Lobachevskii Journal of Mathematics
http://ljm.ksu.ru
ISSN 1818-9962
Vol. 25, 2007, 197–216

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Q-BOUNDED SYSTEMS: COMMON APPROACH TO FISHER-MICCHELLI'S AND BERNSTEIN-WALSH'S TYPE PROBLEMS

(submitted by F. G. Avkhadiev)

Abstract. We have developed a new common method to investigate geometrically fast approximation problems. Fisher-Micchelli's, Bernstein-Walsh's and Batirov-Varga's well known results are obtained as applications.

Introduction: Fisher-Micchelli's & Bernstein-Walsh's type problems

Let K be a compact subset of the open unit disk D, $H^\infty(D)$ be the set of bounded analytic functions on D and $C(K)$ be the set of continuous functions on K.

Then each function $f \in H^\infty(D)$ is approximable by finite linear combinations of the system of powers $\left\{z^k\right\}_0^\infty$ uniformly on K at a rate of geometrical progression and as approximants one can take Taylor polynomials, i.e.

$$\limsup_{n\to\infty} \sqrt[n]{\|f - P_n f\|_{C(K)}} < 1,$$

where $P_n(f)(z) := \sum\limits_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k$.

2000 Mathematical Subject Classification. 41A, 30E10, 30E15.
Key words and phrases. fast approximation, n-widths, biorthogonal systems.

If $f \in H(\mathbb{C})$ then one can obtain faster approximation

$$\lim_{n\to\infty} \sqrt[n]{\|f - P_n f\|_{C(K)}} = 0.$$

From now on approximation which is faster than geometrical progression will be called fast approximation.

In case when logarithmic capacity of K is positive, $\gamma(K) > 0$, Bernstein and Walsh [BW] obtained the following result.

BW type result (Bernstein-Walsh): *The class of functions* $f : K \to \mathbb{C}$ *permitting the fast approximation by finite linear combinations of the system* $\left\{z^k\right\}_0^\infty$ *in* $C(K)$, *coincides with* $H(\mathbb{C})$.

We have seen that to make the fast approximation of the class $H^\infty(D)$ we have to miniaturize it up to the class of entire functions. The natural question arises if we can fast approximate whole class $H^\infty(D)$ using some other system instead of $\left\{z^k\right\}_0^\infty$. Due to Fisher and Micchelli [FM] the answer is negative.

FM type result (Fisher-Micchelli) *There is no system of functions*

$$e_k^{(n)} : K \to \mathbb{C}, \;\; k = 1, 2, \ldots, n; \; n = 1, 2, \ldots$$

such that every function $f \in H^\infty(D)$ *admits the fast approximation by polynomials*

$$\sum_{k=1}^{n} a_k^{(n)} e_k^{(n)}$$

in $C(K)$.

Reasoning from these results, let us consider the following problems named Fisher-Micchelli's and Bernstein-Walsh's type problem, respectively.

Problems:

FM) *For a given space* H, *find a system* $e_k^{(n)}$ *such that each element of* H *admits the fast approximation by linear combinations* $\sum_{k=1}^{n} a_k^{(n)} e_k^{(n)}$.

BW) *For a given system* $e_k^{(n)}$, *find the class of elements permitting the fast approximation by linear combinations* $\sum_{k=1}^{n} a_k^{(n)} e_k^{(n)}$.

We have developed a common method to investigate both problems based on the notion of q-bounded systems.

1. q-bounded systems

Definition 1.1 Let X be a Banach space, $\left(e_k^{(n)}\right)_{k\le n} \subset X$, $n = 1, 2, \dots$ be a triangle matrix, and L_n be the linear span of the finite system $\left\{e_k^{(n)}\right\}_{k=1}^{n}$. For $q > 0$ the matrix $\left(e_k^{(n)}\right)_{k\le n}$ is called

i. *q-lower bounded in X, if for each sequence $P_n = \sum_{k=1}^{n} a_k^{(n)} e_k^{(n)} \in L_n$ one has*

$$\limsup_{n\to\infty} \sqrt[n]{\|P_n\|_X} \ge q \limsup_{n\to\infty} \sqrt[n]{\max_{1\le k\le n} \left|a_k^{(n)}\right|}.$$

ii. *q-upper bounded in X, if*

$$\limsup_{n\to\infty} \sqrt[n]{\max_{1\le k\le n} \left\|e_k^{(n)}\right\|_X} \le q.$$

Definition 1.2 *The matrix $\left(e_k^{(n)}\right)_{k\le n}$ is called 0-lower bounded (∞-upper bounded), if it is q-lower (upper) bounded for some $q \in (0, \infty)$.*

Definition 1.3 *The system $\{e_k\}_{k=1}^{\infty} \subset X$ is called q-lower (upper) bounded, if the matrix $\left(e_k^{(n)}\right)_{k\le n}$ is q-lower (upper) bounded for $e_k^{(n)} := e_k$, $k \le n$.*

Checking of q-upper boundedness is usually easy. As regards checking of q-lower boundedness, it seems difficult. The following lemma shows that checking of q-lower boundedness can be reduced to checking of q-upper boundedness of biorthogonal system.

Lemma 1.1 (Checking lower boundedness, [F]) *If the finite systems $\left\{\varphi_k^{(n)}\right\}_{k=1}^{n}$ and $\left\{e_k^{(n)}\right\}_{k=1}^{n}$ are biorthogonal for all $n \in N$ and the matrix $\left(\varphi_k^{(n)}\right)_{k\le n}$ is q-upper bounded in a normed space Y then the matrix $\left(e_k^{(n)}\right)_{k\le n}$ is $\frac{1}{q}$-lower bounded in the dual space Y^*.*

2. Basic lemma

There is a close relation between Kolmogorov n-widths and q-bounded systems.

Definition 2.1 *Let K be a subset of a normed linear space X. The quantity*

$$d_n(K, X) = \inf_{X_n \subset X} \sup_{x\in K} \inf_{y\in X_n} \|y - x\|_X ,$$

where the leftmost infimum is taken over all subspaces $X_n \subset X$ of dimension n, is called the Kolmogorov n-width of K in X.

Lemma 2.1(q-bounded systems and Kolmogorov n-widths)

Let H, X be Banach spaces and $H \subset X$. If there exists a matrix $\left(\varphi_k^{(n)}\right)_{k\le n} \subset H$ which is q_1-lower bounded in X and q_2-upper bounded in H, then

$$\limsup_{n\to\infty} \sqrt[n]{d_n(B_H, X)} \ge \frac{q_1}{q_2},$$

for the unit ball $B_H = \{x \in H : \|x\|_H \le 1\}$ of H.

Lemma 2.1 could be established by Tikhomirov's well known result.

Lemma (Tikhomirov, [T]) *If B_{n+1} is a unit ball of some $n+1$-dimensional subspace of a Banach space X, then $d_n(B_{n+1}, X) = 1$.*

Instead, we shall give here an elementary proof in the sense that it does not depend on Borsuk's theorem.

Proof. Assuming the converse, there is a positive number δ and the n-dimensional ($n \ge n_0$) subspaces $Y_n \subset X$ such that

$$\sup_{x\in B_H} \inf_{y\in Y_n} \sqrt[n]{\|y-x\|_X} < \frac{q_1}{q_2} - \delta \quad (n \ge n_0).$$

Let $\left\{e_1^{(n)}, \ldots, e_n^{(n)}\right\}$ be a basis of Y_n. Denoting

$$r_n(x) = \inf_{a_k^{(n)}} \sqrt[n]{\left\| x - \sum_{k=1}^{n} a_k^{(n)} e_k^{(n)} \right\|_X},$$

we have

$$\sup_{x\in B_H} r_n(x) < \frac{q_1}{q_2} - \delta \quad (n \ge n_0). \tag{1}$$

For each $x \in B_H$ take coefficients $a_k^{(n)}(x)$ such that

$$\limsup_{n\to\infty} \sqrt[n]{\left\| x - \sum_{k=1}^{n} a_k^{(n)}(x) e_k^{(n)} \right\|_X} = \limsup_{n\to\infty} r_n(x). \tag{2}$$

Since $\left(\varphi_k^{(n)}\right)_{k\le n}$ is q_2-upper bounded in H, it follows that

$$\psi_k^{(n)} := \frac{\varphi_k^{(n)}}{(q_2+\varepsilon)^n} \in B_H, \;\; k = 1, 2, \ldots, n, \tag{3}$$

for each positive ε beginning from some $N(\varepsilon)$.

The linear dependence of the system

$$\left\{ \left\langle a_1^{(n)}(\psi_i^{(n+1)}), a_2^{(n)}(\psi_i^{(n+1)}), \ldots, a_n^{(n)}(\psi_i^{(n+1)}) \right\rangle \right\}_{i=1}^{n+1} \text{ in } \mathbb{C}^n$$

implies the existence of coefficients $c_i^{(n+1)}$, $i = 1, \ldots, n+1$ such that $\sum\limits_{i=1}^{n+1} c_i^{(n+1)} a_k^{(n)}(\psi_i^{(n+1)}) = 0$, $k = 1, 2, \ldots, n$ and

$$\max_{1 \le i \le n+1} \left| c_i^{(n+1)} \right| = 1. \tag{4}$$

Denote $P_{n+1} = \sum\limits_{i=1}^{n+1} c_i^{(n+1)} \psi_i^{(n+1)}$. Combining (1) – (4), we obtain

$$\begin{aligned}
\frac{q_1}{q_2+\varepsilon} &\le \limsup_{n\to\infty} \sqrt[n+1]{\|P_{n+1}\|_X} \\
&= \limsup_{n\to\infty} \sqrt[n+1]{\left\| \sum_{i=1}^{n+1} c_i^{(n+1)} \left[\psi_i^{(n+1)} - \sum_{k=1}^{n} a_k^{(n)}(\psi_i^{(n+1)}) e_k^{(n)} \right] \right\|_X} \\
&\le \limsup_{n\to\infty} \sqrt[n+1]{\sum_{i=1}^{n+1} \left| c_i^{(n+1)} \right|} \\
&\qquad \times \limsup_{n\to\infty} \sqrt[n+1]{\max_{1\le i\le n+1} \left\| \psi_i^{(n+1)} - \sum_{k=1}^{n} a_k^{(n)}(\psi_i^{(n+1)}) e_k^{(n)} \right\|_X} \\
&= \limsup_{n\to\infty} \max_{1\le i\le n+1} r_{n+1}(\psi_i^{(n+1)}) \le \limsup_{n\to\infty} \sup_{x\in B_H} r_{n+1}(x) \le \frac{q_1}{q_2} - \delta.
\end{aligned}$$

This yields that $\frac{q_1}{q_2} - \delta \ge \frac{q_1}{q_2}$ and the contradiction proves the lemma.

The following *basic* lemma shows that checking of q-boundedness of even one system leads to solution of both problems at once.

Lemma 2.2 (Basic lemma)

FM) *Suppose $H \subset X$ are Banach spaces and $\|x\|_X \le C\|x\|_H$, $x \in H$. If exists a matrix $\left(\varphi_k^{(n)}\right)_{k\le n} \subset H$, which is q_1-lower bounded in X and q_2-upper bounded in H, then for every matrix $\left(e_k^{(n)}\right)_{k\le n} \subset X$ there is an element $x \in H$ such that*

$$\limsup_{n\to\infty} \sqrt[n]{\left\| x - \sum_{k=1}^{n} a_k^{(n)} e_k^{(n)} \right\|_X} \ge \frac{q_1}{q_2},$$

for all numerical matrices $\left(a_k^{(n)}\right)_{k\le n}$.

BW) *Let $\{e_k\}_{k=1}^{\infty}$ be a 0-lower and ∞-upper bounded system in a Banach space X. For $x \in X$ there are polynomials $P_n = \sum\limits_{k=1}^{n} a_k^{(n)} e_k$ satisfying $\sqrt[n]{\|x - P_n\|_X} \underset{n\to\infty}{\to} 0$ if and only if $x = \sum\limits_{k=1}^{\infty} x_k e_k$, $\sqrt[k]{|x_k|} \underset{k\to\infty}{\to} 0$.*

Proof. One can find *BW)* proof in [F] under even more general conditions.

FM) Assuming the converse, it is possible to take $\left(e_k^{(n)}\right)_{k\leq n} \subset X$ such that $\limsup\limits_{n\to\infty} r_n(x) < \frac{q_1}{q_2}$ $(\forall x \in H)$ for $r_n(x) := \inf\limits_{a_k^{(n)}} \sqrt[n]{\left\| x - \sum\limits_{k=1}^{n} a_k^{(n)} e_k^{(n)} \right\|_X}$.
Then let $H = \bigcup\limits_{i=1}^{\infty} H_i$, where $H_i := \left\{ x \in H : \limsup\limits_{n\to\infty} r_n(x) < \frac{q_1}{q_2} - \frac{1}{i} \right\}$. Since H is a Banach space, one can take some H_{i_0} being a set of the second category there.

Denote $q_0 = \frac{q_1}{q_2} - \frac{1}{i_0}$ and

$$E_k = \{x \in H : r_n(x) \leq q_0 \ \ for \ \ n \geq k\}.$$

As $\|*\|_H$ is stronger than $\|*\|_X$, the sets E_k are closed in H. Since $H_{i_0} \subset \bigcup\limits_{k=1}^{\infty} E_k$, then some E_{k_0} contains a ball in H, that is the estimates

$$r_n(x_0 + \mu x) \leq q_0 \quad (n \geq k_0)$$

hold for some positive number μ, for some $x_0 \in X$ and for each x chosen from the unit ball $B_H = \{x \in H : \|x\|_H \leq 1\}$. As

$$\begin{aligned} \sqrt[n]{\mu} \cdot r_n(x) &= r_n(\mu x) \\ &\leq \sqrt[n]{\inf_{a_k^{(n)}} \left\| x_0 + \mu x - \sum_{k=1}^{n} a_k^{(n)} e_k^{(n)} \right\|_X + \inf_{a_k^{(n)}} \left\| x_0 - \sum_{k=1}^{n} a_k^{(n)} e_k^{(n)} \right\|_X} \\ &\leq \sqrt[n]{2} q_0 \quad (\forall x \in B_H), \end{aligned}$$

then

$$\limsup_{n\to\infty} \sup_{x\in B_H} r_n(x) \leq q_0.$$

Therefore

$$\limsup_{n\to\infty} \sqrt[n]{d_n(B_H, X)} \leq q_0$$

that contradicts lemma 2.1.

To each pair $\left(X, \varphi_k^{(n)}\right)$, where $\left(\varphi_k^{(n)}\right)_{k\leq n}$ is some matrix of elements from a Banach space X, let's put in correspondence the set $\Im_X\left(\varphi_k^{(n)}\right)$, consisting of elements $x \in X$ that

$$\sum_{j=1}^{n} \sum_{i=1}^{j} a_{ij}^{(n)} \varphi_i^{(j)} \xrightarrow[n\to\infty]{X} x, \ \ \sup_n \sum_{j=1}^{n} \sum_{i=1}^{j} \left| a_{ij}^{(n)} \right| < \infty.$$

Corollary. *Let $\left(\varphi_k^{(n)}\right)_{k\le n}$ be q-lower bounded matrix in the Banach space X and $\left\|\varphi_k^{(n)}\right\|_X \le C,\ k \le n,\ n = 1, 2, \ldots$. Then for each matrix $\left(e_k^{(n)}\right)_{k\le n} \subset X$ there is an element $x \in \Im_X\left(\varphi_k^{(n)}\right)$ such that*

$$\limsup_{n\to\infty} \sqrt[n]{\left\|x - \sum_{k=1}^{n} a_k^{(n)} e_k^{(n)}\right\|_X} \ge q,$$

for all numerical matrices $\left(a_k^{(n)}\right)_{k\le n}$.

Proof. Let M_n be the Banach space of all $n \times n$ triangular numerical matrices $a = (a_{ij})$, $1 \le i \le j \le n$, equipped with the norm $|a|_n = \sum\limits_{j=1}^{n}\sum\limits_{i=1}^{j} |a_{ij}|$. For $a = (a_{ij}) \in M_n$ denote

$$a\varphi = \sum_{j=1}^{n}\sum_{i=1}^{j} a_{ij}\varphi_i^{(j)} \in X.$$

Consider the set M of sequences $A = \left\{a^{(n)}\right\}_{n=1}^{\infty}$, $a^{(n)} \in M_n$ that converges in X,

$$A\varphi = \lim_{n\to\infty} a^{(n)}\varphi,$$

and $\sup\limits_n \left|a^{(n)}\right|_n < \infty$. Then M is a Banach space, with the norm

$$\|A\|_M = \sup_n \left|a^{(n)}\right|_n.$$

Therefore the set $\Im_X\left(\varphi_k^{(n)}\right)$ that coincide with $\{A\varphi : A \in M\}$ turns to a Banach space, equipped with the norm

$$\|x\|_{\Im_X} = \inf_{A: A\varphi = x} \|A\|_M.$$

Indeed, $\Im_X\left(\varphi_k^{(n)}\right)$ is isometrically isomorphic to the factor space M/M_0, which is a Banach space as $M_0 = \{A : A\varphi = 0\}$ is a closed subspace of M.

To complete the proof it is enough to note that

$$\left\|\varphi_k^{(n)}\right\|_{\Im_X} \le 1, \quad \|x\|_X \le C\,\|x\|_{\Im_X},\ x \in \Im_X\left(\varphi_k^{(n)}\right)$$

and use basic lemma FM).

3. Application 1

Let K be a compact subset of the open unit disk D with positive logarithmic capacity $\gamma(K)$, $C(K)$ be the set of continuous functions on K and $H^{\infty}(D)$ be the set of bounded analytic functions on the unit disk D.

Take $H = H^{\infty}(D)$, $X = C(K)$ then $H \subset X$ and $\|*\|_X \leq \|*\|_H$.

It is obvious that the system $\varphi_k^{(n)}(z) = \varphi_k(z) = z^k$ is 1-upper bounded in H. On the other hand, one can easily establish $\frac{\gamma(K)}{4}$-lower boundedness of $\left\{z^k\right\}_0^{\infty}$ in X using the following

Proposition ([F]). *For each polynomial* $P_n(z) = \sum\limits_{k=0}^{n} a_k z^{n-k}$, $a_0 \neq 0$ *there is a polynomial* $Q_n(z) = \sum\limits_{k=0}^{n} b_k z^{n-k}$, $b_0 = 1$ *satisfying*

$$|P_n(z)| \geq \frac{\max\limits_{0\leq k\leq n} |a_k|}{4^n} |Q_n(z)|, \quad z \in \overline{D}.$$

Now applying basic lemma we obtain Fisher-Micchelli's and Bernstein-Walsh's results.

(FM) *Let* $\left(e_k^{(n)}\right)_{k\leq n}$ *be a matrix of continuous functions on* K. *Then there is a function* $f \in H^{\infty}(D)$ *such that*

$$\limsup_{n\to\infty} \sqrt[n]{\left\| f - \sum_{k=1}^{n} a_k^{(n)} e_k^{(n)} \right\|_{C(K)}} \geq \frac{\gamma(K)}{4},$$

for all numerical matrices $\left(a_k^{(n)}\right)_{k\leq n}$.

(BW) *The class of functions* $f : K \to \mathbb{C}$, *permitting the fast approximation by finite linear combinations of the system* $\left\{z^k\right\}_0^{\infty}$ *in* $C(K)$, *coincide with* $H(\mathbb{C})$.

As you could see we have obtained FM and BW results by checking q-boundedness of just one system $\left\{z^k\right\}_{k=0}^{\infty}$.

4. Application 2

It was mentioned above that to make the fast approximation of $H^{\infty}(D)$ we have to miniaturize it up to the class of entire functions. Now let's consider another miniaturization.

Suppose $0 \leq n_1 < n_2 < \ldots < n_k < \ldots$ are integers with density τ, i.e. $\lim\limits_{k\to\infty} \frac{k}{n_k} = \tau$.

Denote

$$H^{\infty}_{\{n_k\}}(D) = \left\{ f \in H^{\infty}(D) : f(z) = \sum_{k=1}^{\infty} a_k z^k, \;\; a_k = 0, \; k \notin \{n_k\} \right\},$$

$$\|f\|_{H^{\infty}_{\{n_k\}}} = \sup_{z\in D} |f(z)| \, .$$

Let K be a compact subset of the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ with positive logarithmic capacity $\gamma\,(K)$.

Theorem 4.1

FM 1) *If* $\tau = 0$ *then* $\lim\limits_{n\to\infty} \sqrt[n]{d_n\,(B, C(K))} = 0$, *for the unit ball* B *of* $H^{\infty}_{\{n_k\}}(D)$.

FM 2) *If* $\tau > 0$ *then for each matrix* $\left(e_k^{(n)}\right)_{k\le n}$ *of continuous functions on* K *there is a function* $f \in H^{\infty}_{\{n_k\}}(D)$ *satisfying*

$$\limsup_{n\to\infty} \sqrt[n]{\left\| f - \sum_{k=1}^{n} a_k^{(n)} e_k^{(n)} \right\|_{C(K)}} \ge \left(\frac{\gamma(K)}{4}\right)^{\frac{1}{\tau}},$$

for all numerical matrices $\left(a_k^{(n)}\right)_{k\le n}$.

BW) *If* $\tau > 0$ *then the class of functions* $f : K \to \mathbb{C}$, *permitting the fast approximation by the finite linear combinations of the system* $\{z^{n_k}\}_{k=1}^{\infty}$ *in* $C(K)$, *coincide with* $H^{\infty}_{\{n_k\}}(D) \cap H(\mathbb{C})$.

Proof. To prove *FM1)* one can take the partial sums of corresponding lacunary series as approximants. Proofs of *FM2)* and *BW)* can be established by the basic lemma.

Indeed, take $H = H^{\infty}_{\{n_k\}}(D)$, $X = C(K)$ and $\varphi_k(z) = z^{n_k} \in H^{\infty}_{\{n_k\}}(D)$. It can be easily checked that $\{z^{n_k}\}_{k=1}^{\infty}$ is 1-upper bounded in H and $\left(\frac{\gamma(K)}{4}\right)^{\frac{1}{\tau}}$-lower bounded in X.

5. Application 3

Consider the system of exponents

$$\left\{e^{-\lambda_k x}\right\}_{k=1}^{\infty}, \tag{5}$$

where λ_k are disjoint numbers, which satisfy

$$\mathrm{Re}\lambda_k \ge a > 0, \;\; |\lambda_k| \le M < \infty, \;\; k = 1, 2, \ldots . \tag{6}$$

To every function $f \in L^2(0, \infty)$ we put in correspondence its approximation error by finite linear combinations of first n elements of (5), i.e. $E_n(f) = \inf_{a_k^{(n)}} \left\| f(x) - \sum_{k=1}^{n} a_k^{(n)} e^{-\lambda_k x} \right\|_{L^2(0,\infty)}$. There is an analogue of Bernstein theorem for exponents.

Theorem (Musoyan, [M1]) *Let $f \in L^2(0,\infty)$. Then the estimate $\limsup_{n\to\infty} \sqrt[n]{E_n(f)} < 1$ holds if and only if there exists an entire function of exponential type with indicator diagram located in the open left half-plane coinciding with f on $(0,\infty)$ almost everywhere.*

In [Z] it has been shown that in a sense such rate of approximation can't be improved.

Theorem ([Z]) *Let D be a set of positive measure $\mu(D)$ and $D \subset \{z : |z| \le M_1,\ \mathrm{Re} z < 0\}$. If (6) takes place then*

$$\limsup_{n\to\infty} \sqrt[n]{E_n(f_\lambda)} \ge \frac{1}{4e\sqrt{\pi}} \frac{\sqrt{\mu(D)}}{\max\{M, M_1\}},$$

where $f_\lambda(z) = e^{-\lambda z}$ for some λ chosen from $-\overline{D} = \{z : -\overline{z} \in D\}$.

For entire function f of exponential type introduce its Borel transform $\beta_f(z) = \sum_{n=0}^{\infty} f^{(n)}(0) z^{-n-1}$. Let K be a compact subset of the open right half-plane with positive logarithmic capacity $\gamma(K)$ and $-K = \{z : -z \in K\}$. For $1 \le p \le \infty$ we denote by L_K^p the class of functions $f \in L^p(0,\infty)$ admitting the extension up to entire function of exponential type with β_f holomorphic on the complement of $-K$.

For $f \in L^p(0,\infty)$ and any matrix $\left(e_k^{(n)}\right)_{k\le n} \subset L^p(0,\infty)$ consider

$$E_n^{(p)}(f) := \inf_{a_k^{(n)}} \left\| f - \sum_{k=1}^{n} a_k^{(n)} e_k^{(n)} \right\|_{L^p(0,\infty)}.$$

Theorem 5.1 (FM) *For each matrix $\left(e_k^{(n)}\right)_{k\le n} \subset L^p(0,\infty)$, $1 \le p \le \infty$, there is a function $f \in L_K^p$ satisfying*

$$\limsup_{n\to\infty} \sqrt[n]{E_n^{(p)}(f)} \ge \frac{\gamma(K)}{d(K)},$$

where $d(K) = \inf\{2M : K \subset \{z : |z| \le M\}\}$.

Proof. There exists a matrix of numbers $(\lambda_{kn})_{k\le n} \subset K$ such that the matrix of exponents $\left(e^{-\lambda_{kn}x}\right)_{k\le n}$ is $\frac{\gamma(K)}{d(K)}$-lower bounded in $L^p(0,\infty)$. To prove this, we introduce Fekete's n-th transfinite diameter and Chebyshev n-th constant of K ([L], p. 606) that are $\tau_n = \max_{z_1,\dots,z_n\in K} \prod_{j\ne k} |z_k - z_j|^{1/n(n-1)}$

and $c_n = \min\limits_{z_1,\ldots,z_n\in\mathbb{C}} \max\limits_{z\in K} \prod\limits_{k=1}^{n} |z_k - z|^{1/n}$ respectively. The theorem of Fekete – Szego states that both τ_n and c_n tend to $\gamma(K)$ as $n \to \infty$.

We take $(\lambda_{kn})_{k\le n}$ in such a way that

$$\prod_{j\neq k} |\lambda_{kn} - \lambda_{jn}|^{1/n(n-1)} = \tau_n, \quad n = 1, 2, \ldots. \tag{7}$$

Consider the finite system of exponents

$$\left\{e^{-\lambda_{1n}x}, \ldots, e^{-\lambda_{nn}x}\right\} \tag{8}$$

and Blashke product $B_n(\lambda) = \prod\limits_{k=1}^{n} \frac{\lambda-\lambda_{kn}}{\lambda+\overline{\lambda_{kn}}}$ for $\{\lambda_{kn}\}_{k=1}^{n}$. The biorthogonal system generated by (8) is the system of functions [M2]

$$\varphi_k^{(n)}(x) = \frac{1}{\overline{B_n'(\lambda_{kn})}} \sum_{m=1}^{n} \frac{e^{-\lambda_{mn}x}}{B_n'(\lambda_{mn})(\lambda_{mn} + \overline{\lambda_{kn}})}, \quad k = 1, \ldots, n; \quad x > 0, \tag{9}$$

that is, $\int\limits_{o}^{\infty} e^{-\lambda_{pn}x}\overline{\varphi_{qn}(x)}dx = \delta_{pq}$ (δ_{pq} is Kronecker's delta) and the linear spans of (8) and (9) coincide. According to lemma 1.1, we just need $\frac{d(K)}{\gamma(K)}$-upper boundedness of $\left(\varphi_k^{(n)}\right)_{k\le n}$ in all spaces $L^p(0,\infty)$, $1 \le p \le \infty$.

The function $\varphi_k^{(n)}(x)$ can be written in the integral form

$$\varphi_k^{(n)}(x) = \frac{1}{\overline{B_n'(\lambda_{kn})}} \frac{1}{2\pi i} \int\limits_{\Gamma_{r,R}} \frac{e^{-\zeta x} d\zeta}{B_n(\zeta)(\zeta + \overline{\lambda_{kn}})}, \tag{10}$$

where $\Gamma_{r,R}$ is the contour consisting of the segment $[r + iR, r - iR]$ and semicircle $\zeta = r + Re^{i\varphi}$, $-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$ running in positive direction. Besides, interior of $\Gamma_{r,R}$ contains the compact set K. For any positive number ε one can fix r such small and R such large that $\left|\frac{\zeta-\lambda}{\zeta+\overline{\lambda}}\right| > 1 - \varepsilon$ when $(\zeta, \lambda) \in \Gamma_{r,R} \times K$. Consequently, (10) implies

$$\left\|\varphi_k^{(n)}\right\|_{L^p(0,\infty)} \le \frac{C}{|B_n'(\lambda_{kn})|} \frac{1}{(1-\varepsilon)^n}, \quad k = 1, 2, \ldots, n; \; n = 1, 2, \ldots,$$

where the constant C does not depend on k and n. As ε was arbitrary, the last estimate leads to

$$\limsup_{n\to\infty} \sqrt[n]{\max_{1\le m\le n} \left\|\varphi_m^{(n)}\right\|_{L^p(0,\infty)}} \le \frac{1}{\liminf\limits_{n\to\infty} \sqrt[n]{\min\limits_{1\le m\le n} |B_n'(\lambda_{mn})|}} \le$$

$$\leq \frac{d(K)}{\liminf\limits_{n\to\infty} \sqrt[n]{\min\limits_{1\leq m\leq n} \prod\limits_{k=1,k\neq m}^{n} |\lambda_{kn}-\lambda_{mn}|}} \leq \frac{d(K)}{\gamma(K)}.$$

Indeed, according to (7) we get

$$\sqrt[n]{\prod_{k=1,k\neq m}^{n} |\lambda_{kn}-\lambda_{mn}|} = \sqrt[n]{\max_{z\in K} \prod_{k=1,k\neq m}^{n} |\lambda_{kn}-z|} \geq c_{n-1}^{1-\frac{1}{n}}$$

for all $m=1,2,\ldots,n$, therefore

$$\liminf_{n\to\infty} \sqrt[n]{\min_{1\leq m\leq n} \prod_{k=1,k\neq m}^{n} |\lambda_{kn}-\lambda_{mn}|} \geq \lim_{n\to\infty} c_{n-1}^{1-\frac{1}{n}} = \gamma(K).$$

Now let's prove $\Im_{L^p(0,\infty)}\left(e^{-\lambda_{kn}x}\right) \subset L_K^p$.

For $f\in \Im_{L^p(0,\infty)}\left(e^{-\lambda_{kn}x}\right)$ consider the exponential polynomials $f_n(z) = \sum\limits_{j=1}^{n}\sum\limits_{i=1}^{j} a_{ij}^{(n)} e^{-\lambda_{ij}z}$ $(z\in\mathbb{C})$ such that

$$\sup_n \sum_{j=1}^{n}\sum_{i=1}^{j} \left|a_{ij}^{(n)}\right| < \infty \qquad \text{and} \qquad \|f_n - f\|_{L^p(0,\infty)} \underset{n\to\infty}{\to} 0.$$

The chosen sequence of polynomials is a normal family of entire functions because of its uniformly boundedness inside of $\mathbb{C}$. Similarly, the sequence of Borel transforms

$$\beta_{f_n}(z) = \sum_{j=1}^{n}\sum_{i=1}^{j} \frac{a_{ij}^{(n)}}{z+\lambda_{ij}}$$

is a normal family on the complement of $-K$. So $f_n(z) \underset{n\to\infty}{\to} \widetilde{f}(z)$ uniformly on each compact subset of the complex plane and $\beta_{f_n}(z) \underset{n\to\infty}{\to} \beta(z)$ uniformly on each compact subset of the complement of $-K$.

As entire function $\widetilde{f}$ is of exponential type and $\widetilde{f}=f$ almost everywhere on $(0,\infty)$, it remains to prove $\beta_{\widetilde{f}}(z)=\beta(z),\ z\in\mathbb{C}\setminus(-K)$. Indeed, if $\mathrm{Re}z\geq\delta>0$ then

$$\beta_{f_n}(z)-\beta_{\widetilde{f}}(z) = \int_0^\infty \left[f_n(t)-f(t)\right] e^{-zt}dt.$$

Therefore

$$\left|\beta_{f_n}(z)-\beta_{\widetilde{f}}(z)\right| = O(1)\,\|f_n-f\|_{L^p(0,\infty)}\ (n\to\infty).$$

Thus $\beta_{\tilde{f}}(z) = \beta(z)$ (Re$z > 0$). Consequently, $\beta_{\tilde{f}}$ and β coincide on the complement of $-K$. Now theorem 5.1 follows from the corollary of basic lemma.

Remark. It is known [J] that the sequence $\{\lambda_k\}_1^\infty$, Re$\lambda_k > 0$ of disjoint numbers satisfies Carleson's separability condition [C]

$$\inf_{1\le j\le n} \prod_{k=1,k\ne j}^{n} \left|\frac{\lambda_k-\lambda_j}{\lambda_k+\overline{\lambda_j}}\right| \ge \delta > 0, \quad n = 1,2,\dots \tag{11}$$

if and only if the system $\left\{e^{-\lambda_k x}\right\}_{k=1}^\infty$ is basis in its closed linear span in the space $L^2(0,\infty)$. If (6) holds then $\left\{e^{-\lambda_k x}\right\}_{k=1}^\infty$ isn't minimal, so (11) doesn't hold. However, a bounded sequence of powers can be taken *geometrically separable,* i.e.

$$\liminf_{n\to\infty} \sqrt[n]{\inf_{1\le j\le n} \prod_{k=1,k\ne j}^{n} \left|\frac{\lambda_k-\lambda_j}{\lambda_k+\overline{\lambda_j}}\right|} \ge \delta > 0. \tag{12}$$

Moreover,

Lemma 5.1 *Let the sequence* $\{\lambda_k\}_1^\infty$ *satisfies the condition* (6)*. Then* (12) *takes place if and only if the system* $\left\{e^{-\lambda_k x}\right\}_{k=1}^\infty$ *is* δ*-lower bounded in* $L^2(0,\infty)$.

Proof. Suppose

$$B_n(\lambda) = \prod_{k=1}^{n} \frac{\lambda-\lambda_k}{\lambda+\overline{\lambda_k}}$$

and

$$\varphi_k^{(n)}(x) = \frac{1}{\overline{B_n'(\lambda_k)}} \sum_{m=1}^{n} \frac{e^{-\lambda_m x}}{B_n'(\lambda_m)(\lambda_m+\overline{\lambda_k})}.$$

The inequality

$$\limsup_{n\to\infty} \sqrt[n]{\max_{1\le j\le n} \left\|\varphi_j^{(n)}\right\|_{L^p(0,\infty)}} \le \frac{1}{\liminf\limits_{n\to\infty} \sqrt[n]{\min\limits_{1\le j\le n} |B_n'(\lambda_j)|}} =$$

$$= \left(\liminf_{n\to\infty} \sqrt[n]{\inf_{1\le j\le n} \prod_{k=1,k\ne j}^{n} \left|\frac{\lambda_k-\lambda_j}{\lambda_k+\overline{\lambda_j}}\right|}\right)^{-1} \tag{13}$$

holds for all $1 \le p \le \infty$. Taking into account lemma 1.1, (13) completes the proof of necessity.

To prove sufficiency we use Fourier transforms of functions $\varphi_k^{(n)} \in L^2(0,\infty)$

$$\widehat{\varphi_k^{(n)}}(\tau) = \frac{1}{\overline{B_n'(\lambda_k)}}\frac{1}{2\pi i}\int_0^\infty e^{-i\tau x}\int_\Gamma \frac{e^{-\zeta x}d\zeta}{B_n(\zeta)(\zeta+\overline{\lambda_k})}dx, \quad \tau \in (-\infty,\infty),$$

where Γ is a contour that lies in the open right half – plane and contains all points $\lambda_k,\ k = 1,2,\dots$ inside. Applying Fubini's and residue theorems, one can obtain

$$\widehat{\varphi_k^{(n)}}(\tau) = \frac{1}{\overline{B_n'(\lambda_k)}}\frac{1}{B_n(-i\tau)(i\tau-\overline{\lambda_k})}.$$

Hence

$$\left\|\varphi_k^{(n)}\right\|_{L^2(0,\infty)} = \sqrt{2\pi}\left\|\widehat{\varphi_k^{(n)}}\right\|_{L^2(-\infty,\infty)}$$
$$= \frac{\sqrt{2\pi}}{|B_n'(\lambda_k)|}\left\{\int_{-\infty}^{\infty}\frac{d\tau}{|i\tau+\lambda_k|^2}\right\}^{1/2} = \frac{\pi\sqrt{2}}{|B_n'(\lambda_k)|\sqrt{\mathrm{Re}\lambda_k}}.$$

On the other hand, since $\varphi_k^{(n)}$ was generated by $\left\{e^{-\lambda_k x}\right\}_{k=1}^n$, it is $1/\delta$-upper bounded in the space $L^2(0,\infty)$ [F]. Therefore

$$\liminf_{n\to\infty}\sqrt[n]{\min_{1\le j\le n}|B_n'(\lambda_j)|} \ge \delta.$$

The lemma is proved.

Combining (13), lemma 1.1 and basic lemma, we get the following BW type theorem.

Theorem 5.2 (BW) *Let the sequence of disjoint numbers $\{\lambda_k\}_1^\infty$ satisfying (6) be geometrically separable. Then the class of functions $f \in L^p(0,\infty)$, $1 \le p \le \infty$, permitting the fast approximation by finite linear combinations of the system $\left\{e^{-\lambda_k x}\right\}_{k=1}^\infty$, coincide with the set of series*

$$\sum_{k=1}^\infty a_k e^{-\lambda_k x}, \quad \sqrt[k]{|a_k|} \underset{k\to\infty}{\to} 0,$$

where the convergence is in the sense of $L^p(0,\infty)$ topology.

6. Application 4

Let $H^p(G^+)$, $1 < p < \infty$, be the Hardy space of functions f analytic on the upper half - plane $G^+ = \{z : \mathrm{Im} z > 0\}$, with the norm

$$\|f\|_{H^p(G^+)} = \sup_{y>0} \left\{ \int_{-\infty}^{\infty} |f(x+iy)|^p \, dx \right\}^{1/p} < \infty.$$

For $f \in H^p(G^+)$ and any matrix $\left(e_k^{(n)}\right)_{k \le n} \subset H^p(G^+)$ consider the approximation error

$$E_n^{(p)}(G^+)(f) := \inf_{a_k^{(n)}} \left\| f - \sum_{k=1}^{n} a_k^{(n)} e_k^{(n)} \right\|_{H^p(G^+)}.$$

The theorem of Paley and Wiener states: the class $H^2(G^+)$ coincides with the set of functions representable in the form

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \widehat{f}(t) e^{izt} dt, \quad z \in G^+,$$

where $\widehat{f} \in L^2(0, \infty)$. If $\widehat{f}(t) = e^{-\lambda t}$, $\mathrm{Re}\lambda > 0$ then the corresponding function $f \in H^2(G^+)$ is

$$f(z) = \frac{i}{2\pi} \frac{1}{z - \overline{\mu}}, \quad \mu = i\overline{\lambda} \in G^+.$$

Reasoning from this, let's consider the system of rational functions

$$e_k(z) = \frac{1}{z - \overline{\lambda_k}}, \quad k = 1, 2, \ldots,$$

where λ_k are disjoint complex numbers chosen from some compact set $K \subset G^+$. Denote $\overline{K} = \{z : \overline{z} \in K\}$. Using Musoyan's theorem, one can establish $\limsup_{n\to\infty} \sqrt[n]{E_n^{(2)}(G^+)(f)} < 1$ for $f \in H^2(G^+) \cap Hol(\mathbb{C} \backslash \overline{K})$, $f(\infty) = 0$.

Then, let $\gamma(K) > 0$ and

$$H_K^p(G^+) = \left\{ f : f \in H^p(G^+) \cap Hol(\mathbb{C} \backslash \overline{K}), \ f(\infty) = 0 \right\}.$$

Theorem 6.1 (FM) *For each matrix* $\left(e_k^{(n)}\right)_{k \le n} \subset H^p(G^+)$ *there is a function* $f \in H_K^p(G^+)$ *satisfying*

$$\limsup_{n\to\infty} \sqrt[n]{E_n^{(p)}(G^+)(f)} \ge \frac{\gamma(K)}{d(K)},$$

where $d(K) = \inf\{2M : K \subset \{z : |z| \le M\}\}$.

Proof. As in theorem 5.1, there exists a matrix of numbers $(\lambda_{kn})_{k\le n} \subset K$ such that $\left(\frac{1}{z-\overline{\lambda_{kn}}}\right)_{k\le n}$ is $\frac{\gamma(K)}{d(K)}$-lower bounded in $H^p(G^+)$ (see [M3] for integral representation of generated biorthogonal system and [F]). The embedding $\Im_{H^p(G^+)}\left(\frac{1}{z-\overline{\lambda_{kn}}}\right) \subset H^p_K(G^+)$ holds as well. To complete the proof it remains to apply the corollary of basic lemma.

Now consider the Hardy space $H^p(D)$, $1 < p < \infty$, of analytic functions on the unit disk $D = \{z : |z| < 1\}$, with the norm

$$\|f\|_{H^p(D)} = \sup_{0<r<1}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f(re^{i\theta})\right|^p d\theta\right\}^{1/p} < \infty.$$

For $f \in H^p(D)$ and any matrix $\left(e_k^{(n)}\right)_{k\le n} \subset H^p(D)$ denote

$$E_n^{(p)}(D)(f) := \inf_{a_k^{(n)}}\left\|f - \sum_{k=1}^{n} a_k^{(n)} e_k^{(n)}\right\|_{H^p(D)}.$$

Let K be a compact subset of the unit disk with positive logarithmic capacity $\gamma(K)$ and $1/K = \{z : 1/z \in K\}$. If $e_k^{(n)}(z) = e_k(z) = z^k$ then

$$\limsup_{n\to\infty}\sqrt[n]{E_n^{(p)}(D)(f)} < 1,$$

for every function f, holomorphic on the complement of $1/K$.

Theorem 6.2 (FM) *For each matrix* $\left(e_k^{(n)}\right)_{k\le n} \subset H^p(D)$ *there is a function* $f \in H^p(D)$*, holomorphic on the complement of* $1/K$ *, satisfying*

$$\limsup_{n\to\infty}\sqrt[n]{E_n^{(p)}(D)(f)} \ge \frac{\gamma(K)}{2}.$$

Proof. First of all, using technique of the theorem 5.1 once more, we find a matrix $(\lambda_{kn})_{k\le n} \subset K$ such that the matrix of rational functions $\left(\frac{1}{1-\overline{\lambda_{kn}}z}\right)_{k\le n}$ is $\frac{\gamma(K)}{2}$-lower bounded in $H^p(D)$ (see [M] for representation of generated biorthogonal system and [F]). Secondly, the checking of embedding $\Im_{H^p(D)}\left(\frac{1}{1-\overline{\lambda_{kn}}z}\right) \subset H(\mathbb{C}\backslash 1/K)$ is trivial. Finally, we use the corollary of basic lemma.

Corresponding BW results are presented below.

Theorem 6.3 (BW) *Let the sequence of disjoint complex numbers* $\{\lambda_k\}_{k=1}^{\infty}$ *be chosen from a compact subset of* G^+ *and be geometrically* *separable, i.e.*

$$\liminf_{n\to\infty} \sqrt[n]{\inf_{1\le j\le n} \prod_{k=1,k\ne j}^{n} \left|\frac{\lambda_k-\lambda_j}{\lambda_k-\overline{\lambda_j}}\right|} \ge \delta > 0.$$

Then the class of functions $f \in H^p(G^+)$, permitting the fast approximation by finite linear combinations of the system $\left\{\frac{1}{z-\overline{\lambda_k}}\right\}_{k=1}^{\infty}$, coincides with the set of series

$$\sum_{k=1}^{\infty} \frac{a_k}{z-\overline{\lambda_k}}, \quad \sqrt[k]{|a_k|} \underset{k\to\infty}{\to} 0,$$

where the convergence is in the sense of $H^p(G^+)$ topology.

Theorem 6.4 (BW) *Let the sequence of disjoint complex numbers $\{\lambda_k\}_{k=1}^{\infty}$ be chosen from a compact subset of the unit disk D and be geometrically separable, i.e.*

$$\liminf_{n\to\infty} \sqrt[n]{\inf_{1\le j\le n} \prod_{k=1,k\ne j}^{n} \left|\frac{\lambda_k-\lambda_j}{1-\overline{\lambda_j}\lambda_k}\right|} \ge \delta > 0.$$

Then the class of functions $f \in H^p(D)$, permitting the fast approximation by finite linear combinations of the system $\left\{\frac{1}{1-\overline{\lambda_k}z}\right\}_{k=1}^{\infty}$, coincides with the set of series

$$\sum_{k=1}^{\infty} \frac{a_k}{1-\overline{\lambda_k}z}, \quad \sqrt[k]{|a_k|} \underset{k\to\infty}{\to} 0,$$

where the convergence is in the sense of $H^p(D)$ topology.

7. Application 5

Let $\hat{\varphi}$ be the Fourier transform of $\varphi \in L^2(\mathbb{R})$. Further, assume that $|\hat{\varphi}(\xi)| \ge m > 0$, $\xi \in (a, a+2\sigma)$ and $\{\lambda_k\}_{k=1}^{\infty} \subset \mathbb{R}$ satisfies *strong separability condition*

$$\sqrt[n]{\inf_{1\le j\le n} \prod_{k=1,k\ne j}^{n} \sin\frac{\sigma}{n}|\lambda_k-\lambda_j|} \ge \delta > 0, \ n \ge n_0. \tag{14}$$

Then the following statement takes place

Theorem 7.1 (BW) *$f \in L^2(\mathbb{R})$ is fast approximable by the system of translates $\{\varphi(x-\lambda_k)\}_{k=1}^{\infty}$,* i.e.

$$\sqrt[n]{\inf_{a_k^{(n)}} \left\| f(x) - \sum_{k=1}^{n} a_k^{(n)} \varphi(x-\lambda_k) \right\|_{L^2(\mathbb{R})}} \underset{n\to\infty}{\to} 0$$

if and only if

$$f(x) \stackrel{L^2(\mathbb{R})}{=} \sum_{k=1}^{\infty} c_k \varphi(x-\lambda_k), \ \sqrt[k]{|c_k|} \to 0.$$

Proof. It is enough to show that under conditions of the theorem the system $\{\varphi(x-\lambda_k)\}_{k=1}^{\infty}$ is 0-lower bounded and ∞-upper bounded in $L^2(\mathbb{R})$. As regards ∞-upper boundedness it is obvious. To prove 0-lower boundedness, at first we denote $P_n(x) = \sum\limits_{k=1}^{n} a_k^{(n)} \varphi(x-\lambda_k)$. Then note that

$$\|P_n(x)\|_{L^2(\mathbb{R})} = \left\|\hat{P}_n(\xi)\right\|_{L^2(\mathbb{R})} = \left\|\hat{\varphi}(\xi) \sum_{k=1}^{n} a_k^{(n)} e^{i\lambda_k \xi}\right\|_{L^2(\mathbb{R})}.$$

So it remains to prove that $\left\{e^{i\lambda_k \xi}\right\}_{k=1}^{\infty}$ is 0-lower bounded in $L^2(a, a+2\sigma)$, that is

$$\limsup_{n\to\infty} \sqrt[n]{\left\|\sum_{k=1}^{n} a_k^{(n)} e^{i\lambda_k \xi}\right\|_{L^2(a,a+2\sigma)}} \geq q \limsup_{n\to\infty} \sqrt[n]{\max_{1\leq k\leq n} \left|a_k^{(n)}\right|} \tag{15}$$

for some positive number q.

As $\left|e^{i\lambda_k(a+\sigma)}\right| = 1, \ k = 1, 2, \ldots$ one can replace (15) by

$$\limsup_{n\to\infty} \sqrt[n]{\left\|\sum_{k=1}^{n} a_k^{(n)} e^{i\lambda_k \xi}\right\|_{L^2(-\sigma,\sigma)}} \geq q \limsup_{n\to\infty} \sqrt[n]{\max_{1\leq k\leq n} \left|a_k^{(n)}\right|}.$$

To establish this inequality we construct biorthogonal matrix $\left(\varphi_k^{(n)}\right)_{k\leq n}$ and use lemma 1.1. Let $\hat{\varphi}_k^{(n)}(\lambda) = \frac{\sin\frac{\sigma(\lambda-\lambda_k)}{n}}{\frac{\sigma(\lambda-\lambda_k)}{n}} \prod\limits_{i=1, i\neq k}^{n} \frac{\sin\frac{\sigma(\lambda-\lambda_i)}{n}}{\sin\frac{\sigma(\lambda_k-\lambda_i)}{n}}$. Then $\hat{\varphi}_k^{(n)}(\lambda) \in PW_\sigma$, where PW_σ is Paley-Wiener class of entire functions of exponential type $\leq \sigma$ which belong to $L^2(\mathbb{R})$. By the theorem of Paley and Wiener one has $\varphi_k^{(n)}(\xi) \in L^2(-\sigma, \sigma)$. Now taking into account

$$\hat{\varphi}_k^{(n)}(\lambda_i) = \begin{cases} 1, \ i = k \\ 0, \ i \neq k \end{cases}$$

we obtain that the systems $\left\{\frac{1}{\sqrt{2\pi}} \varphi_k^{(n)}\right\}_{k=1}^{n}$ and $\left\{e^{i\lambda_k \xi}\right\}_{k=1}^{n}$ are biorthogonal for all natural numbers n. On the other hand, (14) implies

$$\left\|\varphi_k^{(n)}\right\| = \left\|\hat{\varphi}_k^{(n)}\right\| \leq \frac{Cn}{\delta^n},$$

for some positive constant C.

Finally, we apply lemma 1.1 and establish the theorem.

8. Application 6

It is well known (see [B] and [V]) that there is a close relation between the order of entire function and the rate of its approximation by polynomials.

Theorem (Batirov-Varga) *Let $K \subset \mathbb{C}$ be a compact set of positive logarithmic capacity. Then for each entire function f one has*

$$\limsup_{n\to\infty} \frac{n \ln n}{-\ln E_n(f, K)} = \rho,$$

where ρ is the order of function f, $E_n(f, K)$ is the error of approximation of function f by algebraic polynomials in the uniform norm on K.

One can easily obtain an analogy of basic lemma's BW result in this direction.

Theorem 8.1 (BW) *Let $\{e_k\}_{k=1}^{\infty}$ be a 0-lower and ∞-upper bounded system in a Banach space $X, E_n(x)$ be the error of approximation of $x \in X$ by polynomials $\sum\limits_{k=1}^{n} a_k^{(n)} e_k$ in X and, finally, ρ be some non-negative number. Then*

$$\limsup_{n\to\infty} \frac{n \ln n}{-\ln E_n(x)} \le \rho$$

if and only if

$$x = \sum_{k=1}^{\infty} x_k e_k, \quad \limsup_{n\to\infty} \frac{n \ln n}{-\ln |x_k|} \le \rho.$$

Taking $X = C(K)$ and $e_k = z^k$ we immediately establish Batirov-Varga's result.

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Received November 23, 2006