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QUANTIZATIONS OF BRAIDED DERIVATIONS.
1. MONOIDAL CATEGORIES

(submitted by V. V. Lychagin)

Abstract. For monoidal categories we describe braidings and quantizations. We use them to find quantizations of braided symmetric algebras and modules, braided derivations, braided connections, curvatures and differential operators.

1 Introduction

We consider quantizations q, braidings σ and quantizations of braidings σ_q of monoidal categories. We mainly work with braidings that are symmetries.

We consider σ-symmetric algebras A, modules, co- and bialgebras and internal homomorphisms and find quantizations of these.

Internal homomorphisms of σ-symmetric modules has a braided Lie structure with respect to the braided commutator. Quantizations of the internal homomorphisms has the quantized braided Lie structure and can be realized within the original braided Lie structure by what we call dequantization.

We investigate braided derivations of σ-symmetric algebras and modules. The σ-bracket of two braided derivations is a braided derivation. We show that there is a braided Lie structure on the braided derivations.

A quantizations the braided derivations provides an isomorphism of the modules of braided derivations and quantized braided derivations. We also show that the quantizations of braided derivations has the braided Lie structure with respect to the quantizations of the braiding which can be realized within the original braided Lie structure by dequantization.

We define braided connections in modules and braided curvatures. We prove that the braided curvature is A-linear, skew σ-symmetric and is an A-module homomorphism.

We find quantizations of braided connections and braided curvatures. The quantization of the braided curvature is A-linear, skew σ_q-symmetric and an A-module homomorphism with respect to the quantized braiding.

Finally we consider braided differential operators. We show that there is a braided Lie structure on the braided differential operators. A quantization the braided differential operators provides an isomorphism of the original braided differential operators and quantized ones. The quantization of the braided Lie structure can be realized within the original one by dequantization.

This paper is the first in a trilogy.

We have found explicit descriptions of all quantizations and braidings in the monoidal category of modules graded by a finite commutative monoid, [7]. We have proved the same as for any monoidal category for this category, but the picture is somewhat more visible in this case. That is, we have a complete description for braided derivations of graded algebras and graded modules, braided connections, braided curvature, quantizations and so on. This is to be found in the second paper *Quantizations of braided derivations. 2. Graded modules, [8].*

In [7] we showed that the Fourier transform establishes an isomorphism between the categories of $\hat{G}$-graded modules and G-modules where G is a finite abelian group and $\hat{G}$ is the dual of G. Using this we find a description of all quantizations and braiding also for the monoidal category of modules with action by a finite abelian group G. Again, we have a complete and explicit description for σ-derivations of algebras and modules, braided connections, curvature, differential operators and quantizations of these structures. This is to be found in the third paper *Quantizations of braided derivations. 3. Modules with action by a group, [9].*

There are many interesting applications of these results. One of the more interesting applications is quantizations of braided Lie algebras. In the paper [10], which is to be published, we show quantizations of semisimple Lie algebras by quantizations of derivations, for example an alternative quantization of $\mathfrak{sl}_2(\mathbb{C})$.

Note that in all three papers we assume that the associativity constraint is trivial.

2 Quantizations and σ-commutativity

In this section we shall recall some results needed later. We have to define quantizations and σ-commutativity of algebras, modules, co- and bialgebras and internal homomorphisms. Most of this was done by V. V. Lychagin and many results are found in [17].

2.1 Quantizations

A quantization [17] of a monoidal category C is a natural isomorphism of the tensor bifunctor

$$\begin{aligned} q &: \otimes \to \otimes, \\ q_{X,Y} &: X \otimes Y \to X \otimes Y, \end{aligned}$$

$X, Y \in Ob(C)$, which preserves the unit and associativity so that the following diagram

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "\alpha"] \arrow[d, "1 \otimes q_{Y,Z}"'] & (X \otimes Y) \otimes Z \arrow[d, "q_{X,Y} \otimes 1"'] \\
X \otimes (Y \otimes Z) \arrow[d, "q_{X,Y \otimes Z}"'] & (X \otimes Y) \otimes Z \arrow[d, "q_{X \otimes Y,Z}"'] \\
X \otimes (Y \otimes Z) \arrow[r, "\alpha"] & (X \otimes Y) \otimes Z
\end{tikzcd} \tag{1}$$

commutes for all $X, Y, Z \in Ob\,(C)$. We call this the coherence condition for quantizations.

The composition of two quantizations q_1 and q_2 is a quantization and the inverse q^{-1} of a quantization q is a quantization.

2.2 Braidings

A *braiding* [18] of a monoidal category C is a natural isomorphism

$$\begin{array}{rcl} \sigma & : & \otimes \to \otimes \circ \tau \\ \sigma & = & \sigma_{X,Y} : X \otimes Y \to Y \otimes X, \end{array}$$

$X, Y \in Ob\,(C)$, which preserves the unit and associativity such that the following diagrams

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "\alpha"] \arrow[d, "1 \otimes \sigma"'] & (X \otimes Y) \otimes Z \arrow[d, "\sigma"] & X \otimes (Y \otimes Z) \arrow[r, "\alpha"] \arrow[d, "\sigma"'] & (X \otimes Y) \otimes Z \arrow[d, "\sigma \otimes 1"] \\
X \otimes (Z \otimes Y) \arrow[d, "\alpha"'] & Z \otimes (X \otimes Y), & (Y \otimes Z) \otimes X \arrow[d, "\alpha^{(-1)}"'] & (Y \otimes X) \otimes Z \arrow[d, "\alpha^{(-1)}"] \\
(X \otimes Z) \otimes Y \arrow[r, "\sigma \otimes 1"] & (Z \otimes X) \otimes Y \arrow[from=u, "\alpha"] & Y \otimes (Z \otimes X) \arrow[r, "1 \otimes \sigma"] & Y \otimes (X \otimes Z)
\end{tikzcd} \tag{2}$$

commute. This is the coherence condition on braidings.

The braiding σ is a symmetry if

$$\sigma_{Y,X} \circ \sigma_{Y,Z} = Id, \tag{3}$$

and a monoidal category equipped with such is called symmetric. We shall work only with symmetries.

When the associativity constraint is trivial, the coherence condition gives what we call the bihomomorphism conditions for any braiding σ

$$\begin{aligned} (\sigma_{X,Z} \otimes 1) \circ (1 \otimes \sigma_{Y,Z}) &= \sigma_{X \otimes Y,Z}, && \text{(i)} \\ (1 \otimes \sigma_{X,Z}) \circ (\sigma_{X,Y} \otimes 1) &= \sigma_{X,Y \otimes Z}, && \text{(ii)} \end{aligned}$$

$X, Y, Z \in Obj\,(C)$.

The trivial braiding is the twist, $\tau : X \otimes Y \to Y \otimes X$.

Any braiding composed with the twist, $\tau \circ \sigma$, is a quantization since the coherence condition for quantizations then is satisfied.

Quantizations act on the set of braidings as follows

$$(\sigma_q)_{X,Y} = q_{Y,X}^{-1} \circ \sigma_{X,Y} \circ q_{X,Y},$$

and σ_q is also a braiding.

2.3 Algebras

Let A be an algebra in a monoidal category C with multiplication

$$\mu : A \otimes A \to A$$

and unit $\eta : e \to A$.

Given a braiding σ we say that A is *σ-commutative or* σ-symmetric, [16], [12], if

$$\mu = \mu \circ \sigma.$$

Note that when the associativity constraint is trivial, the bihomomorphism conditions for the σ-commutativity of algebras is

$$\begin{aligned} \mu \circ (\mu \otimes 1) \circ (\sigma_{x,z} \otimes 1) \circ (1 \otimes \sigma_{y,z}) &= \mu \circ \sigma_{xy,z} \circ (\mu \otimes 1), && (4) \\ \mu \circ (1 \otimes \mu) \circ (1 \otimes \sigma_{x,z}) \circ (\sigma_{x,y} \otimes 1) &= \mu \circ \sigma_{x,yz} \circ (1 \otimes \mu), && (5) \end{aligned}$$

for $x, y, z \in A$.

Given a quantization q we define a *quantization* A_q of the algebra A as the same object A equipped with a new multiplication

$$\mu_q = \mu \circ q_{A,A} : A \otimes A \to A.$$

$A_q = \left(A, \mu_q, \eta\right)$ is an algebra.

If an algebra A in a monoidal category is σ-commutative, then it's quantization A_q is σ_q-commutative [17].

2.4 Modules

Let (A, μ) be an algebra in a monoidal category C. Let E be a left A-module with action

$$\nu^l : A \otimes E \to E$$

in a monoidal category. If not stated otherwise, assume that all modules are left modules.

By a *quantization* E_q of the A-module E we mean the same object E equipped with a new action

$$\nu_q^l = \nu^l \circ q_{A,E} : A \otimes E \to E.$$

$E_q = \left(E, \nu_q^l\right)$ is also a left A-module in C.

Let A be a σ-symmetric algebra and E be an A-A-bimodule. Given a braiding σ we say that E is σ*-commutative* if $\nu^l = \nu^r \circ \sigma_{A,E}$ and $\nu^r = \nu^l \circ \sigma_{E,A}$ [16], that is,

$$\begin{tikzcd}
A \otimes E \arrow[r, "\nu^l"] \arrow[d, "\sigma"'] & X \arrow[d, "="'] \\
E \otimes A \arrow[r, "\nu^r"] & X
\end{tikzcd} \tag{6}$$

and

$$\begin{tikzcd}
E \otimes A \arrow[r, "\nu^r"] \arrow[d, "\sigma"'] & X \arrow[d, "="'] \\
A \otimes E \arrow[r, "\nu^l"] & X
\end{tikzcd} \tag{7}$$

commutes. If σ is a symmetry then an A-module E is left σ-commutative if and only if it is right σ-commutative, that is (6) and (7) implies each other. This means that when σ is a symmetry we need only to consider left modules, not bimodules.

When the associativity constraint is trivial, the bihomomorphism conditions for any the σ-commutativity of modules is

$$M \circ (M \otimes 1) \circ (\sigma \otimes 1) \circ (1 \otimes \sigma) = M \circ \sigma \circ (M \otimes 1), \tag{8}$$
$$M \circ (1 \otimes M) \circ (1 \otimes \sigma) \circ (\sigma \otimes 1) = M \circ \sigma \circ (1 \otimes M), \tag{9}$$

where M is μ, ν^lor ν^r, depending on the triplet of A and E.

For any A-A-bimodule E the σ*-symmetric* part is

$$E_\sigma = \left\{x \in E | \nu^l (a \otimes x) = \nu^r \circ \sigma_{A,E} (a \otimes x), \nu^r (x \otimes a) = \nu^l \circ \sigma_{E,A} (x \otimes a), \forall a \in A\right\}.$$

The quantization of a right A-module E with the action

$$\nu^r : E \otimes A \to E,$$

is done by giving E the new action

$$\nu_q^r = \nu^r \circ q_{E,A} : E \otimes A \to E.$$

A quantization of a A-A-bimodule is the same object E equipped with two new actions $E_q = \left(E, \nu_q^l, \nu_q^r\right)$. E_q is an $A_q - A_q$-bimodule in C.

E is σ-commutative, then E_q is σ_q-commutative;

$$\nu^l \circ q_{A,E} = \nu_q^l = \nu_q^r \circ (\sigma_q)_{A,E}$$

and similarly $\nu_q^r = \nu_q^l \circ (\sigma_q)_{E,A}$.

Let E_σ be the symmetric part of the A-A-bimodule E. Consider the quotient bimodule $E/E_\sigma = E_\sigma^{(0)}$ and define $E_\sigma^{(1)}$as the preimage of $(E/E_\sigma)_\sigma$ with respect to the canonical projection $E \to E/E_\sigma$. Proceeding, we get a filtration of E by bimodules $E_\sigma^{(i)}, i = -1, 0, 1, \ldots, E_\sigma^{(-1)} = 0$. We call the bimodule

$$E_\sigma^* = \cup E_\sigma^{(i)}$$

a *differential approximation* of the A-A-bimodule E.

2.5 Coalgebras

Let A be a coalgebra in C with comultiplication $\Delta : A \to A \otimes A$ and counit $\varepsilon : A \to e$.

A is *σ-cocommutative* if

$$\Delta = \sigma^{-1} \circ \Delta.$$

Define a *quantization* A_q of the coalgebra A as the same object A equipped with a new comultiplication defined by

$$\Delta_q = q_{A,A}^{-1} \circ \Delta : A \to A \otimes A.$$

$A_q = (A, \Delta_q, \varepsilon)$ is a coalgebra.

2.6 Bialgebras

Let A be an algebra in C. Then the tensor square $A \otimes A$ can be considered as an algebra with multiplication

$$\mu_\sigma^{\otimes 2} = (\mu \otimes \mu) \circ (1 \otimes \sigma \otimes 1).$$

Let A be a coalgebra in C. Then the tensor square $A \otimes A$ can be considered as a coalgebra with comultiplication

$$\Delta_\sigma^{\otimes 2} = (1 \otimes \sigma \otimes 1) \circ \Delta \otimes \Delta.$$

A σ-bialgebra (A, μ, Δ) in a monoidal category C is an algebra (A, μ) and a coalgebra (A, Δ) such that the diagonal

$$\Delta : (A, \mu) \to \left(A \otimes A, \mu_\sigma^{\otimes 2}\right)$$

and counit are algebra morphisms and the multiplication

$$\mu : \left(A \otimes A, \Delta_\sigma^{\otimes 2}\right) \to (A, \Delta)$$

and unit are coalgebra morphisms.

The quantization $A_q = \left(A, \mu_q, \Delta_q\right)$ is a bialgebra in C, [16].

2.7 Internal homomorphisms

For a closed monoidal category C and any two objects X and Y in C there is the internal homomorphism bifunctor hom and the internal homomorphism object hom (X, Y),

$$\mathrm{hom} : X \otimes Y \to \mathrm{hom}\,(X, Y)$$

together with the composition

$$\mu^h : \hom(Y, Z) \otimes \hom(X, Y) \to \hom(X, Z),$$

$$g \otimes f \longmapsto g \circ f = g * f.$$

The collection of internal homomorphisms is an "algebra" with respect to $*$.

If C is equipped with a braiding σ, then the composition of internal homomorphisms is called σ-symmetric if

$$\mu^h = \mu^h \circ \sigma.$$

The morphism

$$ev_{X,Y} : \hom(X, Y) \otimes X \to Y$$

is called the evaluation map if any morphism $f : X \otimes Z \to Y$ can be represented by the composition

$$f = ev_{X,Y} \circ \left(\hat{f} \otimes id_X\right)$$

for a unique morphism $\hat{f} : Z \to \hom(X, Y)$. That is

$$Mor(Z, \hom(X, Y)) \cong Mor(Z \otimes X, Y).$$

The evaluation map also can be considered as a result of the multiplication μ^h,

$$ev_{X,Y}(f \otimes x) = f(x) = f * x,$$

$x \in X$, $f \in \hom(X, Y)$.

Given a quantization q, define a quantization of the internal homomorphisms to be a quantization as an algebra and we equip the internal homomorphisms with a new multiplication. Namely a quantization is a natural isomorphism

$$Q_q : \hom(X, Y) \to \hom(X, Y)$$

defined by

$$Q_q(f) * x = f *_q x$$

for all $f \in \hom(X, Y)$ and $x \in \hom(1, X)$, where

$$f *_q g = \mu^h\left(q_{\hom(X,Y),\hom(Y,Z)}(f \otimes g)\right) = \mu^h_q(f \otimes g).$$

For any quantization we have

$$\begin{aligned} Q_q(f) * Q_q(f) &= Q_q(f *_q g), & (10)\\ Q_q(x) &= x. & (11) \end{aligned}$$

3 Braided Lie structure of internal homomorphisms

We shall describe the module structure and braided Lie algebra structure of the internal homomorphisms. We describe the same for the quantizations of the internal homomorphisms.

3.1 Module structure of $\hom(E, E')$

Let σ be a braiding in a monoidal category C.

Let A be a σ-commutative algebra, let E and E' be left A-modules and let $\hom(E, E')$ be the set of internal homomorphisms.

The set $\hom(E, E')$, is an A-A-bimodule with the left and right multiplications defined by

$$\begin{aligned} \nu_a^l(f)(x) &= \nu^l(a \otimes f)(x) = af(x), \\ \nu_a^r(f)(x) &= \nu^r(f \otimes a)(x) = f(ax), \end{aligned}$$

$a \in A$, $x \in E$.

Proposition 1 *Let σ be a symmetry and E and E' be σ-commutative A-A-modules. Then $\hom(E, E')$ is σ-commutative as a module, that is, the diagrams*

$$\begin{tikzcd} A \otimes hom(X,Y) \arrow[r, "\nu^l"] \arrow[d, "\sigma"'] & hom(X,Y) \arrow[d, "="'] \\ hom(X,Y) \otimes A \arrow[r, "\nu^r"] & hom(X,Y) \end{tikzcd} \tag{12}$$

and

$$\begin{tikzcd} hom(X,Y) \otimes A \arrow[r, "\nu^r"] \arrow[d, "\sigma"'] & hom(X,Y) \arrow[d, "="'] \\ A \otimes hom(X,Y) \arrow[r, "\nu^l"] & hom(X,Y) \end{tikzcd} \tag{13}$$

commute.

Proof. Write the actions as

$$\begin{aligned} ev_{E,E'} \circ \left(\nu^l \otimes 1\right) &= v_{E'}^l \circ (1 \otimes ev_{E,E'}) : A \otimes \hom(E, E') \otimes E \to E', \\ ev_{E,E'} \circ (\nu^r \otimes 1) &= ev_{E,E'} \circ \left(1 \otimes v_E^l\right) : \hom(E, E') \otimes A \otimes E \to E'. \end{aligned}$$

Then

$$\begin{aligned} \nu^l(a \otimes f)(x) &= \nu_{E'}^l \circ (1 \otimes ev_{E,E'})(a \otimes f \otimes x) \\ &= \nu_{E'}^r \circ \sigma \circ (1 \otimes ev_{E,E'})(a \otimes f \otimes x) \\ &= ev_{E,E'} \circ (1 \otimes \nu_E^r) \circ (1 \otimes \sigma) \circ (\sigma \otimes 1)(a \otimes f \otimes x) \\ &= ev_{E,E'} \circ \left(1 \otimes \nu_E^l\right) \circ (\sigma \otimes 1)(a \otimes f \otimes x) \\ &= \nu^r \circ \sigma_{A,\hom(E,E')}(a \otimes f)(x), \end{aligned}$$

and

$$\begin{aligned}
\nu^r(f\otimes a)(x) &= ev_{E,E'}\circ\left(1\otimes v_E^l\right)(f\otimes a\otimes x)\\
&= ev_{E,E'}\circ\left(1\otimes v_E^r\right)\circ(1\otimes\sigma)(f\otimes a\otimes x)\\
&= \nu_{E'}^r\circ(ev_{E,E'}\otimes 1)\circ(1\otimes\sigma)(f\otimes a\otimes x)\\
&= \nu_{E'}^l\circ\sigma\circ(ev_{E,E'}\otimes 1)\circ(1\otimes\sigma)(f\otimes a\otimes x)\\
&= \nu_{E'}^l\circ(1\otimes ev_{E,E'})\circ(\sigma\otimes 1)\circ(1\otimes\sigma)\circ(1\otimes\sigma)(f\otimes a\otimes x)\\
&= \nu_{E'}^l\circ(1\otimes ev_{E,E'})\circ(\sigma\otimes 1)(f\otimes a\otimes x)\\
&= \nu^l\circ\sigma(f\otimes a)(x),
\end{aligned}$$

$a\in A,\ f\in \hom(E,E'),\ x\in E.$ ■

3.2 Braided commutators and Lie structure of $\hom(E,E)$

Let σ be a braiding, A be an σ-symmetric algebra and E be a left σ-symmetric A-module. Consider $\hom(E,E)$.

Definition 2 *Define the σ-bracket or σ-commutator of* $\hom(E,E)$

$$c_\sigma:\hom(E,E)\otimes\hom(E,E)\to\hom(E,E)$$

by

$$c_\sigma=[\_,\_]^\sigma=\mu^h-\mu^h\circ\sigma.$$

Proposition 3 *Let σ be a symmetry. The σ-bracket satisfies the conditions,*

$$\begin{aligned}
c_\sigma\circ\left(\nu^l\otimes 1\right)(a\otimes f\otimes g) &= \left(\nu^l\circ(1\otimes c_\sigma)-\nu^l\circ(ev_{E,E}\otimes 1)\circ\sigma\right)(a\otimes f\otimes g),\\
c_\sigma\circ\left(1\otimes\nu^l\right)(f\otimes a\otimes g) &= \left(\nu^l\circ(1\otimes c_\sigma)\circ(\sigma\otimes 1)+\nu^l\circ(ev_{E,E}\otimes 1)\right)(f\otimes a\otimes g),
\end{aligned}$$

$a\in A,\ f,g\in\hom(E,E).$

Proof.

$$\begin{aligned}
c_\sigma\circ\left(\nu^l\otimes 1\right) &= \mu^h\circ\left(\nu^l\otimes 1\right)-\mu^h\circ\sigma\circ\left(\nu^l\otimes 1\right)\\
&= \nu^l\circ\left(1\otimes\mu^h\right)-\mu^h\circ\left(1\otimes\nu^l\right)\circ(\sigma\otimes 1)\circ(1\otimes\sigma)\\
&= \nu^l\circ\left(1\otimes\mu^h\right)-\mu^h\circ(\nu^r\otimes 1)\circ(\sigma\otimes 1)\circ(1\otimes\sigma)\\
&\quad -\nu^l\circ\left(1\otimes\mu^h\right)\circ(\sigma\otimes 1)\circ(\sigma\otimes 1)\circ(1\otimes\sigma)\\
&= \nu^l\circ\left(1\otimes\mu^h\right)-\mu^h\circ\sigma\circ\left(\nu^l\otimes 1\right)-\nu^l\circ\left(1\otimes\mu^h\right)\circ(1\otimes\sigma)\\
&= \nu^l\circ(1\otimes c_\sigma)-\nu^l\circ(ev_{E,E}\otimes 1)\circ\sigma,
\end{aligned}$$

and

$$\begin{aligned}
c_\sigma\circ\left(1\otimes\nu^l\right) &= \mu^h\circ\left(1\otimes\nu^l\right)-\mu^h\circ\sigma\circ\left(1\otimes\nu^l\right)\\
&= \mu^h\circ(\nu^r\otimes 1)+\nu^l\circ\left(1\otimes\mu^h\right)\circ(\sigma\otimes 1)\\
&\quad -\nu^l\circ\left(1\otimes\mu^h\right)\circ(1\otimes\sigma)\circ(\sigma\otimes 1)\\
&= \nu^l\circ(ev_{E,E}\otimes 1)+\nu^l\circ(1\otimes c_\sigma)\circ(\sigma\otimes 1).
\end{aligned}$$

■

Proposition 4 *The σ-bracket is σ-invariant,*

$$\mu^h \circ \sigma \circ (c_\sigma \otimes 1) = \mu^h \circ (1 \otimes c_\sigma) \circ (\sigma \otimes 1) \circ (1 \otimes \sigma) .$$

Proof.

$$\begin{aligned}
\mu^h \circ \sigma \circ (c_\sigma \otimes 1) &= \mu^h \circ \sigma \circ \big(\big(\mu^h - \mu^h \circ \sigma\big) \otimes 1\big) \\
&= \mu^h \circ \sigma \circ \big(\mu^h \otimes 1\big) - \mu^h \circ \sigma \circ \big(\mu^h \otimes 1\big) \circ (\sigma \otimes 1) ,
\end{aligned}$$

and

$$\begin{aligned}
& \mu^h \circ (1 \otimes c_\sigma) \circ (\sigma \otimes 1) \circ (1 \otimes \sigma) \\
= \; & \mu^h \circ \big(1 \otimes \big(\mu^h - \mu^h \circ \sigma\big)\big) \circ (\sigma \otimes 1) \circ (1 \otimes \sigma) \\
= \; & \mu^h \circ \big(1 \otimes \mu^h\big) \circ (\sigma \otimes 1) \circ (1 \otimes \sigma) - \mu^h \circ \big(1 \otimes \mu^h\big) \circ (1 \otimes \sigma) \circ (\sigma \otimes 1) \circ (1 \otimes \sigma) \\
= \; & \mu^h \circ \big(\mu^h \otimes 1\big) \circ (\sigma \otimes 1) \circ (1 \otimes \sigma) - \mu^h \circ \big(\mu^h \otimes 1\big) \circ (\sigma \otimes 1) \circ (1 \otimes \sigma) \\
= \; & \mu^h \circ \sigma \circ \big(\mu^h \otimes 1\big) - \mu^h \circ \sigma \circ \big(\mu^h \otimes 1\big) \circ (\sigma \otimes 1) .
\end{aligned}$$

■

Proposition 5 *Let σ be a symmetry. The σ-commutator satisfies skew σ-symmetricity, that is, the diagram*

$$\begin{array}{ccc}
hom(X,X) \otimes hom(X,X) & \xrightarrow{c_\sigma} & hom(X,X) \\
\sigma \Big\downarrow & & \Big\downarrow = \\
hom(X,X) \otimes hom(X,X) & \xrightarrow{-c_\sigma} & hom(X,X)
\end{array} \tag{14}$$

commutes, equivalently

$$c_\sigma = -c_\sigma \circ \sigma.$$

Proof.

$$\begin{aligned}
c_\sigma \circ \sigma \, (f \otimes g) &= \mu^h \circ \sigma \, (f \otimes g) - \mu^h \circ \sigma \circ \sigma \, (f \otimes g) \\
&= \mu^h \circ \sigma \, (f \otimes g) - \mu^h \, (f \otimes g) \\
&= -c_\sigma \, (f \otimes g) ,
\end{aligned}$$

$f, g \in \hom(E, E)$. ■

Proposition 6 *Let σ be a symmetry. Then $\hom(E,E)$ equipped with the σ-commutator satisfies the σ-Jacobi identity,*

$$c_\sigma \circ (1 \otimes c_\sigma) = c_\sigma \circ (c_\sigma \otimes 1) + c_\sigma \circ (1 \otimes c_\sigma) \circ (\sigma \otimes 1) .$$

Proof. The Jacobi identity applied to three elements in $\hom(E,E)$ is satisfied when

$$\begin{aligned}
& c_\sigma \circ (1 \otimes c_\sigma) \\
= \; & \mu^h \circ \big(1 \otimes \mu^h\big) && \text{(a)} \\
& - \mu^h \circ \big(1 \otimes \mu^h\big) \circ (1 \otimes \sigma) && \text{(b)} \\
& -\mu^h \circ \sigma \circ \big(1 \otimes \mu^h\big) && \text{(c)} \\
& +\mu^h \circ \sigma \circ \big(1 \otimes \mu^h\big) \circ (1 \otimes \sigma) && \text{(d)}
\end{aligned}$$

is equal to

$$
\begin{array}{rlr}
& c_\sigma \circ (c_\sigma \otimes 1) + c_\sigma \circ (1 \otimes c_\sigma) \circ (\sigma \otimes 1) & \\
= & \mu^h \circ \big(\mu^h \otimes 1\big) & \text{(e)} \\
& -\mu^h \circ \big(\mu^h \otimes 1\big) \circ (\sigma \otimes 1) & \text{(f)} \\
& -\mu^h \circ \sigma \circ \big(\mu^h \otimes 1\big) & \text{(g)} \\
& +\mu^h \circ \sigma \circ \big(\mu^h \otimes 1\big) \circ (\sigma \otimes 1) & \text{(h)} \\
& +\mu^h \circ \big(1 \otimes \mu^h\big) \circ (\sigma \otimes 1) & \text{(i)} \\
& -\mu^h \circ \big(1 \otimes \mu^h\big) \circ (1 \otimes \sigma) \circ (\sigma \otimes 1) & \text{(j)} \\
& -\mu^h \circ \sigma \circ \big(1 \otimes \mu^h\big) \circ (\sigma \otimes 1) & \text{(k)} \\
& +\mu^h \circ \sigma_{g,hf} \circ \big(1 \otimes \mu^h\big) \circ (1 \otimes \sigma) \circ (\sigma \otimes 1)\,, & \text{(l)}
\end{array}
$$

which is the case since
$(a) = (e)\,;$

$$\mu^h \circ \big(1 \otimes \mu^h\big) = \mu^h \circ \big(\mu^h \otimes 1\big)\,,$$

$(b) = (k)\,;$

$$
\begin{aligned}
\mu^h \circ \big(1 \otimes \mu^h\big) \circ (1 \otimes \sigma) &= \mu^h \circ \sigma \circ \big(1 \otimes \mu^h\big) \circ (\sigma \otimes 1) \\
&= \mu^h \circ \big(1 \otimes \mu^h\big) \circ (1 \otimes \sigma) \circ (\sigma \otimes 1) \circ (\sigma \otimes 1)\,,
\end{aligned}
$$

$(c) = (j)\,;$

$$\mu^h \circ \sigma \circ \big(1 \otimes \mu^h\big) = \mu^h \circ \big(1 \otimes \mu^h\big) \circ (1 \otimes \sigma) \circ (\sigma \otimes 1)\,,$$

$(d) = (h)\,;$

$$\mu^h \circ \sigma \circ \big(1 \otimes \mu^h\big) \circ (1 \otimes \sigma) = \mu^h \circ \sigma \circ \big(\mu^h \otimes 1\big) \circ (\sigma \otimes 1)$$

$$
\begin{aligned}
&\mu^h \circ \big(1 \otimes \mu^h\big) \circ (1 \otimes \sigma) \circ (\sigma \otimes 1) \circ (1 \otimes \sigma) \\
&\qquad = \mu^h \circ \big(\mu^h \otimes 1\big) \circ (\sigma \otimes 1) \circ (1 \otimes \sigma) \circ (\sigma \otimes 1)\,,
\end{aligned}
$$

$(f) + (i) = 0;$

$$\mu^h \circ \big(\mu^h \otimes 1\big) \circ (\sigma \otimes 1) = \mu^h \circ \big(1 \otimes \mu^h\big) \circ (\sigma \otimes 1)\,,$$

and $(g) + (l) = 0;$

$$\mu^h \circ \sigma \circ \big(\mu^h \otimes 1\big) = \mu^h \circ \sigma \circ \big(1 \otimes \mu^h\big) \circ (1 \otimes \sigma) \circ (\sigma \otimes 1)\,,$$

since

$$\mu^h \circ \sigma \circ \big(\mu^h \otimes 1\big) = \mu^h \circ \big(\mu^h \otimes 1\big) \circ (\sigma \otimes 1) \circ (1 \otimes \sigma)\,,$$

and

$$\mu^h \circ \sigma \circ \big(1 \otimes \mu^h\big) \circ (1 \otimes \sigma) \circ (\sigma \otimes 1) = \mu^h \circ \big(1 \otimes \mu^h\big) \circ (1 \otimes \sigma) \circ (\sigma \otimes 1) \circ (1 \otimes \sigma) \circ (\sigma \otimes 1)\,.$$

This is shown by using the identities (4), (5) and the fact that, σ satisfies the Yang-Baxter equation,

$$(1 \otimes \sigma) \circ (\sigma \otimes 1) \circ (1 \otimes \sigma) = (\sigma \otimes 1) \circ (1 \otimes \sigma) \circ (\sigma \otimes 1)\,.$$

■

Remark 7 *In the proof of the propositions 5 and 6 we can assume that the internal homomorphisms are equipped with the σ-commutative composition $\mu^h = \mu^h \circ \sigma$ instead of using the fact that σ is a symmetry.*

Definition 8 *A σ-Lie algebra is an algebra equipped with a skew σ-symmetric σ-commutator that is σ-invariant and satisfies the σ-Jacobi identity.*

This is the definition of braided Lie algebras as introduced by D. Gurevich, [4]. By the propositions 4, 5 and 6 we have proved the following.

Theorem 9 *Let σ be a symmetry and an algebra A and a left A-module E be σ-symmetric. Then* $\hom(E, E)$ *is a σ-Lie algebra.*

3.3 Quantization of $\hom(E, E)$

We will define the quantization of the internal homomorphisms and describe the structure on the set of all such, also within the original internal homomorphisms.

Definition 10 *Given a quantization q and $f \in \hom(E, E)$, its quantization is defined by*

$$f_q(x) = Q_q(f)(x) \overset{def}{=} ev_{E,E} \circ q(f \otimes x), \tag{15}$$

$x \in E$.

It is easy to see that $Q_q(f)$ is an internal homomorphism of the quantized A_q-module E_q.

Let f and g be internal homomorphisms. The quantization of composition of internal homomorphisms is

$$f *_q g = \mu_q^h (f \otimes g). \tag{16}$$

The collection of all $Q_q(f)$, $f \in \hom(E, E)$, equipped with the quantization of the composition is denoted by $\hom^q(E_q, E_q)$.

By proposition 1 is $\hom^q(E_q, E_q)$ a σ_q-symmetric module if E is a σ-symmetric A-A-bimodule.

By theorem 9, if σ is a symmetry is $\hom^q(E_q, E_q)$ a σ_q-Lie algebra with respect to the $\sigma_q - q$-bracket (or simply σ_q-bracket when it is clear that the multiplication or composition is the quantized multiplication).

Let γ be a braiding and p any quantization. Define the $\gamma - p$-bracket on internal homomorphisms,

$$c_\gamma^p = [-,-]_p^\gamma = \mu_p^h - \mu_p^h \circ \gamma. \tag{17}$$

Lemma 11 *The $\sigma_q - q$-bracket satisfies*

$$c_{\sigma_q}^q = c_\sigma \circ q.$$

Proof.

$$c_{\sigma_q}^q = \mu_q^h - \mu_q^h \circ \sigma_q = \mu_q^h - \mu^h \circ q \circ q^{-1} \circ \sigma \circ q = \mu^h \circ q - \sigma \circ \mu^h \circ q = c_\sigma \circ q.$$

■

The inverse of the quantization of $f \in \hom^q(E_q, E_q)$ is denoted by

$$Q_q^{-1}(f) = f_c,$$

and is called the *dequantization* of f.

Lemma 12 *The composition of internal homomorphisms satisfies*

$$f_c *_q g_c = (f \circ g)_c\,, \tag{18}$$

$f, g \in \hom^q(E_q, E_q)$.

Proof.

$$\begin{aligned} f_c *_q g_c(x) &= Q_q^{-1}(f) *_q Q_q^{-1}(g)(x) \\ &= ev_{E,E} \circ q^{-1} \circ (1 \otimes ev_{E,E}) \circ \left(1 \otimes q^{-1}\right) \circ (q \otimes 1)(f \otimes g \otimes x) \\ &= ev_{E,E} \circ q^{-1} \circ \left(\mu^h \otimes 1\right) \circ \left(q^{-1} \otimes 1\right) \circ (q \otimes 1)(f \otimes g \otimes x) \\ &= Q_q^{-1}(f \circ g)(x) \\ &= (f \circ g)_c\,, \end{aligned}$$

■

Note,

$$a_c = a,$$

for $a \in A_q$.

The set of all $Q_q^{-1}(f)$, $f \in \hom^q(E_q, E_q)$, is equipped with the bracket $c_{\sigma_q}^q$ and the A-module structure

$$a *_q f_c = \nu^l \circ q(a \otimes f_c)\,, \tag{19}$$

$a \in A$.

The dequantization of a internal homomorphism operates on A in the classical manner, but satisfies somewhat different properties than the classical, as the following theorem states. (See also [14])

Theorem 13 *Let σ be a symmetry. The σ_q-Lie algebra structure of* $\hom^q(E_q, E_q)$ *can be realized within the classical,* $\hom(E, E)$, *by dequantization as follows.*

Let $f, g \in \hom^q(E_q, E_q)$, $a \in A_q$. Then linearity is

$$(f + g)_c = f_c + g_c, \tag{i}$$

A-module structure,

$$(a \circ f)_c = a *_q f_c, \tag{ii}$$

and the braided commutator satisfies

$$Q_q^{-1} \circ c_{\sigma_q} = c_{\sigma_q}^q \circ \left(Q_q^{-1} \otimes Q_q^{-1}\right). \tag{iii}$$

Proof. (i):

$$Q_q^{-1}(f + g) = Q_q^{-1}(f) + Q_q^{-1}(g)\,.$$

(ii): By lemma 12,

$$a_c *_q f_c = a *_q f_c = (a \circ f)_c\,.$$

(iii):

$$\begin{aligned}
& \left[Q_q^{-1}(f), Q_q^{-1}(g)\right]_q^{\sigma_q}(x) \\
= \; & ev_{E,E} \circ q^{-1} \circ (1 \otimes ev_{E,E}) \circ \left(1 \otimes q^{-1}\right) \circ (q \otimes 1)(f \otimes g \otimes x) \\
& - ev_{E,E} \circ q^{-1} \circ (1 \otimes ev_{E,E}) \circ \left(1 \otimes q^{-1}\right) \circ (q \otimes 1) \circ (\sigma_q \otimes 1)(f \otimes g \otimes x) \\
= \; & ev_{E,E} \circ q^{-1} \circ \left(\mu^h \otimes 1\right) \circ \left(q^{-1} \otimes 1\right) \circ (q \otimes 1)(f \otimes g \otimes x) \\
& - ev_{E,E} \circ q^{-1} \circ \left(\mu^h \otimes 1\right) \circ \left(q^{-1} \otimes 1\right) \circ (q \otimes 1) \circ (\sigma_q \otimes 1)(f \otimes g \otimes x) \\
= \; & Q_q^{-1}\left([f,g]^{\sigma_q}\right)(x),
\end{aligned}$$

$x \in E,\ f, g \in \hom^q(E_q, E_q)$. ■

4 Braided derivations in monoidal categories

Let C be a monoidal category equipped with a braiding σ.

Let A be a σ-symmetric algebra in C with the multiplication μ, and E and E' be A-A-modules.

Denote by $\hom_\sigma(E, E') = (\hom(E, E'))_\sigma$ the σ-symmetric part of the $A-A$-bimodule $\hom(E, E')$. The modules

$$(\hom(E, E'))_\sigma^{(i)} = Diff_i^\sigma(E, E')$$

are called the braided or *σ-differential operators*, see [16].

4.1 Braided derivations of algebras

With an additional condition we define σ-derivations of algebras.

Definition 14 *Define σ-derivations or braided derivations of A with values in a A-A-bimodule E as*

$$Der^\sigma(E) = \{f \in Diff_1^\sigma(A, E) \,|\, f(1) = 0\}.$$

An internal homomorphism $f : A \to E$ is a σ-derivation if and only if $f : A \to E_\sigma \subset E$ and f satisfies the *braided or σ-Leibniz rule,* [17],

$$\begin{aligned}
& ev_{A,E} \circ (1 \otimes \mu)(f \otimes a \otimes b) \\
= \; & \left(\nu^r \circ (ev_{A,E} \otimes 1) + \nu^l \circ (1 \otimes ev_{A,E}) \circ (\sigma \otimes 1)\right)(f \otimes a \otimes b),
\end{aligned} \tag{20}$$

$$\begin{aligned}
& ev_{A,E} \circ (1 \otimes \mu) \circ (\sigma \otimes 1)(a \otimes f \otimes b) \\
= \; & \left(\nu^l \circ (1 \otimes ev_{A,E}) + \nu^r \circ (ev_{A,E} \otimes 1) \circ (\sigma \otimes 1)\right)(a \otimes f \otimes b).
\end{aligned} \tag{21}$$

If the braiding is a symmetry, then the two Leibniz rules implies each other.

From now on, assume that every braiding σ is a symmetry. We can then assume E is a left A-module.

Proposition 15 *$Der^\sigma(E)$ has a left A-module structure defined by*

$$\nu^l(a \otimes \partial)(b) \overset{def}{=} \nu_E^l \circ (1 \otimes ev_{A,E})(a \otimes \partial \otimes b) = a\partial(b), \tag{22}$$

$\partial \in Der^\sigma(E)$, $a, b \in A$, ν_E^l is the action of A on E.

Proof. We need to show that $\nu_a^l(\partial)$ is a σ-derivation, that is satisfies the σ-Leibniz rule, and using the σ-Leibniz rule for ∂,

$$a\partial(bc) = \begin{pmatrix} \nu_E^l \circ \left(1 \otimes \nu_E^l\right) \circ (1 \otimes ev_{A,A} \otimes 1) \\ +\nu_E^l \circ \left(1 \otimes \nu_E^l\right)(1 \otimes 1 \otimes ev_{E,E}) \circ (1 \otimes \sigma \otimes 1) \end{pmatrix} (a \otimes \partial \otimes b \otimes c),$$

$a, b, c \in A$, $\partial \in Der^{\sigma}(E)$, and

$$\begin{aligned} & \nu_E^l \circ \left(1 \otimes \nu_E^l\right) \circ (1 \otimes ev_{A,A} \otimes 1) + \nu_E^l \circ \left(1 \otimes \nu_E^l\right)(1 \otimes 1 \otimes ev_{E,E}) \circ (1 \otimes \sigma \otimes 1) \\ = \ & \nu_E^l \circ \left(\nu_E^l \otimes 1\right) \circ (1 \otimes ev_{A,A} \otimes 1) \\ & +\nu_E^l \circ ((\mu \circ \sigma) \otimes 1) \circ (1 \otimes 1 \otimes ev_{E,E}) \circ (1 \otimes \sigma \otimes 1) \\ = \ & \nu_E^l \circ (ev_{A,A} \otimes 1) \circ \left(\nu^l \otimes 1 \otimes 1\right) \\ & +\nu_E^l \circ \left(1 \otimes \nu_E^l\right) \circ (1 \otimes 1 \otimes ev_{E,E}) \circ (\sigma \otimes 1 \otimes 1) \circ (1 \otimes \sigma \otimes 1) \\ = \ & \nu_E^l \circ (ev_{A,A} \otimes 1) \circ \left(\nu^l \otimes 1 \otimes 1\right) \\ & +\nu_E^l \circ (1 \otimes ev_{E,E}) \circ \left(1 \otimes \nu^l \otimes 1\right) \circ (\sigma \otimes 1 \otimes 1) \circ (1 \otimes \sigma \otimes 1), \end{aligned}$$

and the condition,

$$\nu^l(a \otimes \partial)(1) = a\partial(1) = 0,$$

is satisfied. ■

If we consider $Diff_1^{\sigma}(A, E)$ and not $Der^{\sigma}(E)$ then there is a right A-module structure.

Proposition 16 *$Diff_1^{\sigma}(A, E)$ has in addition to the left A-module structure defined by (22), a right A-module structure defined by*

$$\nu^r(\partial \otimes a)(b) \overset{def}{=} ev_{A,E} \circ (1 \otimes \mu)(\partial \otimes a \otimes b) = \partial(ab),$$

$\partial \in Diff_1^{\sigma}(A, E)$, $a, b \in A$.

Proof. The σ-Leibniz rule for ∂a is,

$$\partial a(bc) = \begin{pmatrix} \nu_E^r \circ (ev_{A,E} \otimes 1) \circ (\nu^r \otimes 1 \otimes 1) \\ +\nu_E^l \circ (1 \otimes ev_{A,E}) \circ (\sigma_{\partial a, b} \otimes 1) \circ (\nu^r \otimes 1 \otimes 1) \end{pmatrix} (\partial \otimes a \otimes b \otimes c). \quad (23)$$

Using the σ-Leibniz rule for ∂,

$$\begin{aligned} \partial a(bc) &= ev_{A,E} \circ (1 \otimes \mu) \circ (1 \otimes 1 \otimes \mu)(\partial \otimes a \otimes b \otimes c) \\ &= ev_{A,E} \circ (1 \otimes \mu) \circ (1 \otimes \mu \otimes 1)(\partial \otimes a \otimes b \otimes c) \\ &= \begin{pmatrix} \nu_E^r \circ (ev_{A,E} \otimes 1) \circ (1 \otimes \mu \otimes 1) \\ +\nu_E^l \circ (1 \otimes ev_{A,E}) \circ (\sigma \otimes 1) \circ (1 \otimes \mu \otimes 1) \end{pmatrix} (\partial \otimes a \otimes b \otimes c), \end{aligned}$$

$a, b, c \in A$, $\partial \in Diff_1^\sigma(A, E)$, we see that (23) is satisfied as

$$\begin{aligned}
& \nu_E^l \circ (1 \otimes ev_{A,E}) \circ (\sigma \otimes 1) \circ (1 \otimes \mu \otimes 1) \\
= \; & \nu_E^l \circ (1 \otimes ev_{A,E}) \circ \left(1 \otimes \nu^l \otimes 1\right) \circ (1 \otimes \sigma \otimes 1) \circ (\sigma \otimes 1 \otimes 1) \\
= \; & \nu_E^l \circ \left(1 \otimes \nu^l\right) \circ (1 \otimes 1 \otimes ev_{A,E}) \circ (1 \otimes \sigma \otimes 1) \circ (\sigma \otimes 1 \otimes 1) \\
= \; & \nu_E^l \circ (\mu \otimes 1) \circ (1 \otimes 1 \otimes ev_{A,E}) \circ (1 \otimes \sigma \otimes 1) \circ (\sigma \otimes 1 \otimes 1) \\
= \; & \nu_E^l \circ (\mu \otimes 1) \circ (\sigma \otimes 1) \circ (1 \otimes 1 \otimes ev_{A,E}) \circ (1 \otimes \sigma \otimes 1) \circ (\sigma \otimes 1 \otimes 1) \\
= \; & \nu_E^l \circ (1 \otimes ev_{A,E}) \circ (\mu \otimes 1 \otimes 1) \circ (\sigma \otimes 1) \circ (1 \otimes \sigma \otimes 1) \circ (\sigma \otimes 1 \otimes 1) \\
= \; & \nu_E^l \circ (1 \otimes ev_{A,E}) \circ (\sigma \otimes 1) \circ \left(\nu^l \otimes 1 \otimes 1\right) \circ (\sigma \otimes 1 \otimes 1) \\
= \; & \nu_E^l \circ (1 \otimes ev_{A,E}) \circ (\sigma \otimes 1) \circ (\nu^r \otimes 1 \otimes 1) \circ (\sigma \otimes 1 \otimes 1) \circ (\sigma \otimes 1 \otimes 1) \\
= \; & \nu_E^l \circ (1 \otimes ev_{A,E}) \circ (\sigma \otimes 1) \circ (\nu^r \otimes 1 \otimes 1) .
\end{aligned}$$

■

For the rest of the paper, consider the σ-derivations of a σ-symmetric algebra A with values in A, denoted by

$$Der^\sigma(A) .$$

Proposition 17 *The σ-commutator of two σ-derivations of A is a σ-derivation of A,*

$$c_\sigma : Der^\sigma(A) \otimes Der^\sigma(A) \to Der^\sigma(A) .$$

Proof. The σ-Leibniz rule is satisfied for the σ-commutator of two σ-derivations,

$$\begin{aligned}
c_\sigma(\partial_1 \otimes \partial_2)(ab) &= \mu^h(\partial_1 \otimes \partial_2)(ab) - \mu^h \circ \sigma(\partial_1 \otimes \partial_2)(ab) \\
&= c_\sigma(\partial_1 \otimes \partial_2)(a)\, b + \mu \circ (1 \otimes ev_{A,A})(\sigma(c_\sigma(\partial_1 \otimes \partial_2) \otimes a) \otimes b),
\end{aligned}$$

$a, b \in A$, $\partial_1, \partial_2 \in Der^\sigma(A)$, as we see,

$$\begin{aligned}
& ev_{A,A} \circ (1 \otimes ev_{A,A}) \circ (1 \otimes 1 \otimes \mu) - ev_{A,A} \circ (1 \otimes ev_{A,A}) \circ (\sigma \otimes 1 \otimes 1) \circ (1 \otimes 1 \otimes \mu) \\
& = ev_{A,A} \circ (1 \otimes \mu) \circ (1 \otimes ev_{A,A} \otimes 1) + ev_{A,A} \circ (1 \otimes \mu) \circ (1 \otimes 1 \otimes ev_{A,A}) \circ (1 \otimes \sigma \otimes 1) \\
& - ev_{A,A} \circ (1 \otimes \mu) \circ (1 \otimes ev_{A,A} \otimes 1) \circ (\sigma \otimes 1 \otimes 1) \\
& - ev_{A,A} \circ (1 \otimes \mu) \circ (1 \otimes 1 \otimes ev_{A,A}) \circ (1 \otimes \sigma \otimes 1) \circ (\sigma \otimes 1 \otimes 1) \\
& = \mu \circ (ev_{A,A} \otimes 1) \circ (1 \otimes ev_{A,A} \otimes 1) + \mu \circ (1 \otimes ev_{A,A}) \circ (\sigma \otimes 1) \circ (1 \otimes ev_{A,A} \otimes 1) \\
& + \mu \circ (ev_{A,A} \otimes 1) \circ (1 \otimes 1 \otimes ev_{A,A}) \circ (1 \otimes \sigma \otimes 1) \\
& + \mu \circ (1 \otimes ev_{A,A}) \circ (1 \otimes 1 \otimes ev_{A,A}) \circ (\sigma \otimes 1 \otimes 1) \circ (1 \otimes \sigma \otimes 1) \\
& - \mu \circ (ev_{A,A} \otimes 1) \circ (1 \otimes ev_{A,A} \otimes 1) \circ (\sigma \otimes 1 \otimes 1) \\
& - \mu \circ (1 \otimes ev_{A,A}) \circ (\sigma \otimes 1) \circ (1 \otimes ev_{A,A} \otimes 1) \circ (\sigma \otimes 1 \otimes 1) \\
& - \mu \circ (ev_{A,A} \otimes 1) \circ (1 \otimes 1 \otimes ev_{A,A}) \circ (1 \otimes \sigma \otimes 1) \circ (\sigma \otimes 1 \otimes 1) \\
& - \mu \circ (1 \otimes ev_{A,A}) \circ (1 \otimes 1 \otimes ev_{A,A}) \circ (\sigma \otimes 1 \otimes 1) \circ (1 \otimes \sigma \otimes 1) \circ (\sigma \otimes 1 \otimes 1) \\
& = \mu \circ (ev_{A,A} \otimes 1) \circ (c_\sigma \otimes 1 \otimes 1) + \mu \circ (1 \otimes ev_{A,A}) \circ (\sigma \otimes 1) \circ \left(\mu^h \otimes 1 \otimes 1\right) \\
& - \mu \circ (1 \otimes ev_{A,A}) \circ (\sigma \otimes 1) \circ \left(\mu^h \otimes 1 \otimes 1\right) \circ (\sigma \otimes 1 \otimes 1) \\
& = \mu \circ (ev_{A,A} \otimes 1) \circ (c_\sigma \otimes 1 \otimes 1) + \mu \circ (1 \otimes ev_{A,A}) \circ (\sigma \otimes 1) \circ (c_\sigma \otimes 1 \otimes 1) .
\end{aligned}$$

Furthermore,

$$c_\sigma(\partial_1 \otimes \partial_2)(1) = \mu^h(\partial_1 \otimes \partial_2)(1) - \mu^h \circ \sigma(\partial_1 \otimes \partial_2)(1) = 0.$$

■

Corollary 18 *Let the braiding σ be a symmetry. Then $Der^\sigma(A)$ equipped with the σ-commutator is a σ-Lie algebra.*

4.2 Braided derivations of modules

Let A be a σ-symmetric algebra and let E be a σ-symmetric A-module.

Definition 19 *An operator $\partial : E \to E$ is said to be a σ-derivation of E over $\partial^A \in Der^{\sigma}(A)$ if ∂ satisfy the σ-Leibniz rule with respect to ∂^A*

$$\partial(ax) = \partial^A(a)\,x + \nu_E^l \circ (1 \otimes ev_{A,A}) \circ (\sigma \otimes 1)(\partial \otimes a \otimes x), \tag{24}$$

$x \in E$, $a \in A$ and ν_E^l is the action of A on E.

The pair (∂, ∂_A) is called a σ-derivation of E over A. (We could also call this pair a σ-$\mathfrak{D}$-module, [13].)

The morphism $\pi : (\partial, \partial_A) \to \partial_A$ we call the projection from the σ-derivations of E over A to the σ-derivations of A.

The set of σ-derivations of E over A is denoted by $Der^{(\sigma,A)}(E)$.

Proposition 20 *$Der^{(\sigma,A)}(E)$ has a left A-module structure defined by*

$$\nu_a^l(\partial)(x) \stackrel{def}{=} \nu_E^l \circ (1 \otimes ev_{E,E})(a \otimes \partial \otimes x) = a\partial(x),$$

$\partial \in Der^{(\sigma,A)}(E)$, $a \in A$, $x \in E$.

Proof. Just repeat the proof of proposition 15. ■

If we consider $Diff_1^{\sigma}(E,E)$ and not $Der^{(\sigma,A)}(E)$ then there is a right A-module structure.

Proposition 21 *$Diff_1^{\sigma}(E,E)$ has in addition a right A-module structure defined by*

$$\nu_a^r(\partial)(x) \stackrel{def}{=} ev_{E,E} \circ \left(1 \otimes \nu_E^l\right)(\partial \otimes a \otimes x) = \partial(ax),$$

$\partial \in Diff_1^{\sigma}(E,E)$, $a \in A$, $x \in E$.

Proof. Just repeat the proof of proposition 16. ■

Proposition 22 *The σ-commutator of two (σ,A)-derivations of E is a (σ,A)-derivation of E,*

$$c_{\sigma} : Der^{(\sigma,A)}(E) \otimes Der^{(\sigma,A)}(E) \to Der^{(\sigma,A)}(E).$$

Proof. Simply repeat the proof of proposition 17. ■

Corollary 23 *Let the braiding σ be a symmetry. Then the (σ,A)-derivations of E equipped with the σ-commutator is a σ-Lie algebra.*

We get the following sequence of σ-symmetric left A-modules and σ-Lie algebras

$$0 \to \hom(E,E) \to Der^{(\sigma,A)}(E) \xrightarrow{\pi} Der^{\sigma}(A).$$

4.3 Quantization of braided derivations of algebras

Consider derivations of a σ-symmetric algebra A.

Definition 24 *Given a quantization q and $\partial \in Der^{\sigma}(A)$ define the quantization of ∂ by*

$$Q_q(\partial)(a) = ev_{A,A} \circ q(\partial \otimes a),$$

$a \in A$.

Sometimes we use the notation $\partial_q = Q_q(\partial)$.

$Q_q(\partial)$ is an operator of the quantized algebra A_q. Denote by $Der^{\sigma_q}(A_q)$ the set of all $Q_q(\partial)$, $\partial \in Der^{\sigma}(A)$, equipped with the quantized composition.

Theorem 25 *Given a braiding σ, let σ_q be the quantization of σ. The operator*

$$\begin{array}{rcl} Q_q & : & (Der^{\sigma}(A), c_{\sigma}) \rightarrow \left(Der^{\sigma_q}(A_q), c^q_{\sigma_q}\right), \\ \partial & \in & Der^{\sigma}(A) \longmapsto Q_q(\partial) \in Der^{\sigma_q}(A_q), \end{array} \tag{25}$$

is an isomorphism of modules between the σ-derivations of A and the σ_q-derivations of A_q

Proof. We need to show that the σ_q-Leibniz rule is satisfied

$$Q_q(\partial)(a *_q b) = Q_q(\partial)(a) *_q b + \mu_A \circ q \circ (1 \otimes ev_{A,A}) \circ (1 \otimes q) \circ (\sigma_q \otimes 1)(\partial \otimes a \otimes b)$$

since

$$\begin{aligned} & ev_{A,A} \circ (1 \otimes \mu_A) \circ q \circ (1 \otimes q) \\ & = (\mu_A \circ (ev_{A,A} \otimes 1) + \mu_A \circ (1 \otimes ev_{A,A}) \circ (\sigma \otimes 1)) \circ q \circ (1 \otimes q) \\ & = (\mu_A \circ (ev_{A,A} \otimes 1) + \mu_A \circ (1 \otimes ev_{A,A}) \circ (\sigma \otimes 1)) \circ q \circ (q \otimes 1) \\ & = (\mu_A \circ q \circ (ev_{A,A} \otimes 1) + \mu_A \circ (1 \otimes ev_{A,A}) \circ q \circ (\sigma \otimes 1)) \circ (q \otimes 1) \\ & = \left(\mu_A \circ q \circ (ev_{A,A} \otimes 1) + \mu_A \circ q \circ (1 \otimes ev_{A,A}) \circ (1 \otimes q) \circ \left(q^{-1} \otimes 1\right) \circ (\sigma \otimes 1)\right) \circ (q \otimes 1) \\ & = \mu_A \circ q \circ (ev_{A,A} \otimes 1) \circ (q \otimes 1) + \mu_A \circ q \circ (1 \otimes ev_{A,A}) \circ (1 \otimes q) \circ (\sigma_q \otimes 1) \end{aligned}$$

where

$$q_{A \otimes Der^{\sigma}(A),B} \circ \left(\sigma_{Der^{\sigma}(A),A} \otimes 1\right) = \left(\sigma_{Der^{\sigma}(A),A} \otimes 1\right) \circ q_{Der^{\sigma}(A) \otimes A,B}$$

by naturality.

Note that the σ_q-Leibniz rule is satisfied for the σ_q-commutator of two σ_q-derivations which follows from proposition 22. ■

By proposition 1 is $Der^{\sigma_q}(A_q)$ a σ_q-symmetric module and by theorem 9 a σ_q-Lie algebra with respect to the $\sigma_q - q$-bracket.

By theorem 13 the σ_q-Lie algebra structure of $\left(Der^{\sigma_q}(A_q), c^q_{\sigma_q}\right)$ can be realized within the classical one by dequantization.

4.4 Evaluations and commutators

For both σ- and σ_q-derivations, evaluating a derivation of some element corresponds to taking the braided bracket of the derivation and that element.

Proposition 26 *Let A be σ-commutative algebra, $\partial \in Der^{\sigma}(A)$ and $a \in A$. Then*

$$\partial(a) = c_{\sigma}(\partial \otimes a).$$

Let

$$\partial_q = Q_q(\partial_c) \in Der^{\sigma_q}(A_q)$$

and $a \in A_q$. Then

$$\partial_q(a) = c^q_{\sigma_q}(\partial_q \otimes a).$$

Proof. Let $\partial \in Der^{\sigma}(A)$ and $a \in A, b \in A$. The σ-Leibniz rule is

$$\partial(ab) = \partial(a)\, b + \mu \circ (1 \otimes ev_{A,A}) \circ (\sigma \otimes 1)(\partial \otimes a \otimes b)$$

and since

$$\partial(ab) = ev_{A,A} \circ (\nu^r \otimes 1)(\partial \otimes a \otimes b),$$

when we consider ∂ simply as an internal homomorphism, and

$$\mu \circ (1 \otimes ev_{A,A}) \circ (\sigma \otimes 1)(\partial \otimes a \otimes b) = ev_{A,A} \circ \left(\nu^l \otimes 1\right) \circ (\sigma \otimes 1)(\partial \otimes a \otimes b),$$

clearly, by rearranging the Leibniz rule we get

$$\partial(a) = c_{\sigma}(\partial \otimes a).$$

Let $\partial \in Der^{\sigma}\ (A)$. By the σ_q-Leibniz rule we have

$$\begin{aligned} &ev_{A,A} \circ (1 \otimes \mu)(Q_q(\partial) \otimes a \otimes b) \\ &\qquad = (\mu \circ (ev_{A,A} \otimes 1) + \mu \circ (1 \otimes ev_{A,A}) \circ (\sigma \otimes 1))(\partial \otimes a \otimes b). \end{aligned}$$

Since

$$ev_{A,A} \circ (1 \otimes \mu)(Q_q(\partial) \otimes a \otimes b) = ev_{A,A} \circ (\nu^r \otimes 1)(Q_q(\partial) \otimes a \otimes b)$$

and

$$\mu \circ (1 \otimes ev_{A,A}) \circ (\sigma \otimes 1) = ev_{A,A} \circ \left(\nu^l \otimes 1\right) \circ (\sigma \otimes 1),$$

clearly, by rearranging, the evaluation of $Q_q(\partial)$ on $a \in A_q$ is

$$\begin{aligned} ev_{A,A}(Q_q(\partial) \otimes a) &= \left(\nu^r - \nu^l \circ \sigma\right)(Q_q(\partial) \otimes a) \\ &= c^q_{\sigma_q}(Q_q(\partial) \otimes a). \end{aligned}$$

■

4.5 Quantization of braided derivations of modules

Consider derivations of a σ-symmetric algebra A and a σ-symmetric A-module E. Let $(\partial, \partial_A) \in Der^{(\sigma,A)}(E)$.

Definition 27 *Given a quantization q, define the quantization of (∂, ∂_A) by*

$$Q_q(\partial)(x) = ev_{A,A} \circ q(\partial \otimes x),$$

$x \in E$. If $x \in A$, then this is the quantization of ∂_A defined in section 4.3.

$Q_q(\partial)$ is an operator of the quantized module E_q. Denote by $Der^{(\sigma_q,A_q)}(E_q)$ the set of all $Q_q(\partial)$, $\partial \in Der^{(\sigma,A)}(E)$, equipped with the quantized composition.

Theorem 28 *Given a braiding σ, let σ_q be the quantization of σ. The operator*

$$\begin{aligned} Q_q &: \left(Der^{(\sigma,A)}(E), c_\sigma\right) \to \left(Der^{(\sigma_q,A_q)}(E_q), c^q_{\sigma_q}\right), \\ \partial &\in Der^{(\sigma,A)}(E) \longmapsto Q_q(\partial) \in Der^{(\sigma_q,A_q)}(E_q), \end{aligned} \tag{26}$$

is an isomorphism of modules between the σ-derivations of E over A and the σ_q-derivations of E_q over A_q.

By proposition 1 is $Der^{(\sigma_q,A_q)}(E_q)$ a σ_q-symmetric module and by theorem 9 a σ_q-Lie algebra with respect to the $\sigma_q - q$-bracket.

By theorem 13 the σ-Lie algebra of the structure of $\left(Der^{(\sigma_q,A_q)}(E_q), c^q_{\sigma_q}\right)$ can be realized within the classical one by dequantization.

5 Braided connections and curvatures

Let σ be a symmetry, A be a σ-symmetric algebra and E a σ-commutative A-module.

Definition 29 *A σ-connection in E is a A-module homomorphism ∇*

$$\nabla : Der^\sigma(A) \to Der^{(\sigma,A)}(E),$$

such that

$$\pi \circ \nabla = Id.$$

Definition 30 *A σ-connection ∇ is flat if it is a σ-Lie algebra homomorphism, that is,*

$$\nabla \circ c_\sigma = c_\sigma \circ (\nabla \otimes \nabla).$$

Definition 31 *In general, define the σ-curvature of ∇ to be*

$$K_\nabla : Der^\sigma(A) \otimes Der^\sigma(A) \to \hom(E,E),$$

$$K_\nabla = c_\sigma \circ (\nabla \otimes \nabla) - \nabla \circ c_\sigma.$$

Theorem 32 *The σ-curvature K_∇ is a σ-homomorphism of E, that is,*

$$ev_{E,E} \circ \left(K_\nabla \otimes \nu^l\right) = \nu^l \circ (1 \otimes ev_{E,E}) \circ (\sigma \otimes 1) \circ (K_\nabla \otimes 1 \otimes 1), \tag{i}$$

which maps $Der^\sigma(A) \otimes Der^\sigma(A) \otimes A \otimes E$ to E, and it is skew σ-symmetric,

$$K_\nabla = -K_\nabla \circ \sigma. \tag{ii}$$

Furthermore K_∇ satisfies

$$\begin{aligned} K_\nabla \circ \left(\nu^l \otimes 1\right) &= \nu^l \circ K_\nabla, & \text{(iii)} \\ K_\nabla \circ \left(1 \otimes \nu^l\right) &= \nu^l \circ K_\nabla \circ (\sigma \otimes 1). & \text{(iv)} \end{aligned}$$

Proof. (i): Note that

$$ev_{E,E} \circ \left(K_\nabla \otimes \nu^l\right) = ev_{E,E} \circ \left(1 \otimes \nu^l\right) \circ (K_\nabla \otimes 1 \otimes 1).$$

Then

$$ev_{E,E} \circ \left(K_\nabla \otimes \nu^l\right) = \nu^l \circ (1 \otimes ev_{E,E}) \circ (\sigma \otimes 1) \circ (K_\nabla \otimes 1 \otimes 1)$$

if and only if

$$ev_{E,E} \circ \left(1 \otimes \nu^l\right) = \nu^l \circ (1 \otimes ev_{E,E}) \circ (\sigma \otimes 1) : \hom(E,E) \otimes A \otimes E \to E.$$

By proposition 1 and by the symmetry of σ,

$$v^l_{E'} \circ (1 \otimes ev_{E,E'}) \circ \left(\sigma_{\hom(E,E'),A} \otimes 1\right) = ev_{E,E'} \circ \left(1 \otimes v^l_E\right) : \hom(E,E') \otimes A \otimes E \to E'$$

for σ-commutative A-modules E and E'.

(ii):

$$\begin{aligned} -K_\nabla \circ \sigma &= -c_\sigma \circ (\nabla \otimes \nabla) \circ \sigma + \nabla \circ c_\sigma \circ \sigma \\ &= -c_\sigma \circ \sigma \circ (\nabla \otimes \nabla) + \nabla \circ c_\sigma \circ \sigma \\ &= c_\sigma \circ (\nabla \otimes \nabla) - \nabla \circ c_\sigma \\ &= K_\nabla, \end{aligned}$$

by proposition 5.

(iii):

$$\begin{aligned} K_\nabla \circ \left(\nu^l \otimes 1\right) &= c_\sigma \circ (\nabla \otimes \nabla) \circ \left(\nu^l \otimes 1\right) - \nabla \circ c_\sigma \circ \left(\nu^l \otimes 1\right) \\ &= c_\sigma \circ \left(\nu^l \otimes 1\right) \circ (1 \otimes \nabla \otimes \nabla) - \nabla \circ c_\sigma \circ \left(\nu^l \otimes 1\right) \\ &= \nu^l \circ (1 \otimes c_\sigma) \circ (1 \otimes \nabla \otimes \nabla) - \nu^l \circ (ev_{A,A} \otimes 1) \circ \sigma \circ (1 \otimes \nabla \otimes \nabla) \\ &\quad -\nabla \circ \nu^l \circ (1 \otimes c_\sigma) + \nabla \circ \nu^l \circ (ev_{A,A} \otimes 1) \circ \sigma \\ &= \nu^l \circ (1 \otimes c_\sigma) \circ (1 \otimes \nabla \otimes \nabla) - \nu^l \circ (1 \otimes \nabla) \circ (1 \otimes c_\sigma) \\ &= \nu^l \circ K_\nabla, \end{aligned}$$

if and only if

$$\nu^l \circ (ev_{A,A} \otimes 1) \circ \sigma \circ (1 \otimes \nabla \otimes \nabla) = \nabla \circ \nu^l \circ (ev_{A,A} \otimes 1) \circ \sigma,$$

which clearly is satisfied.

(iv):

$$\begin{aligned}
& K_\nabla \circ \left(1 \otimes \nu^l\right) = c_\sigma \circ (\nabla \otimes \nabla) \circ \left(1 \otimes \nu^l\right) - \nabla \circ c_\sigma \circ \left(1 \otimes \nu^l\right) \\
= \ & c_\sigma \circ \left(1 \otimes \nu^l\right) \circ (\nabla \otimes 1 \otimes \nabla) - \nabla \circ c_\sigma \circ \left(1 \otimes \nu^l\right) \\
= \ & \nu^l \circ (1 \otimes c_\sigma) \circ (\sigma \otimes 1) \circ (\nabla \otimes 1 \otimes \nabla) + \nu^l \circ (ev_{A,A} \otimes 1) \circ (\nabla \otimes 1 \otimes \nabla) \\
& -\nabla \circ \nu^l \circ (1 \otimes c_\sigma) \circ (\sigma \otimes 1) - \nabla \circ \nu^l \circ (ev_{A,A} \otimes 1) \\
= \ & \nu^l \circ (1 \otimes c_\sigma) \circ (1 \otimes \nabla \otimes \nabla) \circ (\sigma \otimes 1) - \nu^l \circ (1 \otimes \nabla) \circ (1 \otimes c_\sigma) \circ (\sigma \otimes 1) \\
= \ & \nu^l \circ K_\nabla \circ (\sigma \otimes 1) .
\end{aligned}$$

Note that

$$\sigma \circ (\nabla \otimes 1) = (1 \otimes \nabla) \circ \sigma$$

by the naturality of braidings. ■

5.1 Quantization of braided connections and curvatures

Let ∇ be a σ-connection in E. Then the quantization of ∇,

$$Q_q(\nabla) = \nabla_q : Der^{\sigma_q}(A_q) \to Der^{(\sigma_q, A_q)}(E_q) .$$

is defined by

$$\nabla_q \overset{def}{=} Q_q \circ \nabla \circ Q_q^{-1}, \tag{27}$$

that is, the following diagram commutes

$$\begin{tikzcd}
Der^{(\sigma,A)}(E) \arrow[d, "Q_q"'] & Der^{\sigma}(A) \arrow[l, "\nabla"'] \arrow[d, "Q_q"] \\
Der^{(\sigma_q,A_q)}(E_q) & Der^{\sigma_q}(A_q) \arrow[l, "\nabla_q"']
\end{tikzcd} .$$

Proposition 33 *The quantization ∇_q is a σ_q-connection in E_q.*

Proof. Clearly,

$$\pi_q \circ \nabla_q = Id,$$

as

$$\pi_q = Q_q \circ \pi \circ Q_q^{-1}.$$

■

Let ∇_q be a σ_q-connection in E_q. Then the $\sigma_q - q$-curvature of ∇_q (or simply σ_q-curvature when it is clear that the multiplication or composition is the quantized multiplication),

$$K^q_{\nabla_q} : Der^{\sigma_q}(A_q) \cdot_{\sigma_q} Der^{\sigma_q}(A_q) \to End_{A_q}(E_q) ,$$

is defined by

$$K^q_{\nabla_q} = c^q_{\sigma_q} \circ (\nabla_q \otimes \nabla_q) - \nabla_q \circ c^q_{\sigma_q} .$$

Let us state the properties of the $\sigma_q - q$-curvature.

Theorem 34 *The $\sigma_q - q$-curvature is a σ_q-homomorphism, that is,*

$$ev_{E,E} \circ \left(K^q_{\nabla_q} \otimes \nu^l_q\right) = \nu^l_q \circ (1 \otimes ev_{E,E}) \circ (\sigma_q \otimes 1) \circ \left(K^q_{\nabla_q} \otimes 1 \otimes 1\right), \tag{i}$$

which maps $Der^{\sigma_q}(A_q) \otimes Der^{\sigma_q}(A_q) \otimes A_q \otimes E_q$ to E_q, and is skew σ_q-symmetric,

$$K^q_{\nabla_q} = -K^q_{\nabla_q} \circ \sigma_q. \tag{ii}$$

Furthermore $K^q_{\nabla_q}$ satisfies

$$\begin{aligned} K^q_{\nabla_q} \circ \left(\nu^l_q \otimes 1\right) &= \nu^l_q \circ \left(1 \otimes K^q_{\nabla_q}\right), && \textit{(iii)} \\ K^q_{\nabla_q} \circ \left(1 \otimes \nu^l_q\right) &= \nu^l_q \circ K^q_{\nabla_q} \circ (\sigma_q \otimes 1), && \textit{(iv)} \end{aligned}$$

$\forall a \in A$.

Proof. (i): See the proof of (i) of theorem 32.

(ii):

$$\begin{aligned} K^q_{\nabla_q} &= c^q_{\sigma_q} \circ (\nabla_q \otimes \nabla_q) - \nabla_q \circ c^q_{\sigma_q} \\ &= -c^q_{\sigma_q} \circ \sigma_q \circ (\nabla_q \otimes \nabla_q) + \nabla_q \circ c^q_{\sigma_q} \circ \sigma_q \\ &= -K_{\nabla_q} \circ \sigma_q. \end{aligned}$$

(iii):

$$\begin{aligned} & K^q_{\nabla_q} \circ \left(\nu^l_q \otimes 1\right) \\ = \; & c^q_{\sigma_q} \circ (\nabla_q \otimes \nabla_q) \circ \left(\nu^l_q \otimes 1\right) - \nabla_q \circ c^q_{\sigma_q} \circ \left(\nu^l_q \otimes 1\right) \\ = \; & c^q_{\sigma_q} \circ \left(\nu^l_q \otimes 1\right) \circ (1 \otimes \nabla_q \otimes \nabla_q) - \nabla_q \circ c^q_{\sigma_q} \circ \left(\nu^l_q \otimes 1\right) \\ = \; & \nu^l_q \circ \left(1 \otimes c^q_{\sigma_q}\right) \circ (1 \otimes \nabla_q \otimes \nabla_q) - \nu^l_q \circ (ev_{A,A} \otimes 1) \circ \sigma_q \circ (1 \otimes \nabla_q \otimes \nabla_q) \\ & -\nabla_q \circ \nu^l_q \circ \left(1 \otimes c^q_{\sigma_q}\right) + \nabla_q \circ \nu^l_q \circ (ev_{A,A} \otimes 1) \circ \sigma_q \\ = \; & \nu^l_q \circ \left(1 \otimes c^q_{\sigma_q}\right) \circ (1 \otimes \nabla_q \otimes \nabla_q) - \nu^l_q \circ (1 \otimes \nabla_q) \circ \left(1 \otimes c^q_{\sigma_q}\right) \\ = \; & \nu^l_q \circ \left(1 \otimes K^q_{\nabla_q}\right) \end{aligned}$$

if and only if,

$$\nu^l_q \circ (ev_{A,A} \otimes 1) \circ \sigma_q \circ (1 \otimes \nabla_q \otimes \nabla_q) = \nabla_q \circ \nu^l_q \circ (ev_{A,A} \otimes 1) \circ \sigma_q,$$

which clearly is satisfied. (iv):

$$\begin{aligned} & K^q_{\nabla_q} \circ \left(1 \otimes \nu^l_q\right) = c^q_{\sigma_q} \circ (\nabla_q \otimes \nabla_q) \circ \left(1 \otimes \nu^l_q\right) - \nabla_q \circ c^q_{\sigma_q} \circ \left(1 \otimes \nu^l_q\right) \\ = \; & \nu^l_q \circ \left(1 \otimes c^q_{\sigma_q}\right) \circ (\sigma_q \otimes 1) \circ (1 \otimes \nabla_q \otimes \nabla_q) + \nu^l_q \circ (ev_{A,A} \otimes 1) \circ (1 \otimes \nabla_q \otimes \nabla_q) \\ & -\nabla_q \circ \nu^l_q \circ \left(1 \otimes c^q_{\sigma_q}\right) \circ (\sigma_q \otimes 1) - \nabla_q \circ \nu^l_q \circ (ev_{A,A} \otimes 1) \\ = \; & \nu^l_q \circ \left(1 \otimes c^q_{\sigma_q}\right) \circ (\sigma_q \otimes 1) \circ (1 \otimes \nabla_q \otimes \nabla_q) - \nabla_q \circ \nu^l_q \circ \left(1 \otimes c^q_{\sigma_q}\right) \circ (\sigma_q \otimes 1) \\ = \; & \nu^l_q \circ K^q_{\nabla_q} \circ (\sigma_q \otimes 1). \end{aligned}$$

■

We have the following condition for the braided curvature of dequantizations of braided derivations.

Theorem 35 *The σ_q-curvature of the connection ∇_q defined by*

$$K_{\nabla_q} = c_{\sigma_q} \circ (\nabla_q \otimes \nabla_q) - \nabla_q \circ c_{\sigma_q}, \tag{28}$$

and the $\sigma_q - q$-curvature K^q_∇ of the σ-connection ∇ of E defined by

$$K^q_\nabla = c^q_{\sigma_q} \circ (\nabla \otimes \nabla) - \nabla \circ c^q_{\sigma_q}, \tag{29}$$

are related as follows,

$$Q_q^{-1} \circ K_{\nabla_q} = K^q_\nabla \circ \left(Q_q^{-1} \otimes Q_q^{-1}\right). \tag{30}$$

Proof. Proof of (30):

$$\begin{aligned}
K_{\nabla_q} &= c_{\sigma_q} \circ (\nabla_q \otimes \nabla_q) - \nabla_q \circ c_{\sigma_q} \\
&= Q_q \circ c^q_{\sigma_q} \circ \left(Q_q^{-1} \otimes Q_q^{-1}\right) \circ \left(\left(Q_q \circ \nabla \circ Q_q^{-1}\right) \otimes \left(Q_q \circ \nabla \circ Q_q^{-1}\right)\right) \\
&\quad - Q_q \circ \nabla \circ Q_q^{-1} \circ Q_q \circ c^q_{\sigma_q} \circ \left(Q_q^{-1} \otimes Q_q^{-1}\right) \\
&= Q_q \circ c^q_{\sigma_q} \circ (\nabla \otimes \nabla) \circ \left(Q_q^{-1} \otimes Q_q^{-1}\right) - Q_q \circ \nabla \circ c^q_{\sigma_q} \circ \left(Q_q^{-1} \otimes Q_q^{-1}\right) \\
&= Q_q \circ K^q_\nabla \circ \left(Q_q^{-1} \otimes Q_q^{-1}\right).
\end{aligned}$$

■

6 Braided differential operators

We shall see how the picture is for braided differential operators.

6.1 Braided differential operators in algebras

Let σ be a braiding and A be a σ-symmetric algebra.

Recall that the module

$$(\hom(A, A))^{(k)}_\sigma = Diff^\sigma_k(A, A)$$

is called the braided or σ-differential operators in A order at most k.

An equivalent and more familiar way to define a σ-differential operator of order at most k is the linear map

$$f : A \to A,$$

such that

$$c_\sigma \circ (1 \otimes c_\sigma) \circ \cdots \circ \left(\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes c_\sigma\right)(a_0 \otimes a_1 \otimes \cdots \otimes a_k \otimes f) = 0 \tag{31}$$

$\forall a_0, \ldots, a_k \in A$. Denote by $Diff^\sigma_k(A, A)$ the set of σ-differential operators of order at most k. Note,

$$f \in Diff^\sigma_k(A, A) \Leftrightarrow c_\sigma(f \otimes a) \in Diff^\sigma_{k-1}(A, A), \forall a \in A.$$

Let $Diff^\sigma(A, A) = \cup Diff^\sigma_k(A, A)$.

From [15] we have the following two results.

Proposition 36 *The σ-commutator of two σ-differential operators:*

$$f \in Diff_i^{\sigma}(A, A) \text{ and } g \in Diff_j^{\sigma}(A, A)$$

is a σ-differential operator of order at most $i + j - 1$,

$$c_\sigma (f \otimes g) \in Diff_{i+j-1}^{\sigma}(A, A).$$

The next result also follows from theorem 9.

Corollary 37 *If σ is a symmetry and an algebra A is σ-symmetric then $Diff^{\sigma}(A, A)$ is a σ-Lie algebra.*

Proposition 38 *There is an $A - A$-module structure on $Diff^{\sigma}(A, A)$ defined by*

$$\nu_a^l (f)(b) = \nu^l (a \otimes f)(b) = af(b), \tag{32}$$
$$\nu_a^r (f)(b) = \nu^r (f \otimes a)(b) = f(ab), \tag{33}$$

$a, b \in A$, $f \in Diff^{\sigma}(A, A)$, and $\nu^l (a \otimes f)$, $\nu^r (f \otimes a) \in Diff_k^{\sigma}(A, A)$, for $f \in Diff_k^{\sigma}(A, A)$.

Proof. Let

$$\sigma^{i,i+1} = \left(\underbrace{1 \otimes \cdots \otimes 1}_{i-1} \otimes \sigma \otimes \underbrace{1 \otimes \cdots \otimes 1}_{(k+3)-(i+1)} \right),$$

than the left action on a braided differential operator again is a braided differential operator,

$$\begin{aligned}
& c_\sigma \circ (1 \otimes c_\sigma) \circ \cdots \circ \left(\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes c_\sigma \right) \circ \left(\underbrace{1 \otimes \cdots \otimes 1}_{k+1} \otimes \nu^l \right) \\
= \ & c_\sigma \circ \cdots \circ \left(\underbrace{1 \otimes \cdots \otimes 1}_{k-1} \otimes c_\sigma \right) \circ \left(\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes \left(\left(\nu^l + ev_{A,A} \circ \sigma \right) \circ \left(1 \otimes \nu^l \right) \right) \right) \\
= \ & c_\sigma \circ \cdots \circ \left(\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes \left(\left(\nu^l \circ (\mu \otimes 1) + \mu \circ (1 \otimes ev_{A,A}) \circ (1 \otimes \sigma) \right) \circ (\sigma \otimes 1) \right) \right) \\
= \ & c_\sigma \circ \cdots \circ \left(\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes \nu^l \right) \circ \left(\underbrace{1 \otimes \cdots \otimes 1}_{k+1} \otimes c_\sigma \right) \circ \left(\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes \sigma \otimes 1 \right) \\
& \vdots \\
= \ & \nu^l \circ (1 \otimes c_\sigma) \circ \cdots \circ \left(\underbrace{1 \otimes \cdots \otimes 1}_{k+1} \otimes c_\sigma \right) \circ \sigma^{1,2} \circ \cdots \circ \sigma^{k+1,k+2} \\
= \ & 0,
\end{aligned}$$

when applied to $a_0 \otimes a_1 \otimes \cdots \otimes a_k \otimes b \otimes f$, and also the right action on a braided differential operator again is a braided differential operator,

$$\begin{aligned}
& c_\sigma \circ (1 \otimes c_\sigma) \circ \cdots \circ \left(\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes c_\sigma \right) \circ \left(\underbrace{1 \otimes \cdots \otimes 1}_{k+1} \otimes \nu^r \right) \\
= \; & c_\sigma \circ \cdots \circ \left(\underbrace{1 \otimes \cdots \otimes 1}_{k-1} \otimes c_\sigma \right) \circ \left(\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes \left(\left(\nu^l + ev_{A,A} \circ \sigma \right) \circ (1 \otimes \nu^r) \right) \right) \\
= \; & c_\sigma \circ \cdots \circ \left(\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes \left(\nu^r \circ \left(\nu^l \otimes 1 \right) + ev_{A,A} \circ (1 \otimes \mu) \circ (1 \otimes \sigma) \circ (\sigma \otimes 1) \right) \right) \\
= \; & c_\sigma \circ \cdots \circ \left(\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes \nu^r \right) \circ \left(\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes c_\sigma \otimes 1 \right) \\
& \vdots \\
= \; & \nu^r \circ (c_\sigma \otimes 1) \circ \cdots \circ \left(\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes c_\sigma \otimes 1 \right) \\
= \; & 0,
\end{aligned}$$

when applied to $a_0 \otimes a_1 \otimes \cdots \otimes a_k \otimes f \otimes b$, $a_0, \ldots, a_k, b \in A$, $f \in Diff_k^\sigma (A, A)$. ■

Consider the symbol of the differential operators which is the leading part with respect to derivatives,

$$Smbl_k^\sigma (A, A) = Diff_k^\sigma (A, A) / Diff_{k-1}^\sigma (A, A) ,$$

then we have the $\mathbb{Z}$-graded object

$$Smbl^\sigma (A, A) = \sum Smbl_k^\sigma (A, A) .$$

The class of $[f, g]^\sigma \in Diff_{i+j-1}^\sigma (A, A)$,

$$\overline{[f, g]^\sigma} \in Smbl_{i+j-1}^\sigma (A, A) ,$$

depends on the class of the two σ-differential operators $f \in Diff_i^\sigma (A, A)$ and $g \in Diff_j^\sigma (A, A)$, hence there is a σ-Poisson structure on the braided symbol algebra, [15].

6.2 Braided differential operators in modules

Let σ be a braiding, A be a σ-symmetric algebra and let E be a σ-symmetric A-module.

Definition 39 *The module*

$$(\hom (E, E))_\sigma^{(k)} = Diff_k^\sigma (E, E)$$

is called the braided or σ-differential operators in order at most k of E.

Denote the σ-differential operators of order at most k in E by $Diff_k^{(\sigma,A)}(E,E)$ and we consider $Diff^{(\sigma,A)}(E,E) = \cup Diff_k^{(\sigma,A)}(E,E)$.

From [15] we have the following two results.

Proposition 40 *The σ-commutator of two σ-differential operators*

$$f \in Diff_i^{(\sigma,A)}(E,E) \text{ and } g \in Diff_j^{(\sigma,A)}(E,E),$$

is given by

$$[f,g]^\sigma \in Diff_{i+j}^{(\sigma,A)}(E,E),$$

and so has order at most $i+j$.

Corollary 41 *If σ is a symmetry and an algebra A and a left A-module E are σ-symmetric then $Diff^\sigma(E,E)$ is a σ-Lie algebra.*

We consider the $A-A$-module structure on $Diff^\sigma(E,E)$.

Proposition 42 *There is an $A-A$-module structure on $Diff^\sigma(E,E)$ defined by*

$$\begin{aligned} \nu_a^l(f)(x) &= \nu^l(a\otimes f)(x) = af(x), \\ \nu_a^r(f)(x) &= \nu^r(f\otimes a)(x) = f(ax), \end{aligned}$$

$a\in A$, $x\in E$, $f\in Diff^\sigma(E,E)$.

Proof. The proof is the same as for proposition 38. ■

Consider the symbol of the differential operators which is the leading part with respect to derivatives,

$$Smbl_k^\sigma(E,E) = Diff_k^\sigma(E,E)/Diff_{k-1}^\sigma(E,E),$$

then we have the $\mathbb{Z}$-graded object

$$Smbl^\sigma(E,E) = \sum Smbl_k^\sigma(E,E).$$

The class of $[f,g]^\sigma \in Diff_{i+j}^\sigma(E,E)$,

$$\overline{[f,g]^\sigma} \in Smbl_{i+j}^\sigma(E,E),$$

depends on the class of the two σ-differential operators $f\in Diff_i^\sigma(E,E)$ and $g\in Diff_j^\sigma(E,E)$, hence there is a σ-Poisson structure on the braided symbol algebra.

6.3 Quantizations of braided differential operators in algebras

We can define quantization of σ-differential operators in algebras. Let A be a σ-commutative algebra.

Definition 43 *Given a quantization q and $f\in Diff^\sigma(A,A)$ define the quantization of f by*

$$Q_q(f)(a) = f_q(a) \overset{def}{=} ev_{A,A}\circ q(f\otimes a),$$

$a\in A$.

$Q_q(f)$ is an operator of the quantized algebra A_q. From [14] we have the following theorem.

Theorem 44 *Given a braiding σ, let σ_q be the quantization of σ. The operator*

$$\begin{aligned} Q_q &: \left(Diff^{\sigma}(A,A), c_\sigma\right) \to \left(Diff^{\sigma_q}(A_q,A_q), c^q_{\sigma_q}\right), \\ f &\in Diff^{\sigma}(A,A) \longmapsto Q_q(f) \in Diff^{\sigma_q}(A_q,A_q), \end{aligned} \tag{34}$$

is an isomorphism of modules.

The symbol of Q_q is an isomorphism of modules

$$\begin{aligned} Smbl(Q_q) &: \left(Smbl^{\sigma}(A,A), c_\sigma\right) \to \left(Smbl^{\sigma_q}(A_q,A_q), c^q_{\sigma_q}\right), \\ f &\in Smbl^{\sigma}(A,A) \longmapsto Smbl(Q_q)(f) \in Smbl^{\sigma_q}(A_q,A_q). \end{aligned} \tag{35}$$

By proposition 38 is $Diff^{\sigma_q}(A_q,A_q)$ a σ_q-symmetric module. By corollary 37, if σ is a symmetry then $Diff^{\sigma_q}(A_q,A_q)$ is a σ_q-Lie algebra with respect to the $\sigma_q - q$-bracket and the quantized composition. Furthermore there is a σ_q-Poisson structure on the quantized braided symbol algebra $Smbl^{\sigma_q}(A_q,A_q)$.

By theorem 13 the σ_q-Lie algebra structure of $\left(Diff^{\sigma_q}(A_q,A_q), c^q_{\sigma_q}\right)$ can be realized within the classical one by dequantization.

6.4 Quantizations of braided differential operators in modules

Let A be a σ-symmetric algebra and let E be a σ-symmetric A-module.

Definition 45 *Given a quantization q and $f \in Diff^{(\sigma,A)}(E,E)$ define the quantization of f by*

$$Q_q(f)(x) = f_q(x) \overset{def}{=} ev_{E,E} \circ q(f \otimes x),$$

$x \in E$.

$Q_q(f)$ is an operator of the quantized module E_q.

Theorem 46 *Given a braiding σ, let σ_q be the quantization of σ. The operator*

$$\begin{aligned} Q_q &: \left(Diff^{(\sigma,A)}(E,E), c_\sigma\right) \to \left(Diff^{(\sigma_q,A_q)}(E_q,E_q), c^q_{\sigma_q}\right), \\ f &\in Diff^{(\sigma,A)}(E,E) \longmapsto Q_q(f) \in Diff^{(\sigma_q,A_q)}(E_q,E_q), \end{aligned} \tag{36}$$

is an isomorphism of modules.

The symbol of Q_q is an isomorphism of modules

$$\begin{aligned} Smbl(Q_q) &: \left(Smbl^{(\sigma,A)}(E,E), c_\sigma\right) \to \left(Smbl^{(\sigma_q,A_q)}(E_q,E_q), c^q_{\sigma_q}\right), \\ f &\in Smbl^{(\sigma,A)}(E,E) \longmapsto Smbl(Q_q)(f) \in Smbl^{(\sigma_q,A_q)}(E_q,E_q). \end{aligned} \tag{37}$$

Proof. The isomorphism as σ-differential operators and σ-symbols is shown in [14]. ■

By proposition 42 is $Diff^{(\sigma_q,A_q)}(E_q,E_q)$ a σ_q-symmetric module. By corollary 41, if σ is a symmetry then $Diff^{(\sigma_q,A_q)}(E_q,E_q)$ is a σ_q-Lie algebra with respect to the $\sigma_q - q$-bracket and the quantized composition. Furthermore there is a σ_q-Poisson structure on the quantized braided symbol algebra, $Smbl^{(\sigma_q,A_q)}(E_q,E_q)$.

By theorem 13 the σ_q-Lie algebra structure of $\left(Diff^{(\sigma_q,A_q)}(E_q,E_q), c^q_{\sigma_q}\right)$ can be realized within the classical one by dequantization.

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