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L^∞-ERROR ESTIMATE FOR A DISCRETE TWO-SIDED OBSTACLE PROBLEM AND MULTILEVEL PROJECTIVE ALGORITHM

(submitted by A. V. Lapin)

Abstract. We are interested in the approximation in the L^∞-norm of variational inequalities with two-sided obstacle. We show that the order of convergence will be the same as that of variational inequalities with one obstacle. We also give multilevel projective algorithm and discuss its convergence.

1. Introduction

Let K be a closed convex set in $H_0^1(\Omega)$ defined by

$$K = \{v \in H_0^1(\Omega) : \Phi \leq v \leq \Psi \ in \ \Omega\},$$

where $\Omega \subset R^2$ is a bounded convex polygon, $\Phi, \Psi \in W^{2,s}(\Omega)(s > 2)$ are two given functions that satisfy $\Phi|_{\bar{\Omega}} < \Psi|_{\bar{\Omega}}$ and $\Phi|_{\partial\Omega} < 0 < \Psi|_{\partial\Omega}$. We consider the following two-sided obstacle problem: find $u \in K$ such that

$$a(u, v - u) \geq f(v - u), \quad \forall v \in K, \tag{1}$$

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where $a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx$, $f(\cdot) = (f, \cdot)$, $f \in L^s(\Omega)$ and $(\cdot,\cdot)$ is the L^2 inner product.

Theorem 1. *Problem* (1) *has a unique solution* $u \in H_0^1(\Omega) \cap W^{2,s}(\Omega)$.

The proof is similar to variational inequalities with one obstacle(see [6]).

Remark 1. *u, Φ, Ψ are Hölder continuous functions by* $W^{2,s}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega)$ *with* $0 < \alpha < 1$.

A lot of results on L^∞-error estimate for the finite element approximation of obstacle problems have been obtained(see [1, 4, 5, 7, 8, 10]). In this paper, we will discuss L^∞-error estimate for the finite element approximation of problem (1) by using the results of variational inequalities with one obstacle. Based on results in [13] we establish a rate of convergence $h^2|logh|$, provided $\Phi, \Psi \in W^{2,\infty}(\Omega)$ and $f \in L^\infty(\Omega)$. This result has be established in a different way (see [4]). At last, we will present a multilevel projective algorithm (see[14]) and discuss its convergence.

2. Preliminaries

In this section, we will construct two variational inequalities with lower and upper obstacles respectively which have the same solution as problem (1). Firstly, we make some preparations.

Set three sets as follows:

$$W_1 = \{x \in \Omega : u(x) = \Phi(x)\},$$

$$W_2 = \{x \in \Omega : u(x) = \Psi(x)\}$$

and

$$W_3 = \Omega - (W_1 \cup W_2).$$

It's easy to see that W_1, W_2 is two disjoint closed sets and then write

$$r = dist(W_1, W_2) > 0. \tag{2}$$

Lemma 1. *There exist three open sets $\tilde{W}_i, i = 1,2,3$, which have the following properties:*

$$W_1 \subset \tilde{W}_1, \quad W_2 \subset \tilde{W}_2, \quad \tilde{W}_3 \subset W_3,$$

$$\tilde{W}_1 \bigcap \tilde{W}_2 = \emptyset$$

and

$$\bigcup_{i=1}^{3} \tilde{W}_i = \Omega.$$

Moreover, there is a partition of unity $\{\theta_i\}_{i=1}^3$ *satisfying* $0 \le \theta_i \in C^\infty(\bar\Omega)$, $\theta_i = 0$ *on* $\Omega\backslash\tilde W_i$ *and* $\sum_{i=1}^3 \theta_i = 1$.

Proof. Let $\Gamma_H = \{\Omega_i'\}_{i=1}^k$ be a set of coarse mesh of Ω with mesh size $H < r/2 - 4\delta$. Here r is defined in (2), $\delta << r$ is a positive number, and $\Omega_i', i = 1, 2, \ldots, k$, are disjoint open sets satisfy $\cup_{i=1}^k \bar\Omega_i' = \bar\Omega$. Then we refine Γ_H to get Γ_h, a set of fine mesh with mesh size $h < \delta/2$. Let $\Omega_i, i = 1, 2, \cdots, k$, be the enlarged subdomains of Ω_i' defined by

$$\Omega_i = \Omega_i' \bigcup \{\tau \bigcup ((\bar\Omega_i' \bigcap \bar\tau)\backslash\partial\Omega) | \tau \in \Gamma_h \text{ and } dist(\tau, \Omega_i') < \delta/2\}.$$

The union of Ω_i cover Ω with overlap size 2δ and $diam(\Omega_i) < r/2$. Following [2], let $\{\tilde\theta_i\}_{i=1}^k$ be a partition of unity satisfying $\tilde\theta_i \in C^\infty(\bar\Omega)$, $\tilde\theta_i \ge 0$, $\tilde\theta_i = 0$ in $\Omega\backslash\Omega_i$ and $\sum_{i=1}^k \tilde\theta_i = 1$. Classify $\{\Omega_i\}_{i=1}^k$ into three groups: for $i = 1, 2$, $\tilde W_i = \bigcup_{\bar\Omega_j \bigcap W_i \neq \emptyset} \{\Omega_j\}$, $\tilde W_3 = \bigcup_{\bar\Omega_j \bigcap (W_1 \bigcup W_2) = \emptyset} \{\Omega_j\}$. Let $\theta_i = \sum_{\Omega_j \subset \tilde W_i} \tilde\theta_j (i = 1, 2, 3)$, the lemma can be easily verified. □

Lemma 2. *Let* u *be the solution of* (1), *then*

$$-\triangle u = f \quad a.e. \quad in\ \tilde W_3, \tag{3}$$

where $\tilde W_3$ *are defined as in Lemma* 1.

Proof. From the definition of $\tilde W_3$ we known that $\Phi < u < \Psi$ in $\tilde W_3$. Let $D(\tilde W_3)$ denote the set of infinitely differentiable functions with compact support $\subset\subset \tilde W_3$. For any $\tilde v \in D(\tilde W_3)$, extend it by zero to $\Omega\backslash\tilde W_3$, we can take positive q sufficiently small such that $v = u \pm q\tilde v \in K$. From (1) we have

$$a(u, \tilde v) = (f, \tilde v).$$

What's more,

$$a(u, \tilde v) = \int_\Omega \nabla u \nabla \tilde v dx = -\int_\Omega u \triangle \tilde v dx = -\int_\Omega \triangle u \tilde v dx,$$

where the last equal follows from the definition of weak derivative. Therefore

$$\int_\Omega (-\triangle u - f)\tilde v dx = \int_{\tilde W_3} (-\triangle u - f)\tilde v dx = 0.$$

By the arbitrary of $\tilde v$, we have $-\triangle u = f$ *in* $\tilde W_3$. □

Remark 2. *Use the same technique in Lemma* 2, *we can also get* $-\triangle u \ge f$ *in* $\tilde W_1$ *and* $-\triangle u \le f$ *in* $\tilde W_2$.

Let $K_1 = \{v \in H_0^1(\Omega) \cap H^2(\Omega) : v \ge \Phi\}$, $K_2 = \{v \in H_0^1(\Omega) \cap H^2(\Omega) : v \le \Psi\}$.

Remark 3. *$v \in K_1$(or K_2) is Hölder continuous function by embedding theorem, therefore v is bounded.*

Lemma 3. *$a(u, w) \geq f(w)$ if $w \in H_0^1(\tilde{W}_1)$ and $u + w \in K_1$, where u is the solution of* (1)*, and w is extended by zero to $\Omega \backslash \tilde{W}_1$.*

Proof. For $0 \leq q \leq 1$, we consider the two cases: if $w(x) \geq 0$, then $(u+qw)(x) \geq u(x) \geq \Phi(x)$; if $w(x) < 0$, then $(u+qw)(x) \geq (u+w)(x) \geq \Phi(x)$. So we have $u + qw \geq \Phi$ for $0 \leq q \leq 1$. Furthermore, from the definition of $\tilde{W}_1$, we know that $dist(\tilde{W}_1, W_2) \geq r/2$ and it is easy to verify that $u < \Psi$ on $\bar{\tilde{W}}_1$. Since w is bounded by Remark 3, there exists a sufficiently small positive $q \leq 1$ such that $\Psi \geq u + qw \in K$. Take $v = u + qw$ in (1), then we have $a(u, w) \geq f(w)$. □

We now construct two variational inequalities with one obstacle. Define

$$f^{(1)} = \begin{cases} -\triangle u & x \in \tilde{W}_2, \\ f & otherwise \end{cases} \tag{4}$$

and

$$f^{(2)} = \begin{cases} -\triangle u & x \in \tilde{W}_1, \\ f & otherwise, \end{cases} \tag{5}$$

where $u \in W^{2,s}(\Omega)$ is the solution of problem (1). It is obvious that $f^{(1)}$ and $f^{(2)} \in L^s(\Omega)$. From Lemma 2 and Remark 2, we have

$$f^{(1)} \leq f \leq f^{(2)} \quad a.e. \ in \ \Omega. \tag{6}$$

Problem I: Find $u_1 \in K_1$ such that

$$a(u_1, v - u_1) \geq f^{(1)}(v - u_1), \quad \forall v \in K_1. \tag{7}$$

Problem II: Find $u_2 \in K_2$ such that

$$a(u_2, v - u_2) \geq f^{(2)}(v - u_2), \quad \forall v \in K_2. \tag{8}$$

Remark 4. *Problems* (7), (8) *have unique solutions in $W^{2,s}(\Omega)$ (see* [6]*).*

Lemma 4. *Problems* (1), (7) *and* (8) *have the same solution.*

Proof. For simplicity, we only prove problems (1) and (7) have the same solution. Let u be solution of (1). For any $v \in K_1$, $v_i = \theta_i(v - u) \in H_0^1(\Omega) \bigcap H^2(\Omega)$, $i = 1, 2, 3$, have supports $\subset \tilde{W}_i$ respectively and the restriction of v_i are in $H_0^1(\tilde{W}_i) \cap H^2(\tilde{W}_i)$. Here θ_i, $i = 1, 2, 3$, are defined as in Lemma 1. Obviously, $v - u = \sum_{i=1}^{3} v_i$ and

$$a(u, v - u) = a(u, v_1) + a(u, v_2) + a(u, v_3). \tag{9}$$

It's easy verify that $u + v_1 \in K_1$. So by Lemma 3 and (4) we have $a(u, v_1) \geq f(v_1) = \int_\Omega f v_1 dx = \int_{\tilde{W}_1} f v_1 dx = \int_{\tilde{W}_1} f^{(1)} v_1 dx = \int_\Omega f^{(1)} v_1 dx = f^{(1)}(v_1)$. By Lemma 2 and (4), we can directly verify that

$$\begin{aligned} a(u, v_3) = (-\triangle u, v_3) = \int_\Omega -\triangle u v_3 dx &= \int_{\tilde{W}_3} -\triangle u v_3 dx \\ &= \int_{\tilde{W}_3} f^{(1)} v_3 dx = \int_\Omega f^{(1)} v_3 dx = f^{(1)}(v_3). \end{aligned}$$

Similarly, $a(u, v_2) = f^{(1)}(v_2)$. So we have

$$a(u, v - u) \geq f^{(1)}(v_1 + v_2 + v_3) = f^{(1)}(v - u).$$

Thereby, u is a solution of problem (7). By the uniqueness of the solution of problem (1) and problem (7), the lemma is proved. $\square$

3. Main Results

Let a triangulation T_h be defined over Ω, satisfying the shape regularity and maximum angle conditions(see [1]). Let $\tilde{V}_h = \tilde{V}_h(T_h)$ denote the space of continuous piecewise linear functions over T_h. Take $V_h = \tilde{V}_h \cap H_0^1(\Omega)$. For $v \in C^0(\bar{\Omega})$, let $\pi_h(v) \in \tilde{V}_h$ be nodal interpolation of v, that is, $v = \pi_h(v)$ holds at each vertex. Let $\Phi_h = \pi_h(\Phi), \Psi_h = \pi_h(\Psi)$. Take

$$K_h = \{v \in V_h : \Phi_h \leq v \leq \Psi_h\},$$

$$K_h^{(1)} = \{v \in V_h : v \geq \Phi_h\}$$

and

$$K_h^{(2)} = \{v \in V_h : v \leq \Psi_h\}.$$

The correspondent discrete problem of (1) is: find $u_h \in K_h$ such that

$$a(u_h, v_h - u_h) \geq f(v_h - u_h), \quad \forall v_h \in K_h. \tag{10}$$

And the correspondent discrete problems of (7) and (8) are: find $u_h^{(i)} \in K_h^{(i)}, i = 1, 2$, such that

$$a(u_h^{(i)}, v_h - u_h^{(i)}) \geq f^{(i)}(v_h - u_h^{(i)}), \quad \forall v_h \in K_h^{(i)}, \tag{11}$$

respectively. We assume, for $i = 1, 2$, that

$$||u - u_h^{(i)}||_\infty \leq C h^\mu |logh|^\gamma, \quad 0 \leq \mu, \gamma \leq 2. \tag{12}$$

We will use the assumption (12) to estimate the error bound of $||u - u_h||_\infty$.

Denote $\{x_i : i = 1, 2, \ldots, m_h\}$ the interior node set and $\{x_i : i = m_h + 1, \ldots, n_h\}$ the boundary node set of T_h. Let $\{\varphi_h^{(i)}\}_{i=1}^{n_h}$ be the nodal basis for $\tilde{V}_h$ with $\varphi_h^{(i)}(x_j) = \delta_{ij}$. P_h be a canonical function from $\tilde{V}_h$ to R^{m_h}. Namely, for any $v_h = \sum_{i=1}^{n_h} y_i \varphi_h^{(i)}$, let $y = P_h v_h$ with $y^T =$

$(y_1, \ldots, y_{m_h})$. Let $A^h = (a_{ij})_{m_h \times m_h}$ be the stiffness matrix given by $a_{ij} = \int_\Omega \nabla\varphi_h^{(j)} \nabla\varphi_h^{(i)} dx$. To obtain our main results, we need the following lemmas.

Lemma 5. (*see* [9, 11]) *If the triangulation T_h satisfies maximal angle condition, A^h is M-matrix.*

Lemma 6. *If the triangulation T_h satisfies maximal angle condition,*

$$A^h e \geq 0,$$

where $e = (1, 1, \ldots, 1)^T \in R^{m_h}$.

Proof. By maximal angle condition, $a(\varphi_i, \varphi_j) \leq 0$ for $i \neq j$(see [3, 13]). So we have for $i = 1, 2, \ldots, m_h$,

$$\begin{aligned}(A^h e)_i = \sum_{j=1}^{m_h} a_{ij} = \sum_{j=1}^{m_h} a(\varphi_j, \varphi_i) \\ \geq \sum_{j=1}^{m_h} a(\varphi_j, \varphi_i) + \sum_{j=m_h+1}^{n_h} a(\varphi_j, \varphi_i) = a(1, \varphi_i) = 0.\end{aligned}$$

The lemma is followed. $\square$

Lemma 7. [12] *Let sets $I \subset \{1, 2, \ldots, m_h\}$, $J = \{1, 2, \ldots, m_h\} \backslash I$. For any $y, \bar{y} \in R^{m_h}$, if $y_I \leq \bar{y}_I$ and $(A^h y)_J \leq (A^h \bar{y})_J$, we have that*

$$y \leq \bar{y},$$

where A^h is the stiffness matrix given before.

Take $z = P_h u_h$, $z^{(i)} = P_h u_h^{(i)} (i = 1, 2)$, $\phi = P_h \Phi_h$ and $\psi = P_h \Psi_h$. Discrete problems (10) and (11) are equivalent to the following three algebraic problems respectively:

$$\begin{cases} (A^h z)_i = g_i & if \;\; \phi_i < z_i < \psi_i, \\ (A^h z)_i \geq g_i & if \;\; z_i = \phi_i, \\ (A^h z)_i \leq g_i & if \;\; z_i = \psi_i, \\ \phi \leq z \leq \psi; \end{cases} \tag{13}$$

$$\begin{cases} (A^h z^{(1)})_i = g_i^{(1)} & if \;\; z_i^{(1)} > \phi_i, \\ (A^h z^{(1)})_i \geq g_i^{(1)} & if \;\; z_i^{(1)} = \phi_i, \\ z^{(1)} \geq \phi; \end{cases} \tag{14}$$

$$\begin{cases} (A^h z^{(2)})_i = g_i^{(2)} & if \;\; z_i^{(2)} < \psi_i, \\ (A^h z^{(2)})_i \leq g_i^{(2)} & if \;\; z_i^{(2)} = \psi_i, \\ z^{(2)} \leq \psi, \end{cases} \tag{15}$$

where

$$g = \Big((f, \varphi_h^{(1)}), \ldots, (f, \varphi_h^{(m_h)})\Big)^T$$

and

$$g^{(i)} = \Big((f^{(i)}, \varphi_h^{(1)}), \ldots, (f^{(i)}, \varphi_h^{(m_h)})\Big)^T, \ i = 1, 2.$$

By (6) we know

$$g^{(1)} \le g \le g^{(2)}. \tag{16}$$

Lemma 8. *If the assumption* (12) *holds, we have*

$$z^{(1)} - Ch^\mu |logh|^\gamma e \le z, \tag{17}$$

where $e = (1, 1, \ldots, 1)^T \in R^{m_h}$.

Proof. By the assumption (12), we have

$$u_h^{(1)} - Ch^\mu |logh|^\gamma \le u \le \Psi,$$

and therefore

$$z^{(1)} - Ch^\mu |logh|^\gamma e \le \psi.$$

Define I, J by

$$I = \{i \mid z_i^{(1)} = \phi_i\} \cup \{i \mid z_i = \psi_i\},$$
$$J = \{1, 2, \ldots, m_h\} \backslash I,$$

respectively. It's easy to verify that

$$z_j^{(1)} - Ch^\mu |logh|^\gamma \le z_j, \quad j \in I. \tag{18}$$

From (13), (14), (16) and Lemma 6, we have that

$$(A^h z)_J \ge g_J \tag{19}$$

and

$$A(z^{(1)} - Ch^\mu |logh|^\gamma e)_J \le (Az^{(1)})_J = g_J^{(1)} \le g_J. \tag{20}$$

The lemma follows from (18), (19), (20) and Lemma 7. □

Lemma 9. *If the assumption* (12) *holds, then*

$$z^{(2)} + Ch^\mu |logh|^\gamma e \ge z.$$

The proof is similar to that of Lemma 8, we omit it here.

By Lemmas 8 and 9, we have

$$u_h^{(1)} - Ch^\mu |logh|^\gamma \le u_h \le u_h^{(2)} + Ch^\mu |logh|^\gamma.$$

Then, the following theorem becomes obvious.

Theorem 2. *If the assumption* (12) *holds, we have*

$$||u - u_h||_\infty \le Ch^\mu |logh|^\gamma.$$

When $s = \infty$, [13] has obtained the estimate

$$||u - u_h^{(i)}||_\infty \le Ch^2|logh|, \quad for\ i = 1, 2.$$

Therefore by Theorem 2, we have the following theorem.

Theorem 3. *If $s = \infty$ in problem* (1)*, we have*

$$||u - u_h||_\infty \le Ch^2|logh|.$$

4. Multilevel Projective Algorithm

In the sequel, we assume $s = \infty$ in problem (1). We consider a sequence of regular triangulations T_{h_k} of the polygonal domain Ω determined as follows. Suppose T_{h_1} is given and let T_{h_k}, $k \ge 2$, be obtained from $T_{h_{k-1}}$ via a systematic subdivision. Edge midpoints in $T_{h_{k-1}}$ are connected by new edges to form T_{h_k}. Let h_k be the mesh size of T_{h_k} and satisfy

$$h_k = 2^{-(k-1)}h_1, \quad k = 1, 2, \dots.$$

Let V_{h_k} denote the space of continuous piecewise linear functions with respect to T_{h_k} that vanish on $\partial\Omega$. Note that

$$T_{h_{k-1}} \subset T_{h_k} \Rightarrow V_{h_{k-1}} \subset V_{h_k}.$$

Similarly, we use m_{h_k} denote the number of the interior nodes of T_{h_k}. Next we will discuss how to solve the discrete problem (13) on the multilevel mesh.

Let $K^{(0)}$ be a vector space associated with K_h defined as follows:

$$K^{(0)} = \{y \in R^{m_h} | \phi \le y \le \psi\}.$$

Let $P_{K^{(0)}}$ be the projective operator from R^{m_h} into $K^{(0)}$. Therefore, for any $y \in R^{m_h}$,

$$(y - P_{K^{(0)}}y, \bar{y} - P_{K^{(0)}}y)_E \le 0, \quad \text{for all } \bar{y} \in K^{(0)},$$

where $(\cdot,\cdot)_E$ is Euclidean inner product. Now we can define an operator $PG^h : R^{m_h} \mapsto R^{m_h}$ by

$$PG^h(y) = P_{K^{(0)}}(y - \rho_h(A^h y - g)),$$

where

$$\rho_h = \frac{1}{\lambda_{max}(A^h)}.$$

Here, $\lambda_{max}(A^h)$ and $\lambda_{min}(A^h)$ are the biggest eigenvalue and the smallest eigenvalue of A^h respectively. Notice that $\lambda_{max}(A^h) = O(1)$, $\lambda_{min}(A^h) = O(h^2)$, and the condition number of A^h, $\Lambda(A^h)$, satisfies

$$\Lambda(A^h) = \frac{\lambda_{max}}{\lambda_{min}} = O(\frac{1}{h^2}) = O(m_h).$$

Denote $||\cdot||$ the Euclidean norm of R^{m_h}, then we have the following theorem.

Theorem 4. *$z \in K^{(0)}$ solves the discrete problem* (13)*, then for any $y \in R^{m_h}$,*

$$||PG^h(y) - z|| \le (1 - \frac{1}{\Lambda(A^h)})||y - z||,$$

$$||(PG^h)^l(y) - z|| \le (1 - \frac{1}{\Lambda(A^h)})^l||y - z||,$$

here l is a positive integer.

The proof of the theorem is similar to that in [14].

Now we define the intergrid transfer operators. Let $I_{h_{k-1}}^{h_k} : R^{m_{h_{k-1}}} \mapsto R^{m_{h_k}}$ defined by $I_{h_{k-1}}^{h_k}(y) = P_{h_k}(\sum_{i=1}^{m_{h_{k-1}}} y_i\varphi_{h_{k-1}}^{(i)})$ for $y \in R^{m_{h_{k-1}}}$. In the following, we will give the multilevel projective algorithm.

Algorithm MP:

Step 1: Solve the exact solution $\tilde{z}_{h_1}$ of problem (13) on the coarsest mesh;

Step 2: For $k \ge 2$, the approximate solution on the kth level is gotten by

$$\tilde{z}_{h_k} = (PG^{h_k})^{\tau_k} I_{h_{k-1}}^{h_k}(\tilde{z}_{h_{k-1}}),$$

where the positive integer τ_k is chosen such that

$$(1 - \frac{1}{\Lambda(A^{h_k})})^{\tau_k} \le \frac{1}{5 + 2\sqrt{1 + m/2}}.$$

Here m is the maximal number of triangles sharing a common vertex.

Now we give the convergence results of algorithm MP.

Theorem 5. *If $\tilde{z}_{h_k}$, $(k = 1, \ldots)$, is generated by algorithm MP, we have*

$$||z_{h_k} - \tilde{z}_{h_k}|| \le Ch|logh|,$$

where $z_{h_k} \in K^{h_k}$ is the exact solution of problem (13) *on k-th level.*

Theorem 6. *If $\tilde{z}_{h_k}$, $(k = 1, \ldots)$, is generated by algorithm MP, we have*

$$||u_{h_k} - Q_{h_k}(\tilde{z}_{h_k})||_{L^2(\Omega)} \le Ch^2|logh|,$$

where $u_{h_k} \in V_{h_k}$ *is the exact solution of problem* (10) *on k-th level,* $Q_{h_k} : R^{m_{h_k}} \mapsto V_{h_k}$, *is defined by*

$$Q_{h_k}(y) = \sum_{i=1}^{m_{h_k}} y_i \varphi_{h_k}^{(i)}, \quad for\ any\ y \in R^{m_{h_k}}.$$

The proofs can be seen in [14]. We omit them here.

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