mathbb{A}$-smooth mapping. The composition of the r-jet j^rF with r-jets of $\mathbb{A}$-smooth germs $(\mathbb{A}^n,0)\to T^{\mathbb{A}}M_n$ defines the $\mathbb{A}\otimes\mathbb{R}(n,r)$-smooth mapping

$$F^{\mathbb{A}\otimes\mathbb{R}(n,r)}:T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M_n\to T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k.\tag{25}$$

In terms of the local coordinates induced by local coordinates on M_n and M'_k, the mapping $F^{\mathbb{A}\otimes\mathbb{R}(n,r)}$ has the form (5):

$$\overline{X}^{i'}=f^{i'}(x^j)+\sum_{|p|=1}^{r+q}\frac{1}{p!}\frac{D^pf^{i'}(x^j)}{Dx^p}\overset{\bullet}{X}{}^p,\tag{26}$$

where $f^{i'}(x^j)$ are the $\mathbb{A}$-valued functions which give the restriction of F to the zero section $M_n\subset T^{\mathbb{A}}M_n$. Using the partial derivatives of the $\mathbb{A}$-smooth functions

$$F^{i'}(X^j)=f^{i'}(x^j)+\sum_{|p|=1}^{q}\frac{1}{p!}\frac{D^pf^{i'}(x^j)}{Dx^p}\overset{\circ}{X}{}^p$$

which give the mapping F in terms of local $\mathbb{A}$-coordinates, one can rewrite (26) in the form

$$\overline{X}^{i'} = \overline{F}^{i'}(\overline{X}^j) = F^{i'}(X^j) + \sum_{|p|=1}^{r} \frac{1}{p!} \frac{D^p F^{i'}(X^j)}{DX^p} \overset{*}{X}{}^p, \tag{27}$$

where

$$\frac{D^p F^{i'}(X^j)}{DX^p} = \frac{D^p F^{i'}(X^j)}{Dx^p} = \frac{D^p f^{i'}(x^j)}{Dx^p} + \sum_{|s|=1}^{q} \frac{1}{s!} \frac{D^{p+s} f^{i'}(x^j)}{Dx^{p+s}} \overset{\circ}{X}{}^p.$$

Let $\overline{X} \in T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M_n$, and let $\overline{X}' \in T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k$ be the image of $\overline{X}$ under the mapping (25). The mapping

$$TF^{\mathbb{A}\otimes\mathbb{R}(n,r)} : TT^{\mathbb{A}\otimes\mathbb{R}(n,r)}M_n \to TT^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k \tag{28}$$

of the tangent bundles induced by (25), in terms of local coordinates, is given by the equations $\overline{X}^{i'} = \overline{F}^{i'}(\overline{X}^j)$ of the form (27) and by the equations

$$\overline{V}^{i'} = \frac{\partial \overline{F}^{i'}}{\partial \overline{X}^j} \overline{V}^j = \left(\frac{\partial F^{i'}}{\partial X^j} + \sum_{|p|=1}^{r} \frac{1}{p!} \frac{D^{p+j} F^{i'}}{DX^{p+j}} \overset{*}{X}{}^p \right) \overline{V}^j. \tag{29}$$

The coordinates $\overline{V}^j$ of a tangent vector $\overline{V} \in T_{\overline{X}}T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M_n$, as well as the coordinates of elements of $T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M_n$, are of the form $\overline{V}^j = V^j + \overset{*}{V}{}^j$, where $V^j \in \mathbb{A}$ and $\overset{*}{V}{}^j \in \mathbb{A} \otimes \mathfrak{m}(\mathbb{R}(n,r))$.

If $V^j = 0$, then the elements $\overset{*}{V}{}^j$ are the coordinates of a vertical vector on the bundle $T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M_n \to T^{\mathbb{A}}M_n$, i. e., of a vector tangent to the fiber $T_X^{\mathbb{A}\otimes\mathbb{R}(n,r)}M_n$. In the case $V^j = 0$, the summands in (29) corresponding to $|p| = r+1$ are equal to zero. Therefore, the mapping $VT_{\overline{X}}T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M_n$ of the vertical tangent spaces is defined by the r-jet $j^r_X F$.

Letting now in (28) $\overline{X}$ be the identity $e = j^r_0(\mathrm{id}_{\mathbb{A}^n})$ of the Lie group $G^r_n(\mathbb{A}) \subset T_0^{\mathbb{R}(n,r)}\mathbb{A}^n \subset T^{\mathbb{A}\otimes\mathbb{R}(n,r)}\mathbb{R}^n$, and $\overline{Y}$ be an arbitrary element

$$\overline{Y} = j^r_0 F, \quad F : (\mathbb{A}^n \cong T^{\mathbb{A}}\mathbb{R}^n, 0) \to (T^{\mathbb{A}}M'_k, Y),$$

of the bundle $T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k$, we obtain the mapping

$$T_e F^{\mathbb{A}\otimes\mathbb{R}(n,r)} : T_e G^r_n(\mathbb{A}) \ni \overline{V}_e \mapsto \overline{V}_{\overline{Y}} \in T_{\overline{Y}}T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k \tag{30}$$

which assigns to an element $\overline{V}_e$ of the Lie algebra $\mathfrak{g}^r_n(\mathbb{A}) \cong T_e G^r_n(\mathbb{A})$ the vector $\overline{V}_{\overline{Y}}$ being the value at $\overline{Y}$ of the fundamental vector field $\overline{V}$ of the action of $G^r_n(\mathbb{A})$ on $T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k$ corresponding to $\overline{V}_e$. To find the equations of a fundamental vector field $\overline{V}$ on $T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k$ in terms of local coordinates, we let $\overset{*}{X}{}^p = \varepsilon^p$ in equations (29), which determine the mapping (30). Then we obtain

$$\overline{V}^{i'}(\overline{Y}) = (Y^{i'}_j + \sum_{|p|=1}^{r-1}(p_j+1)Y^{i'}_{p+j}\varepsilon^p)\overline{V}^j_e, \tag{31}$$

where $Y^{i'}_j, Y^{i'}_{p+j} \in \mathbb{A}$, and it is supposed that the sum is taken over the index j. In particular, the fundamental vector fields on $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ are of the form

$$\overline{V}^i(\overline{X}) = (X^i_j + \sum_{|p|=1}^{r-1}(p_j+1)X^i_{p+j}\varepsilon^p)\overline{V}^j_e = (\sum_{|p|=0}^{r-1}(p_j+1)X^i_{p+j}\varepsilon^p)\overline{V}^j_e. \tag{32}$$

In the case $M'_k = \mathbb{R}$, $k=1$, formula (30) determines the fundamental vector fields of the action of the group $G^r_n(\mathbb{A})$ on the algebra $\mathbb{A}\otimes\mathbb{R}(n,r)$. Let $\overline{V}_e \in T_eG^r_n(\mathbb{A})$ be an element of the Lie algebra $\mathfrak{g}^r_n(\mathbb{A})$, and let $\gamma : (\mathbb{R},0) \ni t \mapsto \gamma(t) \in G^r_n(\mathbb{A})$ be a germ of curve such that $(d\gamma/dt)|_0 = \overline{V}_e$. By (30), the value of the fundamental vector field $\overline{V}$ on $\mathbb{A}\otimes\mathbb{R}(n,r)$ corresponding to $\overline{V}_e$ at $\alpha = j^rF \in \mathbb{A}\otimes\mathbb{R}(n,r)$, $F : (\mathbb{A}^n,0) \to \mathbb{A}$, is of the form

$$\overline{V}(\alpha) = \overline{V}_\alpha = T_ej^rF(\overline{V}_e) = \left.\frac{d}{dt}\right|_0 (j^rF\circ\gamma) = \left.\frac{d}{dt}\right|_0 (F^{\mathbb{A}\otimes\mathbb{R}(n,r)}\circ\gamma). \tag{33}$$

Applying (33) to the product $\alpha_1\cdot\alpha_2 \in \mathbb{A}\otimes\mathbb{R}(n,r)$, we obtain

$$\begin{aligned}\overline{V}(\alpha_1\cdot\alpha_2) &= \left.\frac{d}{dt}\right|_0 ((F_1^{\mathbb{A}\otimes\mathbb{R}(n,r)}\cdot F_2^{\mathbb{A}\otimes\mathbb{R}(n,r)})\circ\gamma) = \\ &= \left.\frac{d}{dt}\right|_0 (F_1^{\mathbb{A}\otimes\mathbb{R}(n,r)}\circ\gamma)\cdot(F_2^{\mathbb{A}\otimes\mathbb{R}(n,r)}\circ\gamma)(0) + \\ &+ (F_1^{\mathbb{A}\otimes\mathbb{R}(n,r)}\circ\gamma)(0)\cdot\left.\frac{d}{dt}\right|_0 (F_2^{\mathbb{A}\otimes\mathbb{R}(n,r)}\circ\gamma) = \overline{V}(\alpha_1)\cdot\alpha_2 + \alpha_1\cdot\overline{V}(\alpha_2).\end{aligned}$$

If $\alpha_1 \in \mathbb{A} \subset \mathbb{A}\otimes\mathbb{R}(n,r)$, then α_1 is the r-jet $\alpha_1 = j^rF_1$ of the constant germ $F_1 : (\mathbb{A}^n,0) \mapsto \alpha_1 \in \mathbb{A}$. Consequently, $\overline{V}(\alpha_1) = 0$ and $\overline{V}(\alpha_1\cdot\alpha_2) = \alpha_1\cdot\overline{V}(\alpha_2)$. Now, by means of similar calculations, one can easily show that, for $\beta_1,\beta_2 \in \mathbb{A}$, $\alpha_1,\alpha_2 \in \mathbb{A}\otimes\mathbb{R}(n,r)$, the following relation holds: $\overline{V}(\beta_1\alpha_1+\beta_2\alpha_2) = \beta_1\overline{V}(\alpha_1)+\beta_2\overline{V}(\alpha_2)$.

An arbitrary $\mathbb{A}$-linear derivation $D : \mathbb{A}\otimes\mathbb{R}(n,r) \to \mathbb{A}\otimes\mathbb{R}(n,r)$ is determined by its values $D(\varepsilon^i) \in \mathfrak{m}(\mathbb{A}\otimes\mathbb{R}(n,r))$ on the generators ε^i, $i=1,\dots,n$, of the algebra $\mathbb{R}(n,r) \subset \mathbb{A}\otimes\mathbb{R}(n,r)$. Since $D(\varepsilon^p) = 0$ for $|p| = r+1$, it follows (as in the case of automorphisms (24) of the algebra $\mathbb{A}\otimes\mathbb{R}(n,r)$) that $D(\varepsilon^i) = \overline{V}^i = \sum_{|p|=1}^r V^i_p\otimes\varepsilon^p$, where $V^i_p \in \mathbb{A}$, i. e.,

$\overline{V}^i \in \mathbb{A} \otimes \mathfrak{m}(\mathbb{R}(n,r))$. Therefore, D coincides with the derivation which is the fundamental vector field $\overline{V}$ generated by the element $\overline{V}_e \in T_e G_n^r(\mathbb{A}) = T_e(\mathbb{A} \otimes \mathfrak{m}(\mathbb{R}(n,r))) \cong \mathbb{A} \otimes \mathfrak{m}(\mathbb{R}(n,r))$ with $\mathbb{A} \otimes \mathbb{R}(n,r)$-coordinates $\overline{V}^i$. In fact, the $\mathbb{A} \otimes \mathbb{R}(n,r)$-coordinates of the identity $e \in G_n^r(\mathbb{A})$ are the elements $\{\varepsilon^i\}$, $i = 1, \ldots, n$, and the values of the coordinates $\overline{V}^i(e)$ of the left-invariant vector field corresponding to $\overline{V}_e \in T_e G_n^r(\mathbb{A})$ at $e \in G_n^r(\mathbb{A})$, by (33), are equal to $\overline{V}^i(e) = \overline{V}(\varepsilon^i)$.

To the Lie bracket of fundamental vector fields on $\mathbb{A} \otimes \mathbb{R}(n,r)$ there corresponds the Lie bracket of derivations (21). In fact, the group $G_n^r(\mathbb{A})$, which acts on the right on $\mathbb{A} \otimes \mathfrak{m}(\mathbb{R}(n,r))$ by linear transformations, and the space $T_e G_n^r(\mathbb{A})$ can be considered as subsets in the automorphism algebra $\mathrm{End}(\mathbb{A} \otimes \mathfrak{m}(\mathbb{R}(n,r)))$ of the vector space $\mathbb{A} \otimes \mathfrak{m}(\mathbb{R}(n,r))$ acting on the right on $\mathbb{A} \otimes \mathfrak{m}(\mathbb{R}(n,r))$. $\square$

3.3. **The Lie bracket of $\mathbb{A}$-smooth vector fields.** The tangent bundle $T\mathbb{A}^n$ of the $\mathbb{A}$-module $\mathbb{A}^n$ is naturally identified with $\mathbb{A}^n \times \mathbb{A}^n$, and if $\{X^i\}$, $i = 1, \ldots, n$, are the standard $\mathbb{A}$-coordinates on $\mathbb{A}^n$, then a vector field on an open subset $O \subset \mathbb{A}^n$ is given by a smooth mapping $V : O \ni \{X^i\} \mapsto \{X^i, V^j(X^i)\} \in \mathbb{A}^n \times \mathbb{A}^n$. A vector field V is $\mathbb{A}$-smooth if the functions $V^j = V^j(X^i)$ are $\mathbb{A}$-smooth. In order that a vector field V be $\mathbb{A}$-smooth it is necessary and sufficient that the functions $v^{jb} = v^{jb}(x^{ia})$ satisfy the conditions (3):

$$\partial_{ia} v^{jb} = \gamma_{ag}^b \partial_{i0} v^{jg}, \quad \text{where} \quad \partial_{ia} v^{jb} = \partial v^{jb} / \partial x^{ia}. \tag{34}$$

Proposition 2. *Let U and V be $\mathbb{A}$-smooth vector fields on a domain $O \subset \mathbb{A}^n$, and let $W = [U, V]$ be the Lie bracket of these fields. Then*

i) *W is an $\mathbb{A}$-smooth vector field.*

ii) *The $\mathbb{A}$-coordinates W^j of W are of the form*

$$W^j = U^i \partial_i V^j - V^i \partial_i U^j. \tag{35}$$

Proof. Statement i) follows from statement ii) since the functions U^j, V^j and their partial derivatives are $\mathbb{A}$-smooth.

Let us prove statement ii). By the definition of the Lie bracket of vector fields, the conditions of $\mathbb{A}$-smoothness of a vector field (34), and by relation (4) for partial derivatives of an $\mathbb{A}$-smooth function, we have

$$\begin{aligned} W^j &= w^{jb} e_b = \\ &= (u^{ia} \partial_{ia} v^{jb} - v^{ia} \partial_{ia} u^{jb}) e_b = (u^{ia} \gamma_{ag}^b \partial_{i0} v^{jg} - v^{ia} \gamma_{ag}^b \partial_{i0} u^{jg}) e_b = \\ &= u^{ia} e_a \partial_{i0} v^{jg} e_g - v^{ia} e_a \partial_{i0} u^{jg} e_g = U^i \partial_i V^j - V^i \partial_i U^j. \end{aligned} \tag{36}$$

$\square$

3.4. **Lie algebras of germs and jets of $\mathbb{A}$-smooth vector fields.** We will use the following notation:

$\mathrm{Vect}_0(\mathbb{R}^n)$ is the Lie algebra of germs of vector fields on $\mathbb{R}^n$ at zero,

$\mathrm{Vect}_0(\mathbb{R}^n, 0)$ is the Lie algebra of germs of vector fields on $\mathbb{R}^n$ at zero which take zero value at zero.

Proposition 2 allows one to consider also the following Lie algebras of germs of $\mathbb{A}$-smooth vector fields:

$\mathrm{Vect}_0(\mathbb{A}^n)$, the Lie algebra of germs of $\mathbb{A}$-smooth vector fields on $\mathbb{A}^n$ at zero,

$\mathrm{Vect}_0(\mathbb{A}^n, 0)$, the Lie algebra of germs of $\mathbb{A}$-smooth vector fields on $\mathbb{A}^n$ at zero which take zero value at zero,

$\mathrm{Vect}_0(\mathbb{A}^n, \mathring{\mathbb{A}}^n)$, the Lie algebra of germs of $\mathbb{A}$-smooth vector fields on $\mathbb{A}^n$ at zero whose value at zero belongs to the submodule $\mathring{\mathbb{A}}^n$.

$\mathrm{Vect}_0(\mathbb{A}^n, \mathbb{I}^n)$, the Lie algebra of germs of $\mathbb{A}$-smooth vector fields on $\mathbb{A}^n$ at zero whose value at zero belongs to the submodule $\mathbb{I}^n \in \mathbb{A}^n$ consisting of the n-tuples with elements from an ideal $\mathbb{I}$ of $\mathbb{A}$.

Proposition 3. *The following Lie algebra isomorphisms take place:*

$$\mathrm{Vect}_0(\mathbb{A}^n) \cong \mathbb{A} \otimes \mathrm{Vect}_0(\mathbb{R}^n), \tag{37}$$

$$\mathrm{Vect}_0(\mathbb{A}^n, 0) \cong \mathbb{A} \otimes \mathrm{Vect}_0(\mathbb{R}^n, 0). \tag{38}$$

Proof. The isomorphisms (37) and (38) follow from the fact that the germs of $\mathbb{A}$-smooth vector fields U, V, and $[U, V]$ on $\mathbb{A}^n$ at zero are completely defined by their restrictions to $\mathbb{R}^n$ (see (36)). The restriction $u = U|\mathbb{R}^n$ of a germ of $\mathbb{A}$-smooth vector field U is a germ of $\mathbb{A}^n$-valued vector field on $\mathbb{R}^n$ at zero. The germ u is of the form $u^j(x^i) = u^{jb}(x^i)e_b$. Since $\mathbb{A}$ is a finite-dimensional algebra, u can be considered as an element $e_b \otimes u^{jb}(x^i)$ of the Lie algebra $\mathbb{A} \otimes \mathrm{Vect}_0(\mathbb{R}^n)$. The germ of $\mathbb{A}$-smooth vector field U is restored from u as its $\mathbb{A}$-prolongation (see (5)) $U^j = u^j(x^i) + \sum_{|p|=1}^{q} \frac{1}{p!}(D^p u^i / Dx^p)\, \mathring{X}^p$. Thus, the isomorphism of Lie algebras $\mathbb{A} \otimes \mathrm{Vect}_0(\mathbb{R}^n) \to \mathrm{Vect}_0(\mathbb{A}^n)$ is realized by passing from germs of $\mathbb{A}^n$-valued vector fields to their $\mathbb{A}$-prolongations. $\square$

Note that the isomorphism (37) allows ones to regard the Lie algebras $\mathrm{Vect}_0(\mathbb{A}^n, \mathring{\mathbb{A}}^n)$ and $\mathrm{Vect}_0(\mathbb{A}^n, \mathbb{I}^n)$ as Lie subalgebras in $\mathbb{A} \otimes \mathrm{Vect}_0(\mathbb{R}^n)$.

Passing in the Lie algebras $\mathrm{Vect}_0(\mathbb{R}^n)$ and $\mathrm{Vect}_0(\mathbb{R}^n, 0)$ from germs of vector fields at zero to the ∞-jets, one obtains the following Lie algebras [1], [7]:

$\mathrm{Vect}_0^\infty(\mathbb{R}^n) = W_n$, the Lie algebra of formal vector fields on $\mathbb{R}^n$, an element $V \in W_n$ is a collection of n formal power series $V^i = \sum_{|p|=0}^{\infty} v^i_p t^p \in$

$\mathbb{R}[[t^1,\dots,t^n]]$, $i=1,\dots,n$, the Lie bracket $[U,V]$ of two formal vector fields is computed by formula (35), where $\partial_i=\partial/\partial t^i$ is the operator of formal differentiation of a power series with respect to t^i, for this reason, $V\in W_n$ can also be written as the linear combination $V^i\partial_i$;

$\mathrm{Vect}_0^\infty(\mathbb{R}^n,0)=L_0(W_n)$, the Lie subalgebra in W_n of formal vector fields V with V^i belonging to the maximal ideal $\mathfrak{m}(\mathbb{R}[[t^1,\dots,t^n]])$ of the algebra $\mathbb{R}[[t^1,\dots,t^n]]$, which consists of the series with zero constant term, i. e., such that $V^i=\sum_{|p|=1}^\infty v_p^i t^p$;

$L_r(W_n)$, $r\ge 0$, the Lie subalgebra in W_n of formal vector fields V with V^i belonging to the $(r+1)$-st power of the ideal $\mathfrak{m}(\mathbb{R}[[t^1,\dots,t^n]])$, i. e., such that $V^i=\sum_{|p|=r+1}^\infty v_p^i t^p$.

$[L_r(W_n),L_s(W_n)]\subset L_{r+s}(W_n)$, therefore, for $r>s$, $L_r(W_n)$ is an ideal in $L_s(W_n)$. In particular, $L_r(W_n)$, $r>0$, is an ideal in $L_0(W_n)$.

Passing in the Lie algebras $\mathrm{Vect}_0(\mathbb{A}^n)$, $\mathrm{Vect}_0(\mathbb{A}^n,0)$, $\mathrm{Vect}_0(\mathbb{A}^n,\mathring{\mathbb{A}}^n)$, and $\mathrm{Vect}_0(\mathbb{A}^n,\mathbb{I}^n)$ from germs of $\mathbb{A}$-smooth vector fields at zero to their ∞-jets, we obtain, respectively, the Lie algebras $\mathrm{Vect}_0^\infty(\mathbb{A}^n)$, $\mathrm{Vect}_0^\infty(\mathbb{A}^n,0)$, $\mathrm{Vect}_0^\infty(\mathbb{A}^n,\mathring{\mathbb{A}}^n)$, and $\mathrm{Vect}_0^\infty(\mathbb{A}^n,\mathbb{I}^n)$.

Denote by $\mathbb{A}[[t^1,\dots,t^n]]=\mathbb{A}(n,\infty)$ the algebra of formal power series $\alpha=\sum_{|p|=0}^\infty \alpha_p t^p$, $\alpha_p\in\mathbb{A}$, with coefficients in a Weil algebra $\mathbb{A}$. Since $\mathbb{A}$ is finite-dimensional, the algebra $\mathbb{A}[[t^1,\dots,t^n]]$ is isomorphic to the tensor product $\mathbb{A}\otimes\mathbb{R}[[t^1,\dots,t^n]]$. The algebra $\mathbb{A}[[t^1,\dots,t^n]]$ has the unique maximal ideal $\mathfrak{m}(\mathbb{A}[[t^1,\dots,t^n]])$ consisting of the series whose constant term α_0 belongs to the maximal ideal $\mathring{\mathbb{A}}$ of $\mathbb{A}$.

By the *Lie algebra of formal vector fields on* $\mathbb{A}^n$ we will mean the Lie algebra $W_n^{\mathbb{A}}$ of formal vector fields $V=V^i\partial_i$ with $V^i\in\mathbb{A}[[t^1,\dots,t^n]]$, $i=1,\dots,n$, with Lie bracket $[U,V]$ defined by (35), where, as in the case of the Lie algebra W_n, $\partial_i=\partial/\partial t^i$ is the operator of formal differentiation of formal power series with respect to t^i.

There are the following Lie subalgebras in $W_n^{\mathbb{A}}$: $L_r(W_n^{\mathbb{A}})$, $r=0,1,\dots$, the Lie subalgebra of formal vector fields $V=V^i\partial_i$ such that all the truncated series $\sum_{|p|=0}^r v_p^i t^p$ of $V^i=\sum_{|p|=0}^\infty v_p^i t^p$, $i=1,\dots,n$, are zero; in particular, $L_0(W_n^{\mathbb{A}})$ is the Lie subalgebra of formal vector fields V with V^i having zero constant terms; $\widetilde{L}_0(W_n^{\mathbb{A}})$, the Lie subalgebra of formal vector fields V with V^i belonging to the maximal ideal $\mathfrak{m}(\mathbb{A}[[t^1,\dots,t^n]])$. From (35) it follows that $L_r(W_n^{\mathbb{A}})$, $r=1,\dots$, is an ideal in $L_0(W_n^{\mathbb{A}})$.

The following propositions are consequences of Propositions 2 and 3.

Proposition 4. *The following Lie algebra isomorphisms take place:*

$$\mathrm{Vect}_0^\infty(\mathbb{A}^n) \cong W_n^{\mathbb{A}} \cong \mathbb{A} \otimes W_n, \tag{39}$$

$$\mathrm{Vect}_0^\infty(\mathbb{A}^n, 0) \cong L_0(W_n^{\mathbb{A}}) \cong \mathbb{A} \otimes L_0(W_n), \tag{40}$$

$$\mathrm{Vect}_0^\infty(\mathbb{A}^n, \overset{\circ}{\mathbb{A}}{}^n) \cong \widetilde{L}_0(W_n^{\mathbb{A}}). \tag{41}$$

Proposition 5. *The vector space* $\mathrm{Vect}_0^r(\mathbb{A}^n, 0)$ *of r-jets of germs of $\mathbb{A}$-smooth vector fields from* $\mathrm{Vect}_0(\mathbb{A}^n, 0)$ *with bracket* $[j^r V_1, j^r V_2] = j^r[V_1, V_2]$ *is a Lie algebra isomorphic to the quotient Lie algebra*

$$L_0(W_n^{\mathbb{A}})/L_{r+1}(W_n^{\mathbb{A}}) \cong \mathbb{A} \otimes (L_0(W_n)/L_{r+1}(W_n)).$$

3.5. **Lifts of vector fields to $B^r(\mathbb{A})T^{\mathbb{A}}M_n$.** A vector field $v : M_n \to TM_n$ on a smooth manifold M_n generates a local flow $\varphi_t : M_n \to M_n$. The Weil functor $T^{\mathbb{A}}$ applied to the flow φ_t gives the flow $T^{\mathbb{A}}\varphi_t : T^{\mathbb{A}}M_n \to T^{\mathbb{A}}M_n$ on the Weil bundle $T^{\mathbb{A}}M_n$, the $\mathbb{A}$-prolongation of the flow φ_t. The flow $T^{\mathbb{A}}\varphi_t$, in turn, generates the vector field $v^C : T^{\mathbb{A}}M_n \to TT^{\mathbb{A}}M_n$ on $T^{\mathbb{A}}M_n$ called the complete lift of v. The vector field v^C can be obtained as the composition of the $\mathbb{A}$-prolongation $v^{\mathbb{A}} : T^{\mathbb{A}}M_n \to T^{\mathbb{A}}TM_n$ of the section $v : M_n \to TM_n$ with the diffeomorphism $T^{\mathbb{A}}TM_n \to TT^{\mathbb{A}}M_n$ which follows from the natural equivalence of functors $T^{\mathbb{A}} \circ T^{\mathbb{R}(\varepsilon)} \cong T^{\mathbb{R}(\varepsilon)} \circ T^{\mathbb{A}}$. In what follows we will not distinguish between the mappings v^C and $v^{\mathbb{A}}$. The local $\mathbb{A}$-coordinates $V^i(X^j)$ of the vector field $v^{\mathbb{A}}$ on $T^{\mathbb{A}}M_n$ in terms of the $\mathbb{A}$-chart $h^{\mathbb{A}}$ generated by a chart h on M_n are the $\mathbb{A}$-prolongations of the coordinates $v^i(x^j)$ of v. In accordance with (5),

$$V^i = v^i(x^j) + \sum_{|p|=1}^{q} \frac{1}{p!} \frac{D^p v^i}{Dx^p} \overset{\circ}{X}{}^p. \tag{42}$$

The complete lift $\overline{V}$ of an $\mathbb{A}$-smooth vector field V on the Weil bundle $T^{\mathbb{A}}M_n$ to the bundle $T^{\mathbb{R}(n,r)}T^{\mathbb{A}}M_n \cong T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M_n$ is given by a formula similar to (42). If $v = V|M_n : M_n \to TT^{\mathbb{A}}M_n$ is the restriction of an $\mathbb{A}$-smooth vector field V to $M_n \subset T^{\mathbb{A}}M_n$ (we identify M_n with zero section of $T^{\mathbb{A}}M_n$), then the complete lift $\overline{V} = V^{\mathbb{A}\otimes\mathbb{R}(n,r)}$ of V to $T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M_n$, in terms of local coordinates $\{\overline{X}{}^i\}$ on $T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M_n$ generated by local coordinates $\{x^i\}$ on M_n is of the form

$$\overline{V}{}^i(\overline{X}{}^j) = v^i(x^j) + \sum_{|p|=1}^{q} \frac{1}{p!} \frac{D^p v^i}{Dx^p} \overset{\bullet}{X}{}^p = V^i(X^j) + \sum_{|p|=1}^{q} \frac{1}{p!} \frac{D^p V^i}{Dx^p} \overset{*}{X}{}^p, \tag{43}$$

where $v^i(x^j)$ are the $\mathbb{A}$-valued smooth functions being the restrictions to $\mathbb{R}^n$ of the $\mathbb{A}$-smooth functions $V^i(X^j)$, $\overset{\bullet}{X}{}^i \in \mathfrak{m}(\mathbb{A} \otimes \mathbb{R}(n,r))$, and $\overline{V}{}^i(\overline{X}{}^j)$ are the $\mathbb{A} \otimes \mathbb{R}(n,r)$-smooth functions being the prolongations of $v^i(x^j)$

and $V^i(X^j)$. The principal bundle $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ is an open submanifold in $T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M_n$. Therefore, formulas (43) give also the complete lift of an $\mathbb{A}$-smooth vector field V from $T^{\mathbb{A}}M_n$ to $B^r(\mathbb{A})T^{\mathbb{A}}M_n$.

Proposition 6. *The complete lift $\overline{V}$ of an $\mathbb{A}$-smooth vector field V from $T^{\mathbb{A}}M_n$ to $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ is invariant with respect to the right action of the Lie group $G^r_n(\mathbb{A})$ on $B^r(\mathbb{A})T^{\mathbb{A}}M_n$.*

Proof. In terms of local coordinates, the right action (20)

$$R_Z : B^r(\mathbb{A})T^{\mathbb{A}}M_n \ni \overline{X} \mapsto \overline{Y} = R_Z(\overline{X}) \in B^r(\mathbb{A})T^{\mathbb{A}}M_n \tag{44}$$

is of the form (19):

$$\overset{*}{Y}{}^i = \sum_{|p|=1}^{r} X^i_p \overset{*}{Z}{}^p, \tag{45}$$

where $\overline{X}^i = x^i + \overset{\circ}{X}{}^i + \overset{*}{X}{}^i$, $\overset{*}{X}{}^i = X^i_p \otimes \varepsilon^p$, $\overline{Y}^i = x^i + \overset{\circ}{X}{}^i + \overset{*}{Y}{}^i$ (see (15)). Then the tangent mapping

$$T_{\overline{X}}R_Z : T_{\overline{X}}B^r(\mathbb{A})T^{\mathbb{A}}M_n \ni \overline{V} \mapsto \overline{W} \in T_{\overline{Y}}B^r(\mathbb{A})T^{\mathbb{A}}M_n \tag{46}$$

is of the form

$$\overset{*}{W}{}^i = \sum_{|p|=1}^{r} V^i_p \overset{*}{Z}{}^p, \tag{47}$$

where $\overset{*}{V}{}^i = V^i_p \otimes \varepsilon^p$. The complete lift $\overline{V}$ of an $\mathbb{A}$-smooth vector field V from $T^{\mathbb{A}}M_n$ to $B^r(\mathbb{A})T^{\mathbb{A}}M_n$, in terms of local coordinates, is given by equations (43)

$$\overset{*}{V}{}^i(\overline{X}{}^j) = \sum_{|p|=1}^{r} V^i_p \overset{*}{X}{}^p. \tag{48}$$

Under the right action (46), the complete lift $\overline{V}$ goes to the vector field $\overline{W}$ whose coordinates $\overset{*}{W}{}^i$ at $\overline{Y} = \overline{X} \circ Z$ are obtained by substituting $\overset{*}{Z}{}^i$ in place of ε^i in the expansion of $\overset{*}{V}{}^i(\overline{X}{}^j)$ from (48) in powers of ε^p: $\overset{*}{V}{}^i(\overline{X}{}^j) = A^i_p\varepsilon^p$. For this, one should replace ε^p by $\overset{*}{Z}{}^p$ in the expansion of each $\overset{*}{X}{}^i$ in the right-hand side of (48), which is equivalent to the replacement in (48) of $\sum_{|p|=1}^{r} V^i_p \overset{*}{X}{}^p$ by $\sum_{|p|=1}^{r} V^i_p \overset{*}{Y}{}^p$, where $\overline{Y} = \overline{X} \circ Z$, which, in turn, gives $\overset{*}{V}{}^i(\overline{Y}{}^j)$. □

Proposition 7. *The Lie algebra of right-invariant vector fields on the Lie group $G^r_n(\mathbb{A})$ is isomorphic to the Lie algebra* $\mathrm{Vect}^r_0(\mathbb{A}^n, 0)$.

Proof. Let $V \in \mathrm{Vect}_0(\mathbb{A}^n, 0)$ be a germ of $\mathbb{A}$-smooth vector field at $0 \in \mathbb{A}^n$ such that $V(0) = 0$. In terms of the coordinates X^i on $\mathbb{A}^n$, the germ V is given by functions $V^i(X^j)$. The $\mathbb{A} \otimes \mathbb{R}(n,r)$-lift $\overline{V}$ of the germ V at $Z \in G^r_n(\mathbb{A}) \subset B^r(\mathbb{A})\mathbb{A}^n$, in terms of the induced coordinates, is of the form (48):

$$\overline{V}^i(Z) = \sum_{|p|=1}^{r} V^i_p \overset{*}{Z}{}^p, \quad V^i_p = \frac{1}{p!}\frac{D^p V^i}{DX^p}. \tag{49}$$

By Proposition 6, the restriction of the germ $\overline{V}$ to the Lie group $G^r_n(\mathbb{A})$ is a right-invariant vector field $\widetilde{V}$ on this group. From (49) it follows that the vector field $\widetilde{V}$ is uniquely determined by the r-jet of the germ V. The correspondence (49) assigning to the r-jet of a germ of $\mathbb{A}$-smooth vector field V the right-invariant vector field $\widetilde{V}$ is bijective since the value of the field $\widetilde{V}$ at the identity $e \in G^r_n(\mathbb{A})$ is of the form $\widetilde{V}^i(e) = \sum_{|p|=1}^r V^i_p \varepsilon^p$. What is more, the correspondence (49) is a Lie algebra isomorphism. In fact, the Lie bracket $[\widetilde{U}, \widetilde{V}]$ of right-invariant vector fields $\widetilde{U}$ and $\widetilde{V}$ is the restriction to $G^r_n(\mathbb{A})$ of the Lie bracket $[\overline{U}, \overline{V}]$ of the complete lifts of germs of $\mathbb{A}$-smooth vector fields U and V, which coincides with the complete lift $\overline{[U,V]}$ of the Lie bracket $[U,V]$, and the bracket $[j^r_0 U, j^r_0 V]$ in the Lie algebra $\mathrm{Vect}^r_0(\mathbb{A}^n, 0)$, by the definition, equals to the r-jet $j^r_0[U,V]$. □

3.6. **Fundamental semivector fields on the bundle** $B^r(\mathbb{A})T^{\mathbb{A}}M_n$**.** Denote by $T^{r_1}B^r(\mathbb{A})T^{\mathbb{A}}M_n$, $0 \le r_1 \le r$, the inverse image of the tangent bundle $TB^{r_1}(\mathbb{A})T^{\mathbb{A}}M_n$ under the projection $\pi^r_{r_1}(\mathbb{A}) : B^r(\mathbb{A})T^{\mathbb{A}}M_n \to B^{r_1}(\mathbb{A})T^{\mathbb{A}}M_n$. An element $\overline{V}_{\overline{X}} \in T^{r_1}_{\overline{X}}B^r(\mathbb{A})T^{\mathbb{A}}M_n$ can be considered as a tangent vector to $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ defined up to a summand belonging to the kernel of the projection $T_{\overline{X}}\pi^r_{r_1}(\mathbb{A})$ or, in terms of the algebra $\mathbb{R}(n,r)$, up to a summand belonging to the submodule

$$\overset{\circ}{\mathbb{R}}(n,r)^{r_1+1}\, T_{\overline{X}}B^r(\mathbb{A})T^{\mathbb{A}}M_n$$

generated by the $(r_1{+}1)$-st power $\overset{\circ}{\mathbb{R}}(n,r)^{r_1+1}$ of the maximal ideal $\overset{\circ}{\mathbb{R}}(n,r)$ of the algebra $\mathbb{R}(n,r)$. Therefore, the fiber $T^{r_1}_{\overline{X}}B^r(\mathbb{A})T^{\mathbb{A}}M_n$ of the bundle $T^{r_1}B^r(\mathbb{A})T^{\mathbb{A}}M_n \to B^r(\mathbb{A})T^{\mathbb{A}}M_n$ can be considered as the quotient module $T_{\overline{X}}B^r(\mathbb{A})T^{\mathbb{A}}M_n / \overset{\circ}{\mathbb{R}}(n,r)^{r_1+1}T_{\overline{X}}B^r(\mathbb{A})T^{\mathbb{A}}M_n$. From this point of view, the bundle $T^{r_1}B^r(\mathbb{A})T^{\mathbb{A}}M_n$ is the quotient bundle of the vector bundle $TB^r(\mathbb{A})T^{\mathbb{A}}M_n$ by the subbundle $\overset{\circ}{\mathbb{R}}(n,r)^{r_1+1}TB^r(\mathbb{A})T^{\mathbb{A}}M_n$ generated by the ideal $\overset{\circ}{\mathbb{R}}(n,r)^{r_1+1}$.

In a similar manner, one can define the quotient bundles

$$T^{r_1}T^{\mathbb{R}(n,r)}T^{\mathbb{A}}M_n \to T^{\mathbb{R}(n,r)}T^{\mathbb{A}}M_n.$$

Sections of the bundles

$$T^{r_1}B^r(\mathbb{A})T^{\mathbb{A}}M_n \to B^r(\mathbb{A})T^{\mathbb{A}}M_n$$

and

$$T^{r_1}T^{\mathbb{R}(n,r)}T^{\mathbb{A}}M_n \to T^{\mathbb{R}(n,r)}T^{\mathbb{A}}M_n$$

will be called, following the terminology of A.M. Vasiliev [36], *semivector fields* on $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ and $T^{\mathbb{R}(n,r)}T^{\mathbb{A}}M_n$ respectively.

An $\mathbb{A}$-smooth mapping $F : T^{\mathbb{A}}M_n \to T^{\mathbb{A}}M'_k$ induces the $\mathbb{A}\otimes\mathbb{R}(n,r)$-smooth mapping (25)

$$F^{\mathbb{A}\otimes\mathbb{R}(n,r)} : T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M_n \to T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k,$$
$$j^r_{0,X}\varphi \mapsto j^r_X F \circ j^r_{0,X}\varphi = j^r_{0,F(X)}(F\circ\varphi),$$

where $\varphi : (\mathbb{R}^n, 0) \to (T^{\mathbb{A}}M_n, X)$, and, in particular, as the restriction of (25), the mapping

$$F^{\mathbb{A}\otimes\mathbb{R}(n,r)} : B^r(\mathbb{A})T^{\mathbb{A}}M_n \to T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k,$$

The mapping (25), in turn, induces the mapping of the tangent bundles (28)

$$TF^{\mathbb{A}\otimes\mathbb{R}(n,r)} : TT^{\mathbb{A}\otimes\mathbb{R}(n,r)}M_n \to TT^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k.$$

If there is given only the jet $j^r_X F$ of F at $X \in T^{\mathbb{A}}M_n$, then the mapping

$$T_{\overline{X}}F^{\mathbb{A}\otimes\mathbb{R}(n,r)} : T_{\overline{X}}T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M_n \to T_{\overline{Y}}T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k,$$
$$\text{where}\quad \overline{Y} = F^{\mathbb{A}\otimes\mathbb{R}(n,r)}(\overline{X}),\quad \pi(\mathbb{A})^r_0(\overline{X}) = X.$$

is not defined, but we have uniquely defined (by equations (29)) the mapping

$$j^r_X F = T^{r-1}_{\overline{X}}F^{\mathbb{A}\otimes\mathbb{R}(n,r)} : T^{r-1}_{\overline{X}}T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M_n \to T^{r-1}_{\overline{Y}}T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k,$$
$$\overline{Y} = F^{\mathbb{A}\otimes\mathbb{R}(n,r)}(\overline{X}),\quad \pi(\mathbb{A})^r_0(\overline{X}) = X.$$

In particular, the jet $Y = j^r_0 F$ of $\mathbb{A}$-smooth germ $F : (\mathbb{A}^n, 0) \to (T^{\mathbb{A}}M'_k, X)$ defines the mapping

$$j^r_0 F : T^{r-1}_e B^r(\mathbb{A})T^{\mathbb{A}}\mathbb{R}^n \to T^{r-1}_{\overline{Y}}T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k, \tag{50}$$

where e is the identity of $G^r_n(\mathbb{A})$. The subspaces $T^{r-1}_e B^r(\mathbb{A})T^{\mathbb{A}}\mathbb{R}^n$ and $T^{r-1}_{\overline{Y}}T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k$ carry structures of $\mathbb{A}\otimes\mathbb{R}(n,r)$-modules isomorphic, respectively, to the modules $\mathbb{A}\otimes\mathbb{R}(n,r-1)^n$ and $\mathbb{A}\otimes\mathbb{R}(n,r-1)^k$, and the mapping (50) is $\mathbb{A}\otimes\mathbb{R}(n,r)$-linear. Fixing $\overline{V}^{(r-1)}_e \in T^{r-1}_e B^r(\mathbb{A})T^{\mathbb{A}}\mathbb{R}^n$ and varying $\overline{Y}$, we obtain a semivector field $\overline{V}^{(r-1)}$ on $T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k$.

This semivector field will be called the *fundamental semivector field* (cf [31]) on $T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k$ corresponding to $\overline{V}_e^{(r-1)}$. Similarly, fixing $\overline{V}_e^{(r')} \in T_e^{r'}B^r(\mathbb{A})T^{\mathbb{A}}\mathbb{R}^n$ for $r' < r$ and varying $\overline{Y}$, we obtain the *fundamental semivector field* on $T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k$ corresponding to $\overline{V}_e^{(r')}$.

In terms of local coordinates, fundamental semivector fields are given by equations similar to (31). The semivector field corresponding to $\overline{V}_e^{(r-1)} = \overline{V}_e^j\partial_j(e)$, $\overline{V}_e^j \in \mathbb{A}\otimes\mathbb{R}(n,r-1)$, is given by the equations

$$\overline{V}^{i'}(\overline{Y}) = (Y_j^{i'} + \sum_{|p|=1}^{r-1}(p_j+1)Y_{p+j}^{i'}\varepsilon^p)\overline{V}_e^j, \tag{51}$$

where the product in relations of the form $(Y_{p+j}^{i'}\varepsilon^p)\overline{V}_e^j$ is understood as the action of the algebra $\mathbb{A}\otimes\mathbb{R}(n,r)$ on the quotient algebra $\mathbb{A}\otimes\mathbb{R}(n,r-1)$.

In particular, fundamental semivector fields on the bundle $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ are of the form (see (32))

$$\overline{V}^i(\overline{X}) = (X_j^i + \sum_{|p|=1}^{r-1}(p_j+1)X_{p+j}^i\varepsilon^p)\overline{V}_e^j = (\sum_{|p|=0}^{r-1}(p_j+1)X_{p+j}^i\varepsilon^p)\overline{V}_e^j. \tag{52}$$

Note 2. The bundles $T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k$ and $T^{r'}B^r(\mathbb{A})T^{\mathbb{A}}M^n$ are, obviously, $\mathbb{A}$-smooth manifolds modeled on $\mathbb{A}$-modules of type $\mathbb{A}^m$ for certain $m \in \mathbb{N}$ (these bundles can also be considered as $\mathbb{A}\otimes\mathbb{R}(n,r)$-smooth manifolds modeled on the corresponding $\mathbb{A}\otimes\mathbb{R}(n,r)$-modules, see [33]). From (51) and (52) it follows that fundamental semivector fields are $\mathbb{A}$-smooth sections of these bundles.

Note 3. The complete lift of a germ of $\mathbb{A}$-smooth vector field V on $\mathbb{A}^n \equiv T^{\mathbb{A}}\mathbb{R}^n$ at zero to the bundle $B^r(\mathbb{A})T^{\mathbb{A}}\mathbb{R}^n$ is a germ of $\mathbb{A}\otimes\mathbb{R}(n,r)$-smooth vector field $\overline{V}$ on $B^r(\mathbb{A})T^{\mathbb{A}}\mathbb{R}^n$ defined along the whole fiber $G_n^r(\mathbb{A})$. This germ has equations (43). If there is given only the $(r-1)$-jet $j_0^{r-1}V$ of V, then formulas (43) define a unique element $\overline{V}_e \in T_e^{r-1}B^r(\mathbb{A})T^{\mathbb{A}}\mathbb{R}^n$, which, in turn, determines a unique $(r-1)$-jet $j_0^{r-1}V$. Thus, there is an isomorphism between the $\mathbb{A}\otimes\mathbb{R}(n,r)$-modules $\mathrm{Vect}_0^{r-1}(\mathbb{A}^n)$ and $T_e^{r-1}B^r(\mathbb{A})\mathbb{A}^n$. Since the bundle $B^r(\mathbb{A})\mathbb{A}^n$ is an open submanifold of $T^{\mathbb{A}\otimes\mathbb{R}(n,r)}\mathbb{R}^n \equiv (\mathbb{A}\otimes\mathbb{R}(n,r))^n$, the $\mathbb{A}\otimes\mathbb{R}(n,r)$-module $T_e^{r-1}B^r(\mathbb{A})\mathbb{A}^n$ is canonically isomorphic to the $\mathbb{A}\otimes\mathbb{R}(n,r)$-module $(\mathbb{A}\otimes\mathbb{R}(n,r-1))^n$. From (51) it follows that the set of all fundamental semivector fields on $T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k$ is also an $\mathbb{A}\otimes\mathbb{R}(n,r-1)$-module isomorphic to $(\mathbb{A}\otimes\mathbb{R}(n,r-1))^n$. We will denote the $\mathbb{A}\otimes\mathbb{R}(n,r')$-module of fundamental semivector fields on $T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k$ by $\mathcal{T}^{r'}T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k$.

In the case $M'_k = \mathbb{R}$, $k = 1$, formula (51) determines the fundamental semivector fields on the algebra $\mathbb{A} \otimes \mathbb{R}(n, r)$. As in the proof of Proposition 1, one can easily verify (see also [31]) that these fundamental semivector fields are $\mathbb{A}$-linear derivations from the algebra $\mathbb{A} \otimes \mathbb{R}(n, r)$ to the algebra $\mathbb{A} \otimes \mathbb{R}(n, r-1)$ with respect to the canonical epimorphism $\pi^r_{r-1} : \mathbb{A} \otimes \mathbb{R}(n, r) \to \mathbb{A} \otimes \mathbb{R}(n, r-1)$, and that the set $\mathfrak{D}^r_{r-1}(\mathbb{A})$ of all such derivations coincides with the set of all fundamental semivector fields. By Note 3, $\mathfrak{D}^r_{r-1}(\mathbb{A})$ is an $\mathbb{A} \otimes \mathbb{R}(n, r-1)$-module isomorphic to $(\mathbb{A} \otimes \mathbb{R}(n, r-1))^n$. The $\mathbb{A}$-linear derivations $\partial_j : \mathbb{A} \otimes \mathbb{R}(n, r) \to \mathbb{A} \otimes \mathbb{R}(n, r-1)$ defined by $\partial_j(\varepsilon^i) = \delta^i_j$ form a basis in this module. Using the derivations ∂_j, one can rewrite (51) as follows

$$\overline{V}^{i'}(\overline{Y}) = \overline{V}^j_e \partial_j(\overline{Y}^{i'}), \quad \text{where} \quad \overline{Y}^{i'} = \sum_{|p|=0}^{r} Y^{i'}_p \otimes \varepsilon^p, \quad Y^{i'}_p \in \mathbb{A}. \tag{53}$$

Relations (53) establish an isomorphism between the $\mathbb{A} \otimes \mathbb{R}(n, r-1)$-module $T^{r-1}_e B^r(\mathbb{A}) T^{\mathbb{A}}\mathbb{R}^n$ and the $\mathbb{A} \otimes \mathbb{R}(n, r-1)$-module of fundamental semivector fields on $T^{\mathbb{A}\otimes\mathbb{R}(n,r)} M'_k$. In particular, fundamental semivector fields on $\mathbb{A} \otimes \mathbb{R}(n, r)$ are of the form

$$\overline{V}(\alpha) = \overline{V}^j \partial_j(\alpha), \quad \overline{V}^j \in \mathbb{A} \otimes \mathbb{R}(n, r-1). \tag{54}$$

Note 4. The multiplication in $\mathbb{A} \otimes \mathbb{R}(n, r)$ induces the action

$$\mathbb{A} \otimes \mathbb{R}(n, r-1) \times \mathbb{A} \otimes \mathfrak{m}(\mathbb{R}(n, r)) \to \mathbb{A} \otimes \mathfrak{m}(\mathbb{R}(n, r)).$$

With this in mind, we can express an arbitrary fundamental vector field on $\mathbb{A} \otimes \mathbb{R}(n, r)$ ($\mathbb{A}$-linear derivation of $\mathbb{A} \otimes \mathbb{R}(n, r)$) in the same form as (54):

$$\overline{V}(\alpha) = \overline{V}^j \partial_j(\alpha), \quad \overline{V}^j \in \mathbb{A} \otimes \mathfrak{m}(\mathbb{R}(n, r)), \tag{55}$$

where the operators ∂_j have the same sense as in (54).

Denote by $\mathfrak{D}^r_{r'}(\mathbb{A})$, $r' < r$, the $\mathbb{A} \otimes \mathbb{R}(n, r-1)$-module of $\mathbb{A}$-linear derivations from $\mathbb{A} \otimes \mathbb{R}(n, r)$ to $\mathbb{A} \otimes \mathbb{R}(n, r')$ with respect to the canonical epimorphism $\pi^r_{r'} : \mathbb{A} \otimes \mathbb{R}(n, r) \to \mathbb{A} \otimes \mathbb{R}(n, r')$. Each derivation $D \in \mathfrak{D}^r_{r-1}(\mathbb{A})$ generates a series of derivations $D^r_{r'} \in \mathfrak{D}^r_{r'}(\mathbb{A})$, $r' < r-1$, defined by $D^r_{r'} = \pi^{r-1}_{r'} \circ D$. We will also denote $D^r_{r'}$ simply by D. Then the relation $[D_1, D_2] = D_2 \circ D_1 - D_1 \circ D_2$ defines the bracket

$$[\,,] : \mathfrak{D}^r_{r-1}(\mathbb{A}) \times \mathfrak{D}^r_{r-1}(\mathbb{A}) \to \mathfrak{D}^r_{r-2}(\mathbb{A}). \tag{56}$$

In terms of the derivations ∂_j, the bracket (56) is completely determined by the relations

$$[\partial_i, \partial_j] = 0, \tag{57}$$

from which it follows that if $[\overline{V}_e^{(r-1)}, \overline{W}_e^{(r-1)}] = \overline{U}_e^{(r-2)}$, then

$$\overline{U}_e^i \partial_i = [\overline{V}_e^j \partial_j, \overline{W}_e^j \partial_j] = (\overline{W}_e^j \partial_j(\overline{V}_e^i) - \overline{V}_e^j \partial_j(\overline{W}_e^i))\partial_i. \tag{58}$$

It will be convenient to identify $\overline{V}_e^{(r-1)} = \{\overline{V}_e^j\} \in T_e^{r-1}B^r(\mathbb{A})\mathbb{R}^n$ with $\overline{V}_e^j \partial_j \in \mathfrak{D}_{r-1}^r(\mathbb{A})$.

The Lie bracket of vector fields on $T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k$ induces the $\mathbb{A}$-linear bracket of fundamental semivector fields

$$[\,,] : \mathcal{T}^{r-1}T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k \times \mathcal{T}^{r-1}T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k \to \mathcal{T}^{r-2}T^{\mathbb{A}\otimes\mathbb{R}(n,r)}M'_k, \tag{59}$$

which corresponds to the bracket (56) since the fundamental semivector fields (53) on $T^{\mathbb{A}\otimes\mathbb{R}(n,r)}\mathbb{R}^k$ can be regarded as $\mathbb{A}$-linear mappings (cf the proof of Proposition 1).

3.7. **The structure form of the bundle $B^r(\mathbb{A})T^{\mathbb{A}}M_n$.** The structure form θ^r of the bundle B^rM_n of r-frames on a real smooth manifold M_n is a 1-form on B^rM_n with values in the space $T_e^{r-1}B^r\mathbb{R}^n$, where e is the identity of the differential group $G_n^r \subset B^r\mathbb{R}^n$. The form θ^r is defined as follows [42]. If $X = j^r f$, where $f : (\mathbb{R}^n, 0) \to (M_n, x)$ is a germ of diffeomorphism, then

$$\theta_X^r(v_X) = j^r(f^{-1})(T\pi_{r-1}^r(v_X)).$$

As the domain of values of the structure form θ^r, one can also take the space $\mathrm{Vect}_0^{r-1}(\mathbb{R}^n)$ of $(r-1)$-jets of vector fields at zero on $\mathbb{R}^n$ [9], [21].

In a similar way, we define the $T_e^{r-1}B^r(\mathbb{A})T^{\mathbb{A}}\mathbb{R}^n$-valued (or, equivalently, $(\mathbb{A}\otimes\mathbb{R}(n, r-1))^n$-valued) structure form $\Theta^r = \Theta^r(T^{\mathbb{A}}M_n)$ of the bundle $B^r(\mathbb{A})T^{\mathbb{A}}M_n$. Let $\overline{X} = j^r F$, where $F : (\mathbb{A}^n, 0) \to (T^{\mathbb{A}}M_n, X)$ is a germ of $\mathbb{A}$-diffeomorphism, and let $\overline{V}_{\overline{X}}$ be a tangent vector to $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ at $\overline{X}$. We let

$$\Theta_{\overline{X}}^r(\overline{V}_{\overline{X}}) = j^r(F^{-1})(T\pi_{r-1}^r(\overline{V}_{\overline{X}})). \tag{60}$$

From (60) it follows that in fact the form Θ^r is defined on elements from $T^{r-1}B^r(\mathbb{A})T^{\mathbb{A}}M_n$ and assigns to the value $\overline{V}_{\overline{X}}^{(r-1)}$ at $\overline{X}$ of the fundamental semivector field $\overline{V}^{(r-1)}$ corresponding to an element $\overline{V}_e^{r-1}$ the same element $\overline{V}_e^{r-1}$.

Proposition 8. *The structure form Θ^r is an $\mathbb{A}$-smooth mapping.*

Proof. In terms of local coordinates, a fundamental semivector field on $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ is given by functions $\overline{V}^i(\overline{X}) = (\sum_{|p|=0}^r \overline{X}_{p+j}^i)\overline{V}_e^j$ (see (52)) of $\mathbb{A}$-valued coordinates $\overline{X}_p^i$, which, obviously, are $\mathbb{A}$-smooth. When the elements $\overline{V}_e^j \in \mathbb{A}\otimes\mathbb{R}(n, r-1)$ run through the all possible values, equations

(52) define an $\mathbb{A}$-smooth trivialization

$$\psi : (\mathbb{A} \otimes \mathbb{R}(n, r-1))^n \times B^r(\mathbb{A})T^{\mathbb{A}}M_n \to T^{r-1}B^r(\mathbb{A})T^{\mathbb{A}}M_n \tag{61}$$

of the bundle $T^{r-1}B^r(\mathbb{A})T^{\mathbb{A}}M_n$. Using the mappings

$$\lambda : (\mathbb{A} \otimes \mathbb{R}(n, r-1))^n \ni \{\overline{V}^j\}_e \partial_j(e) \mapsto \overline{V}^j_e \partial_j(e) \in T^{r-1}_e B^r(\mathbb{A})\mathbb{A}^n$$

and

$$\mathrm{pr}_1 : (\mathbb{A} \otimes \mathbb{R}(n, r-1))^n \times B^r(\mathbb{A})T^{\mathbb{A}}M_n \mapsto (\mathbb{A} \otimes \mathbb{R}(n, r-1))^n,$$

we can represent the structure form Θ^r as the composition

$$\Theta^r = \lambda \circ pr_1 \circ \psi^{-1} \circ \pi^r_{r-1}.$$

Since ψ is an $\mathbb{A}$-diffeomorphism, its inverse ψ^{-1} is also an $\mathbb{A}$-diffeomorphism, whence it follows that the form Θ^r is $\mathbb{A}$-smooth. $\square$

The structure form Θ^r possesses the following property, which is an analogue of the corresponding property of the structure form of the bundle of r-frames of real smooth manifold M_n (see, e. g., [21]).

Theorem 2. *Let $\overline{\Phi} : B^r(\mathbb{A})T^{\mathbb{A}}M_n \to B^r(\mathbb{A})T^{\mathbb{A}}M'_n$ be a local diffeomorphism which maps the structure form Θ^r of the bundle $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ into the the structure form Θ'^r of the bundle $B^r(\mathbb{A})T^{\mathbb{A}}M'_n$. Then in a neighborhood of every point $\overline{X} \in B^r(\mathbb{A})T^{\mathbb{A}}M_n$ the diffeomorphism $\overline{\Phi}$ coincides with the $\mathbb{A} \otimes \mathbb{R}(n, r)$-prolongation of a local $\mathbb{A}$-diffeomorphism $\Phi : T^{\mathbb{A}}M_n \to T^{\mathbb{A}}M'_n$.*

Proof. In the proof of this statement, we will use the scheme applied in [21] (Section 1.3.1). We will suppose that the structure forms take values in the $\mathbb{A} \otimes \mathbb{R}(n, r)$-module $(\mathbb{A} \otimes \mathbb{R}(n, r-1))^n$.

1) If $\overline{\Phi}$ coincides with $\mathbb{A} \otimes \mathbb{R}(n, r)$-prolongation $\Phi^{\mathbb{A}\otimes\mathbb{R}(n,r)}$ of a local $\mathbb{A}$-diffeomorphism $\Phi : T^{\mathbb{A}}M_n \to T^{\mathbb{A}}M'_n$, then $\overline{\Phi}$ maps the fundamental semivector field on $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ corresponding to $\overline{V}^{(r-1)}_e \in T^{r-1}B^r(\mathbb{A})\mathbb{A}^n$ into the fundamental semivector field on $B^r(\mathbb{A})T^{\mathbb{A}}M'_n$ corresponding to the same element $\overline{V}^{(r-1)}_e$. Hence it follows that $\overline{\Phi}$ maps the structure form of $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ into the structure form of $B^r(\mathbb{A})T^{\mathbb{A}}M'_n$.

2) Since the structure form Θ^r establishes an isomorphism between the $\mathbb{A} \otimes \mathbb{R}(n, r)$-modules $T^{r-1}_{\overline{X}} B^r(\mathbb{A})T^{\mathbb{A}}M_n$ and $(\mathbb{A} \otimes \mathbb{R}(n, r-1))^n$, it also establishes an isomorphism between the submodules of these modules generated by the ideals $\mathbb{A} \otimes (\overset{\circ}{\mathbb{R}}(n, r-1))^{r_1}$, $r_1 = 0, 1, \dots, r-1$. Hence it follows that $\overline{\Phi}$ is fibered over the local diffeomorphisms

$$\overline{\Phi}^{r_1} : B^{r_1}(\mathbb{A})T^{\mathbb{A}}M_n \to B^{r_1}(\mathbb{A})T^{\mathbb{A}}M'_n, \quad r_1 = 0, 1, \dots, r-1.$$

In particular, $\overline{\Phi}$ is fibered over $\Phi = \overline{\Phi}^0 : T^{\mathbb{A}}M_n \to T^{\mathbb{A}}M'_n$.

3) In the case $r = 1$, the local diffeomorphism $\overline{\Phi}^1 : B^1(\mathbb{A})T^{\mathbb{A}}M_n \to B^1(\mathbb{A})T^{\mathbb{A}}M'_n$ is fibered over $\Phi : T^{\mathbb{A}}M_n \to T^{\mathbb{A}}M'_n$. Since the map $\overline{\Phi}^1$ induces an isomorphism of $\mathbb{A}$-modules $T^0_{\overline{X}}B^1(\mathbb{A})T^{\mathbb{A}}M_n \equiv T_X T^{\mathbb{A}}M_n$ and $T^0_{\overline{\Phi}(\overline{X})}B^1(\mathbb{A})T^{\mathbb{A}}M'_n \equiv T_{\Phi(X)}T^{\mathbb{A}}M'_n$, the tangent mappings $T_X\Phi$ are $\mathbb{A}$-linear. Hence it follows that the mapping $\Phi : T^{\mathbb{A}}M_n \to T^{\mathbb{A}}M'_n$ is $\mathbb{A}$-smooth.

4) Assume now that the statement of the theorem holds for the bundles of $\mathbb{A}$-smooth p-frames for $p = 1, \dots, r-1$. As has been shown in item 2), the mapping $\overline{\Phi}$ is fibered over $\overline{\Phi}^{r-1} : B^{r-1}(\mathbb{A})T^{\mathbb{A}}M_n \to B^{r-1}(\mathbb{A})T^{\mathbb{A}}M'_n$. Since fundamental semivector fields which are sections of the bundle $T^{r-2}B^r(\mathbb{A})T^{\mathbb{A}}M_n \to B^r(\mathbb{A})T^{\mathbb{A}}M_n$ project into fundamental semivector fields being sections of the bundle $T^{r-2}B^{r-1}(\mathbb{A})T^{\mathbb{A}}M_n \to B^{r-1}(\mathbb{A})T^{\mathbb{A}}M_n$, it follows that the mapping $\overline{\Phi}^{r-1}$ maps the structure form of $B^{r-1}(\mathbb{A})T^{\mathbb{A}}M_n$ into the structure form of $B^{r-1}(\mathbb{A})T^{\mathbb{A}}M'_n$. Hence it follows that the mapping $\overline{\Phi}^{r-1}$ coincides with the $\mathbb{A}\otimes\mathbb{R}(n, r-1)$-prolongation of a local $\mathbb{A}$-diffeomorphism $\Phi : T^{\mathbb{A}}M_n \to T^{\mathbb{A}}M'_n$. Consider the $\mathbb{A}\otimes\mathbb{R}(n,r)$-prolongation $\Phi^{\mathbb{A}\otimes\mathbb{R}(n,r)}$ of Φ and the composition $\Psi = (\Phi^{\mathbb{A}\otimes\mathbb{R}(n,r)})^{-1} \circ \overline{\Phi} : B^r(\mathbb{A})T^{\mathbb{A}}M_n \to B^r(\mathbb{A})T^{\mathbb{A}}M_n$. The local diffeomorphism Ψ preserves the structure form and projects into the identity diffeomorphism of the bundle $B^{r-1}(\mathbb{A})T^{\mathbb{A}}M_n$. Therefore, Ψ is a family of right translations of the bundle $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ over the bundle $B^{r-1}(\mathbb{A})T^{\mathbb{A}}M_n$. Such right translations $Z \in G^r_n(\mathbb{A})$ have coordinates of the form $Z^i_k = \delta^i_k$, $Z^i_p = 0$ for $|p| \le r-1$, $Z^i_p \in \mathbb{A}$ for $|p| = r$ and, therefore, do not change the coordinates of elements of the bundle $T^{r-1}B^r(\mathbb{A})T^{\mathbb{A}}M_n$. Equating the coordinates (52) of the values of the fundamental semivector field corresponding to an arbitrary element $\overline{V}^{(r-1)}_e \in T^{r-1}_e B^r(\mathbb{A})\mathbb{A}^n$ at the points $\overline{X}$ and $R_{\overline{Z}}(\overline{X})$, namely,

$$\overline{V}^i(\overline{X}) = \Big(\sum_{|p|=0}^{r-1}(p_j+1)X^i_{p+j}\varepsilon^p\Big)\overline{V}^j_e \tag{62}$$

and

$$\overline{V}^i(R_{\overline{Z}}(\overline{X})) = \Big(\sum_{|p|=0}^{r-1}(p_j+1)X^i_{p+j}\varepsilon^p\Big)\overline{V}^j_e + \\ + \Big(\sum_{|p|=r-1}(p_j+1)X^i_k Z^k_{p+j}\varepsilon^p\Big)\overline{V}^j_e, \tag{63}$$

we conclude that $Z^k_p = 0$ for $|p| = r$. Thus, $\Psi = \mathrm{id}$ and $\overline{\Phi} = \Phi^{\mathbb{A}\otimes\mathbb{R}(n,r)}$. $\square$

There are the natural embeddings of the bundle B^rM_n of r-frames of a manifold M_n into the bundle $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ of $\mathbb{A}$-smooth r-frames $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ and of the structure group G^r_n of the bundle B^rM_n into the structure group $G^r_n(\mathbb{A})$ of the bundle $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ (see (13)) defined by the correspondence $\iota : j^r_0 f \mapsto j^r_0(T^{\mathbb{A}} f)$, where f is a germ of diffeomorphism and $T^{\mathbb{A}} f$ is the $\mathbb{A}$-prolongation of f. Under these embeddings, the vector space $T^{r-1}_e B^r\mathbb{R}^n$ becomes a subspace in $T^{r-1}_e B^r(\mathbb{A})\mathbb{A}^n$ which generates $T^{r-1}_e B^r(\mathbb{A})\mathbb{A}^n$ as an $\mathbb{A}$-module, and each fundamental semivector field on B^rM_n is the restriction of the corresponding fundamental semivector field on $B^r(\mathbb{A})T^{\mathbb{A}}M_n$. This implies the following proposition.

Proposition 9. *i) Under the embedding (13), the structure form θ^r of B^rM_n coincides with the restriction to B^rM_n of the $\mathbb{R}(n, r-1)^n$-valued part of the structure form Θ^r of $B^r(\mathbb{A})T^{\mathbb{A}}M_n$, and the form Θ^r coincides with the $\mathbb{A}$-prolongation of the form θ^r.*

ii) If a local diffeomorphism $\overline{\Phi} : B^r(\mathbb{A})T^{\mathbb{A}}M_n \to B^r(\mathbb{A})T^{\mathbb{A}}M'_n$ maps the structure form Θ^r of $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ into the structure form Θ'^r of $B^r(\mathbb{A})T^{\mathbb{A}}M'_n$ and maps the subbundle B^rM_n to the subbundle $B^rM'_n$, then, in a neighborhood of every point $\overline{X} \in B^r(\mathbb{A})T^{\mathbb{A}}M_n$, $\overline{\Phi}$ coincides with the $\mathbb{A}\otimes\mathbb{R}(n, r)$-prolongation of a local diffeomorphism $\varphi : M_n \to M'_n$.

Proof. The first part of statement i) follows from the fact that the fundamental semivector fields of the bundle B^rM_n are the restrictions of the fundamental semivector fields of the bundle $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ corresponding to the elements of $T^{r-1}_e B^r\mathbb{R}^n \subset T^{r-1}_e B^r(\mathbb{A})\mathbb{A}^n$. The second part of statement i) then follows from the $\mathbb{A}$-smoothness of the form Θ^r.

If $\overline{\Phi}$ maps the structure form of the bundle $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ into the structure form of the bundle $B^r(\mathbb{A})T^{\mathbb{A}}M'_n$ and the bundle B^rM_n to the bundle $B^rM'_n$, then the restriction $\Phi = \overline{\Phi}|B^rM_n$ maps the structure form of B^rM_n into the structure form of $B^rM'_n$. In this case [21], Φ is the $\mathbb{R}(n, r)$-prolongation of a local diffeomorphism $\varphi : M_n \to M'_n$, and the $\mathbb{A}\otimes\mathbb{R}(n, r)$-diffeomorphism $\overline{\Phi}$ coincides with the $\mathbb{A}\otimes\mathbb{R}(n, r)$-prolongation of $\varphi : M_n \to M'_n$. $\square$

3.8. **Structure equations of the bundle $B^r(\mathbb{A})T^{\mathbb{A}}M_n$.** We define the $T^{r-2}_e B^r(\mathbb{A})T^{\mathbb{A}}\mathbb{R}^n$-valued 1-form $\widetilde{\Theta}^r$ on $B^r(\mathbb{A})T^{\mathbb{A}}M_n$ as follows: $\widetilde{\Theta}^r = \pi^{r-1}_{r-2}\circ\Theta^r$, where $\pi^{r-1}_{r-2} : T^{r-1}_e B^r(\mathbb{A})T^{\mathbb{A}}\mathbb{R}^n \to T^{r-2}_e B^r(\mathbb{A})T^{\mathbb{A}}\mathbb{R}^n$ is the canonical projection. Equivalently, $\widetilde{\Theta}^r$ can be defined as the inverse image of the structure form Θ^{r-1} of $B^{r-1}(\mathbb{A})T^{\mathbb{A}}M_n$. Let $\widetilde{\Theta}^r = \widetilde{\Theta}^j e^{(r-2)}_j$ be the expansion of $\widetilde{\Theta}^r$ in terms of the standard basis in the $\mathbb{A}\otimes\mathbb{R}(n, r-2)$-module

$T_e^{r-2}B^r(\mathbb{A})T^{\mathbb{A}}\mathbb{R}^n \equiv (\mathbb{A}\otimes\mathbb{R}(n,r-2))^n$, and let $\Theta^r = \Theta^j e_j^{(r-1)}$ be the expansion of Θ^r in terms of the standard basis in the $\mathbb{A}\otimes\mathbb{R}(n,r-1)$-module $T_e^{r-1}B^r(\mathbb{A})T^{\mathbb{A}}\mathbb{R}^n \equiv (\mathbb{A}\otimes\mathbb{R}(n,r-1))^n$. For two arbitrary fundamental semivector fields $\overline{V}^{(r-1)}$ and $\overline{W}^{(r-1)}$, using the well-known formula for the exterior differential of a 1-form ([11], p. 36), we have

$$\begin{aligned}
&d\widetilde{\Theta}^r(\overline{V}^{(r-1)},\overline{W}^{(r-1)}) = \\
&= \frac{1}{2}(\overline{V}^{(r-1)}(\widetilde{\Theta}^r(\overline{W}^{(r-1)})) - \overline{W}^{(r-1)}(\widetilde{\Theta}^r(\overline{V}^{(r-1)})) - \widetilde{\Theta}^r([\overline{V}^{(r-1)},\overline{W}^{(r-1)}])) = \\
&= -\frac{1}{2}\widetilde{\Theta}^r([\overline{V}^{(r-1)},\overline{W}^{(r-1)}]) = -\frac{1}{2}([\overline{V}_e^{(r-1)},\overline{W}_e^{(r-1)}]) = \\
&= -\frac{1}{2}(\overline{W}_e^j\partial_j\overline{V}_e^i - \overline{V}_e^j\partial_j\overline{W}_e^i)\partial_i = \\
&= -\frac{1}{2}(\Theta^j(\overline{W}^{(r-1)})\partial_j(\Theta^i(\overline{V}^{(r-1)})) - \Theta^j(\overline{V}^{(r-1)})\partial_j(\Theta^i(\overline{W}^{(r-1)}))) = \\
&= (\Theta^j\wedge\partial_j\Theta^i)(\overline{V}^{(r-1)},\overline{W}^{(r-1)}).
\end{aligned}$$

As a result of the above calculations, we obtain the following proposition.

Proposition 10. *On the bundle $B^r(\mathbb{A})T^{\mathbb{A}}M_n$, the following structure equations hold:*

$$d\widetilde{\Theta}^i = \Theta^j\wedge\partial_j\circ\Theta^i. \tag{64}$$

Note 5. For each r, one can take the inverse image $\widetilde{\Theta}'^r$ of the form $\widetilde{\Theta}^r$ with respect to the projection $\pi_r^\infty : B^\infty(\mathbb{A})T^{\mathbb{A}}M_n \to B^r(\mathbb{A})T^{\mathbb{A}}M_n$. Since $\widetilde{\Theta}'^r = \pi_{r-2}^{r-1}\circ\widetilde{\Theta}'^{r+1}$, where $\pi_{r-2}^{r-1} : T_e^{r-2}B^r(\mathbb{A})T^{\mathbb{A}}\mathbb{R}^n \to T_e^{r-2}B^r(\mathbb{A})T^{\mathbb{A}}\mathbb{R}^n$, in the projective limit, one obtains [1] the $\mathbb{A}(n,\infty)^n$-valued form Θ^∞ on $B^\infty(\mathbb{A})T^{\mathbb{A}}M_n$. For r tending to infinity, the series of equations (64) gives the following expression for the exterior differential of the form Θ^∞:

$$d(\Theta^\infty)^i = (\Theta^\infty)^j\wedge\partial_j\circ(\Theta^\infty)^i, \tag{65}$$

where ∂_j is the $\mathbb{A}$-linear derivation of the algebra $\mathbb{A}(n,\infty)$ defined by $\partial_j(\varepsilon^i) = \delta_j^i$.

For $\mathbb{A} = \mathbb{R}$, the equations (103) coincide with the infinite series of G.F. Laptev structure equations [18], [41], [42].

In a form similar to (103) one can represent the Maurer–Cartan equations for the Lie group $G_n^r(\mathbb{A})$ (see [31]).

4. Weil functors and product preserving functors on the category $\mathcal{M}f_{\mathrm{tr}}^m$.

4.1. **The category of m-parameter-dependent manifolds $\mathcal{M}f_{\mathrm{tr}}^m$.** The category $\mathcal{M}f_{\mathrm{tr}}^m$ is defined as follows. The objects of $\mathcal{M}f_{\mathrm{tr}}^m$ are the trivial fiber bundles $p : M_n \times U \to U$, where M_n is a smooth manifold and U is an open subset of $\mathbb{R}^m$. The morphisms of $\mathcal{M}f^m_{\mathrm{tr}}$ are the commutative diagrams of the form

$$\begin{tikzcd}
M \times U \arrow[r, "f"] \arrow[d] & M' \times U' \arrow[d] \\
U \arrow[r, "\mathrm{tr}_{t_0}"] & U'
\end{tikzcd} \tag{66}$$

where f is a smooth mapping and $\mathrm{tr}_{t_0} : U \ni \{t^a\} \mapsto \{t^a + t^a_0\} \in U'$ is the restriction of a translation $\mathrm{tr}_{t_0} : \mathbb{R}^m \to \mathbb{R}^m$ which embeds U into U'. We will denote a morphism (66) by (f, tr_{t_0}) or simply by f. In terms of local coordinates (x^i, t^a) on $M_n \times U$ and $(x^{i'}, t^{a'})$ on $M'_k \times U'$, a morphism (f, tr_{t_0}) is given by equations $x^{i'} = f^{i'}(x^i, t^a)$, $t^{a'} = t^a + t^a_0$.

The category $\mathcal{M}f^m$ is a subcategory of $\mathcal{M}f^m_{\mathrm{tr}}$.

We also consider the category $\mathcal{FM}^m_{\mathrm{tr}}$ of m-parameter-dependent fibered manifolds from $\mathcal{M}f^m_{\mathrm{tr}}$ whose objects are the commutative diagrams

$$\begin{tikzcd}
E \times U \arrow[r, "p_E"] \arrow[d, "\pi"'] & U \arrow[d, "\mathrm{id}"] \\
M_n \times U \arrow[r, "p"] & U
\end{tikzcd} \tag{67}$$

where $p_E : E \times U \to U$ and $p : M_n \times U \to U$ are objects of $\mathcal{M}f^m_{\mathrm{tr}}$, and whose morphisms are the commutative diagrams

$$\begin{tikzcd}
& E \times U \arrow[rr, "f"] \arrow[dl, "\pi"'] \arrow[dd] & & E' \times U' \arrow[dl, "\pi'"'] \arrow[dd] \\
M_n \times U \arrow[rr, "\overline{f}"] \arrow[dd, "p"'] & & M'_k \times U' \arrow[dd, "p'"] & \\
& U \arrow[rr, "\mathrm{tr}_{t_0}"] \arrow[dl, "\mathrm{id}"'] & & U' \arrow[dl, "\mathrm{id}"'] \\
U \arrow[rr, "\mathrm{tr}_{t_0}"] & & U' &
\end{tikzcd} \tag{68}$$

The base functor $B : \mathcal{FM}^m_{\mathrm{tr}} \to \mathcal{M}f^m_{\mathrm{tr}}$ and the erasing functor $\varepsilon : \mathcal{FM}^m_{\mathrm{tr}} \to \mathcal{M}f^m_{\mathrm{tr}}$ are defined as in the case of the category $\mathcal{M}f^m$.

The pair $(\varphi \times \mathrm{tr}_{t_0}, \mathrm{tr}_{t_0})$, where $\varphi \times \mathrm{tr}_{t_0}$ is the product of a smooth mapping $\varphi : M \to M'$ and a translation $\mathrm{tr}_{t_0} : U \to U'$, is a morphism of $\mathcal{M}f^m_{\mathrm{tr}}$. For brevity, in what follows we will denote such a morphism simply by $\varphi \times \mathrm{tr}_{t_0}$. The objects of the category $\mathcal{M}f^m_{\mathrm{tr}}$ together with all morphisms of the form $\varphi \times \mathrm{tr}_{t_0}$ constitute a subcategory of $\mathcal{M}f^m_{\mathrm{tr}}$, which will be denoted by $(\mathcal{M}f \times \mathbb{R}^m)_{\mathrm{tr}}$. The category $\mathcal{M}f \times \mathbb{R}^m$ is a subcategory of $(\mathcal{M}f \times \mathbb{R}^m)_{\mathrm{tr}}$. Obviously, every $(\mathcal{M}f \times \mathbb{R}^m)_{\mathrm{tr}}$-morphism

$\varphi \times \mathrm{tr}_{t_0} : M_n \times U \to M'_k \times U'$ is the restriction of a morphism $\varphi \times \mathrm{tr}_{t_0} : M_n \times \mathbb{R}^m \to M'_k \times \mathbb{R}^m$. More precisely, the following commutative diagram holds:

$$\begin{array}{ccc} M_n \times U & \xrightarrow{\mathrm{id}_M \times i_U} & M_n \times \mathbb{R}^m \\ {\scriptstyle \varphi\times\mathrm{tr}_{t_0}}\big\downarrow & & \big\downarrow{\scriptstyle \varphi\times\mathrm{tr}_{t_0}} \\ M'_k \times U' & \xrightarrow{\mathrm{id}_{M'} \times i_{U'}} & M'_k \times \mathbb{R}^m \end{array} \tag{69}$$

where $i_U : U \to \mathbb{R}^m$ and $i_{U'} : U' \to \mathbb{R}^m$ are inclusions. Note that an $(\mathcal{M}f \times \mathbb{R}^m)_{\mathrm{tr}}$-morphism $\varphi \times \mathrm{tr}_{t_0} : M_n \times \mathbb{R}^m \to M'_k \times \mathbb{R}^m$ can be represented as the compositions $(\varphi \times \mathrm{id}) \circ (\mathrm{id}_M \times \mathrm{tr}_{t_0}) = (\mathrm{id}_{M'} \times \mathrm{tr}_{t_0}) \circ (\varphi \times \mathrm{id})$, where $\varphi \times \mathrm{id}$ is an $\mathcal{M}f \times \mathbb{R}^m$-morphism.

A covariant functor $F : (\mathcal{M}f \times \mathbb{R}^m)_{\mathrm{tr}} \to \mathcal{FM}^m_{\mathrm{tr}}$ satisfying the *prolongation* condition $B \circ F = \mathrm{id}_{\mathcal{M}f \times \mathbb{R}^m}$ is called a *prolongation functor*.

To a prolongation functor (10), one can associate the functor $F_0 : \mathcal{M}f \to \mathcal{M}f$ defined by

$$F_0 = R_0 \circ \varepsilon \circ F \circ R_{\mathrm{pt}}\,.$$

A prolongation functor $F : (\mathcal{M}f \times \mathbb{R}^m)_{\mathrm{tr}} \to \mathcal{FM}^m_{\mathrm{tr}}$ will be called a *bundle functor* if it satisfies the following conditions:

i) for any open subsets $U' \subset U \subset \mathbb{R}^m$ and any smooth manifold M_n, $F(M_n \times U')$ is the restriction of $F(M_n \times U)$;

ii) the restriction of F to the category $\mathcal{M}f^m$ is a bundle functor.

4.2. **Products in the category $\mathcal{M}f^m_{\mathrm{tr}}$.** The product of two objects $C_1 = p : M_n \times U \to U$ and $C_2 = p' : M'_k \times U' \to U'$ of the category $\mathcal{M}f^m_{\mathrm{tr}}$ in the sense of diagram (11) cannot exist because, first, the objects C_1 and C_2 may have distinct domains U and U', second, the translation components tr_{t_0} and $\mathrm{tr}_{t'_0}$ of morphisms $(f_1, \mathrm{tr}_{t_0}) : D \to C_1$ and $(f_2, \mathrm{tr}_{t'_0}) : D \to C_2$ may also not coincide. For this reason, we define the product in the category $\mathcal{M}f^m_{\mathrm{tr}}$ only for objects $p : M_n \times U \to U$ and $p' : M'_k \times U \to U$ with the same domain U.

By the product of objects $p : M \times U \to U$ and $p' : M' \times U \to U$ in the category $\mathcal{M}f^m_{\mathrm{tr}}$ we will understand a triple $(p'' : M'' \times U \to U, \mathrm{Pr}, \mathrm{Pr}')$, where Pr and Pr$'$ are morphisms in $\mathcal{M}f^m_{\mathrm{tr}}$ satisfying the condition that, for any two morphisms $(f, \mathrm{tr}_{t_0}) : W \times V \to M \times U$ and $(f', \mathrm{tr}_{t_0}) : W \times V \to M' \times U$ with the same translation component tr_{t_0}, there exists a unique morphism $(f'', \mathrm{tr}_{t_0}) : W \times V \to M'' \times U$ such that the following diagram commutes:

$$\begin{tikzcd}
M\times U & M''\times U \arrow[l, "\mathrm{Pr}"'] \arrow[r, "\mathrm{Pr}'"] & M'\times U \\
& W\times V \arrow[ul, "(f,\mathrm{tr}_{t_0})"] \arrow[u, "(f'',\mathrm{tr}_{t_0})"'] \arrow[ur, "(f',\mathrm{tr}_{t_0})"'] &
\end{tikzcd} \tag{70}$$

Obviously, the triple $((M\times M')\times U, \mathrm{pr}\times\mathrm{id}_U, \mathrm{pr}'\times\mathrm{id}_U)$, where $\mathrm{pr}: M\times M'\to M$ and $\mathrm{pr}': M\times M'\to M'$ are the projections of the product $M\times M'$ of two manifolds in the category $\mathcal{M}f$ of smooth manifolds, satisfies the above definition.

By the product of two objects $E\times U\to M\times U$ and $E'\times U\to M'\times U$ (for brevity, we will omit the projections to U in diagrams of the form (67)) of the category $\mathcal{FM}^m_{\mathrm{tr}}$ we will understand, as in the case of the category $\mathcal{M}f^m_{\mathrm{tr}}$, an object $E''\times U\to M''\times U$ such that, for any two morphisms from $Q\times V\to W\times V$ to $E\times U\to M\times U$ and $E'\times U\to M'\times U$ with the same translation component tr_{t_0}, there exists a unique morphism from $Q\times V\to W\times V$ to $E''\times U\to M''\times U$ for which the commutative diagram

$$\begin{tikzcd}
E\times U & E''\times U \arrow[l, "\mathrm{Pr}"'] \arrow[r, "\mathrm{Pr}'"] & E'\times U \\
& Q\times V \arrow[ul, "(f,\mathrm{tr}_{t_0})"] \arrow[u, "(f'',\mathrm{tr}_{t_0})"'] \arrow[ur, "(f',\mathrm{tr}_{t_0})"'] &
\end{tikzcd} \tag{71}$$

over the commutative diagram (70) holds.

Obviously, the object $(E\times E')\times U\to(M\times M')\times U$ with the corresponding projections to $E\times U\to M\times U$ and $E'\times U\to M'\times U$ satisfy the above definition.

4.3. m-**parameter families of Weil functors.** By a smooth m-parameter family of algebras we will mean a vector bundle $\mathbb{V}\times\mathbb{R}^m\to\mathbb{R}^m$ with a smooth fiberwise bilinear multiplication operation $*:\mathbb{V}\times\mathbb{V}\times\mathbb{R}^m\to\mathbb{V}\times\mathbb{R}^m$ which is a morphism of the category $\mathcal{M}f^m$. Suppose that $*$ turns each fiber $\mathbb{V}_t=\mathbb{V}$, $t\in\mathbb{R}^m$, into a local Weil algebra $\mathbb{A}(t)$ such that the unity 1_t smoothly depends on t and the bundle $\mathbb{V}\times\mathbb{R}^m\to\mathbb{R}^m$ can be represented as the Whitney sum

$$\mathbb{V}\times\mathbb{R}^m=(\mathbb{R}\times\mathbb{R}^m)\oplus_{\mathbb{R}^m}(\overset{\circ}{\mathbb{V}}\times\mathbb{R}^m), \tag{72}$$

where $\mathbb{R}\times\mathbb{R}^m\to\mathbb{R}^m$ is the m-parameter family of algebras of real numbers spanned by the unities $1_t\in\mathbb{V}_t$, $t\in\mathbb{R}^m$, and $\overset{\circ}{\mathbb{V}}\times\mathbb{R}^m\to\mathbb{R}^m$ is the m-parameter family of nilpotent algebras whose fiber $\overset{\circ}{\mathbb{V}}(t)$ coincides with the maximal ideal of $\mathbb{A}(t)$, then such a family $\mathbb{A}(t)$ will be called a *family of Weil algebras.* We will denote an m-parameter family of Weil algebras by $\mathbb{A}(t)\times\mathbb{R}^m$ or simply by $\mathbb{A}(t)$. In a similar manner one can define a family of $\mathbb{A}(t)$-modules $\mathbb{A}(t)^n\times\mathbb{R}^m$, where $\mathbb{A}(t)^n=\underbrace{\mathbb{A}(t)\times\cdots\times\mathbb{A}(t)}_{n}$.

To every m-parameter family of Weil algebras $\mathbb{A}(t)\times\mathbb{R}^m$ one can associate a covariant functor $\widetilde{T}^{\mathbb{A}(t)}:\mathcal{M}f\times\mathbb{R}^m\to\mathcal{FM}^m$ called an m-parameter family of Weil functors [3]. There are several descriptions of the Weil functor (see, e. g., [14]). Within the framework of the problem studied in this section, it is convenient to consider the construction of Weil functor based on the use of local charts. Define the action of $\widetilde{T}^{\mathbb{A}(t)}$ on $\mathbb{R}\times\mathbb{R}^m$ as follows: $\widetilde{T}^{\mathbb{A}(t)}(\mathbb{R}\times\mathbb{R}^m)=\mathbb{A}(t)\times\mathbb{R}^m$. For an $\mathcal{M}f\times\mathbb{R}^m$-morphism $f\times\mathrm{id}:\mathbb{R}\times\mathbb{R}^m\to\mathbb{R}\times\mathbb{R}^m$, the morphism $\widetilde{T}^{\mathbb{A}(t)}(f\times\mathrm{id}):\mathbb{A}(t)\times\mathbb{R}^m\to\mathbb{A}(t)\times\mathbb{R}^m$, $(x+\overset{\circ}{X},t)\mapsto(y+\overset{\circ}{Y},t)$ is defined by the equations

$$y+\overset{\circ}{Y}=f(x)+\sum_{p=1}^{\infty}\frac{1}{p!}\frac{d^pf(x)}{dx^p}\,\overset{\circ}{X}{}^p,$$

where the coordinates $(x+\overset{\circ}{X},t)$ correspond to the decomposition (72) and the summation, for each $t\in\mathbb{R}^m$, is over a finite number of summands depending on the height of $\mathbb{A}(t)$.

We let $\widetilde{T}^{\mathbb{A}(t)}(\mathbb{R}^n\times\mathbb{R}^m)=\mathbb{A}(t)^n\times\mathbb{R}^m$ and define $\widetilde{T}^{\mathbb{A}(t)}(f\times\mathrm{id}):\mathbb{A}(t)^n\times\mathbb{R}^m\to\mathbb{A}(t)^k\times\mathbb{R}^m$ by

$$y^j+\overset{\circ}{Y}{}^j=f^j(x^1,\dots,x^n)+\sum_{|p|\geq1}^{\infty}\frac{1}{p!}\frac{\partial^{|p|}f^j(x^1,\dots,x^n)}{\partial x^p}\,\overset{\circ}{X}{}^p,\qquad(73)$$

where $p=(p_1,\dots,p_n)$ is a multiindex.

For an open subset $U\subset\mathbb{R}^n$, we let $\widetilde{T}^{\mathbb{A}(t)}(U\times\mathbb{R}^m)=(\pi^n_{\mathbb{A}(t)})^{-1}(U\times\mathbb{R}^m)$, where $\pi_{\mathbb{A}(t)}:\mathbb{A}(t)\times\mathbb{R}^m\to\mathbb{R}\times\mathbb{R}^m$ is the fiberwise epimorphism of algebras, and define the action of $\widetilde{T}^{\mathbb{A}(t)}$ on morphisms $f\times\mathrm{id}:U\times\mathbb{R}^m\to U\times\mathbb{R}^m$ by the same relations (73).

Let now M_n be an arbitrary n-dimensional smooth manifold with atlas consisting of charts (U_α,h_α), $\alpha\in A$, with transition functions $h_{\alpha\beta}=h_\alpha\circ h_\beta^{-1}:h_\beta(U_{\alpha\beta})\to h_\alpha(U_{\alpha\beta})$. Then the collection $\{(U_\alpha\times\mathbb{R}^m,h_\alpha\times\mathrm{id})\}_{\alpha\in A}$ is an atlas on $M_n\times\mathbb{R}^m$. The collection of $\mathcal{FM}^m$-morphisms $\{\widetilde{T}^{\mathbb{A}(t)}(h_{\alpha\beta}\times\mathrm{id})\}$ allows ones to glue the domains $\{\widetilde{T}^{\mathbb{A}(t)}(U_\alpha\times\mathbb{R}^m)\}_{\alpha\in A}$ and obtain the total space of the bundle $\widetilde{T}^{\mathbb{A}(t)}(M_n\times\mathbb{R}^m)\to M_n\times\mathbb{R}^m$. Let $f:M_n\to M'_k$ be a smooth mapping of an n-dimensional manifold M_n to a k-dimensional manifold M'_k. Then we define the morphism $\widetilde{T}^{\mathbb{A}(t)}(f\times\mathrm{id})$ as the mapping given locally by equations (73). We have $\widetilde{T}^{\mathbb{A}(t)}(f\circ g)=\widetilde{T}^{\mathbb{A}(t)}(f)\circ$

$\widetilde{T}^{\mathbb{A}(t)}(g)$ and $\widetilde{T}^{\mathbb{A}(t)}(\mathrm{id}) = \mathrm{id}$. Therefore, $\widetilde{T}^{\mathbb{A}(t)}$ is a covariant functor from the category $\mathcal{M}f \times \mathbb{R}^m$ to the category $\mathcal{FM}^m$.

The functor $\widetilde{T}^{\mathbb{A}(t)} : \mathcal{M}f \times \mathbb{R}^m \to \mathcal{FM}^m$ constructed above is called an *m-parameter family of Weil functors*.

In the case when $m = 1$ and the algebra $\mathbb{A}(t)$ does not depend on t, the bundle $\widetilde{T}^{\mathbb{A}(t)}(M_n \times \mathbb{R})$ turns out to be a time-dependent Weil bundle $T^{\mathbb{A}} M_n \times \mathbb{R}$ studied by M. Doupovec and I. Kolář [4].

4.4. **Product preserving bundle functors on** $(\mathcal{M}f \times \mathbb{R}^m)_{\mathrm{tr}}$**.** The restriction of a product preserving bundle functor $F : \mathcal{M}f^m_{\mathrm{tr}} \to \mathcal{FM}^m_{\mathrm{tr}}$ to the subcategory $(\mathcal{M}f \times \mathbb{R}^m)_{\mathrm{tr}}$ is a product preserving bundle functor $\overline{F} : (\mathcal{M}f \times \mathbb{R}^m)_{\mathrm{tr}} \to \mathcal{FM}^m_{\mathrm{tr}}$. For this reason, we first consider an arbitrary product preserving bundle functor $G : (\mathcal{M}f \times \mathbb{R}^m)_{\mathrm{tr}} \to \mathcal{FM}^m_{\mathrm{tr}}$. By (69), a morphism $\varphi \times \mathrm{tr}_{t_0} : M_n \times U \to M'_k \times U'$ is the restriction of a morphism $\varphi \times \mathrm{tr}_{t_0} : M_n \times \mathbb{R}^m \to M'_k \times \mathbb{R}^m$, which, as was mentioned above, can be represented in the form $(\varphi \times \mathrm{id}) \circ (\mathrm{id}_M \times \mathrm{tr}_{t_0}) = (\mathrm{id}_{M'} \times \mathrm{tr}_{t_0}) \circ (\varphi \times \mathrm{id})$, where $\varphi \times \mathrm{id}$ is a morphism of the category $\mathcal{M}f \times \mathbb{R}^m$. Then we have the following commutative diagram:

$$\begin{tikzcd} M_n \times \mathbb{R}^m \arrow[r, "\varphi\times\mathrm{id}"] \arrow[d, "\mathrm{id}_{M_n}\times\mathrm{tr}_{t_0}"'] & M'_k \times \mathbb{R}^m \arrow[d, "\mathrm{id}_{M'_k}\times\mathrm{tr}_{t_0}"] \\ M_n \times \mathbb{R}^m \arrow[r, "\varphi\times\mathrm{id}"] & M'_k \times \mathbb{R}^m \end{tikzcd} \tag{74}$$

Applying the functor G to diagram (74), we obtain the commutative diagram

$$\begin{tikzcd} G(M_n \times \mathbb{R}^m) \arrow[r, "G(\varphi\times\mathrm{id})"] \arrow[d, "G(\mathrm{id}_{M_n}\times\mathrm{tr}_{t_0})"'] & G(M'_k \times \mathbb{R}^m) \arrow[d, "G(\mathrm{id}_{M'_k}\times\mathrm{tr}_{t_0})"] \\ G(M_n \times \mathbb{R}^m) \arrow[r, "G(\varphi\times\mathrm{id})"] & G(M'_k \times \mathbb{R}^m) \end{tikzcd} \tag{75}$$

The horizontal arrows of diagram (74) are morphisms of the category $\mathcal{M}f \times \mathbb{R}^m$. Therefore, the restriction of G to the category $\mathcal{M}f \times \mathbb{R}^m$ coincides (up to an equivalence) with some m-parameter family of Weil functors $\widetilde{T}^{\mathbb{A}(t)} : \mathcal{M}f \times \mathbb{R}^m \to \mathcal{FM}^m$. Let $\mathbb{A} = \mathbb{A}(0)$. Passing to the restriction of the upper arrow of diagram (75) to $0 \in \mathbb{R}^m$ (applying the functor R_0) and the restriction of the lower arrow to $t_0 \in \mathbb{R}^m$, we obtain a commutative diagram which establishes the natural equivalence of the Weil functors $T^{\mathbb{A}} \to T^{\mathbb{A}(t_0)}$. The diagram obtained in the same manner for the morphism $\varphi \times \mathrm{tr}_{-t_0}$ gives the inverse natural equivalence.

Consider now the family $\mu = \{G(\mathrm{id}_{M_n} \times \mathrm{tr}_{-t})|_t, t \in \mathbb{R}^m\}$ of natural equivalences. It depends smoothly on $t \in \mathbb{R}^m$. These equivalences and the projections $\mathrm{pr}_1 = \mu$ and $\mathrm{pr}_2 = \overline{F}(p)$ introduce on $G(M_n \times \mathbb{R}^m)$ the structure of the product $T^{\mathbb{A}}M_n \times \mathbb{R}^m$ such that $G(\varphi \times \mathrm{id}) = T^{\mathbb{A}}\varphi \times \mathrm{id}$ and $G(\mathrm{id}_{M_n} \times \mathrm{tr}_{t_0}) = \mathrm{id}_{T^{\mathbb{A}}M_n} \times \mathrm{tr}_{t_0}$. Then $G(\varphi \times \mathrm{tr}_{t_0}) = T^{\mathbb{A}}\varphi \times \mathrm{tr}_{t_0}$ (with respect to the above introduced product structure) for every morphism $\varphi \times \mathrm{tr}_{t_0} \in \mathrm{Mor}((\mathcal{M}f \times \mathbb{R}^m)_{\mathrm{tr}})$. For a local algebra $\mathbb{A}$, denote by $T^{\mathbb{A}} : (\mathcal{M}f \times \mathbb{R}^m)_{\mathrm{tr}} \to (\mathcal{FM} \times \mathbb{R}^m)_{\mathrm{tr}}$ the functor whose action on objects and morphisms of the category $(\mathcal{M}f \times \mathbb{R}^m)_{\mathrm{tr}}$ is defined, respectively, as follows:

$$p : M_n \times U \to U \quad \longmapsto \quad \begin{tikzcd} T^{\mathbb{A}}M_n \times U \arrow[r, "p^{\mathbb{A}}"] \arrow[d, "\pi\times\mathrm{id}"'] & U \arrow[d, "\mathrm{id}"] \\ M_n \times U \arrow[r, "p"] & U \end{tikzcd} \tag{76}$$

$$\begin{tikzcd} M_n \times U \arrow[r, "\varphi\times\mathrm{tr}_{t_0}"] \arrow[d] & M'_k \times U' \arrow[d] \\ U \arrow[r, "\mathrm{tr}_{t_0}"] & U' \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} T^{\mathbb{A}}M_n \times U \arrow[r, "T^{\mathbb{A}}\varphi\times\mathrm{tr}_{t_0}"] \arrow[d, "\pi\times\mathrm{id}"'] & T^{\mathbb{A}}M'_k \times U' \arrow[d, "\pi'\times\mathrm{id}"] \\ M_n \times U \arrow[r, "\varphi\times\mathrm{tr}_{t_0}"] & M'_k \times U' \end{tikzcd} \tag{77}$$

As a result of the above discussion, we obtain the following theorem.

Theorem 3. *A product preserving bundle functor* $G : (\mathcal{M}f \times \mathbb{R}^m)_{\mathrm{tr}} \to \mathcal{FM}^m_{\mathrm{tr}}$ *is naturally equivalent to the functor* $T^{\mathbb{A}} : (\mathcal{M}f \times \mathbb{R}^m)_{\mathrm{tr}} \to (\mathcal{FM} \times \mathbb{R}^m)_{\mathrm{tr}}$ *determined by some Weil algebra* $\mathbb{A}$.

4.5. **The generalized Weil functor $\widehat{T}^{\mathbb{A}}_\sigma$.** Applying the Weil functor $T^{\mathbb{A}}$ defined by a Weil algebra $\mathbb{A}$ to an object $p : M_n \times U \to U$ of the category $\mathcal{M}f^m_{\mathrm{tr}}$, we obtain the bundle $T^{\mathbb{A}}p : T^{\mathbb{A}}(M_n \times U) \to T^{\mathbb{A}}U$. Let $\sigma : U \to T^{\mathbb{A}}U \equiv U \times \mathring{\mathbb{A}}^m$ be a section $\{t^a\} \mapsto (\{t^a\}, \{\mathring{\sigma}^a\})$, $a = 1, \dots, m$, given by m constant elements $\mathring{\sigma}^a$ of $\mathring{\mathbb{A}}$. Denote by $\widehat{T}^{\mathbb{A}}_\sigma(M \times U) \to U$ the pullback of the bundle $T^{\mathbb{A}}p : T^{\mathbb{A}}(M_n \times U) \to T^{\mathbb{A}}U$ under the mapping $\sigma : U \to T^{\mathbb{A}}U$. We have the following commutative diagram:

$$\begin{tikzcd} & M \times U \arrow[rr, "\mathrm{id}"] \arrow[dd] & & M \times U \arrow[dd] \\ \widehat{T}^{\mathbb{A}}_\sigma(M \times U) \arrow[ur, "\pi_M"] \arrow[rr, "\sigma^*_M"] \arrow[dd, "\widehat{T}^{\mathbb{A}}_\sigma(p)"'] & & T^{\mathbb{A}}(M \times U) \arrow[ur, "\pi_M"] \arrow[dd, "T^{\mathbb{A}}(p)"] & \\ & U \arrow[rr, "\mathrm{id}"] & & U \\ U \arrow[ur, "\mathrm{id}"] \arrow[rr, "\sigma"] & & T^{\mathbb{A}}U \arrow[ur, "\pi_{\mathrm{pt}}"] & \end{tikzcd} \tag{78}$$

The left square of diagram (78)

$$\begin{tikzcd}
\widehat{T}^{\mathbb{A}}_{\sigma}(M\times U) \arrow[r] \arrow[d] & M\times U \arrow[d] \\
U \arrow[r] & U
\end{tikzcd} \tag{79}$$

is an object of the category $\mathcal{FM}^m_{\mathrm{tr}}$. Applying the pullback construction to the $\mathbb{A}$-prolongation

$$\begin{tikzcd}
T^{\mathbb{A}}(M\times U) \arrow[r, "T^{\mathbb{A}}f"] \arrow[d] & T^{\mathbb{A}}(M'\times U') \arrow[d] \\
T^{\mathbb{A}}U \arrow[r, "T^{\mathbb{A}}\,\mathrm{tr}_{t_0}"] & T^{\mathbb{A}}U'
\end{tikzcd} \tag{80}$$

of a morphism

$$\begin{tikzcd}
M\times U \arrow[r, "f"] \arrow[d] & M'\times U' \arrow[d] \\
U \arrow[r, "\mathrm{tr}_{t_0}"] & U'
\end{tikzcd} \tag{81}$$

we obtain the diagram

$$\begin{tikzcd}
& M\times U \arrow[rr, "f"] \arrow[dd] & & M'\times U' \arrow[dd] \\
\widehat{T}^{\mathbb{A}}_{\sigma}(M\times U) \arrow[ur, "\pi_M"] \arrow[rr, "\widehat{T}^{\mathbb{A}}_{\sigma}(f)"] \arrow[dd] & & \widehat{T}^{\mathbb{A}}_{\sigma}(M'\times U') \arrow[ur, "\pi_{M'}"] \arrow[dd] & \\
& U \arrow[rr, "\mathrm{tr}_{t_0}"] & & U' \\
U \arrow[ur, "\mathrm{id}"] \arrow[rr, "\mathrm{tr}_{t_0}"] & & U' \arrow[ur, "\mathrm{id}"] &
\end{tikzcd} \tag{82}$$

which is a morphism in $\mathcal{FM}^m_{\mathrm{tr}}$. Obviously, for the composition $f\circ g$ of morphisms g and f of the category we have $\widehat{T}^{\mathbb{A}}_{\sigma}(f\circ g)=\widehat{T}^{\mathbb{A}}_{\sigma}(f)\circ\widehat{T}^{\mathbb{A}}_{\sigma}(g)$ and $\widehat{T}^{\mathbb{A}}_{\sigma}(\mathrm{id})=\mathrm{id}$.

Thus, the correspondence $\widehat{T}^{\mathbb{A}}_{\sigma}$ which assigns to an object $p: M_n\times U\to U$ of $\mathcal{M}f^m_{\mathrm{tr}}$ the object (79) of $\mathcal{FM}^m$ and to a morphism (81) the morphism (82) is a functor $\widehat{T}^{\mathbb{A}}_{\sigma}:\mathcal{M}f^m_{\mathrm{tr}}\to\mathcal{FM}^m$, which will be called a *generalized Weil functor*. Obviously, $\widehat{T}^{\mathbb{A}}_{\sigma}:\mathcal{M}f^m_{\mathrm{tr}}\to\mathcal{FM}^m$ preserves products.

If, in terms of local coordinates, a morphism (81) is given by equations $y^{i'}=f^{i'}(x^j,t^a)$, $t^{a'}=t^a+t^a_0$, then the morphism (80), in terms of the induces $\mathbb{A}$-coordinates $\{X^j, S^a=t^a+\overset{\circ}{S}{}^a\}$ on $T^{\mathbb{A}}(M_n\times U)$ and $\{Y^{i'}, S^{a'}=$

$t^{a'} + \overset{\circ}{S}{}^{a'}\}$ on $T^{\mathbb{A}}(M'_k \times U')$ is of the form (see [33]):

$$Y^{i'} = f^{i'}(x,t) + \sum_{p+s=1}^{q} \frac{1}{p!s!} \frac{\partial^{|p+s|} f^{i'}}{\partial x^p \partial t^s} \overset{\circ}{X}{}^p \overset{\circ}{S}{}^s, \quad S^{a'} = S^a + t_0^a,$$

whence it follows that, in terms of the induced coordinates $\{X^j, t^a\}$ on $\widehat{T}^{\mathbb{A}}_{\sigma}(M_n \times U)$ and $\{Y^{i'}, t^{a'}\}$ on $\widehat{T}^{\mathbb{A}}_{\sigma}(M'_k \times U')$, the morphism (82) is of the form

$$Y^{i'} = f^{i'}(x,t) + \sum_{p+s=1}^{q} \frac{1}{p!s!} \frac{\partial^{|p+s|} f^{i'}}{\partial x^p \partial t^s} \overset{\circ}{X}{}^p \overset{\circ}{\sigma}{}^s. \tag{83}$$

Equations (83) can be rewritten in the form

$$Y^{i'} = f^{i'}(x,t) + \sum_{p=0}^{q} \frac{1}{p!} \frac{\partial^{|p|}}{\partial x^p} \left\{ \sum_{s=1}^{q} \frac{1}{s!} \frac{\partial^{|s|} f^{i'}}{\partial t^s} \overset{\circ}{\sigma}{}^s \right\} \overset{\circ}{X}{}^p =$$

$$= \widehat{f}^{i}(x,t) + \sum_{p=1}^{q} \frac{1}{p!} \frac{\partial^{|p|} \widehat{f}^{i'}}{\partial x^p} \overset{\circ}{X}{}^p,$$

where

$$\widehat{f}^{i'}(x,t) = f^{i'}(x,t) + \sum_{s=1}^{q} \frac{1}{s!} \frac{\partial^{|s|} f^{i'}}{\partial t^s} \overset{\circ}{\sigma}{}^s,$$

whence it follows that the restriction of the morphism $\widehat{T}^{\mathbb{A}}_{\sigma}(f, \mathrm{tr}_{t_0})$ to each fiber over U is an $\mathbb{A}$-smooth mapping.

In what follows it will be convenient to use the following explicit construction for the bundle $\widehat{T}^{\mathbb{A}}_{\sigma}(M_n \times U)$. An element of $\widehat{T}^{\mathbb{A}}_{\sigma}(M_n \times U)$ can be considered as the $\mathbb{A}$-velocity $j^{\mathbb{A}} g$ of a germ $g : (\mathbb{R}^{\ell}, 0) \to M_n \times U$ such that the following diagram is commutative:

$$\begin{tikzcd} (\mathbb{R}^{\ell}, 0) \arrow[r, "g"] \arrow[d, "\mathrm{id}"'] & M_n \times U \arrow[d, "\mathrm{id}"] \\ (\mathbb{R}^{\ell}, 0) \arrow[r, "\mathrm{tr}_{t_0} \circ \widehat{\sigma}"] & U \end{tikzcd} \tag{84}$$

where $\widehat{\sigma} : (\mathbb{R}^{\ell}, 0) \to (\mathbb{R}^m, 0)$ is a germ with $j^{\mathbb{A}}\widehat{\sigma} = \{\overset{\circ}{\sigma}{}^a\}$, $a = 1, \ldots, m$. Then the morphism $\widehat{T}^{\mathbb{A}}_{\sigma}(f)$ defined by (82) can be written in the form

$$\widehat{T}^{\mathbb{A}}_{\sigma}(f) : j^{\mathbb{A}} g \mapsto j^{\mathbb{A}}(f \circ g) \tag{85}$$

4.6. **Product preserving bundle functors on the category $\mathcal{M}f^m_{\mathrm{tr}}$.** As in the case of the category $\mathcal{M}f^m$, we introduce the functor Φ : $\mathcal{M}f^m_{\mathrm{tr}} \to (\mathcal{M}f \times \mathbb{R}^m)_{\mathrm{tr}}$ which assigns to a bundle $M_n \times U \to U$ the bundle $(M_n \times U) \times U \to U$ and to a morphism $(f, \mathrm{tr}_{t_0}) : M_n \times U \to M'_k \times U'$, $(x,t) \mapsto (f(x,t), t+t_0)$, the morphism $\Phi(f, \mathrm{tr}_{t_0}) : (M_n \times U) \times U \to (M'_k \times U') \times U'$, $((x,s),t) \mapsto ((f(x,s), s+t_0), t+t_0)$. The sections $\sigma_M : M_n \times \mathbb{R}^m \to (M_n \times \mathbb{R}^m) \times \mathbb{R}^m$, $\sigma_M(x,t) = ((x,t),t)$, define a natural transformation of functors $\sigma : \mathrm{Id}_{\mathcal{M}f^m_{\mathrm{tr}}} \to \Phi$. In fact, we have the commutative diagram

$$\begin{tikzcd}
M_n \times U \arrow[r, "f"] \arrow[d, "\sigma_M"'] & M'_k \times U' \arrow[d, "\sigma_{M'}"] \\
(M_n \times U) \times U \arrow[r, "\Phi(f)"] & (M'_k \times U') \times U'
\end{tikzcd} \tag{86}$$

$$\begin{tikzcd}
(x,t) \arrow[r, "f"] \arrow[d, "\sigma_M"'] & (y, t+t_0) \arrow[d, "\sigma_{M'}"] \\
((x,t),t) \arrow[r, "\Phi(f)"] & ((y, t+t_0), t+t_0)
\end{tikzcd}$$

Let now $F : \mathcal{M}f^m_{\mathrm{tr}} \to \mathcal{FM}^m_{\mathrm{tr}}$ be a product preserving bundle functor, and let $\overline{F} : (\mathcal{M}f \times \mathbb{R}^m)_{\mathrm{tr}} \to \mathcal{FM}^m_{\mathrm{tr}}$ be the restriction of F to the subcategory $(\mathcal{M}f \times \mathbb{R}^m)_{\mathrm{tr}}$. Applying F to diagram (86), we obtain the diagram

$$\begin{tikzcd}
F(M_n \times U) \arrow[r, "F(f)"] \arrow[d, "F(\sigma_M)"'] & F(M'_k \times U') \arrow[d, "F(\sigma_{M'})"] \\
F((M_n \times U) \times U) \arrow[r, "F\Phi(f)"] & F((M'_k \times U') \times U')
\end{tikzcd} \tag{87}$$

$$\begin{tikzcd}
(X,t) \arrow[r, "F(f)"] \arrow[d, "F(\sigma_M)"'] & (Y, t+t_0) \arrow[d, "F(\sigma_W)"] \\
((X,S),t) \arrow[r, "F\Phi(f)"] & ((Y, S+t_0), t+t_0)
\end{tikzcd}$$

whose lower arrow is the morphism $\overline{F}\Phi(f, \mathrm{tr}_{t_0})$.

In accordance with diagram (70), the section σ_M is the product of the two morphisms $\mathrm{id}_{M_n \times \mathbb{R}^m}$ and $\sigma'_M : M_n \times \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m$, $\sigma'_M : (x,t) =$

(t,t). Thus, we have the commutative diagram

$$\begin{tikzcd} M_n \times \mathbb{R}^m & (M_n \times \mathbb{R}^m) \times \mathbb{R}^m \arrow[l, "\mathrm{Pr}"'] \arrow[r, "\mathrm{Pr}'"] & \mathbb{R}^m \times \mathbb{R}^m \\ & M_n \times \mathbb{R}^m \arrow[ul, "\mathrm{id}"] \arrow[u, "\sigma_M"'] \arrow[ur, "\sigma'_M"'] & \end{tikzcd} \tag{88}$$

Applying to diagram (88) the functor F, we obtain the diagram

$$\begin{tikzcd} F(M_n \times \mathbb{R}^m) & F((M_n \times \mathbb{R}^m) \times \mathbb{R}^m) \arrow[l, "\mathrm{Pr}"'] \arrow[r, "\mathrm{Pr}'"] & F(\mathbb{R}^m \times \mathbb{R}^m) \\ & F(M_n \times \mathbb{R}^m) \arrow[ul, "\mathrm{id}"] \arrow[u, "F(\sigma_M)"'] \arrow[ur, "F(\sigma'_M)"'] & \end{tikzcd} \tag{89}$$

Hence the morphism $F(\sigma_M)$ is determined by $F(\sigma'_M)$. Applying F to the commutative diagram

$$\begin{tikzcd} M_n \times \mathbb{R}^m \arrow[rr, "\mathrm{pt}"] \arrow[dr, "\sigma'_M"'] & & \mathrm{pt} \times \mathbb{R}^m \arrow[dl, "\sigma'_{\mathrm{pt}}"] \\ & \mathbb{R}^m \times \mathbb{R}^m & \end{tikzcd}$$

we conclude that $F(\sigma'_M)$ is completely determined by the morphism $F(\sigma'_{\mathrm{pt}}) : \mathrm{pt} \times \mathbb{R}^m \to T^{\mathbb{A}}\mathbb{R}^m \times \mathbb{R}^m$, which is in fact a section $\mathbb{R}^m \to \mathbb{A}^m \times \mathbb{R}^m$ of the form $F(\sigma'_{pt}) : \{t^a\} \mapsto (\{S^a(t)\}, \{t^a\})$, $S^a(t) = t^a + \mathring{\sigma}^a(t)$, $\mathring{\sigma}^a(t) \in \mathring{\mathbb{A}}$, $a = 1, \dots, m$. From diagram (87), we have $t + t_0 + \mathring{\sigma}(t) = t + t_0 + \mathring{\sigma}(t + t_0)$. Therefore, $\mathring{\sigma}^a(t) = \mathring{\sigma}^a(t + t_0)$. Consequently, the $\mathring{\mathbb{A}}$-valued functions $\mathring{\sigma}^a(t)$ are constants.

The lower arrow of diagram (87) is an $(\mathcal{FM} \times \mathbb{R}^m)_{\mathrm{tr}}$-morphism $T^{\mathbb{A}} f \times \mathrm{tr}_{t_0}$. In terms of local coordinates, it is of the form [33]:

$$Y^{i'} = f^{i'}(x,t) + \sum_{p+s=1}^{q} \frac{1}{p!s!} \frac{\partial^{|p+s|} f^{i'}}{\partial x^p \partial t^s} \mathring{X}^p \, \mathring{S}^s.$$

Then the morphism $F(f, \mathrm{tr}_{t_0})$ (the upper arrow of diagram (87)), in terms of local coordinates, is is given by the equations

$$Y^{i'} = f^{i'}(x,t) + \sum_{p+s=1}^{q} \frac{1}{p!s!} \frac{\partial^{|p+s|} f^{i'}}{\partial x^p \partial t^s} \mathring{X}^p \, \mathring{\sigma}^s,$$

which coincide with (83). Thus, the action of F on morphisms is completely determined by the collection of elements $\mathring{\sigma}^a \in \mathring{\mathbb{A}}$, $a = 1, \dots, m$, which defines the section $F(\sigma'_{pt})$.

As a result of the above discussion, we obtain the following theorem.

Theorem 4. *A product preserving bundle functor $F : \mathcal{M}f^m_{\mathrm{tr}} \to \mathcal{FM}^m_{\mathrm{tr}}$ is naturally equivalent to a generalized Weil functor $\widehat{T}^{\mathbb{A}}_{\sigma} : \mathcal{M}f^m_{\mathrm{tr}} \to \mathcal{FM}^m_{\mathrm{tr}}$ defined by some Weil algebra $\mathbb{A}$ and a collection of elements $\overset{\circ}{\sigma}{}^a \in \overset{\circ}{\mathbb{A}}$, $a = 1, \ldots, m$.*

5. Higher order geometry of manifolds from $\mathcal{M}f^m_{\mathrm{tr}}$.

5.1. **Higher order frame bundles of manifolds from $\mathcal{M}f^m_{\mathrm{tr}}$.** Denote by $\widehat{T}^{\mathbb{R}(n+m,r)}_{\mathrm{pr}}(M'_k \times U)$ the bundle $\widehat{T}^{\mathbb{R}(n+m,r)}_{\sigma}(M'_k \times U)$ corresponding to the section σ defined by $\widehat{\sigma} = \mathrm{pr} : (\mathbb{R}^n \times \mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ (see (84)), i. e., by $\overset{\circ}{\sigma}{}^a = \nu^a$, where $\{\nu^a\}$, $a = 1, \ldots, m$, is the standard pseudobasis in $\mathbb{R}(m, r)$. Let $\widehat{B}^r(M_n \times U)$ be the set of all r-jets of invertible germs of morphisms $(f, \mathrm{tr}_{t_0}) : (\mathbb{R}^n \times \mathbb{R}^m, (0, 0)) \to (M_n \times \mathbb{R}^m, (x, t))$, $(u^j, t^a) \mapsto (x^i(u^j, t^a), t^{a'} = t^a + t^a_0)$, from the category $\mathcal{M}f^m_{\mathrm{tr}}$. The set $\widehat{B}^r(M_n \times U)$ is an open subset in the bundle $\widehat{T}^{\mathbb{R}(n+m,r)}_{\mathrm{pr}}(M_n \times U)$, and so it inherits from $\widehat{T}^{\mathbb{R}(n+m,r)}_{\mathrm{pr}}(M_n \times U)$ the structure of a smooth manifold. In addition, $\widehat{B}^r(M_n \times U)$ is a locally trivial fiber bundle over $M_n \times U$ which is an object of the category $\mathcal{FM}^m_{\mathrm{tr}}$. The standard fiber of $\widehat{B}^r(M_n \times U)$ is the Lie group $\widehat{D}^r(n, m)$ consisting of all r-jets of invertible germs of morphisms $(f, \mathrm{id}) : (\mathbb{R}^n \times \mathbb{R}^m, (0, 0)) \to (\mathbb{R}^n \times \mathbb{R}^m, (0, 0))$. The Lie group $\widehat{D}^r(n, m)$ acts on the right on $\widehat{B}^r(M_n \times U)$ by the law of composition of jets. Therefore, $\widehat{B}^r(M_n \times U)$ is a principal bundle.

The Weil algebra $\mathbb{R}(n + m, r)$ considered as the algebra of r-jets of germs $(\mathbb{R}^n \times \mathbb{R}^m, 0) \to \mathbb{R}$, $(u^j, t^a) \mapsto x = f(u^j, t^a)$, contains the two Weil subalgebras consisting of r-jets of germs which do not depend, respectively, on u^j and t^a. These subalgebras are isomorphic, respectively, to $\mathbb{R}(n, r)$ and $\mathbb{R}(m, r)$. In what follows by $\mathbb{R}(n, r)$ and $\mathbb{R}(m, r)$ we will mean the above mentioned subalgebras of the algebra $\mathbb{R}(n+m, r)$. The subalgebras $\mathbb{R}(n, r)$ and $\mathbb{R}(m, r)$ generate the algebra $\mathbb{R}(n+m, r)$, and $\mathbb{R}(n+m, r) \cong \mathbb{R}(n, r)\widehat{\otimes}\mathbb{R}(m, r) = \mathbb{R}(n, r) \otimes \mathbb{R}(m, r)/\mathfrak{m}(\mathbb{R}(n, r) \otimes \mathbb{R}(m, r))^{r+1}$.

Let $\{\varepsilon^i\}$, $i = 1, \ldots, n$, denote the standard pseudobasis in $\mathbb{R}(n, r)$ and $\{\nu^a\}$, $a = 1, \ldots, m$, the standard pseudobasis in $\mathbb{R}(m, r)$. The total collection $\{\varepsilon^i, \nu^a\}$, $i = 1, \ldots, n$, $a = 1, \ldots, m$, is a pseudobasis in $\mathbb{R}(n + m, r) \cong \mathbb{R}(n, r)\widehat{\otimes}\mathbb{R}(m, r)$. In terms of this pseudobasis, the local coordinates on $\widehat{B}^r(M_n \times U)$ induced by coordinates on $M_n \times U$ can be written in the form

$$X^i = \sum_{|p|+|s|=0}^{r} x^i_{ps}\varepsilon^p\nu^s. \tag{90}$$

In terms of the coordinates (90), the composition $Y = X \circ Z$ in the Lie group $\widehat{D}^r(n, m)$ and the right action $Y = X \circ Z$ of $\widehat{D}^r(n, m)$ on

$\widehat{B}^r(M_n \times U)$ are, respectively, of the form

$$Y^i = \sum_{|p|+|s|=1}^{r} x^i_{ps} \overset{\circ}{Z}{}^p \nu^s \quad \text{and} \quad Y^i = \sum_{|p|+|s|=0}^{r} x^i_{ps} \overset{\circ}{Z}{}^p \nu^s. \tag{91}$$

The principal fiber bundle $(\widehat{B}^r(M_n \times U), \widehat{D}^r(n,m), M_n \times U)$ will be called the *r-frame bundle of* $M_n \times U$.

Thus, to a manifold $M_n \times U$, there is naturally associated the sequence of principal bundles of higher order frames

$$M_n \xleftarrow{\pi^1_0} \widehat{B}^1(M_n \times U) \xleftarrow{\pi^2_1} \dots \xleftarrow{\pi^r_{r-1}} \widehat{B}^r(M_n \times U) \xleftarrow{\pi^{r+1}_r} \dots \\ \dots \longleftarrow \widehat{B}^\infty(M_n \times U), \tag{92}$$

where $\widehat{B}^\infty(M_n \times U)$ is the bundle of infinite order frames of $M_n \times U$, the limit of the projective system $M_n \times U \leftarrow \widehat{B}^1(M_n \times U) \leftarrow \dots$ endowed with the corresponding structure of an infinite-dimensional smooth manifold in the sense of Bernshtein–Rozenfeld [1]. The bundle $\widehat{B}^\infty(M_n \times U)$ is formed by the infinite order jets of germs of morphisms $(f, \mathrm{tr}_t) : (\mathbb{R}^n \times \mathbb{R}^m, (0,0)) \to (M_n \times \mathbb{R}^m, (x,t))$. The group $\widehat{D}^\infty(n,m)$ consisting of the infinite order jets of germs of invertible morphisms $(f, \mathrm{id}) : (\mathbb{R}^n \times \mathbb{R}^m, (0,0)) \to (\mathbb{R}^n \times \mathbb{R}^m, (0,0))$ acts naturally on the right on $\widehat{B}^\infty(M_n \times U)$.

Proposition 11. i) *The Lie group* $\widehat{D}^r(n,m)$ *is isomorphic to the Lie group of* $\mathbb{R}(m,r)$*-linear automorphisms of the algebra* $\mathbb{R}(n,r)\widehat{\otimes}\mathbb{R}(m,r)$.

ii) *The Lie algebra* $\mathfrak{d}^r(n,m)$ *of the Lie group* $\widehat{D}^r(n,m)$ *is isomorphic to the Lie algebra of* $\mathbb{R}(m,r)$*-linear derivations of the algebra* $\mathbb{R}(n,r)\widehat{\otimes}\mathbb{R}(m,r)$ *with bracket*

$$[D_1, D_2] = D_2 \circ D_1 - D_1 \circ D_2. \tag{93}$$

Proof. The proof of this proposition is similar to that of Proposition 1 (see also [31] for the case of algebra $\mathbb{A}(n,q) = \mathbb{R}(n,q)\widehat{\otimes}\mathbb{A}$). The algebra $\mathbb{R}(n,r)\widehat{\otimes}\mathbb{R}(m,r)$ can be considered as the algebra of $\mathbb{R}(n+m,r)$-jets of morphisms $(f, \mathrm{id}) : (\mathbb{R}^n \times \mathbb{R}^m, (0,0)) \to (\mathbb{R} \times \mathbb{R}^m, (x,0))$. Then the action of the $\mathbb{R}(n,r)\widehat{\otimes}\mathbb{R}(m,r)$-jet of a germ of morphism $(g, \mathrm{id}) : (\mathbb{R}^n \times \mathbb{R}^m, (0,0)) \to (\mathbb{R}^n \times \mathbb{R}^m, (0,0))$ on the right on $\mathbb{R}(n,r)\widehat{\otimes}\mathbb{R}(m,r)$ by the law of composition of jets is an automorphism. If a germ $(f, \mathrm{id}) : (\mathbb{R}^n \times \mathbb{R}^m, (0,0)) \to (\mathbb{R} \times \mathbb{R}^m, (x,0))$ does not depend on $u \in \mathbb{R}^n$ (i.e., is of the form $x = f(t^a)$), then $(f, \mathrm{id}) \circ (g, \mathrm{id}) = (f, \mathrm{id})$ and, therefore, the element $j^{\mathbb{R}(n,r)\widehat{\otimes}\mathbb{R}(m,r)} g$ of $\widehat{D}^r(n,m)$ acts on the subalgebra $\mathbb{R}(m,r)$ as the identity transformation.

The identification of the tangent spaces to the algebra $\mathbb{R}(n,r)\widehat{\otimes}\mathbb{R}(m,r)$ with the same algebra converts the fundamental vector fields of the action of $\widehat{D}^r(n,m)$ on $\mathbb{R}(n,r)\widehat{\otimes}\mathbb{R}(m,r)$ into the $\mathbb{R}(m,r)$-linear derivations of $\mathbb{R}(n,r)\widehat{\otimes}\mathbb{R}(m,r)$. $\square$

5.2. **The structure form of the bundle $\widehat{B}^r(M_n \times U)$.** Let $\varphi : (\mathbb{R}^n \times \mathbb{R}^m, (0,0)) \to (M_n \times \mathbb{R}^m, (x,t_0))$ be a germ of invertible morphism, and let $X = j^r\varphi$ be the r-frame from $\widehat{B}^r(M_n \times U)$ defined by φ. The tangent mapping to the germ $\widehat{T}^{\mathbb{R}(n+m,r)}_{\mathrm{pr}}(\varphi)$ at $e \in \widehat{B}^r(\mathbb{R}^n \times \mathbb{R}^m)$ is an isomorphism of tangent spaces

$$T\widehat{T}^{\mathbb{R}(n+m,r)}_{\mathrm{pr}}(\varphi) : T_e\widehat{B}^r(\mathbb{R}^n \times \mathbb{R}^m) \to T_X\widehat{B}^r(M_n \times U).$$

But the r-frame $X = j^r\varphi$ defines only the isomorphism

$$T^{r-1}\widehat{T}^{\mathbb{R}(n+m,r)}_{\mathrm{pr}}(\varphi) : T^{r-1}_e\widehat{B}^r(\mathbb{R}^n \times \mathbb{R}^m) \to T^{r-1}_X\widehat{B}^r(M_n \times U), \tag{94}$$

where $T^{r-1}\widehat{B}^r(M_n \times U)$ is the pullback of the tangent bundle $T\widehat{B}^r(M_n \times U)$ under the projection $\pi^r_{r-1} : \widehat{B}^r(M_n \times U) \to \widehat{B}^{r-1}(M_n \times U)$. An element of $T^{r-1}_X\widehat{B}^r(M_n \times U)$ can be considered as a tangent vector to $\widehat{B}^r(M_n \times U)$ given up to an addend belonging to the kernel of the projection $T_X\pi^r_{r-1}$ or, in terms of the algebra $\mathbb{R}(n+m,r)$, up to an addend belonging to the submodule

$$\mathfrak{m}(\mathbb{R}(n+m,r))^r V_X\widehat{B}^r(M_n \times U)$$

of the vertical tangent $\mathbb{R}(n+m,r)$-module $V_X\widehat{B}^r(M_n \times U)$ (the tangent space to the fiber $t = t_0$ of the projection of $\widehat{B}^r(M_n \times U)$ to U) generated by the r-th power of the maximal ideal $\mathfrak{m}(\mathbb{R}(n+m,r))$ of $\mathbb{R}(n+m,r)$. Thus, a fiber $T^{r-1}_X\widehat{B}^r(M_n \times U)$ of the bundle $T^{r-1}\widehat{B}^r(M_n \times U)$ can be considered as the quotient space

$$T_X\widehat{B}^r(M_n \times U)/\mathfrak{m}(\mathbb{R}(n+m,r))^r V_X\widehat{B}^r(M_n \times U).$$

Then the bundle $T^{r-1}\widehat{B}^r(M_n \times U)$ is the quotient bundle of the vector bundle $T\widehat{B}^r(M_n \times U) \to \widehat{B}^r(M_n \times U)$ by the subbundle

$$\mathfrak{m}(\mathbb{R}(n+m,r))^r V\widehat{B}^r(M_n \times U)$$

of the vertical tangent bundle $V\widehat{B}^r(M_n \times U)$.

Each element $V^{(r-1)}_e$ of $T^{r-1}_e\widehat{B}^r(\mathbb{R}^n \times \mathbb{R}^m)$ defines, by means of the mappings (94), a section $V^{(r-1)}$ of the bundle $T^{r-1}\widehat{B}^r(M_n \times U)$. We will call the section $V^{(r-1)}$ the *fundamental semivector field on $\widehat{B}^r(\mathbb{R}^n \times \mathbb{R}^m)$ corresponding to $V^{(r-1)}_e$*.

The inverse mappings to (94) define the 1-form $\widehat{\theta}^r$ on $\widehat{B}^r(M_n \times U)$ with values in $T^{r-1}_e\widehat{B}^r(\mathbb{R}^n \times \mathbb{R}^m)$:

$$\widehat{\theta}^r_X(V_X) = (T^{r-1}\widehat{T}^{\mathbb{R}(n+m,r)}_{\mathrm{pr}}(\varphi))^{-1}(T\pi^r_{r-1}(V_X)). \tag{95}$$

The form $\widehat{\theta}^r$ will be called the *structure form* of the bundle $\widehat{B}^r(M_n \times U)$.

By a local isomorphism $\varphi : M_n \times U \to M'_n \times U'$ we will mean a morphism of the category $\mathcal{M}f^m_{\mathrm{tr}}$ which is a local diffeomorphism.

Theorem 5. *Let $\Phi : \widehat{B}^r(M_n \times U) \to \widehat{B}^r(M'_n \times U')$ be a local diffeomorphism which maps the structure form $\widehat{\theta}^r$ of the bundle $\widehat{B}^r(M_n \times U)$ into the structure form $\widehat{\theta}'^r$ of the bundle $\widehat{B}^r(M'_n \times U')$. Then in a neighborhood of every point $X \in \widehat{B}^r(M_n \times U)$ the mapping Φ coincides with the $\mathbb{R}(n,r)\widehat{\otimes}\mathbb{R}(m,r)$-prolongation of a local isomorphism $\varphi : M_n \times U \to M'_n \times U'$.*

Proof. The mapping Φ maps the fundamental semivector field on $\widehat{B}^r(M_n \times U)$ corresponding to an element $V_e \in T^{r-1}_e\widehat{B}^r(\mathbb{R}^n \times \mathbb{R}^m)$ into the fundamental semivector field on $\widehat{B}^r(M'_n \times U')$ corresponding to the same element V_e. If the projection of V_e to the tangent space $T_{0,0}(\mathbb{R}^n \times \mathbb{R}^m)$ is the zero vector, then the projections of the corresponding fundamental semivector fields to $M_n \times U$ and $M'_n \times U'$ are the zero vector fields. Hence it follows that the local diffeomorphism Φ projects into a local diffeomorphism $\varphi : M_n \times U \to M'_n \times U'$. For the same reason, Φ and φ project into a local diffeomorphism $\psi : U \to U'$, i. e., in terms of local coordinates, φ is of the form $x^{i'} = \varphi^{i'}(x^j, t^a)$, $t^{a'} = \psi^{a'}(t^b)$. In addition, from (95) and (94) it follows that the projection of a fundamental semivector field to $\mathbb{R}^m$ is a constant vector field. Consequently, constant vector fields on $\mathbb{R}^m$ are invariant under the mapping ψ. But then $\frac{\partial t^{a'}}{\partial t^b} = \delta^a_b$ and $t^{a'} = t^a + t^a_0$, i. e., ψ is a translation of $\mathbb{R}^m$. Thus, $\varphi : M_n \times U \to M'_n \times U'$ is a morphism of the category $\mathcal{M}f^m_{\mathrm{tr}}$.

Now, following the arguments from Section 1.3.1 of [21] (see also the proof of Theorem 2), we consider the composition

$$\Psi^r = (\varphi^{\mathbb{R}(n,r)\widehat{\otimes}\mathbb{R}(m,r)})^{-1} \circ \Phi : \widehat{B}^r(M_n \times U) \to \widehat{B}^r(M_n \times U). \tag{96}$$

The local diffeomorphism Ψ^r preserves the structure form and projects into the identity diffeomorphism of $M_n \times U$ to itself. Therefore, Ψ^r consists of right translations $R_{Z(x,t)}$, $Z \in \widehat{D}^r(n,m)$, of the fibers $(\pi^r_0)^{-1}(x,t)$ of the bundle $\widehat{B}^r(M_n \times U)$. It remains to show that Ψ^r is the identity mapping.

For $r = 1$, in terms of local coordinates, the right translation $R_Z : X \mapsto Y$ is of the form $y^i_j = x^i_k z^k_j$, $y^i_a = x^i_k z^k_a + x^i_a$. Let $v_e \in T^0_e\widehat{B}^1(\mathbb{R}^n \times \mathbb{R}^m)$ be an arbitrary element, and let (v^i_e, v^a_e) be the coordinates of v_e. Equating the coordinates (v^i, v^a) of the values of the fundamental semivector field corresponding to v_e at X and Y, we obtain

$$x^i_j v^j_e + x^i_a v^a_e = x^i_k z^k_j v^j_e + (x^i_k z^k_a + x^i_a) v^a_e,$$

whence it follows that $z^k_j = \delta^k_j$, $z^k_a = 0$, that is, $Z(x,t) = e \in \widehat{D}^1(n,m)$.

Assuming that the statement holds for $r = 1, \dots, r-1$, we conclude that the mapping (96) fibers over the identity mapping $\widehat{B}^{r-1}(M_n \times U) \to \widehat{B}^{r-1}(M_n \times U)$. Then, as in the case $r = 1$, equating the expressions similar to (62), (63), we obtain $Z(X^{r-1}) = e \in \widehat{D}^r(n,m)$. □

5.3. **Structure equations of $\widehat{B}^r(M_n \times U)$.** Let $T^{r'}\widehat{T}^{\mathbb{R}(n+m,r)}_{\mathrm{pr}}(M'_k \times U)$, $r' < r$, be the pullback of the tangent bundle $T\widehat{T}^{\mathbb{R}(n+m,r')}_{\mathrm{pr}}(M'_k \times U)$ under the projection $\pi^r_{r'} : \widehat{T}^{\mathbb{R}(n+m,r)}_{\mathrm{pr}}(M'_k \times U) \to \widehat{T}^{\mathbb{R}(n+m,r')}_{\mathrm{pr}}(M'_k \times U)$. We define *fundamental semivector fields corresponding to elements* $V^{(r')}_e \in T^{r'}_e \widehat{B}^r(\mathbb{R}^n \times \mathbb{R}^m)$ on the bundles $\widehat{T}^{\mathbb{R}(n+m,r)}_{\mathrm{pr}}(M'_k \times U)$ as sections of the bundles $T^{r'}\widehat{T}^{\mathbb{R}(n+m,r)}_{\mathrm{pr}}(M'_k \times U)$ in the following way. Let $Y_0 = j^r f$ be an element of $\widehat{T}^{\mathbb{R}(n+m,r)}_{\mathrm{pr}}(M'_k \times U)$ defined by a germ of morphism $f : (\mathbb{R}^n \times \mathbb{R}^m, (0,0)) \to (M'_k \times U, (y_0, t_0))$, $y^{i'} = f^{i'}(x^j, t^a)$, $t'^a = t^a + t^a_0$. Applying the functor $\widehat{T}^{\mathbb{R}(n+m,r)}_{\mathrm{pr}}$ to f, we obtain the germ

$$F : (\widehat{B}^r(M_n \times U), e) \to (\widehat{T}^{\mathbb{R}(n+m,r)}_{\mathrm{pr}}(M'_k \times U), Y_0)$$

given by equations (83)

$$Y^{i'} = f^{i'}(x,t) + \sum_{p+s=1}^{r} \frac{1}{p!s!} \frac{\partial^{|p+s|} f^{i'}}{\partial x^p \partial t^s} \overset{\circ}{X}{}^p \nu^s. \tag{97}$$

The jet $j^r f$ defines the mapping

$$T^{r-1}\widehat{T}^{\mathbb{R}(n+m,r)}_{\mathrm{pr}}(f) : T^{r-1}_e \widehat{B}^r(\mathbb{R}^n \times \mathbb{R}^m) \to T^{r-1}_{Y_0}\widehat{T}^{\mathbb{R}(n+m,r)}_{\mathrm{pr}}(M'_k \times U),$$

which assigns to $V^{(r-1)}_e$ the value $V^{(r-1)}(Y_0)$ of the corresponding fundamental semivector field $V^{(r-1)}$ at Y_0. In terms of local coordinates, the field $V^{(r-1)}(Y)$ is given by the equations (cf (51)

$$V^{i'}(Y) = \sum_{|p|+|s|=0}^{r-1} (p_j + 1) y^{i'}_{p+j\, s} \varepsilon^p \nu^s) V^j_e + \sum_{|p|+|s|=0}^{r-1} (s_a + 1) y^{i'}_{p\, s+a} \varepsilon^p \nu^s) v^a_e, \tag{98}$$

$$v^a(Y) = v^a_e,$$

where $V^j_e \in \mathbb{R}(n+m,r)$, $j = 1, \dots, n$, and $v^a_e \in \mathbb{R}$, $a = 1, \dots, m$, are the standard coordinates of $V^{(r-1)}_e$. In particular, for $M'_k = \mathbb{R}$, the fundamental semivector field $V^{(r-1)}$ corresponding to $V^{(r-1)}_e$ can be regarded as the mapping

$$\begin{gathered}\mathbb{R}(n+m,r)\oplus\mathbb{R}^m\to\mathbb{R}(n+m,r-1)\oplus\mathbb{R}^m,\\ (\alpha=\alpha_{ps}\varepsilon^p\nu^s,t^a)\mapsto(V_e^j\partial_j(\alpha)+v_e^a\partial_a(\alpha),v_e^a),\end{gathered}\tag{99}$$

where $\partial_j:\mathbb{R}(n+m,r)\to\mathbb{R}(n+m,r-1)$ and $\partial_a:\mathbb{R}(n+m,r)\to\mathbb{R}(n+m,r-1)$ are the derivations defined, respectively, by $\partial_j(\varepsilon^i)=\delta^i_j$, $\partial_j(\nu^b)=0$ and $\partial_a(\varepsilon^i)=0$, $\partial_a(\nu^b)=\delta^a_b$. Using the derivations ∂_j and ∂_a, one can rewrite equations (98) in the form

$$V^{i'}(Y)=V_e^j\partial_j(Y^{i'})+v_e^a\partial_a(Y^{i'}),\quad v^a(Y)=v_e^a.\tag{100}$$

Note that fundamental semivector fields (99) on $\mathbb{R}(n+m,r)\oplus\mathbb{R}^m$ and (100) on $\widehat{T}^{\mathbb{R}(n+m,r)}_{\mathrm{pr}}(\mathbb{R}^k\times\mathbb{R}^m)$ can be regarded as linear mappings, and so the bracket of fundamental semivector fields $[V^{(r-1)},W^{(r-1)}]$ induced by the Lie bracket of vector fields can be calculated as the bracket of linear endomorphisms. Taking into account that $V^{i'}(Y)$ in (100) do not depend on t^a, and $v^a(Y)$ are constant, from (100), we obtain $[V^{(r-1)},W^{(r-1)}]=U^{(r-2)}$, where

$$\begin{gathered}U^{i'}(Y)=((W_e^k\partial_kV_e^j-V_e^k\partial_kW_e^j)+(w_e^b\partial_bV_e^j-v_e^b\partial_bW_e^j))\partial_j(Y^{i'}),\\ u^a(Y)=0.\end{gathered}\tag{101}$$

We define the $T^{r-2}_e\widehat{B}^r(\mathbb{R}^n\times\mathbb{R}^m)$-valued 1-form $\widetilde{\theta}^r$ on $\widehat{B}^r(M_n\times U)$ as follows: $\widetilde{\theta}^r=\pi^{r-1}_{r-2}\circ\theta^r$, where $\pi^{r-1}_{r-2}:T^{r-1}_e\widehat{B}^r(\mathbb{R}^n\times\mathbb{R}^m)\to T^{r-2}_e\widehat{B}^r(\mathbb{R}^n\times\mathbb{R}^m)$ is the canonical projection. Let $\widetilde{\theta}^r=\widetilde{\theta}^je_j^{(r-2)}+\widetilde{\theta}^ae_a^{(r-2)}$ be the expansion of $\widetilde{\theta}^r$ in terms of the standard bases of the direct summands $\mathbb{R}(n+m,r-2)^n$ and $\mathbb{R}^m$ in $T^{r-2}_e\widehat{B}^r(\mathbb{R}^n\times\mathbb{R}^m)\equiv\mathbb{R}(n+m,r-2)^n\oplus\mathbb{R}^m$. Then, by calculations similar to those in Proposition 10, we obtain the following proposition.

Proposition 12. *On the bundle $\widehat{B}^r(M_n\times U)$, the following structure equations hold:*

$$d\widetilde{\theta}^i=\theta^j\wedge\partial_j\circ\theta^i+\theta^a\wedge\partial_a\circ\theta^i,\quad d\widetilde{\theta}^a=0.\tag{102}$$

Note 6. From the above discussion it follows that one can also take the space of $(r-1)$-jets at zero of vector fields on $\mathbb{R}^n\times\mathbb{R}^m$ which project into constant vector fields on $\mathbb{R}^m$ as the domain of values of the form $\widehat{\theta}^r$.

Note 7. As in the case of the bundle $B^\infty(\mathbb{A})T^{\mathbb{A}}M_n$ (see Note 5), passing to the projective limit, one can define the $\mathbb{R}(n+m,\infty)^n\oplus\mathbb{R}^m$-valued structure form $\widehat{\theta}^\infty$ on $\widehat{B}^\infty(M_n\times U)$. For r tending to infinity, the series of equations (102) gives the following expression for the exterior differential of the form $\widehat{\theta}^{\infty}$:

$$d(\widehat{\theta}^{\infty})^i = (\widehat{\theta}^{\infty})^j \wedge \partial_j \circ (\widehat{\theta}^{\infty})^i + (\widehat{\theta}^{\infty})^a \wedge \partial_a \circ (\widehat{\theta}^{\infty})^i, \quad d(\widehat{\theta}^{\infty})^a = 0, \quad (103)$$

where ∂_j and ∂_a are the derivations of the algebra $\mathbb{R}(n+m,\infty)$ of formal power series in $n+m$ variables ε^i and ν^a defined by the above indicated relations $\partial_j(\varepsilon^i) = \delta^i_j$, $\partial_j(\nu^b) = 0$ and $\partial_a(\varepsilon^i) = 0$, $\partial_a(\nu^b) = \delta^a_b$.

5.4. **Connections in $\widehat{B}^r(M_n \times U)$.** A connection Γ in a principal bundle $P = P(M, G, \pi)$ is a G-invariant horizontal distribution on P [11], [14]. Since the horizontal planes at $X \in P$ are in a bijective correspondence with the 1-jets of germs of sections $s : (M, x) \to (P, X)$, $x = \pi(X)$, called also connection elements, a connection Γ can also be defined as a G-equivariant section $\Gamma : P \to J^1P$, where J^1P is the first jet prolongation of P (the set of all connection elements) [14].

Connection elements on $\widehat{B}^r(M_n \times U)$ are 1-jets of germs of sections $\gamma : (M \times U, (x_0, t_0)) \to \widehat{B}^r(M \times U)$. In terms of the local coordinates (90), such a germ of section γ is given by equations

$$\overset{\circ}{X}{}^i = \sum_{|p|+|s|=1}^{r} X^i_{ps}(x^i, t^a)\varepsilon^p \nu^s,$$

and the corresponding connection element has coordinates

$$\Gamma^i_j = \frac{\partial \overset{\circ}{X}{}^i}{\partial x^j}\bigg|_{(x_0,t_0)} = \sum_{|p|+|s|=1}^{r} \frac{\partial \overset{\circ}{X}{}^i_{ps}}{\partial x^j}\bigg|_{(x_0,t_0)} \varepsilon^p \nu^s = \sum_{|p|+|s|=1}^{r} \Gamma^i_{psj}\varepsilon^p \nu^s = \sum_{|p|=0}^{r} \Gamma^i_{pj}\varepsilon^p,$$

$$\Gamma^i_a = \frac{\partial \overset{\circ}{X}{}^i}{\partial t^a}\bigg|_{(x_0,t_0)} = \sum_{|p|+|s|=1}^{r} \frac{\partial \overset{\circ}{X}{}^i_{ps}}{\partial t^a}\bigg|_{(x_0,t_0)} \varepsilon^p \nu^s = \sum_{|p|+|s|=1}^{r} \Gamma^i_{psa}\varepsilon^p \nu^s = \sum_{|p|=0}^{r} \Gamma^i_{pa}\varepsilon^p \quad (104)$$

where the coordinates Γ^i_{pj} and Γ^i_{pa} are elements of the quotient algebra $\mathbb{R}(m,r)/\mathfrak{m}(\mathbb{R}(m,r))^{|p|}$. In what follows, following the standard summation convention, we will omit the sign of sum in sums like (104).

The right action $Y = R_Z(X)$ of the group $\widehat{D}^r(n,m)$ on $\widehat{B}^r(M_n \times U)$ induces the right action $R_Z(j^1\gamma) = j^1(R_Z \circ \gamma)$ of $\widehat{D}^r(n,m)$ on the bundle of connection elements $J^1\widehat{B}^r(M_n \times U)$. In terms of local coordinates, this action is of the form (see (91))

$$R_Z : (X^i, \Gamma^i_{pj}, \Gamma^i_{pa}) \mapsto (Y^i, \Gamma^i_{pj}Z^p, \Gamma^i_{pa}Z^p). \quad (105)$$

Analytically, a connection Γ in $\widehat{B}^r(M_n \times U)$ can be given by the coordinates of the connection elements along the natural local section of

$\widehat{B}^r(M_n \times U)$ defined by a local coordinate chart on $M_n \times U$. These coordinates are functions $\Gamma^i_{pj}(x^i, t^a)$, $\Gamma^i_{pa}(x^i, t^a)$ of the local coordinates on $M_n \times U$ called the *connection coefficients*. The natural section is given by the equations $\overset{\circ}{X}{}^i_{pr} = \delta^i_p \delta^0_r$, where $\delta^i_p = 1$ when $p_i = 1$ and $p_j = 0$ if $i \neq j$, and $\delta^i_p = 0$ in all the other cases. Therefore, the connection elements at an arbitrary $X \in \widehat{B}^r(M_n \times U)$ are of the form

$$\Gamma^i_j(X) = \Gamma^i_{psj}(x^j, t^a)\, \overset{\circ}{X}{}^p \nu^s, \qquad \Gamma^i_a(Y) = \Gamma^i_{psa}(x^j, t^a)\, \overset{\circ}{X}{}^p \nu^s.$$

Coordinate transformations

$$x^{i'} = f^{i'}(x^i, t^a), \quad t^{a'} = t^a + t^a_0$$

induce the following coordinate transformations on $\widehat{B}^r(M_n \times U)$:

$$\overset{\circ}{X}{}^{i'} = \sum_{|p|=0}^{r} A^{i'}_p(x^i, t^a)\, \overset{\circ}{X}{}^p, \tag{106}$$

where $A^{i'}_p(x^i, t^a) \in \mathbb{R}(m, r)/\mathfrak{m}(\mathbb{R}(m, r))^{|p|}$ are defined by

$$A^{i'}_p(x^i, t^a)\varepsilon^p = \sum_{|p|+|s|=1}^{r} \frac{1}{p!s!} \frac{D^{|p|+|s|} f^{i'}}{Dx^p Dt^s} \nu^s \varepsilon^p.$$

Differentiating the coordinate transformations (106), one obtains the transformation law for the connection coefficients

$$\Gamma^{i'}_{uj'}(A^*_s \varepsilon^s)^u = \frac{\partial x^\alpha}{\partial x^{j'}} \left(\frac{\partial A^{i'}_p}{\partial x^\alpha} \varepsilon^p + A^{i'}_p \sum_{i=1}^{n} p_i \Gamma^i_{sj} \varepsilon^{p+s-i} \right), \tag{107}$$

$$\begin{aligned} \Gamma^{i'}_{ua'}(A^*_s \varepsilon^s)^u = {} & \frac{\partial A^{i'}_p}{\partial t^a} \varepsilon^p - \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^{j'}}{\partial t^a} \frac{\partial A^{i'}_p}{\partial x^j} \varepsilon^p + \\ & + A^{i'}_p \sum_{i=1}^{n} p_i \left(\Gamma^i_{sa} - \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^{j'}}{\partial t^a} \Gamma^i_{sj} \right) \varepsilon^{p+s-i}, \end{aligned} \tag{108}$$

where $(A^*_s \varepsilon^s)^u = A^u$ for $A^i = A^i_s \varepsilon^s$ and $u = (u_1, \dots, u_n)$.

The horizontal distribution of a connection in $\widehat{B}^r(M_n \times U)$ is given by the equations

$$d\, \overset{\circ}{Y}{}^i - \Gamma^i_{psj}(x^j, t^a)\, \overset{\circ}{Y}{}^p \nu^s dx^j - \Gamma^i_{psa}(x^j, t^a)\, \overset{\circ}{Y}{}^p \nu^s dt^a = 0\,. \tag{109}$$

5.5. **Associated connections in $\widehat{T}^{\mathbb{A}}_{\sigma}(M_n \times U)$.** If one replaces a morphism (81) by a germ of isomorphism

$$\begin{tikzcd} (\mathbb{R}^n \times \mathbb{R}^m, (0,0)) \arrow[r, "f"] \arrow[d] & (\mathbb{R}^n \times \mathbb{R}^m, (0,0)) \arrow[d] \\ (\mathbb{R}^m, 0) \arrow[r, "\mathrm{id}"] & (\mathbb{R}^m, 0) \end{tikzcd} \tag{110}$$

then the corresponding diagram (82) gives the left action Φ_σ of the group $\widehat{D}^q(n,m)$ on the standard fiber $\widehat{T}^{\mathbb{A}}_{\sigma\,(0,0)}(\mathbb{R}^n \times \mathbb{R}^m) \equiv T^{\mathbb{A}}_0\mathbb{R}^n \equiv \mathring{\mathbb{A}}^n$ of the bundle $\widehat{T}^{\mathbb{A}}_{\sigma}(M_n \times U)$. If we consider $\widehat{T}^{\mathbb{A}}_{\sigma\,(0,0)}(\mathbb{R}^n \times \mathbb{R}^m)$ as the set of $\mathbb{A}$-velocities $j^{\mathbb{A}}g$ of germs of the form
(see (84))

$$\begin{tikzcd} (\mathbb{R}^\ell, 0) \arrow[r, "g"] \arrow[d, "\mathrm{id}"'] & (\mathbb{R}^n \times \mathbb{R}^m, (0,0)) \arrow[d] \\ (\mathbb{R}^\ell, 0) \arrow[r, "\widehat{\sigma}"] & (\mathbb{R}^m, 0) \end{tikzcd}$$

then the action

$$\Phi_\sigma : \widehat{D}^q(n,m) \times \widehat{T}^{\mathbb{A}}_{\sigma\,(0,0)}(\mathbb{R}^n \times \mathbb{R}^m) \to \widehat{T}^{\mathbb{A}}_{\sigma\,(0,0)}(\mathbb{R}^n \times \mathbb{R}^m)$$

is given by (85):

$$\Phi_\sigma : (j^q f, j^{\mathbb{A}} g) \mapsto j^{\mathbb{A}}(f \circ g), \tag{111}$$

where (f, id) is a germ (110) of isomorphism in $\mathcal{M}f^m$. In the standard coordinates z^i_{ps} on $\widehat{D}^q(n,m)$ and $\mathring{X}^i \in \mathring{\mathbb{A}}$ on $\widehat{T}^{\mathbb{A}}_{\sigma\,(0,0)}(\mathbb{R}^n \times \mathbb{R}^m)$, the action (111), $(Z, X) \mapsto Y$, is of the form

$$\mathring{Y}^i = \sum_{|p|+|s|=1}^{q} z^i_{ps}\, \mathring{X}^p\, \mathring{\sigma}^s. \tag{112}$$

Note that, in general, the action Φ_σ is not effective. Relations (112) define an effective action on $\mathring{\mathbb{A}}^n$ of some factor group $D_\sigma(n, \mathbb{A})$ of the group $\widehat{D}^q(n,m)$.

The action Φ_σ allows one to consider the bundle $\widehat{T}^{\mathbb{A}}_{\sigma}(M_n \times U)$ as associated with the frame bundle $\widehat{B}^q(M_n \times U)$. The mapping

$$\Psi_\sigma : \widehat{B}^q(M_n \times U) \times \mathring{\mathbb{A}}^n \to \widehat{T}^{\mathbb{A}}_{\sigma}(M_n \times U), \quad (Y, \mathring{Z}) \mapsto X,$$

which defines on $\widehat{T}^{\mathbb{A}}_{\sigma}(M_n \times U)$ the structure of a bundle associated with $\widehat{B}^q(M_n \times U)$ has the following equations in terms of local coordinates:

$$X^i = \sum_{|p|+|s|=0}^{q} y^i_{ps}\, \mathring{Z}^p\, \mathring{\sigma}^s. \tag{113}$$

Fixing $\overset{\circ}{Z}{}^i$ in (113), from (109), we obtain the following equations of the horizontal distribution of the connection Γ on $\widehat{T}^{\mathbb{A}}_{\sigma}(M_n \times U)$:

$$d\overset{\circ}{X}{}^i - \Gamma^i_{psj}(x^j, t^a)\overset{\circ}{X}{}^p\overset{\circ}{\sigma}{}^s dx^j - \Gamma^i_{psa}(x^j, t^a)\overset{\circ}{X}{}^p\overset{\circ}{\sigma}{}^s dt^a = 0\,.$$

5.6. **Examples.** 1. The functor $\widehat{T}^{\mathbb{A}}_0$ defined by the zero section $\sigma = 0 : U \to T^{\mathbb{A}}U$, corresponding to the germ $\widehat{\sigma} = 0$ (see (84)) at $0 \in \mathbb{R}^\ell$ of the zero mapping $0 : \mathbb{R}^\ell \ni t \mapsto 0 \in \mathbb{R}^m$, is equivalent to the vertical Weil functor $V^{\mathbb{A}}$, the functor which assigns to $p : M_n \times U \to U$ the bundle $V^{\mathbb{A}}(M_n \times U)$ of $\mathbb{A}$-velocities of germs $g : (\mathbb{R}^\ell, 0) \to M_n \times U$ such that $p \circ g$ is a germ of constant mapping.

2. In the case when $\ell = m$ and $\widehat{\sigma}$ in (84) is the germ of the identity mapping $\mathrm{id} : \mathbb{R}^m \to \mathbb{R}^m$ (and $\overset{\circ}{\sigma}{}^a = \tau^a$, $a = 1, \dots, m$, are the elements of the standard pseudobasis of $\mathbb{A} \equiv T^{\mathbb{A}}\mathbb{R}$), the functor $\widehat{T}^{\mathbb{A}}_{\sigma}$ is equivalent to the functor $\widehat{T}^{\mathbb{A}}$ studied in [2] which assigns to $p : M_n \times U \to U$ the bundle $\widehat{T}^{\mathbb{A}}(M_n \times U)$ of $\mathbb{A}$-velocities of germs $g \circ \mathrm{tr}_{t_0}$, where $g : (\mathbb{R}^m, t_0) \to (M_n \times U, (x, t_0))$ is a germ of section. In this case, the action (112) of the group $\widehat{D}^q(n, m)$ on $\overset{\circ}{\mathbb{A}}{}^n$ reduces to the action of the so-called $\mathbb{A}$-affine differential group $D_n(\mathbb{A})$ [31], [34]:

$$\overset{\circ}{Y}{}^i = \sum_{|p|=0}^{q} \varphi^i_p \overset{\circ}{X}{}^p, \quad \varphi^i_p \in \mathbb{A}, \quad \varphi^i_0 = \overset{\circ}{Y}{}^i_0 \in \overset{\circ}{\mathbb{A}}, \quad \det(\varphi^i_j) \neq 0.$$

The principal bundle associated with $\widehat{T}^{\mathbb{A}}(M_n \times U)$ is the bundle of $\mathbb{A}$-affine frames $B(\mathbb{A})(M_n \times U)$.

3. For the algebra $\mathbb{R}(m, q)$, the group $D_n(\mathbb{R}(m, q))$ coincides with $\widehat{D}^q(n, m)$ and the bundle $B(\mathbb{R}(m, q))(M_n \times U)$ coincides with $\widehat{B}^q(M_n \times U)$.

For $m = 1$ and $q = 1$, $\mathbb{R}(1, 1) = \mathbb{R}(\varepsilon)$ is the algebra of dual numbers $x + \dot{x}\varepsilon$, $\varepsilon^2 = 0$.

Local coordinates $(x^i + \dot{x}^i\varepsilon, t)$ on $\widehat{T}^{\mathbb{R}(\varepsilon)}(M_n \times \mathbb{R})$ are transformed as follows:

$$x^{i'} + \dot{x}^{i'}\varepsilon = \varphi^{i'}(x^i, t) + \left(\frac{\partial\varphi^{i'}}{\partial x^i}\dot{x}^i + \frac{\partial\varphi^{i'}}{\partial t}\right)\varepsilon, \quad t' = t + t_0.$$

The standard fiber of $\widehat{T}^{\mathbb{R}(\varepsilon)}(M_n \times \mathbb{R})$ is $\mathbb{R}^n$, and the structure group $D_n(\mathbb{R}(\varepsilon))$ is the group of affine transformations $\dot{x}^{i'} = a^{i'}_i \dot{x}^i + a^{i'}_0$. The horizontal distribution of a connection in $\widehat{B}^1(M_n \times \mathbb{R})$ has equations [2]

$$d\dot{y}^i = (\Gamma^i_{\lambda\alpha}(x, t)dx^\alpha + \Gamma^i_{\lambda 0}(x, t)dt)\dot{y}^\lambda + \Gamma^i_{0\alpha}(x, t)dx^\alpha + \Gamma^i_{00}(x, t)dt, \quad (114)$$

$$dy^i_\mu = (\Gamma^i_{\lambda\alpha}(x, t)dx^\alpha + \Gamma^i_{\lambda 0}(x, t)dt)y^\lambda_\mu\,. \quad (115)$$

Under a coordinate change on $M_n \times \mathbb{R}$, the connection coefficients transform as follows:

$$\Gamma^{i'}_{j'k'} = \frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^k}{\partial x^{k'}}\left(\frac{\partial x^{i'}}{\partial x^i}\Gamma^i_{jk} + \frac{\partial^2 x^{i'}}{\partial x^j \partial x^k}\right),$$
$$\Gamma^{i'}_{0k'} = \frac{\partial x^k}{\partial x^{k'}}\left(\frac{\partial x^{i'}}{\partial x^i}\Gamma^i_{0k} + \frac{\partial^2 x^{i'}}{\partial t \partial x^k} - \frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^{j'}}{\partial t}\left(\frac{\partial x^{i'}}{\partial x^i}\Gamma^i_{jk} + \frac{\partial^2 x^{i'}}{\partial x^j \partial x^k}\right)\right),$$
$$\Gamma^{i'}_{k'0} = \frac{\partial x^k}{\partial x^{k'}}\left(\frac{\partial x^{i'}}{\partial x^i}\Gamma^i_{k0} + \frac{\partial^2 x^{i'}}{\partial t \partial x^k} - \frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^{j'}}{\partial t}\left(\frac{\partial x^{i'}}{\partial x^i}\Gamma^i_{kj} + \frac{\partial^2 x^{i'}}{\partial x^j \partial x^k}\right)\right),$$
$$\Gamma^{i'}_{00} = \frac{\partial x^{i'}}{\partial x^i}\Gamma^i_{00} + \frac{\partial^2 x^{i'}}{\partial^2 t} - \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^{k'}}{\partial t}\left(\frac{\partial x^{i'}}{\partial x^i}(\Gamma^i_{k0} + \Gamma^i_{0k}) + 2\frac{\partial^2 x^{i'}}{\partial t \partial x^k} + \right.$$
$$\left. + \frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^{j'}}{\partial t}\left(\frac{\partial x^{i'}}{\partial x^i}\Gamma^i_{kj} + 2\frac{\partial^2 x^{i'}}{\partial x^j \partial x^k}\right)\right).$$

The equations of the horizontal distribution on $\widehat{T}^{\mathbb{R}(\varepsilon)}(M_n \times \mathbb{R})$ corresponding to (114), (115) are of the form

$$d\dot{x}^i = \big(\Gamma^i_{\lambda\alpha}(x,t)dx^\alpha + \Gamma^i_{\lambda 0}(x,t)dt\big)\dot{x}^\lambda + \Gamma^i_{0\alpha}(x,t)dx^\alpha + \Gamma^i_{00}(x,t)dt\,.$$

Along a section $c : \mathbb{R} \to M_n \times \mathbb{R}$, the above Pfaff system gives the system of linear differential equations with respect to $\dot{x}^i$:

$$\frac{d\dot{x}^i}{dt} = \Big(\Gamma^i_{\lambda\alpha}(x,t)\frac{dx^\alpha}{dt} + \Gamma^i_{\lambda 0}(x,t)\Big)\dot{x}^\lambda + \Gamma^i_{0\alpha}(x,t)\frac{dx^\alpha}{dt} + \Gamma^i_{00}(x,t).$$

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