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THE ORDER OF ALGEBRAS WITH NONTRIVIAL FIXED POINT SUBALGEBRAS

(submitted by M. A. Malakhaltsev)

ABSTRACT. The paper represents an advancement of research the fundamental problem of which is a classification of algebras A (Weil algebras primarily) having a nontrivial fixed point subalgebra (with respect to <u>all</u> algebra automorphisms). The main result is the determination of the algebra order allowing a nontrivial fixed point subalgebra. Moreover, an autonomous importance of some results about socle elements of A and the unipotency of algebra automorphisms is highlighted.

1. INTRODUCTION

We consider local commutative $\mathbb{R}$-algebra A with identity, the nilpotent ideal $\mathfrak{n}_A$ of which has a finite dimension as a vector space and $A/\mathfrak{n}_A = \mathbb{R}$. We call the *order* of A the minimum $\operatorname{ord}(A)$ of the integers r satisfying $\mathfrak{n}_A^{r+1} = 0$ and the *width* $\mathrm{w}(A)$ of A the dimension $\dim_{\mathbb{R}}(\mathfrak{n}_A/\mathfrak{n}_A^2)$. One can assume A is expressed as a finite dimensional factor $\mathbb{R}$-algebra of the algebra $\mathbb{R}[x_1, \ldots, x_n]$ of real polynomials in several indeterminates. Thus, the main example is

$$\mathbb{D}_n^r := \mathbb{R}[x_1, \ldots, x_n]/\mathfrak{m}^{r+1},$$

$\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$ being the maximal ideal of $\mathbb{R}[x_1, \ldots, x_n]$. Evidently, $\operatorname{ord}(\mathbb{D}_n^r) = r$ and $\mathrm{w}(\mathbb{D}_n^r) = n$. As well, every other such an algebra A of the order r can be expressed in a form

$$A = \mathbb{R}[x_1, \ldots, x_n]/\mathfrak{i} + \mathfrak{m}^{r+1},$$

where an ideal $\mathfrak{i}$ satisfies $\mathfrak{m}^{r+1} \subsetneq \mathfrak{i} \subset \mathfrak{m}^2$ and is generated by finite number of polynomials, i.e. $\mathfrak{i} = \langle P_1, \ldots, P_l \rangle$. The fact $\mathfrak{i} \subset \mathfrak{m}^2$ implies that the width of A is n, too. It is evident, that such expressions of algebras in question are not unique after all. Clearly, A can be expressed also in a form

$$A = \mathbb{D}_n^r / \mathfrak{j},$$

where $\mathfrak{j}$ is an ideal in $\mathbb{D}_n^r$ with analogous conditions as above mentioned $\mathfrak{i}$.

In differential geometry, we talk about *Weil algebras*, see e.g. [3]. In this paper, we shall call them shortly algebras. Notwithstanding that many of our results are not binded with the real field only, we investigate this as our main case here.

Let $\mathrm{Aut}\,A$ be a group of automorphisms of the algebra A. By a *fixed point* of A we mean every $a \in A$ satisfying $\phi(a) = a$ for all $\phi \in \mathrm{Aut}\,A$. Let

$$SA = \{a \in A; \phi(a) = a \text{ for all } \phi \in \mathrm{Aut}\,A\}$$

be the set of all fixed points of A. (We opine that nothing but fixed point subalgebras $A^G = \{a \in A; \phi(a) = a \text{ for all } \phi \in G\}$, where $G \subset \mathrm{Aut}\,A$ is a <u>finite</u> group of automorphisms, were studied in detail up to now. See e.g. [2].) It is clear, that SA is a subalgebra of A. As

$$A = \mathbb{R} \cdot (1 \oplus \mathfrak{n}_A)$$

($\mathfrak{n}_A$ being the ideal of nilpotent elements of A), $\mathbb{R} = \mathbb{R} \cdot 1 \subset SA$ certainly holds, because every automorphism sends 1 into 1.

Remark 1. We remark that SA itself is a subalgebra of

$$S^{\mathrm{Fin}}A := \{a \in A \text{ having a finite orbit with respect to all } \phi \in \mathrm{Aut}\,A\},$$

which is another interesting subalgebra of A apt for a computer algorithmization, as we plan explicate in another paper in future.

As to an original motivation of this research, the bijection between all natural operators lifting vector fields from m-dimensional manifolds to bundles of Weil contact elements and the subalgebra of fixed points SA of a Weil algebra A was determined in [5]. Although in the known geometrically motivated examples is usually $SA = \mathbb{R}$ (such SA is called *trivial*), there are some algebras for which $SA \supsetneq \mathbb{R}$ and we suspect related bundles will have remarkably interesting geometry. Thus, the fundamental problem is a classification of algebras having SA nontrivial. See [5], [6] for known results up to now. Especially, we underline the fact that SA is trivial whenever the ideal $\mathfrak{j}$ is homogeneous. Our new results are facilitated after a sort of a computer working, too; we submit the algorithm aspect in the next joint-work. Nevertheless, one can use pencil paper methods (like in [5], [6]) as well.

Remark 2. As our paper is concentrated on the order of A, we remark that in the original paper of André Weil, [8], the order was called the *height* of A. Further, the term *Loevy length* is also sometimes used ($= \mathrm{ord}(A) + 1$).

2. Socle elements and uni potent automorphisms

2.1. **The basis of the ideal $\mathfrak{j}$ and the basis of the vector space A.** We take an algebra A in a form

$$A = \mathbb{D}_n^r / \mathfrak{j}.$$

The set $\mathcal{G}$ of generators of $\mathfrak{j}$ (the *basis* of $\mathfrak{j}$) can be multifarious. We refer to two important possibilities:

(i) *Groebner basis*, very important in computer algebra and implemented in most of computer algebra systems

(ii) *elementary polynomial basis* outgoing from the minimalization of the length (i.e. number of monomials) of the longest generator among $P_1, \ldots, P_l$ and the follow-up minimalization of the number of generators with such length, the idea was described in [6]

Clearly, A is a real vector space. Its *basis* (naturally, fully independent on the mentioned basis of $\mathfrak{j}$) is constituted by equivalence classes $[1]$, $[x_1]$, $\ldots$, $[x_n]$, plus classes containing higher monomials in $x_1, \ldots x_n$: every such a monomial is contained in one equivalence class, but the choice of the representative is not unique. We shall take the least monomial (with coefficient 1) as to the graded lexicographical order. After a fixing of this basis (denoted $\mathcal{B}(A)$ hereafter), we can write elements of A as a linear combination of the basis elements (written ordinarily with omitted brackets $[\ ]$) by a unique way. Of course, the multiplication rules are needful to be added, best in the form of a table.

Example 1. Let
$$A = \mathbb{D}_3^2/\langle x^2 + xy + xz, xy + y^2 + yz\rangle.$$

Then
$$\mathcal{B}(A) = \{1, x, y, z, x^2, xy, y^2, z^2\},$$

hence $\dim_{\mathbb{R}}(A) = 8$ and elements of A can be written in the form
$$k_1 + k_2 x + k_3 y + k_4 z + k_5 x^2 + k_6 xy + k_7 y^2 + k_8 z^2$$

with the following multiplication table.

·	1	x	y	z	x^2	xy	y^2	z^2
1	1	x	y	z	x^2	xy	y^2	z^2
x	x	x^2	xy	$-x^2 - xy$	0	0	0	0
y	y	xy	y^2	$-y^2 - xy$	0	0	0	0
z	z	$-x^2 - xy$	$-y^2 - xy$	z^2	0	0	0	0
x^2	x^2	0	0	0	0	0	0	0
xy	xy	0	0	0	0	0	0	0
y^2	y^2	0	0	0	0	0	0	0
z^2	z^2	0	0	0	0	0	0	0

Proposition 1. *Let $a \in SA$, $a = \sum_i r_i B_i$, $r_i \neq 0$, $B_i \in \mathcal{B}(A)$. In general, B_i may not belong to SA.*

Proof. Let
$$A = \mathbb{D}_2^5/\langle x^2 y^2 + x^5, x^2 y^2 + y^5\rangle.$$

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The basis is $\mathcal{B}(A) = \{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^4, x^3y, x^2y^2, xy^3, y^4, x^4y, xy^4\}$. Aut A has two connected components:

       1st component

$$x \;\mapsto\; x + C_{1,3}x^2 + C_{1,4}xy + C_{1,6}x^3 + C_{1,7}x^2y + C_{1,8}xy^2 + C_{1,9}y^3 + C_{1,10}x^4 +$$
$$C_{1,11}x^3y + C_{1,12}x^2y^2 + C_{1,13}xy^3 + C_{1,14}y^4 + C_{1,15}x^4y + C_{1,16}xy^4$$

$$y \;\mapsto\; y + C_{2,4}xy + C_{2,5}y^2 + C_{2,6}x^3 + C_{2,7}x^2y + C_{2,8}xy^2 + C_{2,9}y^3 + C_{2,10}x^4 +$$
$$C_{2,11}x^3y + C_{2,12}x^2y^2 + C_{2,13}xy^3 + C_{2,14}y^4 + C_{2,15}x^4y + C_{2,16}xy^4;$$

       2nd component

$$x \;\mapsto\; y + C_{1,4}xy + C_{1,5}y^2 + C_{1,6}x^3 + C_{1,7}x^2y + C_{1,8}xy^2 + C_{1,9}y^3 + C_{1,10}x^4 +$$
$$C_{1,11}x^3y + C_{1,12}x^2y^2 + C_{1,13}xy^3 + C_{1,14}y^4 + C_{1,15}x^4y + C_{1,16}xy^4$$

$$y \;\mapsto\; x + C_{2,3}x^2 + C_{2,4}xy + C_{2,6}x^3 + C_{2,7}x^2y + C_{2,8}xy^2 + C_{2,9}y^3 + C_{2,10}x^4 +$$
$$C_{2,11}x^3y + C_{2,12}x^2y^2 + C_{2,13}xy^3 + C_{2,14}y^4 + C_{2,15}x^4y + C_{2,16}xy^4;$$
$$C_{i,j} \in \mathbb{R}.$$

The elements of the form

$$c_1 + c_2x^2y^2 + c_3(x^4y + xy^4)$$

are fixed for all $c_1, c_2, c_3 \in \mathbb{R}$; however, $x^4y \in \mathcal{B}(A)$ is not fixed and $xy^4 \in \mathcal{B}(A)$ is not fixed. $\qquad\square$

2.2. Socle elements.
If an element $a \in A$ has the property $au = 0$ for all $u \in \mathfrak{n}_A$, we call a the *socle element* of A.

Let $a = \sum_i r_i B_i$ be an element of A expressed as a linear combination of basis elements.

Lemma 1. *a is a socle element if and only if all B_i are socle elements.*

Proof. It is clear that if all B_i are socle elements, then $a = \sum_i r_i B_i$ is also a socle element. Let a be a socle element of A and let us suppose that some B_i, say B_1, is not a socle element. It follows there is $u \in \mathfrak{n}_A$, $u = \sum_j \bar{r}_j \bar{B}_j$ ($\bar{B}_j \in \mathcal{B}(A)$), such that $B_1 u \neq 0$. However, it implies there exists some $\bar{B}_j$, say $\bar{B}_1$ for which $B_1 \bar{B}_1 \neq 0$. We can write $\bar{B}_1$ as a monomial of basis elements from $\mathfrak{n}_A/\mathfrak{n}_A^2$, i.e. $\bar{B}_1 = x_{n_1}^{p_1} \ldots x_{n_k}^{p_k}$, where $p_1, \ldots, p_k \in \mathbb{N}$. Thus $B_1 x_{n_1} \neq 0$, too. It means $a x_{n_1} \neq 0$ and a is not a socle element; a contradiction. $\qquad\square$

Lemma 2. *Every algebra A has non-zero socle elements.*

Proof. If $\operatorname{ord} A = 0$, then $A = \mathbb{R}$ and $\mathfrak{n}_A = \{0\}$: that is why all elements of A are socle elements. If $\operatorname{ord} A = r > 0$, then it is sufficient look for socle elements amidst elements of $\mathcal{B}(A)$ (cf. Lemma 1). We take an element $a \in \mathcal{B}(A)$ and multiply it by all $x_1, \ldots, x_n$: if all products equal zero, then a is a socle element, if not, we take some non-zero product $a x_i$ and multiply it by all $x_1, \ldots, x_n$ repeatedly; the number of such multiplications (till then a socle element is found) is maximally r as $\mathfrak{n}_A^{r+1} = 0$. $\qquad\square$

We can consult the Example 1 anew and find that the multiplication table is an elegant tool for the identification of socle elements. Easily, we have observed that all socle elements constitute an ideal called the *socle* of A and denoted by $\operatorname{soc}(A)$. The following assertion specifies the relation between $\operatorname{soc}(A)$ and $\mathfrak{n}_A$.

Lemma 3. *For every algebra A with $\mathrm{ord}(A) = r$,*

$$\mathfrak{n}_A^r \subset \mathrm{soc}(A).$$

Proof. The lemma is very transparent: every "maximal power" is a socle element. It is also clear, that there exist algebras, for which $\mathfrak{n}_A^r \subsetneqq \mathrm{soc}(A)$. □

Moreover, elements of A in the form

$$r_1 + r_2 a, \qquad r_1, r_2 \in \mathbb{R}, \quad a \in \mathrm{soc}(A)$$

form a subalgebra MA of A used e.g. in [4]. The problem of a relation between SA and MA is still open.

2.3. **Unipotent automorphisms.** We denote by G_A the connected identity component of the group $\mathrm{Aut}\,A$ of automorphisms A. Further, we have the morphism

$$\epsilon_A \colon \mathrm{Aut}\,A \to \mathrm{GL}(\mathfrak{n}_A/\mathfrak{n}_A^2).$$

The kernel of ϵ_A is denoted by U_A and it is a subgroup of $\mathrm{Aut}\,A$ having all elements unipotent. A *unipotent* automorphism is such $\phi \in \mathrm{Aut}\,A$ for which $\mathrm{id}_A - \phi$ is a nilpotent endomorphism of A; alternatively, such $\phi \in \mathrm{Aut}\,A$ for which all eigenvalues (over $\mathbb{C}$) of its matrix representation are equal 1. Of course, there are also unipotent automorphisms not belonging to U_A, in general. The subgroup U_A is connected and the inclusions

$$\mathrm{id}_A \in U_A \subset G_A \subset \mathrm{Aut}\,A$$

always hold. For the properties of U_A see [1] and [7].

Immediately, we have the following assertion.

Proposition 2. *Let A be a Weil algebra of the order r. If $U_A = \mathrm{Aut}\,A$, then all socle elements of A belonging to $\mathfrak{n}_A^r$ belong to SA, too.*

Proof. The assertion is clear: all automorphisms have a form

$$
\begin{aligned}
1 &\mapsto 1 \\
x_1 &\mapsto x_1 + P_1 \\
&\cdots \\
x_n &\mapsto x_n + P_n,
\end{aligned}
$$

where P_i are polynomials without absolute and linear terms. □

(We shall not write $1 \mapsto 1$ below in the description of automorphisms.)

Taking Lemma 2 into consideration, we have obtained immediately:

Corollary 1. *If $\mathrm{ord}\,A > 0$ and $U_A = \mathrm{Aut}\,A$, then SA is nontrivial.*

We will not remind the trivial case $\mathrm{ord}(A) = 0$ below. Within a time of this research, some conjectures about an effect of the property $U_A = G_A$ have occurred. In particular, we have proved:

Proposition 3. *If $U_A = G_A$, then SA can be both nontrivial and trivial (i.e. $U_A = G_A$ is not a sufficient condition for a nontrivial SA).*

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Proof. Of course, the nontrivial case is included e.g. in the previous proposition. Examples of algebras with $U_A = \operatorname{Aut} A$ are in the proof of Proposition 4. The trivial case comes in the algebra

$$A = \mathbb{D}_2^5 / \langle xy^2 + x^5, x^2 y + y^5 \rangle.$$

The basis is $\mathcal{B}(A) = \{1, x, y, x^2, xy, y^2, x^3, x^2 y, xy^2, y^3, x^4, y^4\}$.
Aut A has eight connected components:

1st component
$$x \mapsto -x + C_{1,3}x^2 + C_{1,4}xy + C_{1,6}x^3 + C_{1,7}x^2 y + C_{1,8}xy^2 + C_{1,10}x^4 + C_{1,11}y^4$$
$$y \mapsto -y + C_{2,4}xy + C_{2,5}y^2 + C_{2,7}x^2 y + C_{2,8}xy^2 + C_{2,9}y^3 + C_{2,10}x^4 + C_{2,11}y^4;$$

2nd component
$$x \mapsto x + C_{1,3}x^2 + C_{1,4}xy + C_{1,6}x^3 + C_{1,7}x^2 y + C_{1,8}xy^2 + C_{1,10}x^4 + C_{1,11}y^4$$
$$y \mapsto -y + C_{2,4}xy + C_{2,5}y^2 + C_{2,7}x^2 y + C_{2,8}xy^2 + C_{2,9}y^3 + C_{2,10}x^4 + C_{2,11}y^4;$$

3rd component
$$x \mapsto -x + C_{1,3}x^2 + C_{1,4}xy + C_{1,6}x^3 + C_{1,7}x^2 y + C_{1,8}xy^2 + C_{1,10}x^4 + C_{1,11}y^4$$
$$y \mapsto y + C_{2,4}xy + C_{2,5}y^2 + C_{2,7}x^2 y + C_{2,8}xy^2 + C_{2,9}y^3 + C_{2,10}x^4 + C_{2,11}y^4;$$

4th component
$$x \mapsto x + C_{1,3}x^2 + C_{1,4}xy + C_{1,6}x^3 + C_{1,7}x^2 y + C_{1,8}xy^2 + C_{1,10}x^4 + C_{1,11}y^4$$
$$y \mapsto y + C_{2,4}xy + C_{2,5}y^2 + C_{2,7}x^2 y + C_{2,8}xy^2 + C_{2,9}y^3 + C_{2,10}x^4 + C_{2,11}y^4;$$

5th component
$$x \mapsto -y + C_{1,4}xy + C_{1,5}y^2 + C_{1,7}x^2 y + C_{1,8}xy^2 + C_{1,9}y^3 + C_{1,10}x^4 + C_{1,11}y^4$$
$$y \mapsto -x + C_{2,3}x^2 + C_{2,4}xy + C_{2,6}x^3 + C_{2,7}x^2 y + C_{2,8}xy^2 + C_{2,10}x^4 + C_{2,11}y^4;$$

6th component
$$x \mapsto y + C_{1,4}xy + C_{1,5}y^2 + C_{1,7}x^2 y + C_{1,8}xy^2 + C_{1,9}y^3 + C_{1,10}x^4 + C_{1,11}y^4$$
$$y \mapsto -x + C_{2,3}x^2 + C_{2,4}xy + C_{2,6}x^3 + C_{2,7}x^2 y + C_{2,8}xy^2 + C_{2,10}x^4 + C_{2,11}y^4;$$

7th component
$$x \mapsto -y + C_{1,4}xy + C_{1,5}y^2 + C_{1,7}x^2 y + C_{1,8}xy^2 + C_{1,9}y^3 + C_{1,10}x^4 + C_{1,11}y^4$$
$$y \mapsto x + C_{2,3}x^2 + C_{2,4}xy + C_{2,6}x^3 + C_{2,7}x^2 y + C_{2,8}xy^2 + C_{2,10}x^4 + C_{2,11}y^4;$$

8th component
$$x \mapsto y + C_{1,4}xy + C_{1,5}y^2 + C_{1,7}x^2 y + C_{1,8}xy^2 + C_{1,9}y^3 + C_{1,10}x^4 + C_{1,11}y^4$$
$$y \mapsto x + C_{2,3}x^2 + C_{2,4}xy + C_{2,6}x^3 + C_{2,7}x^2 y + C_{2,8}xy^2 + C_{2,10}x^4 + C_{2,11}y^4;$$
$$C_{i,j} \in \mathbb{R}.$$

By a direct application of automorphisms of the whole group Aut A to a general element

$$k_1 + k_2 x + k_3 y + k_4 x^2 + k_5 xy + k_6 y^2 + k_7 x^3 + k_8 x^2 y + k_9 xy^2 + k_{10} y^3 + k_{11} x^4 + k_{12} y^4$$

of A $(k_i \in \mathbb{R})$ we find SA trivial. $\qquad\square$

Remark 3. We remark that in the list of algebras having unipotent G_A (in [1], Theorem 3.11, the case c.ii) is an error. Surely, let us evaluate automorphisms for

$$A = \mathbb{D}_2^4/\langle x^3 + y^4, x^2y + y^4, xy^2 \rangle.$$

The basis is $\mathcal{B}(A) = \{1, x, y, x^2, xy, y^2, x^2y, y^3\}$.
Aut A consists of only one connected component:

$$
\begin{aligned}
x &\mapsto C_{2,2}^4 x + C_{1,3}x^2 + C_{1,4}xy + (C_{2,2}^6 - C_{2,2}^3)y^2 + C_{1,6}x^2y + C_{1,7}y^3 \\
y &\mapsto (C_{2,2}^4 - C_{2,2}^3)x + C_{2,2}^3 y + C_{2,3}x^2 + C_{2,4}xy + C_{2,5}y^2 + C_{2,6}x^2y + C_{2,7}y^3 \\
& \quad C_{i,j} \in \mathbb{R}; \ C_{2,2} \neq 0.
\end{aligned}
$$

3. The order theorem

3.1. The width 2. Let $r \geq 4$ and let

$$\hat{A}_r = \mathbb{D}_2^r/\langle x^{r-2} + y^{r-1}, x^{r-1} + y^r \rangle.$$

We have obtained the following result about $\hat{A}_r$.

Proposition 4.
  i) $S\hat{A}_4$ is trivial.
  ii) If $r \geq 5$, then $S\hat{A}_r$ is nontrivial and $\dim_{\mathbb{R}} S\hat{A}_r \geq r - 2$.

Proof. i) The problem was formulated in [5] as the Exercise 1. It is sufficient to find one automorphism for which only constants are fixed. Nevertheless, we can describe the whole group Aut A. The basis of $\hat{A}_4$ is $\mathcal{B}(\hat{A}_4) = \{1, x, y, x^2, xy, y^2, x^2y, xy^2\}$.

1st component

$$
\begin{aligned}
x &\mapsto -|C_{2,2}|^3 x + \frac{1}{128}(2|C_{2,2}|^3 - 80C_{1,4} + 3C_{2,2}^4 - C_{2,2}^6 + 96C_{2,2}^2 C_{2,5} + \\
& \quad 48|C_{2,2}|(C_{1,4} - 4C_{2,4} - 2C_{2,5}))x^2 + C_{1,4}xy - \frac{3}{8}(|C_{2,2}|^3 + C_{2,2}^4)y^2 + \\
& \quad C_{1,6}x^2y + C_{1,7}xy^2 \\
y &\mapsto -\frac{1}{4}(|C_{2,2}|^3 + C_{2,2}^2)x + C_{2,2}^2 y + C_{2,3}x^2 + C_{2,4}xy + C_{2,5}y^2 + C_{2,6}x^2y + \\
& \quad C_{2,7}xy^2;
\end{aligned}
$$

2nd component

$$
\begin{aligned}
x &\mapsto |C_{2,2}|^3 x + \frac{1}{128}(-2|C_{2,2}|^3 - 80C_{1,4} + 3C_{2,2}^4 - C_{2,2}^6 + 96C_{2,2}^2 C_{2,5} - \\
& \quad 48|C_{2,2}|(C_{1,4} - 4C_{2,4} - 2C_{2,5}))x^2 + C_{1,4}xy + \frac{3}{8}(|C_{2,2}|^3 - C_{2,2}^4)y^2 + \\
& \quad C_{1,6}x^2y + C_{1,7}xy^2 \\
y &\mapsto \frac{1}{4}(|C_{2,2}|^3 - C_{2,2}^2)x + C_{2,2}^2 y + C_{2,3}x^2 + C_{2,4}xy + C_{2,5}y^2 + C_{2,6}x^2y + \\
& \quad C_{2,7}xy^2; \\
& \quad C_{i,j} \in \mathbb{R}.
\end{aligned}
$$

The example of an automorphism precluding nontrivial elements of $S\hat{A}_4$ is e.g. (we take $C_{2,2} = 2$ and all other $C_{i,j} = 0$ in the 1st component of Aut A)

$$
\begin{aligned}
x &\mapsto -8x - 9y^2 \\
y &\mapsto -3x + 4y.
\end{aligned}
$$

ii) For $r \geq 5$, polynomials x^r, $x^{r-1}y$, $x^{r-2}y^2$, $x^{r-1}+y^r$, $x^{r-2}+y^{r-1}$, $x^{r-1}+xy^{r-1}$, $x^{r-2}y + y^r$ are belonging to $\mathfrak{j} = \langle x^{r-2} + y^{r-1}, x^{r-1} + y^r \rangle$. The automorphisms have a form

$$
\begin{aligned}
x &\mapsto Ax + By + K(x,y) \\
y &\mapsto Cx + Dy + L(x,y),
\end{aligned}
$$

where K and L are polynomials without absolute and linear terms.

The condition $x^{r-2} + y^{r-1} = 0$ gives

$$B^{r-2} = 0,$$

looking at y^{r-2}. It follows $B = 0$. The same condition gives

$$CD^{r-2} = 0,$$

looking at xy^{r-2} (this is obtained as the image of y^{r-1} only and the condition $r > 4$ is essential). It follows $C = 0$ (because $D \neq 0$ as $B = 0$). The same condition also gives

$$A^{r-2} = D^{r-1},$$

looking at x^{r-2}.

The condition $x^{r-1} + y^r = 0$ gives

$$A^{r-1} = D^r,$$

looking at x^{r-1}. The last two conditions imply $A = D = 1$.

We have $U_{\hat{A}_r} = \operatorname{Aut} \hat{A}_r$ and, as to Corollary 1, $S\hat{A}_r$ is nontrivial. The dimension $\dim_{\mathbb{R}} S\hat{A}_r$ must be greater or equal to the number of (linearly independent) basis elements of the socle. $\square$

Remark 4. As to $\hat{A}_4$, Professor Manuel Saorín had proposed in our e-mail communication another way to the squaring up to the group of automorphisms of this algebra: *The algebra* $\hat{A}_4 = \mathbb{D}_2^4/\langle x^2+y^3, x^3+y^4\rangle$ *is isomorphic to the algebra* $\mathbb{D}_2^4/\langle X^2 - Y^3, Y^4\rangle$ *through the isomorphisms of ideals given by*

$$
\begin{aligned}
X &\mapsto -8x + xy - 3y^2 + \frac{19}{56}y^3 \\
Y &\mapsto x - 4y - \frac{1}{14}xy + \frac{1}{4}y^2.
\end{aligned}
$$

3.2. **The general width.**

Lemma 4. *If* $\operatorname{ord}(A) \leq 2$*, then* SA *is trivial.*

Proof. That was recalled in Introduction, that the fulfillment of $\mathfrak{j} \subset \mathfrak{m}^2$ in the expression $A = \mathbb{D}_n^r/\mathfrak{j}$ is presumed. Thus $A = \mathbb{D}_n^2$ or $\mathbb{D}_n^2$ factorized through an ideal generated by homogeneous polynomials of the second order; however, it means that we have only algebras with trivial SA, cf. Introduction and [5]. $\square$

If we know an algebra with nontrivial fixed point subalgebra, then one can construct an algebra with nontrivial fixed point subalgebra of a greater width and the same order as the existing. (It was showed in [6].) If $w(A) = 2$, then the opening order for the possibility of non-trivial SA is 4; we have just proved in Proposition 4 that for all orders greater than 4 it is possible, too. Consequently, the following Proposition 5 completes the classification.

Proposition 5. *There are algebras of the order 3 with nontrivial fixed point subalgebras.*

Proof. It follows from Proposition 1 in [6], that we must look for it in the width greater then 2. Truly, for

$$A = \mathbb{D}_3^3/\langle x^2 + y^3, xy + z^3, y^2z + yz^2\rangle$$

we have obtained $\mathcal{B}(A) = \{1, x, y, z, x^2, xy, y^2, xz, yz, z^2, xz^2, y^2z\}$ and $\mathrm{Aut}\,A$ with two connected components:

      1st component

$$x \;\mapsto\; -x + C_{1,4}x^2 + C_{1,5}xy + y^2 + C_{1,7}xz + C_{1,10}xz^2 + C_{1,11}yz^2$$
$$y \;\mapsto\; y + C_{2,4}x^2 + C_{2,5}xy + C_{2,6}y^2 + C_{2,7}xz + C_{2,8}yz + 3C_{3,1}z^2 +$$
$$C_{2,10}xz^2 + C_{2,11}y^2z$$
$$z \;\mapsto\; C_{3,1}x - y - z + C_{3,4}x^2 + C_{3,5}xy + C_{3,6}y^2 + C_{3,7}xz + C_{3,8}yz + C_{3,9}z^2 +$$
$$C_{3,10}xz^2 + C_{3,11}y^2z;$$

      2nd component

$$x \;\mapsto\; x + C_{1,4}x^2 + C_{1,5}xy + C_{1,7}xz + C_{1,10}xz^2 + C_{1,11}yz^2$$
$$y \;\mapsto\; y + C_{2,4}x^2 + C_{2,5}xy + C_{2,6}y^2 + C_{2,7}xz + C_{2,8}yz - 3C_{3,1}z^2 + C_{2,10}xz^2 +$$
$$C_{2,11}y^2z$$
$$z \;\mapsto\; C_{3,1}x + z + C_{3,4}x^2 + C_{3,5}xy + C_{3,6}y^2 + C_{3,7}xz + C_{3,8}yz +$$
$$C_{3,9}z^2 + C_{3,10}xz^2 + C_{3,11}y^2z;$$
$$C_{i,j} \in \mathbb{R}.$$

The elements of the form

$$c_1 + c_2x^2$$

are fixed for all $c_1, c_2 \in \mathbb{R}$. $\qquad\qquad\square$

    The example implies immediately:

Corollary 2. *$U_A = G_A$ is not a necessary condition for a nontrivial SA.*

Proof. It is evident that G_A for $A = \mathbb{D}_3^3/\langle x^2 + y^3, xy + z^3, y^2z + yz^2\rangle$ has elements (with $C_{3,1} \neq 0$) not belonging to the kernel of ϵ_A; nevertheless, all elements of G_A are still unipotent here (we left a verification of this fact to the reader). $\qquad\square$

    Finally, we summarize to the following "order theorem".

Theorem. *There is no algebra A with $\mathrm{w}(A) = 1$ and with nontrivial fixed point subalgebra. There exist algebras A with $\mathrm{w}(A) = 2$ with a nontrivial fixed point subalgebra if and only if $\mathrm{ord}(A) \geq 4$. For all $\mathrm{w}(A) > 2$, there exist algebras A with a nontrivial fixed point subalgebra if and only if $\mathrm{ord}(A) \geq 3$.*

Proof. The case $\mathrm{w}(A) = 1$ is trivial. The case $\mathrm{w}(A) = 2$ is solved in [6], Proposition 1 and Proposition 2. In particular, nontrivial $SA = \{c_1 + c_2x^2y\}$ was proved for $A = \mathbb{D}_2^4/\langle x^2y + y^4, x^3 + xy^2\rangle$ in [6]. Finally, the case $\mathrm{w}(A) > 2$ is solved by Proposition 4, Lemma 4 and Proposition 5 listed above. $\qquad\square$

    The result can be recapitulated by the tabular form:

Algebras with a nontrivial fixed point subalgebra					
width\order	$\operatorname{ord}(A)=1$	$\operatorname{ord}(A)=2$	$\operatorname{ord}(A)=3$	$\operatorname{ord}(A)=4$	$\operatorname{ord}(A)=5$ ...
$\operatorname{w}(A)=1$	∄	∄	∄	∄	∄ ∄
$\operatorname{w}(A)=2$	∄	∄	∄	✓	✓ ✓
$\operatorname{w}(A)=3$	∄	∄	✓	✓	✓ ✓
$\operatorname{w}(A)=4$	∄	∄	✓	✓	✓ ✓
...	∄	∄	✓	✓	✓ ✓

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