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ON THE ABSTRACT THEOREM OF PICARD

(submitted by D. Kh. Mushtari)

ABSTRACT. Let A be a complex Banach algebra with unit. It was shown by Williams [1] that elements $\mathbf{a}$, $\mathbf{b} \in A$ commute if and only if $\sup\limits_{\lambda \in \mathbf{C}} \| \exp(\lambda\mathbf{b})\mathbf{a} \exp(-\lambda\mathbf{b})\| < \infty$. This result allows us to obtain an analog of the von Neumann-Fuglede-Putnam theorem in case of normal elements in a complex Banach algebra. In the present paper the results by Williams [1] and Khasbardar et Thakare [2] are refined by using [3, 4, 5]. An abstract version of Picard theorem is obtained in this context.

The set $P(A)$ of all normalized states $\varphi : A \to \mathbf{C}$ [6, 7] on a complex Banach algebra A with unit is a weak-star compact convex subset in the space $A^\star$ (conjugate space). The set $V(\mathbf{a}) = \{\varphi(\mathbf{a}) : \varphi \in P(A)\}$ is called *(algebraic) numerical range* of $\mathbf{a} \in A$. In particular, if A is an algebra $B(\mathbf{H})$ of all bounded linear operators defined in the complex Hilbert space $\mathbf{H}$, and $T \in B(\mathbf{H})$, then its (algebraic) numerical range $V(T)$ coincides with the closure of the general numerical range of the operator T [8]. An element $\mathbf{h} \in A$ is called Hermitian if $V(\mathbf{h}) \subset \mathbf{R}$, were $\mathbf{R}$ is the field of reals. This condition is equivalent to $\| \exp(it\mathbf{h})\| = 1$ for all $t \in \mathbf{R}$. We also note that an element $\mathbf{h} \in A$ is called *quasi-Hermitian*[5] if $\| \exp(it\mathbf{h})\| = o(\mid t \mid^{\frac{1}{2}})$ when $\mid t \mid \to \infty$, $t \in \mathbf{R}$. This condition provides $sp(\mathbf{h}) \subset \mathbf{R}$, where $sp(\mathbf{h})$ is the spectrum of$\mathbf{h}$, but the numerical range $V(\mathbf{h})$ may be not contained in the real axis. It is easy to see that for any $\mathbf{a} \in A$ there holds $sp(\mathbf{a}) \subset V(\mathbf{a})$. If an element $\mathbf{a} \in A$ admits representation $\mathbf{a} = \mathbf{h} + i\mathbf{k}$, where $\mathbf{h}$, $\mathbf{k}$ are Hermitian elements, then $\mathbf{a}$ is called *Hermitian decomposable*, and $\mathbf{a}^+ = \mathbf{h} - i\mathbf{k}$ is called the Hermitian conjugate of $\mathbf{a}$. When $[\mathbf{h}, \mathbf{k}] = \mathbf{hk} - \mathbf{kh} = 0$, then the element

$\mathbf{a} = \mathbf{h} + i\mathbf{k}$ is called *normal* element, if $\mathbf{h}, \mathbf{k}$ are quasi-Hermitian elements and $[\mathbf{h}, \mathbf{k}] = 0$, then the element $\mathbf{a}$ is called *quasi-normal*.

Let J be a closed bi-ideal in A. Then the factor-algebra A/J is a Banach algebra with respect to the factor-norm $\|| \bullet \||$, and $\tilde{\mathbf{a}} = \mathbf{a} + J = \pi_J(\mathbf{a}) \in A/J$, where $\pi_J : \quad A \to A/J$ is a canonical homomorphism generated by J-ideal.

Let $\{\mathbf{a}_n\}_0^\infty$, $\{\mathbf{b}_n\}_0^\infty$ be sequences of elements in Banach algebra A such that $\varlimsup_{n\to\infty} \|\mathbf{a}_n\|^{\frac{1}{n}} = \rho_a < \infty$, $\varlimsup_{n\to\infty} \|\mathbf{b}_n\|^{\frac{1}{n}} = \rho_b < \infty$, and let $\mathbf{c} \in A$. Then the function $f(\lambda) = \mathbf{A}(\lambda)\mathbf{c}\mathbf{B}(\lambda) = \sum_0^\infty \frac{\mathbf{c}_n \lambda^n}{n!}$, where $\mathbf{A}(\lambda) = \sum_0^\infty \frac{\mathbf{a}_n \lambda^n}{n!}$, $\mathbf{B}(\lambda) = \sum_0^\infty \frac{\mathbf{b}_n \lambda^n}{n!}$, is an A-valued entire function of exponential type $\sigma_f \leq \rho_a + \rho_b$, and $\mathbf{c}_n = \sum_{p=0}^n < \mathbf{a}_p \mathbf{c} \mathbf{b}_{n-p} >$ is a generalized commutator of the corresponding elements. If the elements $\mathbf{a}, \mathbf{b}, \mathbf{c} \in A$, then one writes $\mathbf{c}\mathbf{a} = \mathbf{b}\mathbf{c} (mod J)$ if $\mathbf{c}\mathbf{a} - \mathbf{b}\mathbf{c} \in J$.

Theorem 1. *Let A be a complex Banach algebra with unit, J be a closed bi-ideal in A, and $\{\mathbf{a}_n\}_0^\infty$, $\{\mathbf{b}_n\}_0^\infty \in \mathbf{A}$ be such that $\varlimsup_{n\to\infty} \|\mathbf{a}_n\|^{\frac{1}{n}} < \infty$, $\varlimsup_{n\to\infty} \|\mathbf{b}_n\|^{\frac{1}{n}} < \infty$. Then $\mathbf{c}_n = \sum_{p=0}^n < \mathbf{a}_p \mathbf{c} \mathbf{b}_{n-p} > \in J$ for all $n \geq 1$ if and only if $\||\tilde{f}(\lambda)\|| = o(|\lambda|)$ for $|\lambda| \to \infty$, $\lambda \in \mathbf{C}$.*

Proof: Let $\pi_J : A \to A/J$ be a canonical homomorphism generated by J. Since, $\mathbf{c}_n \in J$, $\tilde{\mathbf{c}}_n = \tilde{0}$. Hence $\tilde{f}(\lambda) = \sum_0^\infty \frac{\tilde{\mathbf{c}}_n \lambda^n}{n!} = \tilde{\mathbf{c}}_0$, i.e., $\||\tilde{f}(\lambda)\|| = \||\tilde{\mathbf{c}}_0\||$, and so $\frac{\||\tilde{f}(\lambda)\||}{|\lambda|} \to 0$, $|\lambda| \to \infty$.

Conversely assume that $\||\tilde{f}(\lambda)\|| = o(|\lambda|)$ for $|\lambda| \to \infty$, $\lambda \in \mathbf{C}$. Then for a A/J-valued entire function $\tilde{f}(\lambda) = \sum_0^\infty \frac{\tilde{\mathbf{c}}_n \lambda^n}{n!}$, there holds $\tilde{f}(\lambda) \equiv \mathbf{c}_0$ by the Liouville theorem. Hence, for all $n \geq 1$ one has $\tilde{\mathbf{c}}_n = 0$, and therefore $\mathbf{c}_n \in J$. This completes the proof.

In particular, when $\mathbf{a}_n = \mathbf{a}^n$ and $\mathbf{b}_n = \mathbf{b}^n$, where $\mathbf{a}, \mathbf{b} \in A$ the following result holds:

Theorem 2. *Let A be a complex Banach algebra with unit, J be a closed bi-ideal in A, and $\mathbf{a}, \mathbf{b}, \mathbf{c} \in A$. Then $\mathbf{c}\mathbf{a} = \mathbf{b}\mathbf{c} (mod J)$ if and only if $\|| \exp(\lambda\tilde{\mathbf{b}})\tilde{\mathbf{c}} \exp(-\lambda\tilde{\mathbf{a}})\|| = o(|\lambda|)$, $|\lambda| \to \infty$, $\lambda \in \mathbf{C}$.*

The proof is similar to that of Theorem 1.

Combination of Theorem 2 with the generalized theorem of von Neumann-Fuglede-Putnam [3, 4] gives:

Corollary 3. *Let* $\mathbf{a}, \mathbf{b} \in A$ *be quasinormal elements,* $\mathbf{c} \in A$, *and* $J \in A$ *be a closed bi-ideal. If* $\|\|\exp(\lambda\tilde{\mathbf{b}})\tilde{\mathbf{c}}\exp(-\lambda\tilde{\mathbf{a}})\|\| = o(|\lambda|)$, $|\lambda| \to \infty$, $\lambda \in \mathbf{C}$, *then* $\mathbf{a}^{+}\mathbf{c} = \mathbf{c}\mathbf{b}^{+}(mod J)$[9].

Corollary 4. *Let* A *be a complex Banach algebra with unit,* $\mathbf{a}, \mathbf{b} \in A$ *be quasinormal elements and* $\mathbf{c} \in A$. *If* $\|\exp(\lambda\mathbf{a})\mathbf{c}\exp(-\lambda\mathbf{b})\| = o(|\lambda|)$, $|\lambda| \to \infty$, $\lambda \in \mathbf{C}$, *then* $\mathbf{a}^{+}\mathbf{c} = \mathbf{c}\,\mathbf{b}^{+}$.

Let $D : A \to A$ be a continuous A-derivation (i.e., $D \in BL(A)$ and $D(\mathbf{ab}) = (D\mathbf{a})\mathbf{b} + \mathbf{a}(D\mathbf{b})$, where $\mathbf{a}, \mathbf{b} \in A$). Then in the factor-algebra A/J the A/J-derivation $D_J : A/J \to A/J$ is defined by the formula $D_J(\tilde{\mathbf{a}}) = \pi_J(D\mathbf{a}) = \tilde{D}\mathbf{a}$. It is obvious that $D_J \in BL(A/J)$. We denote $aut(A)$ the group of all automorphisms of the algebra A. Then, $\exp(\lambda D) \in aut(A)$ for each $\lambda \in \mathbf{C}$, and $\exp(\lambda D_J) \in aut(A/J)$.

Theorem 5. *Let* A *be a complex Banach algebra with unit,* J *be a closed bi-ideal in* A, *and* $D : A \to A$ *be a continuous* A-*derivation. Then* $D\mathbf{a} \in J$, *where* $\mathbf{a} \in A$, *if and only if* $\|\|(\exp\lambda D_J)(\tilde{\mathbf{a}})\|\| = o(|\lambda|)$, $|\lambda| \to \infty$, $\lambda \in \mathbf{C}$.

The proof of this theorem is similar to those of Theorems 1 and 2.

Corollary 6. *Let* A *be a complex Banach algebra with unit and* $D : A \to A$ *be a continuous* A-*derivation. Then* $\mathbf{a} \in ker(D)$ *if and only if* $\|(\exp\lambda D)(\mathbf{a})\| = o(|\lambda|)$, $|\lambda| \to \infty$, $\lambda \in \mathbf{C}$.

The following result is a generalization of the Picard's theorem for entire functions.

Theorem 7. *Let* A *be a complex Banach algebra with unit,* J *be a closed bi-ideal in* A *and* $D : A \to A$ *be a continuous* A-*derivation. If for the element* $\mathbf{a} \in A$ *one has* $D\mathbf{a} \notin J$, *then* $\bigcup_{\lambda \in \mathbf{C}} V((\exp\lambda D_J)(\tilde{\mathbf{a}})) = \mathbf{C}$.

Proof. Let $\varphi \in P(A/J)$. We consider an entire function $f_\varphi(\lambda) = \varphi(\exp(\lambda D_J)(\tilde{\mathbf{a}}))$. Then the range of the f_φ is contained in the set $U_J = \bigcup_{\lambda \in \mathbf{C}} V((\exp\lambda D_J)(\tilde{\mathbf{a}}))$. Let us assume that $\mathbf{C} \setminus U_J$ contains at least two points. Then according to the Picard's theorem for the entire functions one has $f_\varphi(\lambda) \equiv const$ Hence, for all natural $n \geq 1$ there holds $\varphi(D_J^n(\tilde{\mathbf{a}})) = 0$, and in particular, $D_J(\tilde{\mathbf{a}}) = 0$, i.e., $D\mathbf{a} \in J$. However, this contradicts the condition of the theorem. So, $\mathbf{C} \setminus U_J$ contains at most one point. It remains to show that $U_J = \mathbf{C}$. Since for any fixed $\lambda \in \mathbf{C}$ the operator $\exp(\lambda D_J)$ belongs to $aut(A/J)$, for any element $\tilde{\mathbf{a}} \in A/J$ one has $sp(\tilde{\mathbf{a}}) = sp((\exp\lambda D_J)(\tilde{\mathbf{a}}))$.

Let $\zeta_0 \in sp(\tilde{\mathbf{a}})$ and $\zeta \in \mathbf{C}$ be a complex number. Then on the line passing through the points ζ_0 and ζ there exists a point ζ_1 such that $\zeta_1 \in V((\exp \lambda D_J)(\tilde{\mathbf{a}}))$ for some $\lambda \in \mathbf{C}$. However, $[\zeta_0, \zeta_1] \subset V((\exp \lambda D_J)(\tilde{\mathbf{a}}))$, and since $V((\exp \lambda D_J)(\tilde{\mathbf{a}}))$ is a convex set, we have $\zeta \in V((\exp \lambda D_J)(\tilde{\mathbf{a}}))$, and hence, $U_J = \mathbf{C}$. This completes the proof.

In the case when the ideal J equals $\{0\}$ one has:

Corollary 8. *Let A be a complex Banach algebra with unit and $D : A \to A$ is a continuous A-derivation. If the element $\mathbf{a} \notin ker(D)$, then $\bigcup_{\lambda \in \mathbf{C}} V((\exp \lambda D)(\mathbf{a})) = \mathbf{C}$.*

Corollary 9. *Let A be a complex Banach algebra with unit and a, b, $c \in A$ are such elements that $ac \neq cb \,(mod J)$, where J is a closed bi-ideal in A. Then $\bigcup_{\lambda \in \mathbf{C}} V(\exp(\lambda \tilde{a}) \tilde{c} \exp(-\lambda \tilde{b})) = \mathbf{C}$.*

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