Lobachevskii Journal of Mathematics
http://ljm.ksu.ru
ISSN 1818-9962
Vol. 24, 2006, 135-142

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POSITIVE SOLUTIONS FOR A SINGULAR SECOND ORDER ORDINARY DIFFERENTIAL EQUATION

(submitted by A. V. Lapin)

Abstract. This paper is concerned with the positive solutions for a singular second order ordinary differential equation. Under appropriate conditions, by the classical method of elliptic regularization, we prove the existence of position solutions.

1. Introduction

This paper is concerned with the existence of positive solutions for a singular second order ordinary differential equation

$$\frac{\varphi''}{\sqrt{1+|\varphi'|^2}} - \lambda\frac{|\varphi'|^2}{\varphi(1+|\varphi'|^2)^m} + f(t) = 0, \qquad 0 < t < 1, \tag{1}$$

with one of the following boundary conditions

$$\varphi(1) = \varphi(0) = 0, \tag{2}$$

$$\varphi(1) = \varphi(0) = \varphi'(1) = \varphi'(0) = 0, \tag{3}$$

where $\lambda > 0, m \geq \frac{1}{2}, f(t) \in C^1[0,1]$ and $f(t) > 0$ on $[0,1]$.

2000 Mathematical Subject Classification. 34B15.
Key words and phrases. singular differential equation, positive solution, existence.
Supported by 985 Program of Jilin University.

It is well known that boundary value problems (BVPs) for singular second order ordinary differential equations arise in the field of gas dynamics, flow mechanics, theory of boundary layer, and so on. In recent years, singular ordinary differential equation with dependence on the first order derivative have been studied extensively, see for example [1–8] and references therein where some general existence results were obtained. We point out that the case considered here is not in their considerations since it does not satisfy some sufficient conditions of those papers. Our considerations were motivated by [9] in which the authors studied the following singular differential equation

$$\varphi'' - \lambda \frac{|\varphi'|^2}{\varphi} + 1 = 0,\ 0 < t < 1,$$

with the boundary conditions: $\varphi(1) = \varphi'(0) = 0$. By ordinary differential equation theories, they obtained a decreasing positive solution. In the present paper, we consider (1) and will use the classical method of elliptic regularization to obtain positive solutions to BVP (1), (2) and BVP (1), (3). However, it is easy to see from the boundary conditions (2) (or (3)) that any positive solution to BVP (1), (2) (or BVP (1), (3)) must not be decreasing. Thus the existence results obtained here are not a simple extension of [9]

We say $\varphi \in C^2(0,1) \cap C[0,1]$ is a solution to BVP (1),(2) if it is positive in (0,1) and satisfies (1) and (2). Similarly, we say $\varphi \in C^2(0,1) \cap C^1[0,1]$ is a solution to BVP (1),(3) if it is positive in (0,1) and satisfies (1) and (3).

The main purpose of this paper is to prove the following theorems.

Theorem 1 Let $m \geq \frac{1}{2}, \lambda > 0, f(t) \in C^1[0,1]$ and $0 < \min_{[0,1]} f \leq \max_{[0,1]} f < 1$. Then BVP (1),(2) admits at least a solution.

Theorem 2 Let $m \geq \frac{1}{2}, f(t) \in C^1[0,1], 0 < \min_{[0,1]} f \leq \max_{[0,1]} f < 1$ and $\lambda > \frac{1}{2}[1 - (\max_{[0,1]} f)^2]^{1/2-m}$. Then BVP (1),(3) admits at least a solution.

2. Proofs of Theorems

We will use the classical method of elliptic regularization to prove Theorem 1. For this, we consider the following regularized problem:

$$\frac{\varphi''}{\sqrt{1+|\varphi'|^2}} - \lambda \frac{|\varphi'|^2}{(1+|\varphi'|^2)^m} \frac{\operatorname{sgn}_\varepsilon(\varphi)}{I_\varepsilon(\varphi)} + f(t) = 0,\ 0 < t < 1,$$

$$\varphi(1) = \varphi(0) = \varepsilon,$$

where $\varepsilon \in (0,1)$, $I_\varepsilon(s)$ and $\mathrm{sgn}_\varepsilon(s)$ are defined as follows:

$$I_\varepsilon(s) = \begin{cases} s, & s \geq \varepsilon, \\ \dfrac{s^2+\varepsilon^2}{2\varepsilon}, & |s| < \varepsilon, \\ -s, & s \leq -\varepsilon, \end{cases} \quad \mathrm{sgn}_\varepsilon(s) = \begin{cases} 1, & s \geq \varepsilon, \\ \dfrac{2s}{\varepsilon} - \dfrac{s^2}{\varepsilon^2}, & 0 \leq s < \varepsilon, \\ \dfrac{2s}{\varepsilon} + \dfrac{s^2}{\varepsilon^2}, & -\varepsilon \leq s < 0, \\ -1, & s < -\varepsilon. \end{cases}$$

Clearly, $I_\varepsilon(s), \mathrm{sgn}_\varepsilon(s) \in C^1(\mathbb{R})$, and $I_\varepsilon(s) \geq \varepsilon/2, 1 \geq |\mathrm{sgn}_\varepsilon(s)|, \mathrm{sgn}_\varepsilon(s) \cdot \mathrm{sgn}(s) \geq 0$ in $\mathbb{R}$.

For any $m \geq \frac{1}{2}$, it follows from Theorem 4.1 of Chapter 7 in [10] that for any fixed $\varepsilon \in (0,1)$, the above regularized problem admits a classical solution $\varphi_\varepsilon \in C^2(0,1) \cap C^1[0,1]$. By the maximal principle, it is easy to see that $\varphi_\varepsilon(t) \geq \varepsilon$ on $[0,1]$. Thus φ_ε satisfies

$$\frac{\varphi_\varepsilon''}{\sqrt{1+|\varphi_\varepsilon'|^2}} - \lambda \frac{|\varphi_\varepsilon'|^2}{\varphi_\varepsilon(1+|\varphi_\varepsilon'|^2)^m} + f(t) = 0, \ 0 < t < 1, \tag{4}$$

$$\varphi_\varepsilon(0) = \varphi_\varepsilon(1) = \varepsilon.$$

Note that (4) is equivalent to

$$\left(\int_0^{\varphi_\varepsilon'(t)} \frac{1}{\sqrt{1+s^2}} ds\right)' - \lambda \frac{|\varphi_\varepsilon'|^2}{\varphi_\varepsilon(1+|\varphi_\varepsilon'|^2)^m} + f(t) = 0, \ 0 < t < 1. \tag{5}$$

Lemma 1 Under the assumptions of Theorem 1, for all $\varepsilon \in (0,1)$ there holds

$$|\varphi_\varepsilon'(t)| \leq \frac{\Lambda}{\sqrt{1-\Lambda^2}}, \ t \in [0,1],$$

where $\Lambda \triangleq \max_{[0,1]} f < 1$.

Proof. Noticing $\varphi_\varepsilon(1) = \varphi_\varepsilon(0) = \varepsilon$ and $\varphi_\varepsilon(t) \geq \varepsilon$ for all $t \in [0,1]$, we have

$$\varphi_\varepsilon'(0) = \lim_{t \to 0^+} \frac{\varphi_\varepsilon(t) - \varepsilon}{t} \geq 0,$$

$$\varphi_\varepsilon'(1) = \lim_{t \to 1^-} \frac{\varphi_\varepsilon(t) - \varepsilon}{t-1} \leq 0.$$

On the other hand, it follows from (5) that

$$\left(\int_0^{\varphi_\varepsilon'} \frac{1}{\sqrt{1+s^2}} ds\right)' + \Lambda \geq 0, \ 0 < t < 1,$$

i.e.

$$\left(\int_0^{\varphi_\varepsilon'(t)} \frac{1}{\sqrt{1+s^2}} ds + \Lambda t\right)' \geq 0, \ 0 < t < 1.$$

Thus the function $\int_0^{\varphi_\varepsilon'(t)} \frac{1}{\sqrt{1+s^2}}ds+\Lambda t$ is non-decreasing on $[0,1]$, therefore

$$\begin{aligned}\Lambda &\geq \int_0^{\varphi_\varepsilon'(1)} \frac{1}{\sqrt{1+s^2}}ds+\Lambda\\ &\geq \int_0^{\varphi_\varepsilon'(t)} \frac{1}{\sqrt{1+s^2}}ds+\Lambda t\\ &\geq \int_0^{\varphi_\varepsilon'(0)} \frac{1}{\sqrt{1+s^2}}ds \geq 0,\ t\in[0,1],\end{aligned}$$

and hence

$$\left|\int_0^{\varphi_\varepsilon'(t)} \frac{1}{\sqrt{1+s^2}}ds\right| \leq \Lambda,\ t\in[0,1].$$

From this and using the inequality

$$\left|\frac{z}{\sqrt{1+z^2}}\right| \leq \left|\int_0^{z} \frac{1}{\sqrt{1+s^2}}ds\right|,\ z\in\mathbb{R},$$

we obtain

$$\left|\frac{\varphi_\varepsilon'(t)}{\sqrt{1+|\varphi_\varepsilon'(t)|^2}}\right| \leq \Lambda,\ t\in[0,1],$$

Noticing $\Lambda<1$, we obtain

$$|\varphi_\varepsilon'(t)| \leq \frac{\Lambda}{\sqrt{1-\Lambda^2}},\ t\in[0,1].$$

This completes the proof of Lemma 1.

Denote Λ_m by $\Lambda_m=(1-\Lambda^2)^{1/2-m}$, where Λ is the same as that of Lemma 1. From (4) and Lemma 1 we obtain obtain

$$-\varphi_\varepsilon''+\lambda\frac{|\varphi_\varepsilon'|^2}{\varphi_\varepsilon}-\min_{[0,1]} f\geq 0,\ t\in(0,1). \tag{6}$$

$$-\varphi_\varepsilon''+\frac{\lambda}{\Lambda_m}\frac{|\varphi_\varepsilon'|^2}{\varphi_\varepsilon}-\frac{\Lambda}{(1-\Lambda^2)^{1/2}}\leq 0,\ t\in(0,1). \tag{7}$$

To obtain the uniform bounds of φ_ε, the following comparison theorem will be proved to be very useful.

Proposition 2 Let $\varphi_i\in C^2(0,1)\cap C[0,1]$ and $\varphi_i>0$ on $[0,1](i=1,2)$. If $\varphi_2\geq\varphi_1$ for $t=0,1$, and

$$-\varphi_2''+\varrho\frac{|\varphi_2'|^2}{\varphi_2}-\theta\geq 0,\ t\in(0,1), \tag{8}$$

$$-\varphi_1''+\varrho\frac{|\varphi_1'|^2}{\varphi_1}-\theta\leq 0,\ t\in(0,1), \tag{9}$$

where ϱ and θ are positive constants, then

$$\varphi_2(t)\geq\varphi_1(t),\ t\in[0,1].$$

Proof. From (8) and (9), we have

$$\Big(\frac{\varphi_2^{1-\varrho}}{1-\varrho}\Big)'' \leq -\frac{\theta}{\varphi_2^{\varrho}},\ (\varrho \neq 1)$$

$$\Big(\ln(\varphi_2)\Big)'' \leq -\frac{\theta}{\varphi_2}.\ (\varrho = 1)$$

and

$$\Big(\frac{\varphi_1^{1-\varrho}}{1-\varrho}\Big)'' \geq -\frac{\theta}{\varphi_1^{\varrho}},\ (\varrho \neq 1)$$

$$\Big(\ln(\varphi_1)\Big)'' \geq -\frac{\theta}{\varphi_1}.\ (\varrho = 1)$$

Combining the above inequalities, we obtain

$$w'' \leq \theta\Big(\frac{1}{\varphi_1^{\varrho}} - \frac{1}{\varphi_2^{\varrho}}\Big),\ 0 < t < 1, \tag{10}$$

where $w : [0,1] \to \mathbb{R}$ is defined by

$$w = \begin{cases} \dfrac{\varphi_2^{1-\varrho}}{1-\varrho} - \dfrac{\varphi_1^{1-\varrho}}{1-\varrho}, & (\varrho \neq 1) \\ \ln(\varphi_2) - \ln(\varphi_1). & (\varrho = 1) \end{cases}$$

Clearly, $w \in C^2(0,1) \cap C[0,1]$.

To prove the proposition, we argue by contradiction and assume that there exists a point t_0 of $(0,1)$ such that $\varphi_2(t_0) - \varphi_1(t_0) < 0$. From the assumption, it is easy to see that w reaches a minimum at some point t_* of $(0,1)$ such that

$$w(t_*) = \min_{t\in[0,1]} w(t) < 0, \tag{11}$$

$$w''(t_*) \geq 0. \tag{12}$$

Combining (12) with (10), we have

$$\theta\Big(\frac{1}{\varphi_1^{\varrho}(t_*)} - \frac{1}{\varphi_2^{\varrho}(t_*)}\Big) \geq 0.$$

This implies $\varphi_2(t_*) \geq \varphi_1(t_*)$. However, from (11) we find that $\varphi_2(t_*) < \varphi_1(t_*)$, a contradiction. Thus the proof of Proposition 2 is completed.

Lemma 3 Under the assumptions of Theorem 1, for all $\varepsilon \in (0,1)$ there exists a positive constant C independent of ε such that

$$\varphi_\varepsilon(t) \geq C[t(1-t) + \varepsilon^{1/2}]^2,\ t \in [0,1].$$

Proof. Let $w_\varepsilon = C[t(1-t) + \varepsilon^{1/2}]^2$, where $C \in (0,1]$ will be determined later. By Proposition 2 and noticing (6), it suffices to show that

$$-w_\varepsilon'' + \lambda\frac{|w_\varepsilon'|^2}{w_\varepsilon} - \min_{[0,1]} f \leq 0,\ t \in (0,1), \tag{13}$$

for some sufficiently small positive constant C independent of ε. Simple calculation shows that

$$w_\varepsilon' = 2C[t(1-t)+\varepsilon^{1/2}](1-2t),$$
$$w_\varepsilon'' = 2C(1-2t)^2 - 4C[t(1-t)+\varepsilon^{1/2}],$$

and

$$\begin{aligned}-w_\varepsilon'' + \lambda\frac{|w_\varepsilon'|^2}{w_\varepsilon} - \min_{[0,1]} f =& - 2C(1-2t)^2 + 4C[t(1-t)+\varepsilon^{1/2}]\\ &+ 4C\lambda(1-2t)^2 - \min_{[0,1]} f\\ \le& 4C\Big(2+\lambda\Big) - \min_{[0,1]} f,\ 0<t<1.\end{aligned}$$

Choosing a positive constant C such that

$$C \le \min\left\{1, \frac{\min_{[0,1]} f}{4(2+\lambda)}\right\},$$

we find that (13) holds. Thus the proof of Lemma 3 is completed.

From (6), (7), Lemma 1 and Lemma 3, we derive that for any $\delta \in (0,1/2)$ there exists a positive constant C_δ independent of ε such that

$$|\varphi_\varepsilon''(t)| \le C_\delta,\ \delta \le t \le 1-\delta. \tag{14}$$

From (4), we have

$$\varphi_\varepsilon'' = \lambda\frac{(1+|\varphi_\varepsilon'|^2)^{1/2-m}|\varphi_\varepsilon'|^2}{\varphi_\varepsilon} - f(t)(1+|\varphi_\varepsilon'|^2)^{1/2},\ 0<t<1.$$

Differentiating the above equation with respect to t we get

$$\begin{aligned}\varphi_\varepsilon''' =& \frac{2\lambda\varphi_\varepsilon'\varphi_\varepsilon''}{\varphi_\varepsilon}\Big[(1+|\varphi_\varepsilon'|^2)^{1/2-m} + \Big(\frac{1}{2}-m\Big)(1+|\varphi_\varepsilon'|^2)^{-m-1/2}|\varphi_\varepsilon'|^2\Big]\\ &- \frac{\lambda(1+|\varphi_\varepsilon'|^2)^{1/2-m}(\varphi_\varepsilon')^3}{\varphi_\varepsilon^2} - f(t)(1+|\varphi_\varepsilon'|^2)^{-1/2}\varphi_\varepsilon'\varphi_\varepsilon''\\ &- f'(t)(1+|\varphi_\varepsilon'|^2)^{1/2},\ 0<t<1.\end{aligned}$$

By (14), Lemma 1 and Lemma 3, we derive that for any $\delta \in (0,1/2)$, there exists a positive constant C_δ independent of ε such that

$$|\varphi_\varepsilon'''(t)| \le C_\delta,\ \delta \le t \le 1-\delta.$$

From this and Lemma 1 and using Alzelá-Ascoli theorem and diagonal sequential process, we see that there exists a subsequence $\{\varphi_{\varepsilon_n}\}$ of $\{\varphi_\varepsilon\}$ and a function $\varphi \in C^2(0,1)\cap C[0,1]$ such that, as $\varepsilon_n \to 0$,

$$\varphi_{\varepsilon_n} \to \varphi,\ \text{uniformly in } C[0,1],$$
$$\varphi_{\varepsilon_n} \to \varphi,\ \text{uniformly in } C^2[\delta, 1-\delta].$$

Combining these with (4) (or (5)) and the boundary conditions satisfied by φ_{ε_n}, we find that φ satisfies (1) and (2). By Lemma 3, we have

$$\varphi(t) \geq C[t(1-t)]^2, \ t \in [0,1], \tag{15}$$

therefore $\varphi > 0$ in (0,1), and thus φ is a solution to BVP (1), (2). This completes the proof of Theorem 1.

Proof of Theorem 2. From Theorem 1, we see that for any $\lambda > 0$, BVP (1),(2) admits a solution φ which can be approximated by φ_{ε_n} satisfying (4) (or (5)) with $\varepsilon = \varepsilon_n$. Hence it suffices to show φ satisfies $\varphi'(1) = \varphi'(0) = 0$ for $\lambda > \frac{1}{2}[1 - (\max_{[0,1]} f)^2]^{1/2-m} = \frac{1}{2}(1-\Lambda^2)^{1/2-m} = \frac{1}{2}\Lambda_m$. We claim that if $\lambda > \frac{1}{2}\Lambda_m$, then there exist positive constants C independent of ε_n such that

$$\varphi_{\varepsilon_n}(t) \leq C(1 + \varepsilon_n^{1/2} - t)^2 \text{ on } [0,1], \tag{16}$$

$$\varphi_{\varepsilon_n}(t) \leq C(t + \varepsilon_n^{1/2})^2 \text{ on } [0,1]. \tag{17}$$

We first show (16). Let $v_{\varepsilon_n} = C(1 + \varepsilon_n^{1/2} - t)^2$, where $C \geq 1$ will be determined later. A calculation shows that

$$\begin{aligned} &- v''_{\varepsilon_n} + \frac{\lambda}{\Lambda_m}\frac{|v'_{\varepsilon_n}|^2}{v_{\varepsilon_n}} - \frac{\Lambda}{(1-\Lambda^2)^{1/2}} \\ &= 2C\Big(\frac{2\lambda}{\Lambda_m} - 1\Big) - \frac{\Lambda}{(1-\Lambda^2)^{1/2}}, \ 0 < t < 1. \end{aligned}$$

Choosing a positive constant C such that

$$C \geq \max\left\{1, \frac{\Lambda_m \Lambda}{2(2\lambda - \Lambda_m)(1-\Lambda^2)^{1/2}}\right\}$$

and noticing $\lambda > \frac{1}{2}\Lambda_m$, we find that

$$-v''_{\varepsilon_n} + \frac{\lambda}{\Lambda_m}\frac{|v'_{\varepsilon_n}|^2}{v_{\varepsilon_n}} - \frac{\Lambda}{(1-\Lambda^2)^{1/2}} \geq 0, \ 0 < t < 1,$$

and then, by Proposition 2 and noticing (7), we obtain (16). Similarly we can prove the claim (17). Letting $\varepsilon_n \to 0$ in (16) and (17) to yield

$$\varphi(t) \leq C\min\{t^2, (1-t)^2\}, \ t \in [0,1].$$

Combining this with (15) we immediately obtain $\varphi'(1) = \varphi'(0) = 0$. Thus the proof of Theorem 2 is completed.

References

[1] R.P. Agarwal, D. O'Regan, Singular boundary value problems for superlinear second order ordinary and delay differential equations, J. Differential Equations, 130(1996), 333-355.

[2] D. O'Regan, Theory of Singular Boundary Value Problems, World Scientific, Singapore, 1994.

[3] D. O'Regan, Existence Theory for Nonlinear Differential Equations, Kluwer Acad., Dordrecht/Boston/London 1997.

[4] A. Tineo, Existence theorems for a singular two points Dirichlet problem, Nonlinear Anal. 19(1992), 323-333.

[5] J. Wang, J. Jiang, The existence of positive solutions to a singular nonlinear boundary value problem, J. Math. Anal. Appl. 176(1993), 322-329.

[6] S. Staněk, Positive solutions for singular semipositive boundary value problems, Math. Comput. Model, 33(2001), 341-351.

[7] D.Q. Jiang, Upper and lower solutions method and a singular boundary value problem, Z. Angew. Math. Mech. 82(7) (2002), 481-490.

[8] D. Bonheure, J. M. Gomes, L. Sanchez, Positive solutions of a second order singular ordinary differential equation, Nonlinear Anal. TMA. 61(2005), 1383-1399.

[9] M. Bertsch, M. Ughi, Positivity properties of viscosity solutions of a degenerate parabolic equation, Nonlinear Anal., 14(1990) 571-592.

[10] Y.Z. Chen, L.C. Wu, Second Order Elliptic Equations and Elliptic Systems, Science Press, Beijing, 1997. (in Chinese) English edition is translated from the 1991 Chinese original by Bei Hu. Translations of Mathematical Monographs, 174. American Mathematical Society, Providence, RI, 1998.

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Received August 26, 2006