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H. L. Huru

QUANTIZATIONS OF BRAIDED DERIVATIONS. 2. GRADED MODULES

(submitted by V. V. Lychagin)

Abstract. For the monoidal category of graded modules we find braidings and quantizations. We use them to find quantizations of braided symmetric algebras and modules, braided derivations, braided connections, curvatures and differential operators.

1. Introduction

We consider quantizations q, braidings σ and quantizations of braidings σ_q of the monoidal category of graded modules. The grading is by a finite commutative monoid. We work with braidings that are symmetries, in fact, in this category are all braidings symmetries.

We consider σ-symmetric graded algebras A, graded modules, graded co- and bialgebras and graded internal homomorphisms and find quantizations of these.

We have found explicit descriptions of all quantizations and braidings in the monoidal category of modules graded by a finite commutative monoid and they depend only on the grading, [8].

That is, first of all we have found explicit formulas for σ-symmetric graded algebras A, graded modules, graded co- and bialgebras, graded internal homomorphisms, braided derivations, braided connections and curvature. Then we have found explicit formulas for quantizations of these structures.

From [9] we have the following results. Graded internal homomorphisms has a graded braided Lie structure with respect to the braided commutator. Quantizations of the graded internal homomorphisms has the quantized braided Lie structure and can be realized within the original braided Lie structure by what we call dequantization. We shall go through this in details for graded braided derivations.

We investigate graded braided derivations in σ-symmetric graded algebras and modules. The braided bracket of two braided derivations is a braided derivation. We show that there is a braided Lie structure on the braided derivations.

A quantization the braided derivations provides an isomorphism of the graded modules of braided derivations and quantized braided derivations. We also show that the quantizations of braided derivations has the braided Lie structure with respect to the quantizations of the braiding which can be realized within the original braided Lie structure by dequantization.

We define braided connections in graded modules and braided curvatures. We prove that the braided curvature is A-linear, skew σ-symmetric and is an A-module homomorphism.

We find quantizations of braided connections and braided curvatures. The quantization of the braided curvature is A-linear, skew σ_q-symmetric and an A-module homomorphism with respect to the quantized braiding.

Finally we consider braided differential operators, their symbols and quantizations of these. Because of the $\mathbb{Z}$-grading of the braided symbols we can extend the notion of braiding and quantization of these to include the $\mathbb{Z}$-grading.

This paper is the second in a trilogy.

The results here are all proved for any monoidal category, except for when grading is explicitly involved. That is, we have shown that all the results here also are true for braided derivations of algebras and modules, braided connections, braided curvature, quantizations and so on of any monoidal category. This is found in the first paper *Quantizations of braided derivations. 1. Monoidal categories,* [9].

In [8] we showed that the Fourier transform establishes an isomorphism between the categories of $\hat{G}$-graded modules and G-modules where G is a finite abelian group and $\hat{G}$ is the dual of G. Using this we find a description of all quantizations and braiding also for the monoidal category of modules with action by G. Again, we have a complete and explicit description for braided derivations of algebras and modules, braided connections, curvature, differential operators and quantizations of these structures. This is to be found in the third paper *Quantizations of braided derivations. 3. Modules with action by a group,* [10].

There are many interesting applications of these results. One of the more interesting applications is quantizations of braided Lie algebras. In the paper [11], which is to be published, we show quantizations of semisimple Lie algebras by quantizations of derivations, for example an alternative quantization of $\mathfrak{sl}_2(\mathbb{C})$. To find this quantization we use the fact that $\mathfrak{sl}_2(\mathbb{C})$ is graded by $\mathbb{Z}$ and consider the exterior algebra, hence there is a $\mathbb{Z}\times\mathbb{Z}$-grading which gives nontrivial quantizers.

Note that in all three papers we assume that the associativity constraint is trivial.

As noted, most of the proofs are found for general monoidal categories in [9], but almost all proofs will be repeated for clarity.

2. Graded modules

Let M be a finite commutative monoid. Let R be a commutative ring with unit.

Denote by $\mathrm{mod}_R(M)$ the strict monoidal category [19] of M-graded R-modules,

$$X = \oplus_{m\in M} X_m.$$

Denote the grading of a homogeneous element $x \in X$ either by $|x| \in M$, or write x_m, $m \in M$. Throughout the paper is everything stated in terms of homogeneous elements.

The arrows of $\mathrm{mod}_R(M)$ are grading preserving morphisms.

The tensor product $X \otimes_R X'$ of two objects in $\mathrm{mod}_R(M)$ is defined,

$$(X \otimes_R X')_m = \oplus_{i+j=m}(X_i \otimes_R X'_j).$$

Quantizations and braidings of this category is described in [12] and [8]. Recall that any quantization of the monoidal category $\mathrm{mod}_R(M)$ is realized by a 2-cocycle $q \in Z^2(M, U(R))$,

$$q(i,j)\,q^{-1}(i,j+k)\,q(i+j,k)\,q^{-1}(j,k) = 1, \tag{1}$$

$i,j,k \in M$. When we factor out the trivial quantizations we are left with $H^2(M, U(R))$. For homogeneous elements $x \in X$, $y \in Y$ in the M-graded modules X and Y a quantization has the form

$$q : x \otimes y \longmapsto q(|x|,|y|)\, x \otimes y.$$

Note that quantizations preserve associativity constraints.

Any braiding in $\mathrm{mod}_R(M)$ is realized by $\sigma : M \times M \to \mathbf{U}(R)$ which is a bihomomorphism,

$$\begin{aligned}
\sigma(i+j,k) &= \sigma(i,k)\,\sigma(j,k),\\
\sigma(i,j+k) &= \sigma(i,j)\,\sigma(i,k),
\end{aligned}$$

and a symmetry,

$$\sigma(i,j)\,\sigma(j,i) = 1,$$

$i,j,k \in M$. For homogeneous elements $x \in X$, $y \in Y$ in the M-graded modules X and Y a braiding has the form

$$\sigma : x \otimes y \longmapsto \sigma(|x|,|y|)\, y \otimes x.$$

Any braiding σ is also a 2-cocycle gives a quantization when composed with the twist, $\tau \circ \sigma$.

A quantization by q of a braiding σ is

$$\sigma_q(i,j) = q^{-1}(j,i)\,\sigma(i,j)\,q(i,j),$$

$i,j \in M$.

2.1. **Quantizations of graded algebras.** An algebra A in $\mathrm{mod}_R(M)$ is called an *M-graded R-algebra* and is equipped with multiplication

$$\mu : A \otimes A \to A$$

which maps $A_i \otimes A_j$ to A_{i+j}, $i,j \in M$.

Given a quantization q, a quantization of an algebra A in $\mathrm{mod}_R(M)$ is a new multiplication μ_q defined as follows

$$\mu_q(a \otimes b) = a *_q b = q(|a|,|b|)\, ab$$

where $a,b \in A$ are homogeneous and the multiplication on the right hand side is the old multiplication.

Let σ be a braiding and let A be a σ-commutative algebra in $\mathrm{mod}_R(M)$,

$$ab = \sigma(|a|,|b|)\, ba$$

for all homogeneous $a,b \in A$

Let q be a quantization and σ_q be the quantized braiding. Let A be a σ-commutative algebra in $\mathrm{mod}_R(M)$. Then A_q is σ_q-commutative,

$$a *_q b = \sigma_q(|a|,|b|)\, b *_q a$$

for homogeneous $a,b \in A_q$.

2.2. **Quantizations of graded modules.** An A-module E in $\mathrm{mod}_R(M)$ is called a (left) *M-graded A-module* and is equipped with an action

$$\nu : A \otimes E \to E$$

which maps $A_i \otimes E_j$ to E_{i+j}, $i,j \in M$ and similarly for right modules.

For the category $\mathrm{mod}_R(M)$ a quantization of a left, respectively right, A-module E is

$$a *^l x = q(|a|,|x|)\, ax,$$

respectively

$$x *^r a = q(|x|, |a|)\, xa,$$

for homogeneous $a \in A$, $x \in E$.

A A-A-bimodule E is σ-symmetric if

$$\begin{aligned} ax &= \sigma(|a|, |x|)\, xa, \\ xa &= \sigma(|x|, |a|)\, ax, \end{aligned}$$

for homogeneous $a \in A$, $x \in E$.

Note that since the braidings are symmetries will left σ-symmetric A-module structure imply right σ-symmetric structure, hence we need only to consider left A-modules.

Let E and E' be two σ-symmetric A-A-bimodules and $E \otimes E'$ be their tensor product. Define the action of A on $E \otimes E'$,

$$\begin{aligned} a(x \otimes x') &= ax \otimes x' + \sigma(|a|, |x|)\, x \otimes ax', \\ (x \otimes x')\, a &= x \otimes x'a + \sigma(|x'|, |a|)\, xa \otimes x', \end{aligned}$$

for homogeneous $a \in A$, $x \in E$, $x' \in E'$, and $E \otimes E'$ is σ-symmetric.

2.3. **Quantizations of graded coalgebras.** A coalgebra A with comultiplication $\Delta : A \to A \otimes A$ in the monoidal category $\mathrm{mod}_R(M)$ is called an *M-graded R-coalgebra.* The comultiplication Δ maps

$$A_m \to \sum_{i+j=m} A_i \otimes A_j.$$

Let σ be a braiding and let A be a σ-cocommutative coalgebra in $\mathrm{mod}_R(M)$, that is,

$$\Delta(x) = \sum_{|x'|+|x''|=|x|} x' \otimes x'' = \sum_{|x'|+|x''|=|x|} \sigma^{-1}(|x'|, |x''|)\, x'' \otimes x' = \sigma^{-1} \circ \Delta(x)$$

for homogeneous $x, x', x'' \in A$.

A quantization of a coalgebra A in $\mathrm{mod}_R(M)$ is equipping $A_q = A$ with a new comultiplication Δ_q, defined by

$$\begin{aligned} \Delta_q(x) &= q^{-1}_{A,A}\left(\sum_{|x'|+|x''|=|x|} x' \otimes x''\right) \\ &= \sum_{|x'|+|x''|=|x|} q^{-1}(|x'|, |x''|)\, x' \otimes x'', \end{aligned}$$

for homogeneous $x, x', x'' \in A$.

If A is σ-cocommutative, then A_q is σ_q-cocommutative. The σ_q-cocommutativity is

$$\sum_{|x'|+|x''|=|x|} x' \otimes_q x'' = \sigma_q^{-1}\left(\sum_{|x'|+|x''|=|x|} x'' \otimes_q x'\right)$$
$$= \sum_{|x'|+|x''|=|x|} q(|x''|,|x'|)\,\sigma^{-1}(|x'|,|x''|)\,q^{-1}(|x'|,|x''|)\,q^{-1}(|x''|,|x'|)\,x''\otimes x'$$
$$= \sum_{|x'|+|x''|=|x|} \sigma^{-1}(|x'|,|x''|)\,q^{-1}(|x'|,|x''|)\,x''\otimes x',$$

for homogeneous $x', x'' \in A_q$. The comultiplication in the two last lines is the comultiplication of A.

2.4. **Quantizations of graded internal homomorphisms.** The category $\mathrm{mod}_R(M)$ is a closed, that is internal homomorphism $\hom(X,Y)$ exists for all objects X, Y.

Remark 1. *If $M = G$ is rather a group than a monoid, then for any two objects X and Y in $\mathrm{mod}_R(G)$ there exist a grading on $\hom(X,Y)$ as $f \in \hom(X,Y)$, which maps X_i to Y_j, is given the grading $j-i \in G$ and the internal homomorphisms can be considered as objects in $\mathrm{mod}_R(G)$.*

A quantization q_h of all hom is a new multiplication defined by

$$\mu_q^h = \mu^h \circ q_{\hom(Y,Z),\hom(X,Y)} : \hom(Y,Z) \otimes \hom(X,Y) \to \hom(X,Z)$$

and

$$g *_q f = q(|g|,|f|)\, g \circ f,$$
$$f *_q x = q(|f|,|x|)\, f * x,$$

for homogeneous f, g and x of grading $|f|, |g|, |x| \in G$.

From [9] we have the following results. Internal homomorphisms of a σ-symmetric graded module E over a σ-symmetric algebra A, $\hom(E,E)$, has a braided Lie structure with respect to the braided commutator,

$$[,]^\sigma = \mu^h - \mu^h \circ \sigma.$$

Quantizations of the internal homomorphisms has the quantized braided Lie structure and can be realized within the original braided Lie structure by what we call dequantization. We shall go through this in details for braided derivations in the next section.

3. Braided derivations in graded algebras

In this section we shall discuss braided derivations in σ-commutative graded algebras and quantizations of these.

By remark 1, let from now on $M = G$ be a finite abelian group.

Let R be a field, σ be a braiding in the monoidal category of G-graded modules and A be a G-graded σ-commutative R-algebra.

Definition 2. *A σ-derivation of A of degree $|\partial| \in G$ is an R-linear operator $\partial : A \to A$ such that*

$$\partial : A_g \to A_{g+|\partial|},$$

$g \in G$, that satisfies the σ-Leibniz rule,

$$\partial (ab) = \partial (a) b + \sigma (|\partial|, |a|) a\partial (b), \tag{2}$$

where $a, b \in A$ are homogeneous and a is of grading $|a| \in G$.

The set of σ-derivations of degree k is denoted by $Der_k^\sigma (A)$ and the set of all σ-derivations by $Der^\sigma (A)$.

A left A-module structure on $Der^\sigma (A)$ is defined by

$$(a\partial) (b) = a (\partial (b)),$$

for homogeneous $a, b \in A$, $\partial \in Der_{|\partial|}^\sigma (A)$, and

$$a\partial \in Der_{|a|+|\partial|}^\sigma (A).$$

Definition 3. *A σ-commutator (or σ-bracket) on homogeneous elements $\partial_1, \partial_2 \in Der^\sigma (A)$ of degree $|\partial_1|$ and $|\partial_2|$ respectively, is defined by*

$$[\partial_1, \partial_2]^\sigma = \partial_1\partial_2 - \sigma (|\partial_1|, |\partial_2|) \partial_2\partial_1.$$

Proposition 4. *The σ-commutator of two σ-derivations is a σ-derivation of the combined degree,*

$$[\partial_1, \partial_2]^\sigma \in Der_{|\partial_1|+|\partial_2|}^\sigma (A),$$

$\partial_1, \partial_2 \in Der^\sigma (A)$.

Proof. The σ-commutator of two derivations satisfies the σ-Leibniz rule,

$$([\partial_1, \partial_2]^\sigma) (ab) = [\partial_1, \partial_2]^\sigma (a) b + \sigma (|\partial_1| + |\partial_2|, |a|) a [\partial_1, \partial_2]^\sigma (b),$$

for homogeneous $\partial_1, \partial_2 \in Der^\sigma(A)$, $a, b \in A$, which is easily proved,

$$\begin{aligned}
&[\partial_1, \partial_2]^\sigma (ab) = \partial_1\partial_2 (ab) - \sigma (|\partial_1|, |\partial_2|) \partial_2\partial_1 (ab) \\
&= \partial_1 (\partial_2 (a) b + \sigma (|\partial_2|, |a|) a\partial_2 (b)) \\
&- \sigma (|\partial_1|, |\partial_2|) \partial_2 (\partial_1 (a) b + \sigma (|\partial_1|, |a|) a\partial_1 (b)) \\
&= \partial_1\partial_2 (a) b + \sigma (|\partial_1|, |\partial_2| + |a|) \partial_2 (a) \partial_1 (b) \\
&+ \sigma (|\partial_2|, |a|) \partial_1 (a) \partial_2 (b) + \sigma (|\partial_1|, |a|) \sigma (|\partial_2|, |a|) a\partial_1\partial_2 (b) \\
&- \sigma (|\partial_1|, |\partial_2|) \partial_2\partial_1 (a) b - \sigma (|\partial_1|, |\partial_2|) \sigma (|\partial_2|, |\partial_1| + |a|) \partial_1 (a) \partial_2 (b) \\
&- \sigma (|\partial_1|, |\partial_2|) \sigma (|\partial_1|, |a|) \partial_2 (a) \partial_1 (b) \\
&- \sigma (|\partial_1|, |\partial_2|) \sigma (|\partial_1|, |a|) \sigma (|\partial_2|, |a|) a\partial_2\partial_1 (b) \\
&= \partial_1\partial_2 (a) b + \sigma (|\partial_1| + |\partial_2|, |a|) a\partial_1\partial_2 (b) \\
&- \sigma (|\partial_1|, |\partial_2|) \partial_2\partial_1 (a) b - \sigma (|\partial_1|, |\partial_2|) \sigma (|\partial_1| + |\partial_2|, |a|) a\partial_2\partial_1 (b) \\
&= [\partial_1, \partial_2]^\sigma (a) b + \sigma (|\partial_1| + |\partial_2|, |a|) a [\partial_1, \partial_2]^\sigma (b) .
\end{aligned}$$

∎

Proposition 5. *The σ-bracket satisfies the conditions,*

$$[a\partial_1, \partial_2]^\sigma = a [\partial_1, \partial_2]^\sigma - \sigma (|a| + |\partial_1|, |\partial_2|) \partial_2 (a) \partial_1, \tag{3}$$

$$[\partial_1, a\partial_2]^\sigma = \sigma (|\partial_1|, |a|) a [\partial_1, \partial_2]^\sigma + \partial_1 (a) \partial_2, \tag{4}$$

$\forall a \in A$.

Proof.

$$\begin{aligned}
&[a\partial_1, \partial_2]^\sigma (b) = a\partial_1\partial_2 (b) - \sigma (|a| + |\partial_1|, |\partial_2|) \partial_2 (a\partial_1) (b) \\
&= a\partial_1\partial_2 (b) - \sigma (|a| + |\partial_1|, |\partial_2|) (\partial_2 (a) \partial_1 (b) + \sigma (|\partial_2|, |a|) a\partial_2\partial_1 (b)) \\
&= (a [\partial_1, \partial_2]^\sigma - \sigma (|a| + |\partial_1|, |\partial_2|) \partial_2 (a) \partial_1) (b) , \\
&[\partial_1, a\partial_2]^\sigma (b) = \partial_1 (a\partial_2) (b) - \sigma (|\partial_1|, |a| + |\partial_2|) a\partial_2\partial_1 (b) \\
&= \partial_1 (a) \partial_2 (b) + \sigma (|\partial_1|, |a|) a\partial_1\partial_2 (b) - \sigma (|\partial_1|, |a| + |\partial_2|) a\partial_2\partial_1 (b) \\
&c = (\sigma (|\partial_1|, |a|) a [\partial_1, \partial_2]^\sigma + \partial_1 (a) \partial_2) (b) .
\end{aligned}$$

∎

The braided derivations is a braided Lie algebra as defined in [5].

Theorem 6. *$Der^\sigma(A)$ is a G-graded σ-Lie algebra with respect to the σ-bracket, that is, the following properties are satisfied,*

$$[Der^\sigma(A), Der^\sigma(A)]^\sigma \subseteq Der^\sigma(A), \tag{i}$$

$$\left[Der_i^\sigma(A), Der_j^\sigma(A)\right]^\sigma \subseteq Der_{i+j}^\sigma(A), \tag{i'}$$

$i, j \in G$, skew σ-symmetricity,

$$[\partial_1, \partial_2]^\sigma = -\sigma(|\partial_1|, |\partial_2|)[\partial_2, \partial_1]^\sigma, \tag{ii}$$

the σ-Jacobi identity for derivations,

$$[\partial_1, [\partial_2, \partial_3]^\sigma]^\sigma = [[\partial_1, \partial_2]^\sigma, \partial_3]^\sigma + \sigma(|\partial_1|, |\partial_2|)[\partial_2, [\partial_1, \partial_3]^\sigma]^\sigma, \tag{iii}$$

for $\partial_1, \partial_2, \partial_3 \in Der^\sigma(A)$ of degree $|\partial_1|, |\partial_2|, |\partial_3|$ respectively.

Proof. Let's do the proof for skew σ-symmetricity,

$$\begin{aligned} [\partial_1, \partial_2]^\sigma &= \partial_1\partial_2 - \sigma(|\partial_1|, |\partial_2|)\partial_2\partial_1 \\ &= \sigma(|\partial_1|, |\partial_2|)\partial_2\partial_1 - \sigma(|\partial_1|, |\partial_2|)\sigma(|\partial_2|, |\partial_1|)\partial_2\partial_1 \\ &= -\sigma(|\partial_1|, |\partial_2|)[\partial_2, \partial_1]^\sigma. \end{aligned}$$

∎

3.1. **Quantizations of braided derivations in graded algebras.** Let A be a σ-commutative G-graded algebra. Given a quantization q and an operator $\partial : A \to A$ of degree $|\partial|$ define it's quantization

$$\partial_q(a) = Q_q(\partial)(a) \overset{def}{=} q(|\partial|, |a|)\partial(a), \tag{5}$$

for homogeneous $a \in A_{|a|}$.

$Q_q(\partial)$ is an operator of the quantized M-graded algebra A_q.

The quantization of composition is

$$\partial_1 *_q \partial_2 = q(|\partial_1|, |\partial_2|)\partial_1 \circ \partial_2. \tag{6}$$

Denote by $Der^{\sigma_q}(A_q)$ set set of all $Q_q(\partial)$, $\partial \in Der^\sigma(A)$, equipped with the quantization of the composition.

Theorem 7. *Given a braiding σ, let σ_q be the quantization of σ. The operator*

$$\begin{aligned} Q_q &: (Der^\sigma(A), [-,-]^\sigma) \to \left(Der^{\sigma_q}(A_q), [-,-]_q^{\sigma_q}\right), \\ \partial &\in Der^\sigma_{|\partial|}(A) \longmapsto Q_q(\partial) \in Der^{\sigma_q}_{|\partial|}(A_q). \end{aligned} \tag{7}$$

is an isomorphism of modules between the σ-derivations of A and the σ_q-derivations of A_q.

Proof. The σ_q-Leibniz rule is satisfied

$$\begin{aligned} Q_q(\partial)(a *_q b) &= q(|a|, |b|)q(|\partial|, |a| + |b|)\partial(ab) \\ &= q(|\partial|, |a|)q(|\partial| + |a|, |b|)(\partial(a)b + \sigma(|\partial|, |a|)a\partial(b)) \\ Q_q(\partial)(a *_q b) &= Q_q(\partial)(a) *_q b + \sigma_q(|\partial|, |a|)a *_q Q_q(\partial)(b) \end{aligned}$$

where

$$
\begin{aligned}
&Q_q(\partial)(a) *_q b \\
&= q(|\partial|,|a|)\, q(|\partial|+|a|,|b|)\, \partial(a)\, b \sigma_q(|\partial|,|a|)\, a *_q Q_q(\partial)(b) \\
&= q^{-1}(|a|,|\partial|)\, \sigma(|\partial|,|a|)\, q(|\partial|,|a|)\, a *_q Q_q(\partial)(b) \\
&= q^{-1}(|a|,|\partial|)\, \sigma(|\partial|,|a|)\, q(|\partial|,|a|)\, q(|a|,|\partial|+|b|)\, q(|\partial|,|b|)\, a\partial(b) \\
&= q(|\partial|,|a|)\, q(|\partial|+|a|,|b|)\, \sigma(|\partial|,|a|)\, a\partial(b)\,.
\end{aligned}
$$

The σ_q-commutator of two derivations satisfies the σ_q-Leibniz rule,

$$
\begin{aligned}
&[\partial_1,\partial_2]_q^{\sigma_q}(ab) = \partial_1 *_q \partial_2(ab) - \sigma_q(\!|\partial_1|,|\partial_2|\!)\, \partial_2 *_q \partial_1(ab) \\
&= (\partial_1 *_q \partial_2)(a)\, b + q(|\partial_1|,|\partial_2|)\, \sigma_q(|\partial_1|,|\partial_2|+|a|)\, \partial_2(a)\, \partial_1(b) \\
&+ q(|\partial_2|,|\partial_1|)\, \sigma_q(|\partial_2|,|a|)\, \partial_1(a)\, \partial_2(b) \\
&\qquad + \sigma_q(|\partial_1|+|\partial_2|,|a|)\, a\partial_1 *_q \partial_2(b) - \sigma_q(\!|\partial_1|,|\partial_2|\!)\, \partial_2 *_q \partial_1(a)\, b \\
&\qquad - q(|\partial_2|,|\partial_1|)\, \sigma_q(\!|\partial_1|,|\partial_2|\!)\, \sigma_q(|\partial_2|,|\partial_1|+|a|)\, \partial_1(a)\, \partial_2(b) \\
&\qquad - q(|\partial_1|,|\partial_2|)\, \sigma_q(\!|\partial_1|,|\partial_2|\!)\, \sigma_q(|\partial_1|,|a|)\, \partial_2(a)\, \partial_1(b) \\
&\qquad - \sigma_q(\!|\partial_1|,|\partial_2|\!)\, \sigma_q(|\partial_1|+|\partial_2|,|a|)\, a\partial_2 *_q \partial_1(b) \\
&= (\partial_1 *_q \partial_2)(a)\, b + \sigma_q(|\partial_1|+|\partial_2|,|a|)\, a\,(\partial_1 *_q \partial_2)(b) \\
&\qquad - \sigma_q(\!|\partial_1|,|\partial_2|\!)\, (\partial_2 *_q \partial_1)(a)\, b \\
&\qquad - \sigma_q(\!|\partial_1|,|\partial_2|\!)\, \sigma_q(|\partial_1|+|\partial_2|,|a|)\, a\,(\partial_2 *_q \partial_1)(b) \\
&= [\partial_1,\partial_2]^{\sigma_q}(a)\, b + \sigma_q(|\partial_1|+|\partial_2|,|a|)\, a\, [\partial_1,\partial_2]^{\sigma_q}(b)\,,
\end{aligned}
$$

for homogeneous $\partial_1, \partial_2 \in Der^{\sigma_q}(A_q)$. ∎

$Der^{\sigma_q}(A_q)$ is a σ_q-symmetric A_q-module,

$$
\begin{aligned}
[a\partial_1,\partial_2]_q^{\sigma_q} &= a\,[\partial_1,\partial_2]_q^{\sigma_q} - \sigma_q(|a|+|\partial_1|,|\partial_2|)\, \partial_2(a)\, \partial_1, && (8) \\
[\partial_1,a\partial_2]_q^{\sigma_q} &= \sigma_q(|\partial_1|,|a|)\, a\,[\partial_1,\partial_2]_q^{\sigma_q} + \partial_1(a)\, \partial_2, && (9)
\end{aligned}
$$

for homogeneous $\partial_1, \partial_2 \in Der^{\sigma_q}(A_q)$, $a \in A_q$.

Furthermore, $Der^{\sigma_q}(A_q)$ is a σ_q-Lie algebra with respect to the $(\sigma_q{-}q)$-bracket, $[-,-]_q^{\sigma_q}$.

Let q_1, q_2 and q be quantizations, then

$$Q_{q_1 q_2} = Q_{q_1} \circ Q_{q_2}$$

and

$$Q_{q^{-1}} = Q_q^{-1}.$$

The inverse of the quantization of an operator ∂ of A_q is denoted by

$$Q_q^{-1}(\partial) = \partial_c.$$

As an object, $(A_q)_c = A_q = A$.

Proposition 8. *The composition satisfies*

$$(\partial_1)_c *_q (\partial_2)_c = (\partial_1 \circ \partial_2)_c , \tag{10}$$

for graded operators ∂_1 and ∂_2 on A_q.

Proof.

$$\begin{aligned} &Q_q^{-1}(\partial_1) *_q Q_q^{-1}(\partial_2)(a) \\ &\quad = q(|\partial_1|, |\partial_2|)\, q^{-1}(|\partial_1|, |\partial_2| + |a|)\, q^{-1}(|\partial_2|, |a|)\, \partial_1\partial_2(a) \\ &\qquad = q^{-1}(|\partial_1| + |\partial_2|, |a|)\, \partial_1\partial_2(a) = Q_q^{-1}(\partial_1 \circ \partial_2)(a) , \end{aligned}$$

for homogeneous $a \in A_q$. ∎

Let γ be any braiding and p any quantization. Define the $\gamma{-}p$-bracket,

$$[-,-]_p^\gamma ,$$

on operators, for which the composition between the operators is $*_p$,

$$[\partial_1, \partial_2]_p^\gamma = \partial_1 *_p \partial_2 - \gamma(|\partial_1|, |\partial_2|)\, \partial_2 *_p \partial_1. \tag{11}$$

Proposition 9. *Let $\partial_1 \in Der^\sigma_{|\partial_1|}(A)$, $\partial_2 \in Der^\sigma_{|\partial_2|}(A)$, then*

$$\left([\partial_1, \partial_2]^{\sigma_q}\right)_c = [(\partial_1)_c , (\partial_2)_c]_q^{\sigma_q} .$$

Proof.

$$\begin{aligned} &[\partial_1, \partial_2]_q^{\sigma_q} = \partial_1 *_q \partial_2 - \sigma_q(|\partial_1|, |\partial_2|)\, \partial_2 *_q \partial_1 \\ &\quad = q(|\partial_1|, |\partial_2|)\, \partial_1\partial_2 \\ &\qquad - q^{-1}(|\partial_2|, |\partial_1|)\, \sigma(|\partial_1|, |\partial_2|)\, q(|\partial_1|, |\partial_2|)\, q(|\partial_2|, |\partial_1|)\, \partial_2\partial_1 \\ &\qquad\qquad = q(|\partial_1|, |\partial_2|)\, [\partial_1, \partial_2]^\sigma . \end{aligned}$$

∎

Definition 10. *Define dequantization of $Der^{\sigma_q}(A_q)$ as the inverse of the set of all $Q_q^{-1}(\partial)$, $\partial \in Der^{\sigma_q}(A_q)$, equipped with the $[-,-]_q^{\sigma_q}$ bracket and the A-module structure*

$$a *_q \partial_c = q(|a|, |\partial|)\, a\partial_c, \tag{12}$$

for homogeneous $\partial \in Der^{\sigma_q}(A_q)$ and $a \in A_{|a|}$.

The dequantization of the braided derivations operates on A in the classical manner, but satisfies somewhat different properties than the classical, as the following theorem states.

Theorem 11. *The braided Lie algebra structure of $Der^{\sigma_q}(A_q)$ can be realized within the classical, $Der^{\sigma}(A)$, by dequantization.*

For homogeneous elements $\partial_1, \partial_2 \in Der^{\sigma_q}(A_q)$, $a \in A_q$, the following linearity is satisfied,

$$(\partial_1 + \partial_2)_c = (\partial_1)_c + (\partial_2)_c\,, \tag{i}$$

A-module structure,

$$(a \circ \partial_1)_c = a *_q (\partial_1)_c\,, \tag{ii}$$

and for the commutator,

$$([\partial_1, \partial_2]^{\sigma_q})_c = [(\partial_1)_c, (\partial_2)_c]_q^{\sigma_q}\,. \tag{iii}$$

Proof. (i):

$$Q_q^{-1}(\partial_1 + \partial_2) = Q_q^{-1}(\partial_1) + Q_q^{-1}(\partial_2)\,,$$

(ii), A-module structure: By proposition 8,

$$a_c *_q (\partial_1)_c = a *_q (\partial_1)_c = (a \circ \partial_1)_c\,,$$

(iii), σ_q-bracket:

$$Q_q^{-1}([\partial_1, \partial_2]^{\sigma_q})(a) = q^{-1}(|\partial_1| + |\partial_2|, |a|)(\partial_1\partial_2 - \sigma_q(|\partial_1|, |\partial_2|)\partial_2\partial_1)(a)$$

is equal to

$$\begin{aligned}
[(\partial_1)_c, (\partial_2)_c]_q^{\sigma_q} &= q^{-1}(|\partial_1|, |\partial_2| + |a|)\, q^{-1}(|\partial_2|, |a|)\, q(|\partial_1|, |\partial_2|)\,\partial_1\partial_2(a) \\
&\quad - q^{-1}(|\partial_2|, |\partial_1| + |a|)\, q^{-1}(|\partial_1|, |a|)\, q(|\partial_2|, |\partial_1|)\,\sigma_q(|\partial_1|, |\partial_2|)\,\partial_2\partial_1(a) \\
&= q^{-1}(|\partial_1| + |\partial_2|, |a|)\, q^{-1}(|\partial_1|, |\partial_2|)\, q(|\partial_1|, |\partial_2|)\,\partial_1\partial_2(a) \\
&\quad - q^{-1}(|\partial_1| + |\partial_2|, |a|)\, q^{-1}(|\partial_2|, |\partial_1|)\, q(|\partial_2|, |\partial_1|)\,\sigma_q(|\partial_1|, |\partial_2|)\,\partial_2\partial_1(a) \\
&= q^{-1}(|\partial_1| + |\partial_2|, |a|)(\partial_1\partial_2 - \sigma_q(|\partial_1|, |\partial_2|)\partial_2\partial_1)(a)\,,
\end{aligned}$$

for homogeneous $a \in A_q$, $\partial_1, \partial_2 \in Der^{\sigma_q}(A_q)$. ∎

3.2. **Evaluations and commutators.** For both σ- and σ_q-derivations, evaluating a derivation of some element corresponds to taking the braided bracket of the derivation and that element.

Proposition 12. *Let A be σ-commutative algebra and $\partial \in Der^{\sigma}_{|\partial|}(A)$, $a \in A_{|a|}$. Then the evaluation of ∂_c on homogeneous $a \in A$ is*

$$\partial(a) = [\partial, a]^{\sigma}\,.$$

Let

$$\partial_q \in Der^{\sigma_q}_{|\partial|}(A_q)\,.$$

Then the evaluation of the derivation on some homogeneous $a \in (A_q)_{|a|}$ is equal to taking the $\sigma_q - q$-bracket of ∂_q and a,

$$\partial_q(a) = [\partial_q, a]_q^{\sigma_q}.$$

Proof. Let $\partial \in Der^{\sigma}_{|\partial|}(A)$ and $a \in A_{|a|}, b \in A_{|b|}$. By the σ-Leibniz rule

$$\partial(ab) = \partial(a)b + \sigma(|\partial|, |a|)a\partial(b),$$

and clearly, by rearranging,

$$\partial(a) = [\partial, a]^{\sigma} = \partial a - \sigma(|\partial|, |a|)a\partial.$$

For proof of the second half of the proposition let $\partial_q \in Der^{\sigma_q}_{|\partial|}(A_q)$ and $a \in A_{|a|}, b \in A_{|b|}$. By the σ-Leibniz rule

$$\partial_q(ab) = \partial_q(a) *_q b + \sigma_q(|\partial|, |a|)a *_q \partial_q(b),$$

and by rearranging,

$$\partial_q(a) = [\partial_q, a]_q^{\sigma_q} = \partial_q *_q a - \sigma_q(|\partial|, |a|)a *_q \partial_q.$$

∎

4. Braided derivations in graded modules

Let R be a field, σ be a braiding in the monoidal category of graded modules, A be a G-graded σ-commutative R-algebra and E a G-graded σ-symmetric A-module. Let

$$\partial_A : A \to A$$

be a G-graded σ-derivation of A.

Definition 13. *An operator of E, $\partial : E \to E$ is said to be a graded σ-derivation over ∂_A of degree $|\partial| \in G$ if ∂ is R-linear,*

$$\partial : E_g \to E_{g+|\partial|},$$

$g \in G$

$$|\partial| = |\partial_A|,$$

and satisfy the σ-Leibniz rule with respect to ∂_A,

$$\partial(ax) = \partial_A(a)x + \sigma(|\partial|, |a|)a\partial(x),$$

for homogeneous $x \in E$ and $a \in A$.

The pair (∂, ∂_A) is called a σ-derivation of E over A.

The morphism $\pi : (\partial, \partial_A) \to \partial_A$ we call the projection from the σ-derivations of E over A to the σ-derivations of A.

The set of all σ-derivations of E over A of degree $g \in G$ is denoted by $Der_g^{(\sigma,A)}(E)$ and the set of all σ-derivations of E over A (equipped with the quantization of the composition) is denoted by $Der^{(\sigma,A)}(E)$.

A left A-module structure on $Der^{(\sigma,A)}(E)$ is defined by

$$(a\partial)(b) = a(\partial(x)), \tag{13}$$

and

$$a\partial \in Der_{|a|+|\partial|}^{(\sigma,A)}(E), \tag{14}$$

for homogeneous $a \in A$, $x \in E$ $\partial \in Der_{|\partial|}^{(\sigma,A)}(E)$.

The σ-commutator is defined as for σ-derivations of A.

Proposition 14. *The σ-commutator of two σ-derivations is a σ-derivation of the combined degree,*

$$[\partial_1, \partial_2]^\sigma \in Der_{|\partial_1|+|\partial_2|}^{(\sigma,A)}(E),$$

for homogeneous $\partial_1, \partial_2 \in Der^{(\sigma,A)}(E)$.

Proof. The σ-commutator of two derivations satisfies the σ-Leibniz rule over A,

$$([\partial_1, \partial_2]^\sigma)(ax) = [(\partial_1)_A, (\partial_2)_A]^\sigma(a)\, x + \sigma(|\partial_1| + |\partial_2|, |a|)\, a\, [\partial_1, \partial_2]^\sigma(x),$$

for homogeneous $a \in A$, which is proved as follows,

$$\begin{aligned}
[\partial_1, \partial_2]^\sigma(ax) &= \partial_1\partial_2(ax) - \sigma(|\partial_1|, |\partial_2|)\, \partial_2\partial_1(ax) \\
&= (\partial_1)_A(\partial_2)_A(a)\, x + \sigma(|\partial_1|, |\partial_2| + |a|)\, (\partial_2)_A(a)\, \partial_1(x) \\
&+ \sigma(|\partial_2|, |a|)\, (\partial_1)_A(a)\, \partial_2(x) + \sigma(|\partial_1|, |a|)\, \sigma(|\partial_2|, |a|)\, a\partial_1\partial_2(x) \\
&- \sigma(|\partial_1|, |\partial_2|)\, (\partial_2)_A(\partial_1)_A(a)\, x \\
&- \sigma(|\partial_1|, |\partial_2|)\, \sigma(|\partial_2|, |\partial_1| + |a|)\, (\partial_1)_A(a)\, \partial_2(x) \\
&- \sigma(|\partial_1|, |\partial_2|)\, \sigma(|\partial_1|, |a|)\, (\partial_2)_A(a)\, \partial_1(x) \\
&- \sigma(|\partial_1|, |\partial_2|)\, \sigma(|\partial_1|, |a|)\, \sigma(|\partial_2|, |a|)\, a\partial_2\partial_1(x) \\
= {} & (\partial_1)_A(\partial_2)_A(a)\, x + \sigma(|\partial_1| + |\partial_2|, |a|)\, a\partial_1\partial_2(x) \\
- {} & \sigma(|\partial_1|, |\partial_2|)\, (\partial_2)_A(\partial_1)_A(a)\, x - \sigma(|\partial_1|, |\partial_2|)\, \sigma(|\partial_1| + |\partial_2|, |a|)\, a\partial_2\partial_1(x) \\
= {} & [(\partial_1)_A, (\partial_2)_A]^\sigma(a)\, x + \sigma(|\partial_1| + |\partial_2|, |a|)\, a\, [\partial_1, \partial_2]^\sigma(x).
\end{aligned}$$

∎

Proposition 15. *The σ-bracket satisfies*

$$\begin{aligned} [a\partial_1, \partial_2]^\sigma &= a[\partial_1, \partial_2]^\sigma - \sigma(|a| + |\partial_1|, |\partial_2|)\, \partial_2(a)\, \partial_1, && (15) \\ [\partial_1, a\partial_2]^\sigma &= \sigma(|\partial_1|, |a|)\, a[\partial_1, \partial_2]^\sigma + \partial_1(a)\, \partial_2, && (16) \end{aligned}$$

for homogeneous $\partial_1, \partial_2 \in Der^{(\sigma,A)}(E)$ and $a \in A$.

Theorem 16. *$Der^{(\sigma,A)}(E)$ is a G-graded σ-Lie algebra with respect to the σ-bracket. That is, the following properties are satisfied,*

$$\left[Der^{(\sigma,A)}(E), Der^{(\sigma,A)}(E)\right]^\sigma \subseteq Der^{(\sigma,A)}(E), \tag{i}$$

$$\left[Der_i^{(\sigma,A)}(E), Der_j^{(\sigma,A)}(E)\right]^\sigma \subseteq Der_{i+j}^{(\sigma,A)}(E), \tag{i'}$$

and $[\partial_1, \partial_2]^\sigma$ is a σ-derivation over $[(\partial_1)_A, (\partial_2)_A]^\sigma$, the σ-bracket is skew σ-symmetric,

$$[\partial_1, \partial_2]^\sigma = -\sigma(|\partial_1|, |\partial_2|)\, [\partial_2, \partial_1]^\sigma, \tag{ii}$$

and the σ-Jacobi identity for derivations is satisfied,

$$[\partial_1, [\partial_2, \partial_3]^\sigma]^\sigma = [[\partial_1, \partial_2]^\sigma, \partial_3]^\sigma + \sigma(|\partial_1|, |\partial_2|)\, [\partial_2, [\partial_1, \partial_3]^\sigma]^\sigma, \tag{iii}$$

for homogeneous $\partial_1, \partial_2, \partial_3 \in Der^{(\sigma,A)}(E)$.

We get the exact sequence of graded A-modules and G-graded Lie algebras

$$0 \to End_A^\sigma(E) \to Der^{(\sigma,A)}(E) \xrightarrow{\pi} Der^\sigma(A) \tag{17}$$

where $End_A^\sigma(E)$ is the σ-symmetric (graded) endomorphisms of E over A.

4.1. **Quantizations of braided derivations in graded modules.** Let A be a σ-commutative G-graded algebra and E a σ-commutative G-graded A-module

Given a quantization q and an operator

$$\partial : E \to E$$

of degree $|\partial|$, define the quantization of ∂,

$$\partial_q(x) = Q_q(\partial)(x) = q(|\partial|, |x|)\, \partial(x), \tag{18}$$

$x \in E$.

$Q_q(\partial)$ is an operator of the quantized module E_q.

Denote by $Der^{(\sigma_q, A_q)}(E_q)$ set set of all $Q_q(\partial)$, $\partial \in Der^{(\sigma,A)}(E)$, equipped with the quantization of the composition.

Theorem 17. *The operator Q_q is an A-module isomorphism between the σ-derivations of E over A and the σ_q-derivations of E_q over A_q,*

$$\begin{aligned} Q_q &: \left(Der^{(\sigma,A)}(E), [-,-]^{\sigma}\right) \to \left(Der^{(\sigma_q,A_q)}(E_q), [-,-]_q^{\sigma_q}\right), \\ \partial &\in Der^{(\sigma,A)}_{|\partial|}(E) \longmapsto Q_q(\partial) \in Der^{(\sigma_q,A_q)}_{|\partial|}(E_q). \end{aligned} \tag{19}$$

That is, $Q_q(\partial)$ satisfies the σ_q-Leibniz rule with respect to $Q_q(\partial_A)$

$$Q_q(\partial)(ax) = Q_q(\partial_A)(a)\, x + \sigma_q(|\partial|, |a|)\, a Q_q(\partial)(x), \tag{20}$$

for homogeneous $\partial \in Der^{(\sigma,A)}(E)$, $x \in E_q$, $a \in A_q$.

$Der^{(\sigma_q,A_q)}(E_q)$ is a σ_q-symmetric module, satisfying (8) and (9), and is a σ_q-Lie algebra with respect to the $\sigma_q - q$-bracket, $[-,-]_q^{\sigma_q}$.

Theorem 18. *The Lie algebra structure of $Der^{(\sigma_q,A)}(E_q)$ can be realized within the classical, $Der^{(\sigma,A)}(E)$, by dequantization. That is the following is satisfied. The linearity,*

$$(\partial_1 + \partial_2)_c = (\partial_1)_c + (\partial_2)_c, \tag{i}$$

for homogeneous $\partial_1, \partial_2 \in Der^{(\sigma_q,A)}(E_q)$, A-module structure,

$$(a \circ \partial_1)_c = a *_q (\partial_1)_c, \tag{ii}$$

and the commutator,

$$([\partial_1, \partial_2]^{\sigma_q})_c = [(\partial_1)_c, (\partial_2)_c]_q^{\sigma_q}, \tag{iii}$$

for homogeneous $\partial_1, \partial_2 \in Der^{(\sigma_q,A)}(E_q)$, $a \in A_q$.

We get the following commutative diagram exact sequences of graded A-modules and G-graded Lie algebras

(21)

where

$$\pi_q \overset{def}{=} Q_q \circ \pi \circ Q_q^{-1}. \tag{22}$$

5. Braided connections and curvature in graded modules

Let $\partial_1, \partial_2 \in Der^{\sigma}(A)$ be homogeneous.

Definition 19. *A σ-connection in a σ-symmetric graded module E is a σ-symmetric graded module homomorphism ∇ of degree 0*

$$\nabla : Der^{\sigma}_{|\partial_1|}(A) \to Der^{(\sigma,A)}_{|\partial_1|}(E)$$

such that

$$\pi \circ \nabla = Id.$$

Definition 20. *A σ-connection ∇ is flat if it is a σ-Lie algebra homomorphism, that is,*

$$\nabla\left([\partial_1, \partial_2]^\sigma\right) = [\nabla\partial_1, \nabla\partial_2]^\sigma ,$$

for all $\partial_1, \partial_2 \in Der^\sigma(A)$.

Definition 21. *In general, define the σ-curvature of ∇ to be*

$$K_\nabla(\partial_1, \partial_2) = [\nabla\partial_1, \nabla\partial_2]^\sigma - \nabla\left([\partial_1, \partial_2]^\sigma\right).$$

Theorem 22. *Given homogeneous ∂_1 and ∂_2, the σ-curvature*

$$K_\nabla : Der^\sigma(A) \otimes Der^\sigma(A) \to End_A(E),$$

applied to ∂_1 and ∂_2 is a σ-symmetric endomorphism of E, that is,

$$K_\nabla(\partial_1, \partial_2)(a\partial_1) = \sigma\left(|\partial_1| + |\partial_2|, a\right) aK_\nabla(\partial_1, \partial_2)(\partial_1), \tag{i}$$

and K_∇ is skew σ-symmetric,

$$K_\nabla(\partial_1, \partial_2) = -\sigma\left(|\partial_1|, |\partial_2|\right) K_\nabla(\partial_2, \partial_1). \tag{ii}$$

Furthermore K_∇ satisfies the σ-symmetric A-module homomorphisms

$$K_\nabla(a\partial_1, \partial_2) = aK_\nabla(\partial_1, \partial_2), \tag{iii}$$

$$K_\nabla(\partial_1, a\partial_2) = \sigma\left(|\partial_1|, |a|\right) aK_\nabla(\partial_1, \partial_2), \tag{iv}$$

$a \in A$.

Proof. (i):

$$\begin{aligned}
&K_\nabla(\partial_1, \partial_2)(ax) = [\nabla\partial_1, \nabla\partial_2]^\sigma(ax) - \nabla\left([\partial_1, \partial_2]^\sigma\right)(ax)\\
&= \left[(\nabla\partial_1)_A, (\nabla\partial_2)_A\right]^\sigma(a)\,x + \sigma\left(|\partial_1| + |\partial_2|, a\right) a\,[\nabla\partial_1, \nabla\partial_2]^\sigma(x)\\
&\qquad - \left(\nabla\left([\partial_1, \partial_2]^\sigma\right)\right)_A(a)\,x - \sigma\left(|\partial_1| + |\partial_2|, |a|\right) a\left(\nabla\left([\partial_1, \partial_2]^\sigma\right)\right)(x)\\
&= [\partial_1, \partial_2]^\sigma(a)\,x + \sigma\left(|\partial_1| + |\partial_2|, a\right) a\,[\nabla\partial_1, \nabla\partial_2]^\sigma(x)\\
&\qquad\qquad - [\partial_1, \partial_2]^\sigma(a)\,x - \sigma\left(|\partial_1| + |\partial_2|, |a|\right) a\left(\nabla\left([\partial_1, \partial_2]^\sigma\right)\right)(x)\\
&= \sigma\left(|\partial_1| + |\partial_2|, a\right)\left(a\,[\nabla\partial_1, \nabla\partial_2]^\sigma(x) - a\left(\nabla\left([\partial_1, \partial_2]^\sigma\right)\right)(x)\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad = \sigma\left(|\partial_1| + |\partial_2|, a\right) aK_\nabla(\partial_1, \partial_2)(x).
\end{aligned}$$

(ii):

$$\begin{aligned}
K_\nabla(\partial_1, \partial_2) &= [\nabla\partial_1, \nabla\partial_2]^\sigma - \nabla\left([\partial_1, \partial_2]^\sigma\right)\\
&= -\sigma\left(|\partial_1|, |\partial_2|\right)[\nabla\partial_2, \nabla\partial_1]^\sigma - \nabla\left(-\sigma\left(|\partial_1|, |\partial_2|\right)[\partial_2, \partial_1]^\sigma\right)\\
&= -\sigma\left(|\partial_1|, |\partial_2|\right) K_\nabla(\partial_2, \partial_1).
\end{aligned}$$

(iii):

$$K_{\nabla}(a\partial_1,\partial_2) = [\nabla(a\partial_1),\nabla\partial_2]^{\sigma} - \nabla([a\partial_1,\partial_2]^{\sigma})$$
$$= a\begin{pmatrix} [(\nabla\partial_1),\nabla\partial_2]^{\sigma} - \sigma(|a|+|\partial_1|,|\partial_2|)\nabla(\partial_2)(a)\nabla(\partial_1) \\ -a\nabla[\partial_1,\partial_2]^{\sigma} + \sigma(|a|+|\partial_1|,|\partial_2|)\nabla(\partial_2(a)\partial_1) \end{pmatrix}$$
$$= a([(\nabla\partial_1),\nabla\partial_2]^{\sigma} - \nabla[\partial_1,\partial_2]^{\sigma}).$$

(iv):

$$\begin{aligned} K_{\nabla}(\partial_1,a\partial_2) &= [\nabla\partial_1,\nabla(a\partial_2)]^{\sigma} - \nabla([\partial_1,a\partial_2]^{\sigma}) \\ &= [\nabla\partial_1,a(\nabla\partial_2)]^{\sigma} - \sigma(|\partial_1|,|a|)\nabla(a[\partial_1,\partial_2]^{\sigma}) \\ &= \sigma(|\partial_1|,|a|)a[\nabla\partial_1,(\nabla\partial_2)]^{\sigma} + \nabla(\partial_1)(a)\nabla(\partial_2) \\ &\quad -\sigma(|\partial_1|,|a|)a\nabla([\partial_1,\partial_2]^{\sigma}) + \nabla(\partial_1(a)\partial_2) \\ &= \sigma(|\partial_1|,|a|)a([\nabla\partial_1,\nabla\partial_2]^{\sigma} - \nabla[\partial_1,\partial_2]^{\sigma}). \end{aligned}$$

▮

5.1. **Quantization of braided connections and curvature.**

Definition 23. *Let ∇ be a σ-connection in E. The quantization of ∇,*

$$\nabla_q : Der^{\sigma_q}(A_q) \to Der^{(\sigma_q,A_q)}(E_q).$$

is defined by

$$\nabla_q \overset{def}{=} Q_q \circ \nabla \circ Q_q^{-1}, \tag{23}$$

that is, the following diagram commutes

$$\begin{tikzcd} Der^{(\sigma,A)}(E) \arrow[d, "Q_q"'] & Der^{\sigma}(A) \arrow[l, "\nabla"'] \arrow[d, "Q_q"'] \\ Der^{(\sigma_q,A_q)}(E_q) & Der^{\sigma_q}(A_q) \arrow[l, "\nabla_q"'] \end{tikzcd}.$$

Hence, ∇_q is a splitting of the lower sequence in (21).

Proposition 24. *The quantization of a connection ∇, ∇_q, is a σ_q-connection in E_q.*

Let ∇_q be a σ_q-connection in E_q. Then the $\sigma_q - q$-curvature of ∇_q

$$K^q_{\nabla_q} : Der^{\sigma_q}(A_q) \otimes Der^{\sigma_q}(A_q) \to End_{A_q}(E_q),$$

is defined by

$$K^q_{\nabla_q}(\partial_1,\partial_2) = [\nabla_q\partial_1,\nabla_q\partial_2]_q^{\sigma_q} - \nabla_q\left([\partial_1,\partial_2]_q^{\sigma_q}\right). \tag{24}$$

Theorem 25. *The $\sigma_q - q$-curvature satisfies*

$$K^q_{\nabla_q}\left(\partial_1,\partial_2\right)(ax) = \sigma_q\left(|\partial_1| + |\partial_2|, a\right) a K^q_{\nabla_q}\left(\partial_1,\partial_2\right)(x), \tag{i}$$

and is skew σ_q-symmetric,

$$K^q_{\nabla_q}\left(\partial_1,\partial_2\right) = -\sigma_q\left(|\partial_1|, |\partial_2|\right) K^q_{\nabla_q}\left(\partial_2,\partial_1\right).$$

Furthermore, the $\sigma_q - q$-curvature satisfies the σ_q-symmetric A_q-module homomorphisms

$$K^q_{\nabla_q}\left(a\partial_1,\partial_2\right) = a K^q_{\nabla_q}\left(\partial_1,\partial_2\right), \tag{ii}$$

$$K^q_{\nabla_q}\left(\partial_1,a\partial_2\right) = \sigma_q\left(|\partial_1|, |a|\right) a K^q_{\nabla_q}\left(\partial_1,\partial_2\right), \tag{iii}$$

$a \in A_q$.

We get the following picture for dequantizations of braided derivations.

Theorem 26. *The $\sigma_q - q$-curvature K^q_∇ of the σ-connection ∇ of E defined by*

$$K^q_\nabla\left(\left(\partial_1\right)_c, \left(\partial_2\right)_c\right) = \left[\nabla\left(\partial_1\right)_c, \nabla\left(\partial_2\right)_c\right]^{\sigma_q}_q - \nabla\left(\left[\left(\partial_1\right)_c, \left(\partial_2\right)_c\right]^{\sigma_q}_q\right), \tag{25}$$

and the σ_q-curvature of the σ_q-connection ∇_q defined by

$$K_{\nabla_q}\left(\partial_1,\partial_2\right) = \left[\nabla_q\partial_1, \nabla_q\partial_2\right]^{\sigma_q} - \nabla_q\left(\left[\partial_1,\partial_2\right]^{\sigma_q}\right) \tag{26}$$

are related as follows,

$$\left(K_{\nabla_q}\left(\partial_1,\partial_2\right)\right)_c = K^q_\nabla\left(\left(\partial_1\right)_c, \left(\partial_2\right)_c\right), \tag{27}$$

$\partial_1, \partial_2 \in Der^{\sigma_q}\left(A_q\right)$. If ∇ is flat σ-connection in E with respect to the σ-curvature K_∇, then is ∇ is flat in E with respect to the $\sigma_q - q$-curvature K^q_∇ and ∇_q is a flat σ_q-connection in E_q with respect to the σ_q-curvature K_{∇_q}.

Proof. Proof of (27):

$$\begin{aligned}
&K_{\nabla_q}\left(\partial_1,\partial_2\right)\\
&= \left[\nabla_q\partial_1, \nabla_q\partial_2\right]^{\sigma_q} - \nabla_q\left(\left[\partial_1,\partial_2\right]^{\sigma_q}\right)\\
&= \left[Q_q \circ \nabla \circ Q_q^{-1}\left(\partial_1\right), Q_q \circ \nabla \circ Q_q^{-1}\left(\partial_2\right)\right]^{\sigma_q} - Q_q \circ \nabla \circ Q_q^{-1}\left(\left[\partial_1,\partial_2\right]^{\sigma_q}\right)\\
&\qquad = Q_q\left(\left[\nabla\left(\left(\partial_1\right)_c\right), \nabla\left(\left(\partial_2\right)_c\right)\right]^{\sigma_q}_q\right) - Q_q \circ \nabla\left(\left[\left(\partial_1\right)_c, \left(\partial_2\right)_c\right]^{\sigma_q}_q\right)\\
&\qquad\qquad = Q_q\left(\left[\nabla\left(\left(\partial_1\right)_c\right), \nabla\left(\left(\partial_2\right)_c\right)\right]^{\sigma_q}_q - \nabla\left(\left[\left(\partial_1\right)_c, \left(\partial_2\right)_c\right]^{\sigma_q}_q\right)\right)\\
&\qquad\qquad\qquad = Q_q\left(K^q_\nabla\left(\left(\partial_1\right)_c, \left(\partial_2\right)_c\right)\right).
\end{aligned}$$

If $K_{\nabla}(\partial_1, \partial_2) = 0$, then

$$K^q_{\nabla}((\partial_1)_c, (\partial_2)_c) = [\nabla(\partial_1)_c, \nabla(\partial_2)_c]^{\sigma_q}_q - \nabla\left([(\partial_1)_c, (\partial_2)_c]^{\sigma_q}_q\right)$$
$$= q(|\partial_1|, |\partial_2|)([\nabla(\partial_1)_c, \nabla(\partial_2)_c]^{\sigma} - \nabla([(\partial_1)_c, (\partial_2)_c]^{\sigma})) = 0.$$

∎

The formula (27) means that

$$\begin{aligned} K_{\nabla_q}(\partial_1, \partial_2)(x) &= Q_q(K^q_{\nabla}((\partial_1)_c, (\partial_2)_c))(x) \\ &= q(|\partial_1|, |\partial_2|) Q_q(K_{\nabla}((\partial_1)_c, (\partial_2)_c))(x) \\ &= q(|\partial_1| + |\partial_2|, |x|) q(|\partial_1|, |\partial_2|) K_{\nabla}((\partial_1)_c, (\partial_2)_c)(x), \end{aligned}$$

for $x \in E$.

6. Application to γ-densities and γ-forms on $\mathbb{R}$.

Consider the real line, $\mathbb{R}$, and γ-densities on $\mathbb{R}$,

$$\theta = f(x)(|dx|)^{\gamma},$$

and γ-forms on $\mathbb{R}$

$$\theta' = f(x)(dx)^{\gamma},$$

$\gamma \in \mathbb{R}$ and f is a function on $\mathbb{R}$. Denote by Ω_{γ} the set of all γ-densities on $\mathbb{R}$.

If $\gamma = 1$, then we have differential 1-forms on $\mathbb{R}$, if $\gamma = -1$, then we have vector fields on $\mathbb{R}$ and if $\gamma = 0$, then we have functions on $\mathbb{R}$.

For a function on y, $g(y)$, a γ-change of variable is

$$g(y)\, dy \longmapsto g(F(x))\, |F'|^{\gamma}(dx)^{\gamma},$$

if $y = F(x)$.

$\Omega = \oplus_{\gamma \in \mathbb{R}} \Omega_{\gamma}$ is an algebra with the multiplication

$$\begin{aligned} \Omega_{\beta} \otimes \Omega_{\gamma} &\to \Omega_{\beta+\gamma}, \\ f(x)(|dx|)^{\alpha} \otimes g(x)(|dx|)^{\beta} &\mapsto fg(x)(|dx|)^{\alpha+\beta}. \end{aligned}$$

Definition 27. *Define a bracket on Ω by*

$$\left[f_{\beta}|dx|^{\beta}, g_{\gamma}|dx|^{\gamma}\right] = [f_{\beta}, g_{\gamma}]\, |dx|^{\beta+\gamma-1},$$

where

$$[g_{\beta}, f_{\gamma}] = \gamma g'_{\beta} f_{\gamma} - \beta g_{\beta} f'_{\gamma},$$

$g_{\beta} \in \Omega_{\beta}$ *and* $f_{\gamma} \in \Omega_{\gamma}$.

Clearly,

$$[\Omega_{\beta}, \Omega_{\gamma}] \subseteq \Omega_{\beta+\gamma-1},$$

since $f'_{\gamma} \in \Omega_{\gamma-1}$ when $f_{\gamma} \in \Omega_{\gamma}$.

Proposition 28. *Ω is a $\mathbb{R}$-graded Lie algebra. That is,*

$$[g_\beta, f_\gamma] = -[f_\beta, g_\gamma],$$

and the Jacobi identity is satisfied,

$$[h_\alpha, [g_\beta, f_\gamma]] = [[h_\alpha, g_\beta], f_\gamma] + [g_\beta, [h_\alpha, f_\gamma]],$$

$h_\alpha \in \Omega_\alpha, g_\beta \in \Omega_\beta, f_\gamma \in \Omega_\gamma$.

Now consider a collection of n-tuples

$$\Gamma = \{\gamma = (\gamma_1, \ldots, \gamma_n)\} \subset \mathbb{R}^n,$$

such that for $\gamma, \gamma' \in \Gamma$ the sum $\gamma + \gamma' \in \Gamma$ and $\gamma + \gamma' - 1_i \in \Gamma$ for all $i = 1, \ldots, n$, where $1_i = (0, \ldots, 1, \ldots, 0)$, 1 in the ith place.

For each $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Gamma$ consider $\Omega_\gamma = \oplus_{\gamma_i} \Omega_{\gamma_i}$ and the Γ-graded module $\Omega_\Gamma = \oplus_\gamma \Omega_\gamma$. Clearly is Ω_Γ a Γ-graded Lie algebra.

Any quantization of Ω_Γ is given by

$$q(\gamma, \gamma') = \exp(i \langle P\gamma, \gamma' \rangle),$$

where P is a skew symmetric $n \times n$-matrix and the quantization of the standard twist, $\tau_q = q^{-1} \circ \tau \circ q$, is realized by

$$q^{-1}(\gamma', \gamma)\, q(\gamma, \gamma') = \exp(2i \langle P\gamma, \gamma' \rangle).$$

Then the quantization of Ω_Γ by q is a Γ-graded τ_q-Lie algebra, that is,

$$[g_\gamma, f_{\gamma'}] = -\exp(2i \langle P\gamma, \gamma' \rangle) [f_{\gamma'}, g_\gamma],$$

and the Jacobi identity is satisfied,

$$[h_\gamma, [g_{\gamma'}, f_{\gamma''}]] = [[h_\gamma, g_{\gamma'}], f_{\gamma''}] + \exp(2i \langle P\gamma, \gamma' \rangle) [g_{\gamma'}, [h_\gamma, f_{\gamma''}]],$$

$h_\gamma \in \Omega_\gamma, g_{\gamma'} \in \Omega_{\gamma'}, f_{\gamma''} \in \Omega_{\gamma''}$.

Note that even if Γ is infinite there is no problem to extend the theory in this paper to this case as long as there is some minimal grading ς such that there is no $f \in \Omega_\Gamma$ with grading by $\gamma < \varsigma$.

7. Braided differential operators

We shall see how the picture is for braided differential operators. Let G be a finite abelian group.

7.1. **Braided differential operators in graded algebras.** Let A be a σ-symmetric G-graded algebra.

Define a graded σ-differential operator f of degree $|f| \in G$ and order at most k as the linear map $f : A \to A$, such that

$$f : A_g \to A_{g+|f|},$$

$g \in G$, and

$$[a_0, [a_1, \dots [a_k, f]^\sigma \cdots]^\sigma]^\sigma = 0, \tag{28}$$

$\forall a_0, \dots, a_k \in A$.

Denote by $Diff_{k,i}^\sigma (A, A)$ the σ-differential operators of order at most k and degree i and by $Diff_k^\sigma (A, A)$ the set of all of order at most k.

Let's consider $Diff^\sigma (A, A) = \cup Diff_k^\sigma (A, A)$.

From [16] we have the two following results. The σ-commutator of two σ-differential operators $f_1 \in Diff_i^\sigma (A, A)$ and $f_2 \in Diff_j^\sigma (A, A)$ is a σ-differential operator of order at most $i + j - 1$,

$$[f_1, f_2]^\sigma \in Diff_{i+j-1}^\sigma (A, A).$$

and $Diff^\sigma (A, A)$ is a σ-Lie algebra. Furthermore, clearly,

$$[f_1, f_2]^\sigma \in Diff_{i+j-1,|f_1|+|f_2|}^\sigma (A, A),$$

for homogeneous $f_1 \in Diff_{i,|f_1|}^\sigma (A, A)$ and $f_2 \in Diff_{j,|f_2|}^\sigma (A, A)$.

Proposition 29. *There is an $A - A$-module structure on $Diff^\sigma (A, A)$ defined by*

$$\begin{aligned} \nu^l (a \otimes f)(b) &= af(b), \\ \nu^r (f \otimes a)(b) &= f(ab), \end{aligned}$$

$a, b \in A$, $f \in Diff^\sigma (A, A)$ and $af, fa \in Diff_{k,|f|+|a|}^\sigma (A, A)$, for homogeneous $f \in Diff_{k,|f|}^\sigma (A, A)$.

Proof.

$$\begin{aligned}
&[a_0, [a_1, \dots [a_k, bf]^\sigma \cdots]^\sigma]^\sigma \\
&= [a_0, [a_1, \dots [a_{k-1}, (a_k (bf) - \sigma (|a_k|, |b| + |f|) (bf) a_k)]^\sigma \cdots]^\sigma]^\sigma \\
&= [a_0, [a_1, \dots [a_{k-1}, (\sigma (|a_k|, |b|) (ba_k) f - \sigma (|a_k|, |b| + |f|) bf (a_k))]^\sigma \cdots]^\sigma]^\sigma \\
&= \sigma (|a_k|, |b|) \left[a_0, [a_1, \dots [a_{k-1}, b [a_k, f]^\sigma]^\sigma \cdots]^\sigma\right]^\sigma \\
&\vdots \\
&= \sigma (|a_k| + \cdots + |a_0|, |b|) \, b \, [a_0, [a_1, \dots [a_k, f]^\sigma \cdots]^\sigma]^\sigma = 0
\end{aligned}$$

Also the right action on a braided differential operator again is a braided differential operator,

$$\begin{aligned}
& [a_0, [a_1, \dots [a_k, fb]^\sigma \cdots]^\sigma]^\sigma \\
= \; & [a_0, [a_1, \dots [a_{k-1}, (a_k (fb) - \sigma (|a_k|, |f| + |b|) (fb) a_k)]^\sigma \cdots]^\sigma]^\sigma \\
= \; & [a_0, [a_1, \dots [a_{k-1}, ((a_k f) b - \sigma (|a_k|, |f|) f (a_k b))]^\sigma \cdots]^\sigma]^\sigma \\
= \; & \left[a_0, [a_1, \dots [a_{k-1}, [a_k, f]^\sigma b]^\sigma \cdots]^\sigma \right]^\sigma \\
& \vdots \\
= \; & [a_0, [a_1, \dots [a_k, f]^\sigma \cdots]^\sigma]^\sigma b = 0,
\end{aligned}$$

for homogeneous $a_0, \dots, a_k, b \in A$, $f \in Diff_k^\sigma (A, A)$. ∎

Consider the symbol of the differential operators which is the leading part with respect to derivatives,

$$Smbl_k^\sigma (A, A) = Diff_k^\sigma (A, A) / Diff_{k-1}^\sigma (A, A),$$

then we have the $\mathbb{Z}$-graded object

$$Smbl^\sigma (A, A) = \sum_{k \in \mathbb{Z}} Smbl_k^\sigma (A, A).$$

The class of $[f_1, f_2]^\sigma \in Diff^\sigma_{i+j-1, |f_1|+|f_2|} (A, A)$,

$$\overline{[f_1, f_2]^\sigma} \in Smbl^\sigma_{i+j-1, |f_1|+|f_2|} (A, A),$$

depends on the class of the two homogeneous σ-differential operators $f_1 \in Diff^\sigma_{i, |f_1|} (A, A)$ and $f_2 \in Diff^\sigma_{j, |f_2|} (A, A)$, and there is a graded σ-Poisson structure on the braided symbol algebra.

7.2. **Braided differential operators in graded modules.** Let A be a σ-symmetric algebra and let E be a σ-symmetric A-module.

Define a graded σ-differential operator f of E of degree $|f| \in G$ and order at most k as the linear map $f : E \to E$, such that

$$f : E_g \to E_{g+|f|},$$

$g \in G$, and

$$[x, [a_0, \dots [a_{k-1}, f]^\sigma \cdots]^\sigma]^\sigma = 0, \tag{29}$$

$\forall a_0, \dots, a_{k-1} \in A$, $x \in E$.

Denote by $Diff^\sigma_{k,i} (E, E)$ the σ-differential operators of order at most k and degree i, the σ-differential operators in order at most k of E by $Diff_k^{(\sigma, A)} (E, E)$ and we consider $Diff^{(\sigma, A)} (E, E) = \cup Diff_k^{(\sigma, A)} (E, E)$.

From [16] we have the two following results. The σ-commutator of two σ-differential operators $f_1 \in Diff_i^\sigma(E,E)$ and $f_2 \in Diff_j^\sigma(E,E)$ is a σ-differential operator of order at most $i+j$,

$$[f_1,f_2]^\sigma \in Diff_{i+j}^\sigma(E,E).$$

and $Diff^\sigma(E,E)$ is a σ-Lie algebra. Furthermore,

$$[f_1,f_2]^\sigma \in Diff_{i+j,|f_1|+|f_2|}^\sigma(E,E),$$

for homogeneous $f_1 \in Diff_{i,|f_1|}^\sigma(E,E)$ and $f_2 \in Diff_{j,|f_2|}^\sigma(E,E)$.

Proposition 30. *There is an $A-A$-module structure on $Diff^\sigma(E,E)$ defined by*

$$\begin{aligned}\nu^l(a\otimes f)(x) &= af(x),\\ \nu^r(f\otimes a)(x) &= f(ax),\end{aligned}$$

and $af, fa \in Diff_{k,|f|+|a|}^\sigma(E,E)$, for homogeneous $a \in A$ and $f \in Diff_{k,|f|}^\sigma(E,E)$.

Proof. The proof is the same as for proposition 29. ∎

Consider the symbol of the differential operators which is the leading part with respect to derivatives,

$$Smbl_k^\sigma(E,E) = Diff_k^\sigma(E,E)/Diff_{k-1}^\sigma(E,E),$$

then we have the $\mathbb{Z}$-graded object

$$Smbl^\sigma(E,E) = \sum_{k\in\mathbb{Z}} Smbl_k^\sigma(E,E).$$

The class of $[f_1,f_2]^\sigma \in Diff_{i+j,|f_1|+|f_2|}^\sigma(E,E)$,

$$\overline{[f_1,f_2]^\sigma} \in Smbl_{i+j,|f_1|+|f_2|}^\sigma(E,E),$$

depends on the class of the two homogeneous σ-differential operators $f_1 \in Diff_{i,|f_1|}^\sigma(E,E)$ and $f_2 \in Diff_{j,|f_2|}^\sigma(E,E)$, and there is a graded σ-Poisson structure on the braided symbol algebra.

7.3. **Quantizations of braided differential operators in algebras.** We define quantization of σ-differential operators in algebras.

Definition 31. *Given a quantization q and $f \in Diff_{|f|}^\sigma(A,A)$ define the quantization of f by*

$$Q_q(f)(a) = q(|f|,|a|)\,f(a),$$

for homogeneous $a \in A$. Sometimes we use the notation $f_q = Q_q(f)$.

$Q_q(f)$ is an operator of the quantized graded algebra A_q. Denote by $Diff^{\sigma_q}(A_q, A_q)$ the quantization of all σ-differential operators of A equipped with the quantization of composition.

From [15] we have the following result. Given a braiding σ, let A be a σ-commutative algebra. Let σ_q be the quantization of σ. The operator

$$\begin{aligned} Q_q &: \left(Diff^{\sigma}(A,A), [,]^{\sigma}\right) \to \left(Diff^{\sigma_q}(A_q,A_q), [,]_q^{\sigma_q}\right), \\ f &\in Diff^{\sigma}(A,A) \longmapsto Q_q(f) \in Diff^{\sigma_q}(A_q,A_q), \end{aligned} \tag{30}$$

is an isomorphism of modules. The symbol of Q_q is an isomorphism of modules

$$\begin{aligned} Smbl(Q_q) &: \left(Smbl^{\sigma}(A,A), [,]^{\sigma}\right) \to \left(Smbl^{\sigma_q}(A_q,A_q), [,]_q^{\sigma_q}\right), \\ f &\in Smbl^{\sigma}(A,A) \longmapsto Smbl(Q_q)(f) \in Smbl^{\sigma_q}(A_q,A_q). \end{aligned} \tag{31}$$

By proposition 29 is $Diff^{\sigma_q}(A_q, A_q)$ a σ_q-symmetric module and a σ_q-Lie algebra with respect to the $\sigma_q - q$-bracket and the quantized composition.

Furthermore, there is a σ_q-Poisson structure on the quantized braided symbol algebra.

The braided differential operators of A satisfy theorem 11 with *Der* replaced by *Diff* and the σ_q-Lie algebra structure of $Diff^{\sigma_q}(A_q, A_q)$ can be realized within the classical one by dequantization.

7.4. **Quantizations of braided differential operators in modules.** Let A be a σ-symmetric algebra and let E be a σ-symmetric A-module.

Definition 32. *Let q be a quantization and $f \in Diff_{|f|}^{(\sigma,A)}(E,E)$ be homogeneous. Then the quantization of f is defined by*

$$Q_q(f)(x) = q(|f|,|x|)\, f(x),$$

where $x \in E$ is homogeneous.

Sometimes we use the notation $f_q = Q_q(f)$.

$Q_q(f)$ is an operator of the quantized graded module E_q. Denote by $Diff^{(\sigma_q,A_q)}(E_q, E_q)$ the quantization of all σ-differential operators of A equipped with the quantization of composition.

Given a braiding σ, let E be a σ-commutative A-module. Let σ_q be the quantization of σ. The operator

$$\begin{gathered} Q_q : \left(Diff^{(\sigma,A)}(E,E), [,]^{\sigma}\right) \to \left(Diff^{(\sigma_q,A_q)}(E_q,E_q), [,]_q^{\sigma_q}\right), \\ f \in Diff^{(\sigma,A)}(E,E) \longmapsto Q_q(f) \in Diff^{(\sigma_q,A_q)}(E_q,E_q), \end{gathered} \tag{32}$$

is an isomorphism of modules. The symbol of Q_q is an isomorphism of modules

$$Smbl\left(Q_q\right) : \left(Smbl^{(\sigma,A)}\left(E,E\right), [,]^{\sigma}\right) \to \left(Smbl^{(\sigma_q,A_q)}\left(E_q,E_q\right), [,]_q^{\sigma_q}\right), \quad (33)$$
$$f \in Smbl^{(\sigma,A)}\left(E,E\right) \longmapsto Smbl\left(Q_q\right)(f) \in Smbl^{(\sigma_q,A_q)}\left(E_q,E_q\right).$$

This is shown in [15].

$Diff^{(\sigma_q,A_q)}\left(E_q,E_q\right)$ a σ_q-symmetric module and a σ_q-Lie algebra with respect to the $\sigma_q - q$-bracket and the quantized composition. Furthermore, there is a σ_q-Poisson structure on the quantized braided symbol algebra, $Smbl^{(\sigma_q,A_q)}\left(E_q,E_q\right)$.

The braided differential operators of E satisfy theorem 18 with Der replaced by $Diff$ so the σ_q-Lie algebra structure of $Diff^{(\sigma_q,A_q)}\left(E_q,E_q\right)$ can be realized within the classical one by dequantization.

7.5. **Braided symbol and $G \oplus \mathbb{Z}$-grading.** Any G-graded differential operator has a fibration by $\mathbb{Z}$. However, the symbol of the braided differential operators of G-graded algebras A and modules E, $Smbl^{\sigma}\left(A,A\right)$ and $Smbl^{(\sigma,A)}\left(E,E\right)$, is $\mathbb{Z}$-graded and so there is a grading by $G \oplus \mathbb{Z}$.

Instead of only considering quantizations and braidings with respect to the G-grading, we consider such with respect to the grading $G \oplus \mathbb{Z}$. In [11] we consider such quantizations and braidings in connection with exterior and symmetric algebras. We recall the following description of symmetries and quantizations for this case.

Let $\bar{G} = G \oplus \mathbb{Z}$ and denote its elements by $\bar{g} = (g, g_{\mathbb{Z}})$.

Any symmetry

$$\bar{\sigma} : (G \oplus \mathbb{Z}) \times (G \oplus \mathbb{Z}) \to U\left(\mathbb{C}\right)$$

of the monoidal category of $\bar{G} = G \oplus \mathbb{Z}$-graded modules is defined by

$$\bar{\sigma}\left(\bar{g}, \bar{g}'\right) = \sigma\left(g, g'\right) \tau\left(g_{\mathbb{Z}}, g'_{\mathbb{Z}}\right) \gamma\left(g, g'_{\mathbb{Z}}\right) \gamma^{-1}\left(g', g_{\mathbb{Z}}\right), \quad (34)$$

where we have a symmetry of $\bar{G}$-graded modules,

$$\bar{\sigma}|_{(G\oplus\{0\})\times(G\oplus\{0\})} = \sigma : G \times G \to U\left(R\right),$$

a symmetry of $\mathbb{Z}$-graded modules,

$$\bar{\sigma}|_{(\{0\}\oplus\mathbb{Z})\times(\{0\}\oplus\mathbb{Z})} = \tau : \mathbb{Z} \times \mathbb{Z} \to U\left(R\right),$$

and a bihomomorphism,

$$\bar{\sigma}|_{(G\oplus\{0\})\times(\{0\}\oplus\mathbb{Z})} = \gamma : G \times \mathbb{Z} \to U\left(R\right).$$

A quantization $\bar{q}$ of $\bar{G} = G \times \mathbb{Z}$-graded modules is of the form

$$\begin{aligned} \bar{q}\left(\bar{g}, \bar{g}'\right) &= q\left(g, g'\right) \varkappa\left(g, g'_{\mathbb{Z}}\right) \varkappa^{-1}\left(g', g_{\mathbb{Z}}\right) p\left(g_{\mathbb{Z}}, g'_{\mathbb{Z}}\right) \\ &= q\left(g, g'\right) \varkappa\left(g, g'_{\mathbb{Z}}\right) \varkappa^{-1}\left(g', g_{\mathbb{Z}}\right), \end{aligned} \tag{35}$$

where $\bar{g}, \bar{g}' \in \bar{G}$,

$$\bar{q}|_{(G\oplus\{0\})\times(\{0\}\oplus\mathbb{Z})} = \varkappa : G \times \mathbb{Z} \to U\left(R\right)$$

is a bihomomorphism,

$$\bar{q}|_{(G\oplus\{0\})\times(G\oplus\{0\})} = q : G \times G \to U\left(R\right),$$

is quantization of G-graded modules and p, which is a representative of the second cohomology of $\mathbb{Z}$, is trivial.

Considering $Smbl^{\sigma}\left(A, A\right)$ and $Smbl^{(\sigma,A)}\left(E, E\right)$ as $\bar{G}$-graded, they are equipped with a symmetry $\bar{\sigma}$ of $\bar{G}$, where $\bar{\sigma}|_{(\{0\}\oplus\mathbb{Z})\times(\{0\}\oplus\mathbb{Z})} = \tau$ and $\bar{\sigma}|_{(G\oplus\{0\})\times(\{0\}\oplus\mathbb{Z})} = \gamma$ trivial, that is $\bar{\sigma} = \sigma$.

Remark 33. *If we quantize $Smbl^{\sigma}\left(A, A\right)$ and $Smbl^{(\sigma,A)}\left(E, E\right)$ by the quantizer $\gamma = \bar{\sigma}|_{(G\oplus\{0\})\times(\{0\}\oplus\mathbb{Z})}$ then the resulting algebra is $Smbl^{\bar{\sigma}}\left(A, A\right)$ and $Smbl^{(\bar{\sigma},A)}\left(E, E\right)$ that are $\bar{\sigma}$-Poisson algebras with respect to the braiding*

$$\bar{\sigma}\left(\bar{g}, \bar{g}'\right) = \sigma\left(g, g'\right) \gamma\left(g, g'_{\mathbb{Z}}\right) \gamma^{-1}\left(g', g_{\mathbb{Z}}\right).$$

We show the quantized braided Poisson structure for the quantization of $Smbl^{\bar{\sigma}}\left(A, A\right)$ and $Smbl^{(\bar{\sigma},A)}\left(E, E\right)$ for a general braiding $\bar{\sigma}$ in theorems 34 and 35.

Note that $\bar{\sigma}|_{(\{0\}\oplus\mathbb{Z})\times(\{0\}\oplus\mathbb{Z})} = \tau$ always will be trivial since the structure arises from $Smbl^{\sigma}\left(A, A\right)$ and $Smbl^{(\sigma,A)}\left(E, E\right)$.

Assume we have a braided symbols $Smbl^{\bar{\sigma}}\left(A, A\right)$ and $Smbl^{(\bar{\sigma},A)}\left(E, E\right)$ with respect to a symmetry $\bar{\sigma}$, $\bar{\sigma}|_{(\{0\}\oplus\mathbb{Z})\times(\{0\}\oplus\mathbb{Z})} = \tau = 1$.

We use quantizations of the form (35). A quantization of a symbol $f \in Smbl^{\bar{\sigma}}_{k,|f|}\left(A, A\right)$ or $f \in Smbl^{(\bar{\sigma},A)}_{k,|f|}\left(E, E\right)$ is

$$f_{\bar{q}}\left(x\right) = \bar{q}\left(\left(|f|, k\right), \left(|x|, 1\right)\right) f\left(x\right),$$

where the homogeneous x (in either E or A) is given the grading $(|x|, 1)$, $|x| \in G$.

The $Q_{\bar{q}}$ is an isomorphism of modules

$$\begin{gathered} Smbl\left(Q_{\bar{q}}\right) : \left(Smbl^{\bar{\sigma}}\left(A, A\right), [,]^{\bar{\sigma}}\right) \to \left(Smbl^{\bar{\sigma}_{\bar{q}}}\left(A_{\bar{q}}, A_{\bar{q}}\right), [,]^{\bar{\sigma}_{\bar{q}}}_{\bar{q}}\right), \qquad (36) \\ f \in Smbl^{\bar{\sigma}}\left(A, A\right) \longmapsto Smbl\left(Q_{\bar{q}}\right)\left(f\right) \in Smbl^{\bar{\sigma}_{\bar{q}}}\left(A_{\bar{q}}, A_{\bar{q}}\right), \end{gathered}$$

and

$$Smbl\left(Q_{\bar{q}}\right):\left(Smbl^{(\bar{\sigma},A)}\left(E,E\right),[,]^{\bar{\sigma}}\right)\to\left(Smbl^{(\bar{\sigma}_{\bar{q}},A_{\bar{q}})}\left(E_{\bar{q}},E_{\bar{q}}\right),[,]_{\bar{q}}^{\bar{\sigma}_{\bar{q}}}\right) \quad (37)$$
$$f\in Smbl^{(\bar{\sigma},A)}\left(E,E\right)\longmapsto Smbl\left(Q_{\bar{q}}\right)(f)\in Smbl^{(\bar{\sigma}_{\bar{q}},A_{\bar{q}})}\left(E_{\bar{q}},E_{\bar{q}}\right).$$

We obtain the following properties for the quantization of $Smbl^{\bar{\sigma}}\left(A,A\right)$.

Theorem 34. *$Smbl^{\bar{\sigma}_{\bar{q}}}\left(A_{\bar{q}},A_{\bar{q}}\right)$ is a $\bar{G}=G\oplus\mathbb{Z}$-graded $\bar{\sigma}_{\bar{q}}$-Poisson algebra with respect to the $\bar{\sigma}_{\bar{q}}-\bar{q}$-bracket, that is the following properties are satisfied:*

$$\left[Smbl^{\bar{\sigma}_{\bar{q}}}\left(A_{\bar{q}},A_{\bar{q}}\right),Smbl^{\bar{\sigma}_{\bar{q}}}\left(A_{\bar{q}},A_{\bar{q}}\right)\right]_{\bar{q}}^{\bar{\sigma}_{\bar{q}}}\subseteq Smbl^{\bar{\sigma}_{\bar{q}}}\left(A_{\bar{q}},A_{\bar{q}}\right), \quad \text{(i)}$$
$$\left[Smbl_{i,|f|}^{\bar{\sigma}_{\bar{q}}}\left(A_{\bar{q}},A_{\bar{q}}\right),Smbl_{j,|g|}^{\bar{\sigma}_{\bar{q}}}\left(A_{\bar{q}},A_{\bar{q}}\right)\right]_{\bar{q}}^{\bar{\sigma}_{q}}$$
$$\subseteq Smbl_{i+j-1,|f|+|g|}^{\bar{\sigma}_{\bar{q}}}\left(A_{\bar{q}},A_{\bar{q}}\right), \quad \text{(i')}$$

skew $\bar{\sigma}_{\bar{q}}$-symmetricity,

$$\left[f_1,f_2\right]_{\bar{q}}^{\bar{\sigma}_{\bar{q}}}=-\bar{\sigma}_{\bar{q}}\left(\left(|f_1|,i\right),\left(|f_2|,j\right)\right)\left[f_2,f_1\right]_{\bar{q}}^{\bar{\sigma}_{\bar{q}}}, \quad \text{(ii)}$$

the $\bar{\sigma}_{\bar{q}}$-Jacobi identity,

$$\left[f_1,\left[f_2,f_3\right]_{\bar{q}}^{\bar{\sigma}_{\bar{q}}}\right]_{\bar{q}}^{\bar{\sigma}_{\bar{q}}}$$
$$=\left[\left[f_1,f_2\right]_{\bar{q}}^{\bar{\sigma}_{\bar{q}}},f_3\right]_{\bar{q}}^{\bar{\sigma}_{\bar{q}}}+\bar{\sigma}_{\bar{q}}\left(\left(|f_1|,i\right),\left(|f_2|,j\right)\right)\left[f_2,\left[f_1,f_3\right]_{\bar{q}}^{\bar{\sigma}_{\bar{q}}}\right]_{\bar{q}}^{\bar{\sigma}_{\bar{q}}}, \quad \text{(iii)}$$

and

$$\left[f_1,f_2f_3\right]_{\bar{q}}^{\bar{\sigma}_{\bar{q}}}=\left[f_1,f_2\right]_{\bar{q}}^{\bar{\sigma}_{\bar{q}}}f_3+\bar{\sigma}_{\bar{q}}\left(\left(|f_1|,i\right),\left(|f_2|,j\right)\right)f_2\left[f_1,f_3\right]_{\bar{q}}^{\bar{\sigma}_{\bar{q}}}, \quad \text{(iv)}$$

for $f_1\in Smbl_{i,|f_1|}^{\bar{\sigma}_{\bar{q}}}\left(A_{\bar{q}},A_{\bar{q}}\right)$, $f_2\in Smbl_{j,|f_2|}^{\bar{\sigma}_{\bar{q}}}\left(A_{\bar{q}},A_{\bar{q}}\right)$, $f_3\in Smbl_{k,|f_3|}^{\bar{\sigma}_{\bar{q}}}\left(A_{\bar{q}},A_{\bar{q}}\right)$.

Except for (i') we obtain the same properties for the quantization of $Smbl^{(\bar{\sigma},A)}\left(E,E\right)$.

Theorem 35. *$Smbl^{(\bar{\sigma}_{\bar{q}},A_{\bar{q}})}\left(E_{\bar{q}},E_{\bar{q}}\right)$ is a $\bar{G}=G\oplus\mathbb{Z}$-graded $\bar{\sigma}_{\bar{q}}$-Poisson algebra with respect to the $\bar{\sigma}_{\bar{q}}-\bar{q}$-bracket, that is the properties (i), (ii), (iii) and (iv) of theorem 34 are satisfied when replacing $A_{\bar{q}}$ by $E_{\bar{q}}$ and $Smbl^{\bar{\sigma}_{\bar{q}}}$ by $Smbl^{(\bar{\sigma}_{\bar{q}},A_{\bar{q}})}$, and*

$$\left[Smbl_{i,g}^{(\bar{\sigma}_{\bar{q}},A_{\bar{q}})}\left(E_{\bar{q}},E_{\bar{q}}\right),Smbl_{j,h}^{(\bar{\sigma}_{\bar{q}},A_{\bar{q}})}\left(E_{\bar{q}},E_{\bar{q}}\right)\right]_{\bar{q}}^{\bar{\sigma}_{\bar{q}}}$$
$$\subseteq Smbl_{i+j,g+h}^{(\bar{\sigma}_{\bar{q}},A_{\bar{q}})}\left(E_{\bar{q}},E_{\bar{q}}\right), \quad \text{(i')}$$

$g,h\in G$.

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Department of Mathematics, The University of Tromsoe, N-9037 Tromsoe, Norway

E-mail address: Hilja.Huru@matnat.uit.no

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