Lobachevskii Journal of Mathematics
http://ljm.ksu.ru
Vol. 18, 2005, 21–32

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ON INNERNESS OF DERIVATIONS ON $\mathcal{S}(\mathcal{H})$

(submitted by D. Kh. Mushtari)

Abstract. We consider general bounded derivations on the Banach algebra of Hilbert-Schmidt operators on an underlying complex infinite dimensional separable Hilbert space $\mathcal{H}$. Their structure is described by means of unique infinite matrices. Certain classes of derivations are identified together in such a way that they correspond to a unique matrix derivation. In particular, Hadamard derivations, the action of general derivations on Hilbert-Schmidt and nuclear operators and questions about innerness are considered.

1. Introduction

Throughout this article $\mathcal{H}$ will be a separable infinitely dimensional complex Hilbert space. Let $\mathcal{B}(\mathcal{H})$, $\mathcal{K}(\mathcal{H})$, $\mathcal{S}(\mathcal{H})$ and $\mathcal{N}(\mathcal{H})$ be the classes of bounded, compact, Hilbert-Schmidt and nuclear operators on $\mathcal{H}$ respectively. As it is well known, $\mathcal{K}(\mathcal{H})$ is the only non-trivial closed self-adjoint two-sided ideal in $\mathcal{B}(\mathcal{H})$ (cf. [5]). Furthermore, by the *polar decomposition theorem* any $A \in \mathcal{B}(\mathcal{H})$ can be written uniquely as $A = U \circ |A|$, where U is a *partial isometry* and $|A|$ is a *positive operator* (see [8], Vol. 2, Th. 6.1.2, p. 401). Remember that if $A \in \mathcal{K}(\mathcal{H})$ and $\{\rho_n\}$ is the sequence of eigenvalues of $|A|$ then A is said to be a Hilbert-Schmidt operator (resp. a nuclear operator) if $\sum \rho_n^2 < \infty$ (resp. if $\sum \rho_n < \infty$). Moreover, an operator A is of Hilbert-Schmidt type if

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2000 Mathematical Subject Classification. 46H05, 46J45, 47B47.
Key words and phrases. Hilbert-Schmidt and nuclear operator, Nearly-inner matrices, Hadamard products.

and only if the series $\sum \|Af_n\|^2$ converges for at least one orthonormal basis $\{f_n\}$ of $\mathcal{H}$. In that case it is readily seen that $\sum \rho_n^2 = \sum \|Ag_n\|^2$ if $\{g_n\}$ is *any* orthonormal basis of $\mathcal{H}$. So, if $\{e_n\}$ is an orthonormal basis of $\mathcal{H}$ and $A, B \in \mathcal{S}(\mathcal{H})$ then

$$\langle A, B\rangle_2 = \sum \langle Ae_n, Be_n\rangle$$

defines an inner product on $\mathcal{S}(\mathcal{H})$. If $\|A\|_2 = \langle A, A\rangle_2^{1/2}$ then $(\mathcal{S}(\mathcal{H}), \langle \circ, \circ\rangle_2)$ becomes a Hilbert space. Indeed, $(\mathcal{S}(\mathcal{H}), \|\circ\|_2)$ is a Banach $*$ - algebra without unit. Analogously, if $A \in \mathcal{N}(\mathcal{H})$ and $\{e_n\}$ is an orthonormal basis of $\mathcal{H}$ then $\sum \langle Ae_n, e_n\rangle = \sum \rho_n$, i.e. the sum of the former series does not depend on the choice of $\{e_n\}$. This value is known as the trace of A and it is denote as $\operatorname{tr}(A)$. Further, if we let $\|A\|_1 = \operatorname{tr}(|A|)$ then $(\mathcal{N}(\mathcal{H}), \|\circ\|_1)$ is a Banach algebra. If $A \in \mathcal{B}(\mathcal{H})$ then $A \in \mathcal{N}(\mathcal{H})$ if and only if $|A|^{1/2} \in \mathcal{S}(\mathcal{H})$. For more details on this matter the reader can see [7], Ch. I. §2. In this article, by a derivation on a Banach algebra $\mathfrak{U}$ we will mean any linear operator $\delta : \mathfrak{U} \to \mathfrak{U}$ so that the usual Leibnitz rule $\delta(a \cdot b) = \delta(a) \cdot b + a \cdot \delta(b)$ holds for all $a, b \in \mathfrak{U}$. In particular, given $a \in \mathfrak{U}$ the maps $[a, b] = a \cdot b - b \cdot a$ defined for all $b \in \mathfrak{U}$ are the so called *inner derivations*. We will say that a derivation is *outer* if it is not inner.

The authors previously studied the existence and structure of general derivations in the frame of weight sequence Banach algebras (cf. [1]). Our matter in this article is to consider questions concerning to innerness of bounded derivations on $\mathcal{S}(\mathcal{H})$. This problem has been solved in other contexts; for instance in the frame of von Neumann algebras every bounded derivation is inner (cf. [10], [11]). In Section 2 we consider the structure of general (bounded) derivations on $\mathcal{S}(\mathcal{H})$. We develop, in Th. 2 and Prop. 5, the intrinsic relationship between bounded derivations on $\mathcal{S}(\mathcal{H})$ and derivations on a Hilbert space of infinite complex matrices (cf. [2], [3]). The precise structure of derivations on $\mathcal{S}(\mathcal{H})$ is given in Prop. 7 and Prop. 8 allows us to define an equivalence relation on them. So, in Corollary 11 we see how each infinite matrix derivation determines a unique equivalence class of bounded derivations on $\mathcal{S}(\mathcal{H})$. In particular, from this development it follows the simple structure of the so called *Hadamard derivations*. Finally, in Section 3 we describe the action of bounded derivations on self-adjoint Hilbert-Schmidt operators. Hadamard derivations and their restrictions to nuclear operators on $\mathcal{H}$ are considered in Prop. 15. In Prop. 16 we realize certain derivations on $\mathcal{S}(\mathcal{H})$, in general not inner, as certain series of inner derivations on $\mathcal{S}(\mathcal{H})$.

Notation 1. *Throughout this article ω will be an infinite countable set and, if $\mathbb{A}$ is a Banach algebra, $\mathcal{D}(\mathbb{A})$ will denote the class of bounded derivations on $\mathbb{A}$. Let $l^2(\omega \times \omega)$ be the Hilbert space of infinite matrices $a = (a_{n,m})_{n,m\in\omega}$ endowed with the norm $\|a\|_2 = \left(\sum_{n,m\in\omega} |a_{n,m}|^2\right)^{1/2}$. We will write by means of $\overline{a}$, a^t and a^* the conjugate, the transpose and the adjoint of a respectively. If $m, n \in \omega$ it is easy to see that $a^*_{n,m} = \overline{a_{m,n}}$, as usual $a^t_{n,m} = a_{m,n}$ and $\overline{a} = a^{*t}$.*

2. On the structure of general derivations on $\mathcal{S}(\mathcal{H})$

Theorem 2. (cf. [2], [4]) *A bounded linear endomorphism Δ of $l^2(\omega \times \omega)$ is a derivation if and only if there are matrices $\alpha = \{\alpha_{n,m}\}_{n,m\in\omega}$ and $\beta = \{\beta_{n,m}\}_{n,m\in\omega}$ of complex numbers uniquely determined so that*

(i) *For any $n \in \omega$, $\alpha_{n,n} = 0$.*

(ii) $\sup\limits_{n,m\in\omega} |\beta_{n,m}| < \infty$.

(iii) *The matrix α is nearly-inner, i.e. the formal mapping $\mathcal{L}_\alpha : z \to \alpha \cdot z - z \cdot \alpha$ defines a bounded linear operator on $l^2(\omega \times \omega)$.*

(iv) *For any $n, m, p \in \mathbb{N}$ the identities $\beta_{n,m} + \beta_{m,p} = \beta_{n,p}$ hold.*

Then

$$\Delta(z) = \sum_{n,m\in\omega} \left(\sum_{p\in\omega} (z_{p,m}\ \alpha_{n,p} - \alpha_{p,m}\ z_{n,p}) + z_{n,m}\ \beta_{n,m} \right) e_{n,m} \tag{1}$$

for $z \in l^2(\omega \times \omega)$, where for all $n, m \in \omega$ we are writing $e_{n,m} = \left(\delta^{p,q}_{n,m}\right)_{p,q\in\omega}$ and $\delta^{p,q}_{n,m}$ denotes the usual Kronecker´s function. In particular, we can denote $\Delta = \Delta_{\alpha,\beta}$.

Remark 3. *Let $\Delta \in \mathcal{D}(l^2(\omega \times \omega))$. So, the corresponding entries of the matrices α and β related to Δ according to* Th.2 *are defined by the relations*

$$\Delta(e_{n,m}) = \sum_{p\in\omega} (\alpha_{p,n}\ e_{p,m} - \alpha_{m,p}\ e_{n,p}) + \beta_{n,m}\ e_{n,m},\ n, m \in \omega.$$

In particular, observe that

$$\sup_{n,m\in\omega} \sum_{p\in\omega} \left(|\alpha_{p,n}|^2 + |\alpha_{m,p}|^2\right) + |\beta_{n,m}|^2 \leq \|\Delta\|^2 < \infty,$$

i.e. all rows and columns of a nearly-inner matrix are bounded and square summable.

Proposition 4. *Let $\alpha, \alpha_1, \alpha_2$ be nearly-inner matrices with null diagonals and let β, β_1, β_2 be bounded matrices that verify condition* (iv) *of* Th.2. *So, the following formulae hold:*

(i) $[\Delta_{\alpha_1,0}, \Delta_{\alpha_2,0}] = \Delta_{\alpha(\alpha_1,\alpha_2),\beta(\alpha_1,\alpha_2)}$, *where for each* $n, m \in \omega$

$$\begin{aligned} \alpha(\alpha_1,\alpha_2)_{n,m} &= (1-\delta_{n,m}) \cdot [\alpha_1,\alpha_2]_{n,m}\,, \\ \beta(\alpha_1,\alpha_2)_{n,m} &= [\alpha_1,\alpha_2]_{n,n} + [\alpha_2,\alpha_1]_{m,m}\,. \end{aligned} \tag{2}$$

(ii) $[\Delta_{\alpha,0}, \Delta_{0,\beta}] = -\Delta_{\alpha\odot\beta,0}$, *where* $\alpha \odot \beta = \{\alpha_{n,m} \cdot \beta_{n,m}\}_{n,m\in\omega}$ *denotes the Hadamard product of the matrices* α *and* β.

(iii) $[\Delta_{0,\beta_1}, \Delta_{0,\beta_2}] = 0$.

(iv) $[\Delta_{\alpha_1,\beta_1}, \Delta_{\alpha_2,\beta_2}] = \Delta_{\alpha(\alpha_1,\alpha_2)-\alpha_1\odot\beta_2+\alpha_2\odot\beta_1,\beta(\alpha_1,\alpha_2)}$.

Proof. Given $n, m \in \omega$ we obtain that

$$\begin{aligned} &[\Delta_{\alpha_1,0}, \Delta_{\alpha_2,0}](e_{n,m}) \\ &\quad = \sum_{p\in\omega} \left(\delta_n^p \ [\alpha_1,\alpha_2]_{p,n} \cdot e_{p,m} + \delta_m^p \ [\alpha_2,\alpha_1]_{m,p} \cdot e_{n,p} \right) \\ &\qquad\qquad + \left([\alpha_1,\alpha_2]_{n,n} + [\alpha_2,\alpha_1]_{m,m} \right) \cdot e_{n,m}. \end{aligned} \tag{3}$$

By Remark 3 the infinite matrix $\left\{ [\alpha_1,\alpha_2]_{n,m} \right\}_{n,m\in\omega}$ is defined. Hence, relations (2) will be established if we show that $\alpha(\alpha_1,\alpha_2)$ and $\beta(\alpha_1,\alpha_2)$ satisfy the conditions of Th.2. For, by Remark 3 and the definitions in (2) they do for $\beta(\alpha_1,\alpha_2)$. By definition $\alpha(\alpha_1,\alpha_2)$ has null diagonal. In order to see that $\alpha(\alpha_1,\alpha_2)$ is nearly-inner let $z \in l^2(\omega\times\omega)$ be a matrix with only a finite number of non zero entries. If $n, m \in \omega$ we have

$$\begin{aligned} &(\mathcal{L}_{\alpha_1} \circ \mathcal{L}_{\alpha_2} - \mathcal{L}_{\alpha_2} \circ \mathcal{L}_{\alpha_1})(z)_{n,m} \\ &\qquad = \mathcal{L}_{\alpha(\alpha_1,\alpha_2)}(z)_{n,m} + z_{n,m} \cdot \beta(\alpha_1,\alpha_2)_{n,m}\,. \end{aligned} \tag{4}$$

In consequence, the formal operator $\mathcal{L}_{\alpha(\alpha_1,\alpha_2)}$ is defined and obviously linear on the dense subspace of matrices with finite support of $l^2(\omega\times\omega)$. Since α_1 and α_2 are nearly-inner matrices by (4) the restriction of $\mathcal{L}_{\alpha(\alpha_1,\alpha_2)}$ to this subspace is continuous. Thus by completeness it extends to a unique bounded operator on $l^2(\omega\times\omega)$, i.e. the matrix $\alpha(\alpha_1,\alpha_2)$ is nearly-inner and (i) follows. Since $\mathcal{L}_{\alpha\odot\beta} = -[\mathcal{L}_\alpha, \Delta_{0,\beta}]$ and $\mathcal{L}_\alpha, \Delta_{0,\beta} \in \mathcal{B}(l^2(\omega\times\omega))$ then $\mathcal{L}_{\alpha\odot\beta}$ is also bounded. So $\alpha\odot\beta$ is a nearly-inner matrix and as it has null diagonal (ii) holds. Now, the proofs of (iii) and (iv) are straightforward. ■

Proposition 5. *Let* $e = \{e_n\}_{n\in\omega}$, $f = \{f_m\}_{m\in\omega}$ *be orthonormal basis of* $\mathcal{H}$, *let* $U \in \mathcal{B}(\mathcal{H})$ *be the unitary operator so that* $Ue_n = f_n$ *if* $n \in \omega$ *and let* $A \in \mathcal{S}(\mathcal{H})$.

(i) If $\mathfrak{J}_{e,f}(A) = \{\langle Ae_n, f_m\rangle\}_{n,m\in\omega}$ then $\mathfrak{J}_{e,f}$ defines an isometric isomorphism of $\mathcal{S}(\mathcal{H})$ onto $l^2(\omega\times\omega)$.

(ii) $\mathfrak{J}_{e,f}(A) = \mathfrak{J}_{e,e}(U^*A) = \overline{\mathfrak{J}_{f,f}(AU^*)}$.
(iii) The map $\mathfrak{S}_{e,f}(A) = \overline{\mathfrak{J}_{e,f}(UA^*)}$ is an $*$-algebraic isometric isomorphism of $\mathcal{S}(\mathcal{H})$ onto $l^2(\omega \times \omega)$.

Proof. Since

$$\sum_{n,m\in\omega} |\langle Ae_n, f_m\rangle|^2 = \sum_{n\in\omega} \|Ae_n\|^2 = \|A\|_2^2 < \infty$$

then $\mathfrak{J}_{e,f}$ is well defined and it is clearly an $*$- isometry from $\mathcal{S}(\mathcal{H})$ into $l^2(\omega \times \omega)$. If $a = (a_{n,m})_{n,m\in\omega}$ belongs to $l^2(\omega \times \omega)$ it is easy to see that

$$h \to Ah = \sum_m f_m \sum_n a_{n,m} \langle h, e_n\rangle$$

defines a Hilbert - Schmidt operator on $\mathcal{H}$ so that $\mathfrak{J}_{e,f}(A) = a$. Hence by the open mapping theorem (i) holds. For (ii) it suffices to observe that for all $n, m \in \omega$ is

$$\langle Ae_n, f_m\rangle = \langle Ae_n, Ue_m\rangle = \langle AU^* f_n, f_m\rangle .$$

Moreover,

$$\begin{aligned} \mathfrak{S}_{e,f}(A)^*_{n,m} &= \overline{\mathfrak{S}_{e,f}(A)_{m,n}} = \langle UA^* e_m, f_n\rangle \\ &= \langle e_m, Ae_n\rangle = \overline{\langle UAe_n, f_m\rangle} = \overline{\mathfrak{J}_{e,f}(UA)_{n,m}} \\ &= \mathfrak{S}_{e,f}(A^*)_{n,m} \end{aligned}$$

It is clear that $\mathfrak{S}_{e,f}$ is linear and as $\|UA^*\|_2 = \|A\|_2$ if $A \in \mathcal{S}(\mathcal{H})$ by (i) $\mathfrak{S}_{e,f}$ becomes an isometry. On the other hand, if $a \in l^2(\omega \times \omega)$ it is easily seeing that $\mathfrak{S}^{-1}_{e,f}(a) = \mathfrak{J}^{-1}_{e,f}(\overline{a})^* \circ U$ and $\mathfrak{S}^{-1}_{e,f}$ becomes also isometric. Finally, if $A, B \in \mathcal{S}(\mathcal{H})$ and $m, n \in \omega$ we have

$$\begin{aligned} (\mathfrak{S}_{e,f}(A) \cdot \mathfrak{S}_{e,f}(B))_{n,m} &= \sum_{p\in\omega} \mathfrak{S}_{e,f}(A)_{n,p} \cdot \mathfrak{S}_{e,f}(B)_{p,m} \\ = \sum_{p\in\omega} \langle f_p, UA^* e_n\rangle \cdot \langle f_m, UB^* e_p\rangle &= \sum_{p\in\omega} \langle Ae_p, e_n\rangle \cdot \langle Be_m, e_p\rangle \\ = \left\langle \sum_{p\in\omega} \langle Be_m, e_p\rangle Ae_p, e_n \right\rangle &= \langle ABe_m, e_n\rangle = \langle U^* f_m, (AB)^* e_n\rangle \\ &= \overline{\langle U(AB)^* e_n, f_m\rangle} = \mathfrak{S}_{e,f}(AB)_{n,m} \end{aligned}$$

and (iii) follows. ■

Remark 6. *In what follows, if $a, b \in \mathcal{H}$ we will write $a^* \otimes b$ to denote the vector map*

$$(a^* \otimes b)(h) = \langle h, a\rangle \cdot b, \quad (h \in \mathcal{H}).$$

It is easy to see that

(1) $a^* \otimes b$ *is a Hilbert-Schmidt if* $a, b \in \mathcal{H}$ *since it is a finite rank operator.*
(2) $\langle Aa, b\rangle = \langle A, a^* \otimes b\rangle_2$ *if* $a, b \in \mathcal{H}$ *and* $A \in \mathcal{S}(\mathcal{H})$.
(3) $(a_1 + a_2)^* \otimes b = a_1^* \otimes b + a_2^* \otimes b$ *if* $a_1, a_2, b \in \mathcal{H}$.
(4) $a^* \otimes (b_1 + b_2) = a^* \otimes b_1 + a^* \otimes b_2$ *if* $a, b_1, b_2 \in \mathcal{H}$.
(5) $(\lambda a)^* \otimes b = \bar{\lambda}\,(a^* \otimes b) = a^* \otimes \left(\bar{\lambda} b\right)$ *if* $a, b \in \mathcal{H}$ *and* $\lambda \in \mathbb{C}$.
(6) $(a^* \otimes b)^* = b^* \otimes a$ *if* $a, b \in \mathcal{H}$.
(7) *If* $e = \{e_n\}_{n\in\omega}$ *and* $f = \{f_m\}_{m\in\omega}$ *are orthonormal basis of* $\mathcal{H}$ *the set* $\{e_n^* \otimes f_m\}_{n,m\in\omega}$ *becomes orthonormal basis of* $\mathcal{S}(\mathcal{H})$. *In fact, the class of finite rank operators is dense in* $\mathcal{S}\left(\mathcal{H}\right)$, (cf. [7], p. 36).

Proposition 7. *Let* $\delta \in \mathcal{D}\left(\mathcal{S}\left(\mathcal{H}\right)\right)$, $A \in \mathcal{S}\left(\mathcal{H}\right)$ *and let* $e = \{e_n\}_{n\in\omega}$ *be an orthonormal basis of* $\mathcal{H}$. *There exist a unique set of bounded linear forms* $\{\gamma_e^{n,m}\}_{n,m\in\omega}$ *on* $\mathcal{S}\left(\mathcal{H}\right)$ *so that* $\delta\left(A\right)$ *can be written in* $\mathcal{S}\left(\mathcal{H}\right)$ *as*

$$\delta\left(A\right) = \sum_{m,n\in\omega} \gamma_e^{n,m}\left(A\right)\ e_n^* \otimes e_m. \tag{5}$$

Further, there exist unique matrices α *and* β *as in Th. 2 so that for each* $n, m \in \omega$ *is* $\gamma_e^{n,m}(A) = \langle A, B_e^{n,m}\rangle_2$,

$$B_e^{n,m} = e_n^* \otimes \left[e(\bar{\alpha}_m) + \bar{\beta}_{m,n} \cdot e_m\right] - e(\alpha^n)^* \otimes e_m,$$

$e(\overline{\alpha}_m) = \sum_{p\in\omega} \overline{\alpha_{m,p}} \cdot e_p$ *and* $\alpha^n = \sum_{p\in\omega} \alpha_{p,n} \cdot e_p$.

Proof. Given two orthonormal basis $e = \{e_n\}_{n\in\omega}$, $f = \{f_m\}_{m\in\omega}$ of $\mathcal{H}$, by (iii) of Prop. 5 there is a 1-1 correspondence

$$\Psi_{e,f} : \mathcal{D}\left(l^2\left(\omega \times \omega\right)\right) \to \mathcal{D}(\mathcal{S}(\mathcal{H})),\ \Psi_{e,f}\left(\Delta\right) = \mathfrak{S}_{e,f}^{-1} \circ \Delta \circ \mathfrak{S}_{e,f}. \tag{6}$$

So, if $\delta \in \mathcal{D}(\mathcal{S}(\mathcal{H}))$ there are unique matrices $\alpha = \left(\alpha_{m,n}\right)_{m,n\in\omega}$, $\beta = \left(\beta_{m,n}\right)_{m,n\in\omega}$ in the conditions of Th. 2 so that $\mathfrak{S}_{e,f} \circ \delta \circ \mathfrak{S}_{e,f}^{-1} = \Delta_{\alpha,\beta}$. Now, with the above notation if $A \in \mathcal{S}\left(\mathcal{H}\right)$ then

$$\begin{aligned}\delta\left(A\right) &= \left(\mathfrak{S}_{e,f}^{-1} \circ \Delta\right)\left(\overline{\mathfrak{J}_{e,f}(UA^*)}\right) \\ &= \sum_{m,n\in\omega}\left(\sum_{p\in\omega}(\langle Ae_n, e_p\rangle\ \alpha_{m,p} - \alpha_{p,n}\ \langle Ae_p, e_m\rangle) + \langle Ae_n, e_m\rangle\ \beta_{m,n}\right) \\ &\qquad\qquad \cdot \mathfrak{S}_{e,f}^{-1}\left(e_{m,n}\right) \end{aligned} \tag{7}$$

$$= \sum_{m,n\in\omega} \left(\left\langle Ae_n, e\left(\overline{\alpha}_m\right)\right\rangle - \left\langle Ae\left(\alpha^n\right), e_m\right\rangle + \left\langle Ae_n, e_m\right\rangle \ \beta_{m,n}\right) \cdot e_n^* \otimes e_m$$

$$= \sum_{m,n\in\omega} \left\langle A, B_e^{n,m}\right\rangle_2 \cdot e_n^* \otimes e_m$$

and so (5) follows. ■

Proposition 8. *Let* $e = \{e_n\}_{n\in\omega}$, $f = \{f_m\}_{m\in\omega}$, $g = \{g_n\}_{n\in\omega}$, $h = \{h_m\}_{m\in\omega}$ *be orthonormal basis of* $\mathcal{H}$, $\Delta \in \mathcal{D}\left(l^2\left(\omega\times\omega\right)\right)$, $A \in \mathcal{S}(\mathcal{H})$. *Then*

$$\Psi_{e,f}\left(\Delta\right)\left(V^*AU\right) = V^*\Psi_{g,h}\left(\Delta\right)\left(A\right)U, \tag{8}$$

where $U, V \in \mathcal{B}(\mathcal{H})$ *are the unitary operators so that* $Ue_n = g_n$ *and* $Vf_m = h_m$ *if* $n, m \in \omega$.

Proof. Observe that both sides in (8) are defined because $\mathcal{S}(\mathcal{H})$ is a two-sided ideal of $\mathcal{B}(\mathcal{H})$ (cf. [6], §15.4.8, p. 335). The proof follows observing that

$$e_n^* \otimes f_m = V^*\left(g_n^* \otimes h_m\right)U \text{ and } \gamma_{e,f}^{n,m}\left(V^*AU\right) = \gamma_{g,h}^{n,m}(A)$$

for each $m, n \in \omega$ and all $A \in \mathcal{S}(\mathcal{H})$. ■

Notation 9. *Let* $\delta_1, \delta_2 \in \mathcal{D}\left(\mathcal{S}\left(\mathcal{H}\right)\right)$. *We will write* $\delta_1 \sim \delta_2$ *if and only if there are unitary operators* U, V *on* $\mathcal{H}$ *so that* $\delta_1\left(V^*AU\right) = V^*\delta_2\left(A\right)U$ *if* $A \in \mathcal{S}\left(\mathcal{H}\right)$. *It is readily seeing that* $\sim$ *defines an equivalence relation on* $\mathcal{D}\left(\mathcal{S}\left(\mathcal{H}\right)\right)$.

Corollary 10. *Given* $\delta_1, \delta_2 \in \mathcal{D}(\mathcal{S}(\mathcal{H}))$, $\delta_1 \sim \delta_2$ *if and only if for all orthonormal basis* $e = \{e_n\}_{n\in\omega}$, $f = \{f_m\}_{m\in\omega}$ *of* $\mathcal{H}$ *there are orthonormal basis* $g = \{g_n\}_{n\in\omega}$, $h = \{h_m\}_{m\in\omega}$ *of* $\mathcal{H}$ *so that* $\Psi_{e,f}^{-1}\left(\delta_1\right) = \Psi_{g,h}^{-1}(\delta_2)$.

Proof.

(⇒): Let U, V unitary operators on $\mathcal{H}$ so that $\delta_1\left(V^*AU\right) = V^*\delta_2\left(A\right)U$ for all $A \in \mathcal{S}\left(\mathcal{H}\right)$. Given orthonormal basis $e = \{e_n\}_{n\in\omega}$, $f = \{f_m\}_{m\in\omega}$ of $\mathcal{H}$ there exists $\Delta \in \mathcal{D}\left(l^2\left(\omega\times\omega\right)\right)$ so that $\Psi_{e,f}\left(\Delta\right) = \delta_1$. If we write $Ue_n = g_n$ and $Vf_m = h_m$ if $n, m \in \omega$ then $g = \{g_n\}_{n\in\omega}$, $h = \{h_m\}_{m\in\omega}$ are orthonormal basis of $\mathcal{H}$. So, by Prop. 8 we have $\Psi_{g,h}\left(\Delta\right) = \delta_2$ and the condition is necessary.

(⇐): Let $e = \{e_n\}_{n\in\omega}$, $g = \{g_n\}_{n\in\omega}$, $h = \{h_m\}_{m\in\omega}$ be fixed orthonormal basis of $\mathcal{H}$ so that $\Psi_{e,e}^{-1}\left(\delta_1\right)$ and $\Psi_{g,h}^{-1}(\delta_2)$ determine a same element Δ in $\mathcal{D}\left(l^2\left(\omega\times\omega\right)\right)$. Hence $\delta_1 \sim \delta_2$ since by Prop. 8 there are unitary operators $U, V \in \mathcal{B}(\mathcal{H})$ so that for all $A \in \mathcal{S}(\mathcal{H})$ is

$$\delta_1\left(V^*AU\right) = \Psi_{e,e}\left(\Delta\right)\left(V^*AU\right) = V^*\Psi_{g,h}\left(\Delta\right)\left(A\right)U = V^*\Psi_{g,h}\left(\Delta\right)\left(A\right)U.$$

■

Corollary 11. *There is a 1-1 correspondence between* $\mathcal{D}(\mathcal{S}(\mathcal{H}))/\sim$ *onto* $\mathcal{D}\left(l^2\left(\omega\times\omega\right)\right)$, *i.e.* $\mathcal{D}(\mathcal{S}(\mathcal{H}))/\sim\,\equiv\mathcal{D}\left(l^2\left(\omega\times\omega\right)\right)$.

Proof. Let us write

$$\Psi:\mathcal{D}\left(l^2\left(\omega\times\omega\right)\right)\to\mathcal{D}(\mathcal{S}(\mathcal{H}))/\sim,\ \Psi\left(\Delta\right)=\left[\Psi_{e,f}\left(\Delta\right)\right]^{\sim},$$

where $\Delta\in\mathcal{D}\left(l^2\left(\omega\times\omega\right)\right)$, e and f are orthonormal basis of $\mathcal{H}$ and $\left[\Psi_{e,f}\left(\Delta\right)\right]^{\sim}$ denotes the $\sim$ equivalence class of $\Psi_{e,f}\left(\Delta\right)$ in $\mathcal{D}(\mathcal{S}(\mathcal{H}))/\sim$. By Prop. 8 the function Ψ is well defined, i.e. $\Psi\left(\Delta\right)$ does not depend on the choice of the orthonormal basis e nor f. If $\Delta_1,\Delta_2\in\mathcal{D}\left(l^2\left(\omega\times\omega\right)\right)$ and $\Psi\left(\Delta_1\right)=\Psi\left(\Delta_2\right)$ then $\Psi_{e,e}\left(\Delta_1\right)=\Psi_{e,e}\left(\Delta_2\right)$. By Prop. 8 we have $\Delta_1=\Delta_2$, i.e.Ψ is injective. We have already seen that for any orthonormal basis e and f of $\mathcal{H}$ the map $\Psi_{e,f}$, introduced in (6), defines a bijection between $\mathcal{D}\left(l^2\left(\omega\times\omega\right)\right)$ and $\mathcal{D}(\mathcal{S}(\mathcal{H}))$. Therefore Ψ is also surjective. ■

Notation 12. *From now on we'll denote any bounded derivation* δ *on* $\mathcal{D}(\mathcal{S}(\mathcal{H}))$ *as* $\delta=\delta_{\alpha,\beta}$, *where* α,β *are the unique infinite matrices determined by* δ *as we have pointed out in* Prop. 7. *This matrices are uniquely by means of* Th. 2 *and they identify the corresponding* $\sim$ *equivalence class of* δ *as in* Corollary 11. *So,* δ *is intrinsically determined by these matrices while its coordinate representation changes according to the rules of* Prop. 4 *and* (8) *of* Prop. 8. *In particular, we shall say that any bounded derivation* $\delta_{0,\beta}$ *on* $\mathcal{S}\left(\mathcal{H}\right)$ *is a Hadamard derivation.*

Remark 13. *Any Hadamard derivation is induced by bounded sequences of complex numbers. For, if* $\left\{b_n\right\}_{n\in\omega}$ *is such a sequence and*

$$b=\left\{b_n-b_m\right\}_{n,m\in\omega},$$

then $\delta_{o,b}$ *is a Hadamard derivation. On the other hand, given* $\delta_{0,\beta}$ *we already know that* $\beta_{n,m}=\beta_{n,l}+\beta_{l,m}$ *for all* $n,l,m\in\omega$. *In consequence* β *has null diagonal and* $\beta_{n,m}=\beta_{n,1}+\beta_{1,m}=-\beta_{1,n}+\beta_{1,m}$, *i.e. the first row determines the whole* β. *Of course, although* β *defines* $\delta_{0,\beta}$ *uniquely it does not give rise to a unique bounded sequence.*

3. On certain particular derivations

Proposition 14. *Let* $\delta_{\alpha,\beta}\in\mathcal{D}\left(\mathcal{S}\left(\mathcal{H}\right)\right)$ *and* A *be a self-adjoint Hilbert-Schmidt operator on* $\mathcal{H}$. *If* $e=\left\{e_n\right\}_{n\in\omega}$ *is the orthonormal basis of* $\mathcal{H}$

induced by the sequence $\{\rho_n\}_{n\in\omega}$ *of eigenvalues of* A *then*

$$\delta_{\alpha,\beta}(A) = \sum_{n\in\omega} e_n^* \otimes (A - \rho_n \cdot Id_{\mathcal{H}})\, e(\alpha_n)$$
$$= -\sum_{n\in\omega} ((A - \rho_n \cdot Id_{\mathcal{H}})\, e(\overline{\alpha}^n))^* \otimes e_n$$

Proof. If $n \in \omega$ then

$$\delta_{\alpha,\beta}(e_n^* \otimes e_n) = \Psi_{e,e}(\Delta_{\alpha,\beta})(e_n^* \otimes e_n) \tag{9}$$
$$= \mathfrak{S}_{e,e}^{-1}(\Delta_{\alpha,\beta}(e_{n,n})) = \sum_{m\in\omega} (\alpha_{m,n}\ e_n^* \otimes e_m - \alpha_{n,m}\ e_m^* \otimes e_n)$$

Since $A = \sum_{n\in\omega} \rho_n\ e_n^* \otimes e_n$ in $\mathcal{S}(\mathcal{H})$ by (9) we obtain that

$$\delta_{\alpha,\beta}(A) = \sum_{n,m\in\omega} \alpha_{n,m} \cdot (\rho_m - \rho_n)\ e_n^* \otimes e_m. \tag{10}$$

Now from (10) our claim follows easily. ■

Proposition 15. (i) *Any Hadamard derivation on* $\mathcal{S}(\mathcal{H})$ *is the restriction of an inner one on* $\mathcal{B}(\mathcal{H})$.

(ii) *The restriction of any Hadamard derivation to* $\mathcal{N}(\mathcal{H})$ *belongs to* $\mathcal{D}(\mathcal{N}(\mathcal{H}))$.

Proof.

(i) *Let* $\delta_{0,\beta}$ *be a Hadamard derivation and let* $e = \{e_n\}_{n\in\omega}$ *be an orthonormal basis in* $\mathcal{H}$. *By* Remark (13) *we can assume that* $\beta = \{\beta_m - \beta_n\}_{m,n\in\omega}$, *where* $\{\beta_n\}_{n\in\omega}$ *is a bounded sequence in* $\mathbb{C}$. *Let us prove that if* $A \in \mathcal{S}(\mathcal{H})$ *then*

$$\delta_{0,\beta}(A) = \left[\sum_{p\in\omega} \left(\overline{\beta}_p \cdot e_p\right)^* \otimes e_p, A\right]. \tag{11}$$

For, if $n, m \in \omega$ *then*

$$\left[\sum_{p\in\omega} \left(\overline{\beta}_p \cdot e_p\right)^* \otimes e_p, e_n^* \otimes e_m\right] = (\beta_m - \beta_n) \cdot (e_n^* \otimes e_m)$$
$$= \sum_{p,q\in\omega} \gamma_e^{p,q}(e_n^* \otimes e_m)\ e_p^* \otimes e_q$$
$$= \delta_{0,\beta}(e_n^* \otimes e_m),$$

i.e. (11) holds by Prop.7. Indeed, $\delta_{0,\beta}(\circ) = [\mathfrak{h}(\beta), \circ]$, *where* $\mathfrak{h}(\beta) \in \mathcal{B}(\mathcal{H})$ *is the diagonal operator* $\mathfrak{h}(\beta) = \sum_{p\in\omega} \beta_p \cdot \left(e_p^* \otimes e_p\right)$.

(ii) *With the notation of (i), since $\mathcal{N}(\mathcal{H})$ is an ideal our claim is clear if $\mathfrak{h}(\beta) \in \mathcal{N}(\mathcal{H})$ (for instance, if $\{\beta_n\}_{n\in\omega} \in l^1(\omega)$). For the general case, let $A \in \mathcal{N}(\mathcal{H})$. Actually, let $A = U \circ |A|$ be the polar decomposition of A and $\{\rho_n\}_{n\in\omega}$ the sequence of eigenvalues of $|A|$ associated to an orthonormal basis $e = \{e_n\}_{n\in\omega}$. Since $|A| = \sum_{n\in\omega} \rho_n \cdot (e_n^* \otimes e_n)$ in $\mathcal{S}(\mathcal{H})$, we get*

$$\delta_{0,\beta}(|A|) = \sum_{n\in\omega} \rho_n \cdot \delta_{0,\beta}(e_n^* \otimes e_n) = 0$$

as follows by Prop. 14. *Consequently we see that*

$$\begin{aligned}\delta_{0,\beta}(A) &= [\mathfrak{h}(\beta), U \circ |A|] \\ &= [\mathfrak{h}(\beta), U] \circ |A| + U \circ [\mathfrak{h}(\beta), |A|] = [\mathfrak{h}(\beta), U] \circ |A|. \quad (12)\end{aligned}$$

But for all $S, T \in \mathcal{B}(\mathcal{H})$ we have $\|SAT\|_1 \leq \|S\| \, \|A\|_1 \, \|T\|$ (cf. [9], §3.34 (xi), p. 133). *Accordingly we obtain*

$$\|\delta_{0,\beta}(A)\|_1 \leq \|[\mathfrak{h}(\beta), U]\| \cdot \|A\|_1 \leq 2\,\|\mathfrak{h}(\beta)\| \cdot \|A\|_1$$

and $\|\mathfrak{h}(\beta)\| = \sup_{n\in\omega} |\beta_n| < \infty$, i.e. $\delta_{0,\beta}\,|_{\mathcal{N}(\mathcal{H})} \in \mathcal{D}(\mathcal{N}(\mathcal{H}))$.

■

Proposition 16. *Let α be a nearly-inner matrix with null diagonal. Given an orthonormal basis $e = \{e_n\}_{n\in\omega}$ of $\mathcal{H}$ and $A \in \mathcal{S}(\mathcal{H})$ then*

$$\delta_{\alpha,0}(A) = \sum_{p\in\omega} \left[e_p^* \otimes e(\alpha^p), A\right]. \quad (13)$$

Proof. By Prop. 7, (7) we have

$$\delta_{\alpha,0}(A) = \sum_{n,m\in\omega} \sum_{p\in\omega} (\langle Ae_n, e_p\rangle \ \alpha_{m,p} - \alpha_{p,n} \ \langle Ae_p, e_m\rangle) \cdot e_n^* \otimes e_m. \quad (14)$$

Hence, if A belongs to the finite linear span of $\{e_n^* \otimes e_m\}_{n,m\in\omega}$ by (14) we obtain that

$$\delta_{\alpha,0}(A) = \sum_{p\in\omega} \left\{(A^* e_p)^* \otimes e(\alpha^p) - e_p^* \otimes Ae(\alpha^p)\right\}. \quad (15)$$

Since $\{e_n^* \otimes e_m\}_{n,m\in\omega}$ is a basis of the complete space $\mathcal{S}(\mathcal{H})$ and $\delta_{\alpha,0} \in \mathcal{B}(\mathcal{S}(\mathcal{H}))$ then (15) holds for all $A \in \mathcal{S}(\mathcal{H})$. Now, it is straightforward to see that each summand in (15) defines a bounded derivation on $\mathcal{S}(\mathcal{H})$. Indeed, those derivations are all inner, if $p \in \omega$ is

$$(A^* e_p)^* \otimes e(\alpha^p) - e_p^* \otimes Ae(\alpha^p) = \left[e_p^* \otimes e(\alpha^p), A\right]$$

and we get (13). ■

Example 17. *Let* $\mathcal{H} = l^2(\mathbb{N})$ *and let* α *be the nearly-inner matrix so that* $\alpha_{n,m}$ *is* 0 *or* 1*, according as* $m \neq n+1$ *or* $m = n+1$ *respectively. On identifying* e_0^* *with the zero form on* $\mathcal{H}$ *then* α *induces the bounded derivation*

$$\delta_{\alpha,0}(A) = \sum_{n\geq 1, m\geq 1} \left\langle A, e_n^* \otimes e_{m+1} - e_{n-1}^* \otimes e_m \right\rangle_2 \cdot e_n^* \otimes e_m$$

defined for $A \in \mathcal{S}(\mathcal{H})$*. Let us assume that* $\delta_{\alpha,0}$ *is inner, say* $\delta_{\alpha,0}(\circ) = [C, \circ]$ *for same* $C \in \mathcal{S}(\mathcal{H})$*. So, if* $r, s \in \mathbb{N}$ *we get*

$$[C, e_r^* \otimes e_s] = e_r^* \otimes Ce_s - (C^* e_r)^* \otimes e_s \tag{16}$$

and

$$\delta_{\alpha,0}(e_r^* \otimes e_s) = \begin{cases} -e_{r+1}^* \otimes e_1 & if \quad s = 1, \\ e_r^* \otimes e_{s-1} - e_{r+1}^* \otimes e_s & if \quad s > 1. \end{cases} \tag{17}$$

From (16) and (17), if $r \geq 1$ *and* $s > 1$ *we deduce that*

$$[C, e_r^* \otimes e_s](e_r) = Ce_s - \langle Ce_r, e_r \rangle \cdot e_s = e_{s-1}.$$

Hence if $r \neq s$ *is* $\langle Ce_s, e_r \rangle = \delta_r^{s-1}$*. Thus* $\{\langle Ce_s, e_r \rangle\}_{s,r\in\mathbb{N}}$ *becomes not square summable, which contradicts* Prop. 5*. So* $\delta_{\alpha,0}$ *is not inner, although it is realized as a generalized series of inner derivations as stated in* Prop. 16*.*

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Received June 1, 2005; revised version August 8, 2005