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QUANTIZATIONS OF BRAIDED DERIVATIONS. 3. MODULES WITH ACTION BY A GROUP.

(submitted by V. V. Lychagin)

Abstract. For the monoidal category of modules with action by a group we find braidings and quantizations. We use them to find quantizations of braided symmetric algebras and modules, braided derivations, braided connections, curvatures and differential operators.

1. Introduction

We consider quantizations q, braidings or symmetries σ and quantizations of braidings σ_q of the monoidal category of G-modules where G is a finite abelian group.

In [8] we showed that the Fourier transform establishes an isomorphism between the categories of $\hat{G}$-graded modules and G-modules where $\hat{G}$ is the dual of G. We have found explicit descriptions of all quantizations and braidings in the monoidal category of modules graded by $\hat{G}$ and they depend only on the grading.

Using this we find explicit descriptions of all quantizations and braiding also for the monoidal category of modules with action by G.

We consider σ-symmetric G-algebras A, G-modules and find quantizations of these.

We investigate braided derivations of σ-symmetric G-algebras and G-modules. The σ-bracket of two braided derivations is a braided derivation. We show that there is a braided Lie structure on the braided derivations.

A quantization the braided derivations provides an isomorphism of the modules of braided derivations and quantized braided derivations. We also show that the quantizations of braided derivations has the braided Lie structure with respect to the quantizations of the braiding which can be realized within the original braided Lie structure by dequantization.

We define braided connections in G-modules and braided curvatures. We prove that the braided curvature is A-linear, skew σ-symmetric and is an A-module homomorphism.

We find quantizations of braided connections and braided curvatures. The quantization of the braided curvature is A-linear, skew σ_q-symmetric and an A-module homomorphism with respect to the quantized braiding.

We consider braided differential operators of σ-symmetric G-algebras and G-modules. There is a braided Lie structure on the braided differential operators. The quantizations of braided differential operators has the quantized braided Lie structure, which can be realized within the original braided Lie structure by dequantization.

Finally we increase the number of quantizations of braided symbols of differential operators by exploiting the $\mathbb{Z}$-grading of the symbols.

This paper is the third in a trilogy.

As mentioned, we have found explicit descriptions of all quantizations and braidings in the monoidal category of modules graded by a finite commutative monoid, [8]. We have proved the same for this category, but the picture is somewhat more visible in this case. That is, we have a complete and explicit description for braided derivations of graded algebras and graded modules, braided connections, braided curvature, braided differential operators and quantizations of these structures. This is found in the second paper *Quantizations of braided derivations. 2. Graded modules,* [10].

All results and properties that does not concern the correspondence between the monoidal categories of $\hat{G}$-graded modules and G-modules are proved in general for any monoidal category in the first paper, *Quantizations of braided derivations. 1. Monoidal categories,* [9]. Because of this we shall not repeat any of these proofs. We do however repeat the relevant results and show the complete and explicit description we get in the monoidal category of G-modules.

There are many interesting applications of these results. One of the more interesting applications is quantizations of braided Lie algebras. In the paper [11], which is to be published, we show quantizations of semisimple Lie algebras by quantizations of derivations, for example an alternative quantization of $\mathfrak{sl}_2(\mathbb{C})$.

Note that in all three papers we assume that the associativity constraint is trivial.

2. Modules with group action

Let G be a finite abelian group and let $\hat{G} = Hom\left(G, T^1\right)$ be the dual of G. We shall consider the monoidal categories of $\hat{G}$-graded modules and G-modules. The modules are over $\mathbb{C}$.

Let $\mathbb{C}\left[G\right]$ be the group algebra of G, with the basis of Dirac δ-functions, $\left\{\delta_g\right\}_{g\in G}$, and $\mathbb{C}\left(\hat{G}\right)$ be the function algebra of $\hat{G}$, with the basis $\left\{\theta_\chi\right\}_{\chi\in\hat{G}}$,

$$\theta_\chi\left(\phi\right) = \begin{cases} 1, \chi = \phi \\ 0, \chi \neq \phi \end{cases}, \chi, \phi \in \hat{G}.$$

Then the Fourier transform F is a bialgebra isomorphism

$$F : \mathbb{C}\left[G\right] \to \mathbb{C}\left(\hat{G}\right),$$

given by

$$F\left(\delta_g\right)\left(\chi\right) = \chi\left(g^{-1}\right),$$

and the inverse is,

$$F^{-1}\left(\theta_\chi\right) = \frac{1}{|G|}\sum_{g\in G}\chi\left(g\right)\delta_g,$$

$g \in G$ and $\chi \in \hat{G}$.

We consider the two monoidal categories of G-modules and $\hat{G}$-graded modules.

We know that a $\mathbb{C}\left[G\right]$-module is also a G-module, that $\mathbb{C}\left(\hat{G}\right)$-module is a $\hat{G}$-graded module, and the other way around for both cases. Then F^{-1} constructs an isomorphism of categories of between the monoidal categories of $\hat{G}$-graded modules and G-modules as follows.

Let X be a G-module. Then there is a $\hat{G}$-grading on X by the projection

$$\pi_\chi\left(x\right) = F^{-1}\left(\chi\right)\left(x\right) = \frac{1}{|G|}\sum_{g\in G}\chi\left(g\right)g\left(x\right),$$

$x \in X$, $\chi \in \hat{G}$. Let $Y = \sum_{\chi\in\hat{G}} Y_\chi$ be a $\hat{G}$-graded $\mathbb{C}$-module. There is a G-module structure on Y given by

$$gy = \sum_{\chi\in\hat{G}} F\left(\delta_g\right)\left(\chi\right)y_\chi = \sum_{\chi\in\hat{G}}\chi\left(g^{-1}\right)y_\chi,$$

$y = \sum_{\chi\in\hat{G}} y_\chi \in Y$, $g \in G$.

We use this result to find quantizations and braidings for G-modules, which can be found in [8], and in other ways compare G-modules with $\hat{G}$-graded modules.

2.1. **Quantizations of G-modules.** Any quantization of the monoidal category of G-modules is a realized by a 2-cocycle $q \in Z^2(G, U(R))$,

$$q(g,h) = (F \otimes F)^{-1}(\hat{q})(g,h) = \frac{1}{|G|^2} \sum_{\phi,\chi \in \hat{G}} \hat{q}(\phi,\chi)\,\phi(g)\,\chi(h), \qquad (1)$$

$g, h \in G$, where $\hat{q}$ is a quantization of the monoidal category of $\hat{G}$-graded modules. When we factor out the trivial quantizations we are left with $H^2(M, U(R))$. The quantization as an operator $X \otimes Y \to X \otimes Y$ of G-modules X and Y q is of the form

$$q = \frac{1}{|G|^2} \sum_{\substack{g,h \in G \\ \phi,\chi \in \hat{G}}} \hat{q}(\phi,\chi)\,\phi(g)\,\chi(h)\,\delta_g \otimes \delta_h.$$

We use the notation

$$q = \sum_{g,h \in G} Q_{gh} \delta_g \otimes \delta_h.$$

Note that δ_g, $g \in G$, applied to an element is the action by g.

2.2. **Braidings of G-modules.** Any braiding in the monoidal category of G-modules is a 2-cochain $\sigma : G \times G \to U(\mathbb{C})$,

$$\begin{aligned}\sigma(g,h) &= (F \otimes F)^{-1}(\hat{\sigma})(g,h) \\ &= \tau \circ \left(\frac{1}{|G|^2} \sum_{\phi,\chi \in \hat{G}} \hat{\sigma}(\phi,\chi)\,\phi(g)\,\chi(h) \right), \qquad (2)\end{aligned}$$

$g, h \in G$, where $\hat{\sigma}$ is a braiding of the monoidal category of $\hat{G}$-graded modules. As an operator $X \otimes Y \to Y \otimes X$ of G-modules X and Y σ is of the form

$$\sigma = \tau \circ \left(\frac{1}{|G|^2} \sum_{\substack{g,h \in G \\ \phi,\chi \in \hat{G}}} \hat{\sigma}(\phi,\chi)\,\phi(g)\,\chi(h)\,\delta_g \otimes \delta_h \right).$$

Given a quantization q then the quantization of the braiding σ is

$$\sigma_q = \tau \circ \left(\frac{1}{|G|^2} \sum_{\substack{g,h \in G \\ \phi,\chi \in \hat{G}}} \hat{\sigma}_q (\phi, \chi) \phi (g) \chi (h) \delta_g \otimes \delta_h \right),$$

where $\hat{\sigma}_q = q^{-1} \circ \hat{\sigma} \circ q$ is the quantization of the braiding $\hat{\sigma}$ of the monoidal category of $\hat{G}$-graded modules.

We use the notation

$$\sigma = \sum_{g,h \in G} S_{g,h} \delta_g \otimes \delta_h,$$

and given a quantization q, denote the quantization of σ by

$$\sigma_q = \tau \circ \sum_{g,h \in G} S^q_{gh} \delta_g \otimes \delta_h.$$

Using these formulas obtained for quantizations and braidings in the monoidal category of G-modules we get an explicit description of σ-symmetric G-algebras, G-modules, G-co- and bimodules and internal homomorphisms. We shall not describe quantizations of σ-symmetric G-co- and bimodules, these are described for any monoidal category in [9].

2.3. **Quantizations of braided G-algebras.** First the description of quantization of σ-symmetric G-algebras.

A G-algebra A is called σ-symmetric or σ-commutative if

$$ab = \sum_{g,h \in G} S_{g,h} h (b) g (a), \tag{3}$$

$a, b \in A$.

A quantization of A is the same object $A_q = A$ with the new multiplication

$$a *_q b = \sum_{g,h \in G} Q_{gh} g (a) h (b), \tag{4}$$

$a, b \in A$.

If A is σ-symmetric then A_q is σ^q-symmetric,

$$a *_q b = \sum_{g,h \in G} S^q_{g,h} h (b) *_q g (a), \tag{5}$$

$a, b \in A$.

2.4. **Quantizations of braided G-modules.** We also get an explicit description of σ-symmetric G-modules over σ-symmetric G-algebras.

Let E be a σ-symmetric G-module over a σ-symmetric G-algebra A. E is σ-symmetric if

$$ax = \sum_{g,h \in G} S_{g,h} h(x) g(a), \tag{6}$$

$a \in A$, $x \in E$.

A quantizations of E is the same object $E = E_q$ with the new action by $A_q = A$,

$$a *_q x = \sum_{g,h \in G} Q_{gh} g(a) h(x). \tag{7}$$

If E is σ-commutative, then E_q is σ_q-commutative,

$$a *_q x = \sum_{g,h \in G} S^q_{g,h} h(x) *_q g(a), \tag{8}$$

$a \in A, x \in E$.

2.5. **Operators of G-modules.** Let E be a G-module and $W : E \to E$ be an operator of E. There is a induced G-action on W, defined by

$$g(W) = g \circ W \circ g^{-1},$$

$g \in G$.

W is called a $(G, 0) = G$-*(intertwining) operator* if

$$W \circ g = g \circ W,$$

and we say $W \in Hom^0_G(E, E)$.

W is called a (G, α)-*operator* if

$$W \circ g = \alpha(g)(g \circ W),$$

$\alpha \in \hat{G}$, that is, W is a twisted α-homomorphism, denoted by $W \in Hom^\alpha_G(E, E)$.

2.6. **Quantizations of operators of G-modules.** Let E be a G-module and let $W : E \to E$ be a G-operator of E.

Given a quantization q define the quantization of W, $W_q = Q_q(W)$ by

$$Q_q(W)(x) = \sum_{g,h \in G} Q_{gh} g(W)(h(x)),$$

$x \in E_q$.

$Q_q(W)$ is an operator of the quantized G-module E_q.

The quantization of all operators of E is equipped with the quantized composition of operators,

$$W *_q Z = \sum_{g,h \in G} S_{g,h} h\,(W) \circ g\,(Z)\,. \tag{9}$$

The inverse quantization

$$Q_q^{-1}\,(W) = W_c,$$

is called the dequantization of W. The dequantization of W is an operator of the original or classical module E.

3. Braided derivations of G-algebras

We shall describe braided derivations of G-algebras, but first we will show how braided derivations of non-homogeneous elements of graded algebras look like and what the notion of degree is for braided derivations of G-algebras.

3.1. **Braided derivations of non-homogeneous elements of graded algebras.** To find derivations of G-algebras a description of non-homogeneous derivations of non-homogeneous elements of graded algebras (over a commutative ring R) is needed.

For a $\hat{G}$-graded module E and an operator $W = \sum_{\chi \in \hat{G}} W_\chi$ of E we have,

$$\begin{aligned} W\,(x) &= \sum_{\phi,\chi \in \hat{G}} W_\phi\,(x_\chi)\,, \\ (W\,(x))_\chi &= \sum_{\substack{\phi,\psi \in \hat{G} \\ \phi+\psi=\chi}} W_\phi\,(x_\psi)\,, \end{aligned}$$

$x = \sum_{\chi \in \hat{G}} x_\chi \in E$.

Similarly, for a $\hat{\sigma}$-symmetric $\hat{G}$-graded algebra $A = \sum_{\chi \in \hat{G}} A_\chi$, and a $\hat{\sigma}$-derivation of A,

$$\hat{\partial} = \sum_{\chi \in \hat{G}} \hat{\partial}_\chi : A \to A$$

where each $\hat{\partial}_\chi$ is a derivation of degree $\chi \in \hat{G}$, we have

$$\begin{aligned} \hat{\partial}\,(a) &= \sum_{\phi,\chi \in \hat{G}} \hat{\partial}_\phi\,(a_\chi)\,, \\ \left(\hat{\partial}\,(a)\right)_\chi &= \sum_{\substack{\phi,\psi \in \hat{G} \\ \phi+\psi=\chi}} \hat{\partial}_\phi\,(a_\psi)\,, \end{aligned}$$

$a \in A$, $a = \sum_{\chi \in \hat{G}} a_\chi$.

Proposition 1. *Let $a, b \in A$, $a = \sum_{\chi \in \hat{G}} a_\chi$, $b = \sum_{\chi \in \hat{G}} b_\chi$. Then the $\hat{\sigma}$-derivation $\hat{\partial}$ satisfies the following $\hat{\sigma}$-Leibniz rule*

$$\begin{aligned} \hat{\partial}(ab) &= \hat{\partial}(a)\, b + \mu \circ \hat{\sigma}\left(\hat{\partial} \otimes a\right)(b) \\ &= \hat{\partial}(a)\, b + \sum_{\chi, \phi, \psi \in \hat{G}} \hat{\sigma}(\chi, \phi)\, a_\phi \hat{\partial}_\chi(b_\psi). \end{aligned}$$

Proof.

$$\begin{aligned} \hat{\partial}(ab) &= \sum_{\chi, \phi \in \hat{G}} \hat{\partial}_\chi\left((ab)_\phi\right) \\ &= \sum_{\chi, \phi \in \hat{G}} \hat{\partial}_\chi\left(\sum_{\substack{\psi, \varkappa \in \hat{G} \\ \psi + \varkappa = \phi}} a_\psi b_\varkappa\right) \\ &= \sum_{\chi, \psi, \varkappa \in \hat{G}} \hat{\partial}_\chi(a_\psi b_\varkappa) \\ &= \sum_{\chi, \psi, \varkappa \in \hat{G}} \left(\hat{\partial}_\chi(a_\psi)\, b_\varkappa + \hat{\sigma}(\chi, \psi)\, a_\psi \hat{\partial}_\chi(b_\varkappa)\right) \\ &= \hat{\partial}(a)\, b + \hat{\sigma}\left(\hat{\partial} \otimes a\right)(b). \end{aligned}$$

∎

3.2. **Degrees of braided derivations.** Also, it is nice to know the equivalent to degrees for σ-derivations of G-algebras.

Let A be a $\hat{G}$-graded $\hat{\sigma}$-symmetric algebra over $\mathbb{C}$ with respect to a braiding $\hat{\sigma}$ in the monoidal category of $\hat{G}$-graded modules.

Let $\hat{\partial}$ be a $\hat{\sigma}$-derivations of degree $|\hat{\partial}|$ of A, $\hat{\partial} : A_\chi \to A_{\chi + |\hat{\partial}|}$, that is, satisfies

$$\hat{\partial} \circ \theta_\chi = \hat{\partial} \circ \pi_\chi = \pi_{\chi + |\hat{\partial}|} \circ \hat{\partial} = \theta_{\chi + |\hat{\partial}|} \circ \hat{\partial},$$

for $a \in A$, $\chi, |\hat{\partial}| \in \hat{G}$, θ_χ and $\theta_{\chi + |\hat{\partial}|}$, are in the basis of $\mathbb{C}\left(\hat{G}\right)$, and

$$\begin{tikzcd} A \arrow[r, "\hat{\partial}"] \arrow[d, "\theta_\chi"'] & A \arrow[d, "\theta_{\chi + |\hat{\partial}|}"'] \\ A \arrow[r, "\hat{\partial}"] & A \end{tikzcd}$$

commutes.

Proposition 2. *Let $\hat{\partial}$ be a derivation of a $\hat{G}$-graded algebra A of degree $|\hat{\partial}| \in \hat{G}$. Then the operator $\partial = F^{-1}\left(\hat{\partial}\right)$ of A as a G-algebra satisfies*

$$\partial \circ g = |\hat{\partial}|\,(g)\, g \circ \partial,$$

that is, ∂ is an $\left(G, |\hat{\partial}|\right)$-operator.

Proof. By the commutativity of the diagram

$$\begin{tikzcd} A \arrow[r, "\partial"] \arrow[d, "F^{-1}(\theta_\chi)"'] & A \arrow[d, "F^{-1}(\theta_{\chi+|\hat{\partial}|})"] \\ A \arrow[r, "\partial"] & A \end{tikzcd},$$

we have that

$$\begin{aligned} \partial \circ F^{-1}(\theta_\chi) &= F^{-1}\left(\theta_{\chi+|\hat{\partial}|}\right) \circ \partial, \\ \frac{1}{|G|}\sum_{g\in G} \chi(g)\,\partial \circ \rho_g &= \frac{1}{|G|}\sum_{g\in G} \left(\chi + |\hat{\partial}|\right)(g)\,\rho_g \circ \partial \\ &= \frac{1}{|G|}\sum_{g\in G} \chi(g)\,|\hat{\partial}|\,(g)\,\rho_g \circ \partial, \end{aligned}$$

that is, if and only if

$$\partial \circ g = |\hat{\partial}|\,(g)\, g \circ \partial.$$

∎

3.3. **Braided derivations of G-algebras.** Consider a braiding σ in the monoidal category of G-modules,

$$\sigma = (F \otimes F)^{-1}(\hat{\sigma}),$$

where $\hat{\sigma}$ is a braiding in the monoidal category of $\hat{G}$-graded modules. Let A be a σ-symmetric G-algebra.

By the two preceding sections, 3.1 and 3.2, we give the following definitions.

Definition 3. *A σ-derivation ∂ of A is an operator $A \to A$ that satisfies the σ-Leibniz rule*

$$\partial(ab) = \partial(a)\,b + \sum_{g,h\in G} S_{g,h} h(a)\,(g \circ \partial)(b)$$

$a, b \in A$.

Definition 4. *A σ-derivation ∂ of A is said to be of degree $|\partial| \in \hat{G}$ if it is an $(G, |\partial|)$-operator $A \to A$, that is*

$$\partial \circ g = |\partial| (g) \, g \circ \partial. \tag{10}$$

We call ∂ homogeneous if it has a degree.

A σ-derivation ∂_χ of degree $\chi_\partial \in \hat{G}$ satisfies the simplified σ-Leibniz rule

$$\partial (ab) = \partial (a) \, b + \frac{1}{|G|^2} \sum_{\substack{g,h \in G \\ \phi \in \hat{G}}} \hat{\sigma} (\chi_\partial, \phi) \, \phi (h) \, h (a) \, (\partial \circ g) (b) ,$$

$a, b \in A$.

The set of σ-derivations of degree χ is denoted by $Der^\sigma_\chi (A)$ and the set of all σ-derivations by $Der^\sigma (A)$.

A left A-module structure on $Der^\sigma (A)$ is defined by

$$(a\partial) (b) = a (\partial (b)) ,$$

for $a, b \in A$, $\partial \in Der^\sigma (A)$.

A *σ-commutator* (or σ-bracket) of elements $\partial_1, \partial_2 \in Der^\sigma (A)$ is defined by

$$[\partial_1, \partial_2]^\sigma = \partial_1 \partial_2 - \sum_{g,h \in G} S_{g,h} h (\partial_2) \, g (\partial_1) .$$

The σ-bracket satisfies the A-module conditions,

$$[a\partial_1, \partial_2]^\sigma = a [\partial_1, \partial_2]^\sigma - \sum_{g,h \in G} S_{g,h} (h\partial_2 (ga)) \, g\partial_1, \tag{11}$$

$$[\partial_1, a\partial_2]^\sigma = \sum_{g,h \in G} S_{g,h} h (a) \, [g\partial_1, \partial_2]^\sigma + \partial_1 (a) \, \partial_2, \tag{12}$$

$\forall a \in A$, $\partial_1, \partial_2 \in Der^\sigma (A)$.

Theorem 5. *$Der^\sigma (A)$ is a σ-Lie G-algebra with respect to the σ-bracket, that is, the following properties are satisfied;*

$$[Der^\sigma (A) , Der^\sigma (A)]^\sigma \subseteq Der^\sigma (A) , \tag{i}$$

$$\left[Der^\sigma_\phi (A) , Der^\sigma_\chi (A)\right]^\sigma \subseteq Der^\sigma_{\phi+\chi} (A) , \tag{i'}$$

$\phi, \chi \in \hat{G}$, skew σ-symmetricity,

$$[\partial_1, \partial_2]^\sigma = - \sum_{g,h \in G} S_{gh} [h\partial_2, g\partial_1]^\sigma , \tag{ii}$$

and the σ-Jacobi identity for σ-derivations,

$$[\partial_1, [\partial_2, \partial_3]^\sigma]^\sigma = [[\partial_1, \partial_2]^\sigma , \partial_3]^\sigma + \sum_{g,h \in G} S_{gh} [h\partial_2, [g\partial_1, \partial_3]^\sigma]^\sigma , \tag{iii}$$

for braided derivations ∂_1, ∂_2 *and* ∂_3.

Proof. Except from (i') all those above are proved for any monoidal category in [9]. Now

$$\left[Der^{\sigma}_{\phi}(A), Der^{\sigma}_{\chi}(A)\right]^{\sigma} \subseteq Der^{\sigma}_{\phi+\chi}(A)$$

if and only if

$$[\partial_1, \partial_2]^{\sigma} \circ g = (\phi + \chi)(g)\, g \circ [\partial_1, \partial_2]^{\sigma},$$

which is is easy to prove,

$$\begin{aligned}
[\partial_1, \partial_2]^{\sigma} \circ g &= \partial_1 \circ \partial_2 \circ g - \sum_{h,h' \in G} S_{h,h'} h'(\partial_2) \circ h(\partial_1) \circ g \\
&= \chi(g)\,\phi(g)\, g \circ \partial_1 \circ \partial_2 - \phi(g)\,\chi(g)\, g \circ \sum_{h,h' \in G} S_{h,h'} h'(\partial_2) \circ h(\partial_1) \\
&= (\phi + \chi)(g)\, g \circ [\partial_1, \partial_2]^{\sigma}, \qquad \partial_1 \in Der^{\sigma}_{\phi}(A), \partial_2 \in Der^{\sigma}_{\chi}(A).
\end{aligned}$$

∎

3.4. **Quantizations of σ-derivations of G-algebras.** Consider derivations of a G-algebra A.

Given a quantization q define the quantization of a σ-derivation of A, ∂, by

$$Q_q(\partial)(a) = \partial_q(a) \overset{def}{=} \sum_{g,h \in G} Q_{gh}\, g(\partial)(h(a)),$$

$a \in A$.

$Q_q(\partial)$ is an operator of the quantized G-algebra A_q. Let us denote by $Der^{\sigma_q}(A_q)$ the quantization of all σ-derivations of A equipped with the quantization of composition,

$$\partial_1 *_q \partial_2 = \sum_{g,h \in G} Q_{gh}\, g(\partial_1) \circ h(\partial_2),$$

$\partial_1, \partial_2 \in Der^{\sigma_q}(A_q)$. We define the $\sigma_q - q$-bracket by

$$[\partial_1, \partial_2]^{\sigma_q}_q = \partial_1 *_q \partial_2 - \sum_{g,h \in G} S^q_{g,h} h(\partial_2) *_q g(\partial_1).$$

The operator Q_q

$$\begin{aligned}
Q_q &: \left(Der^{\sigma}(A), [,]^{\sigma}\right) \to \left(Der^{\sigma_q}(A_q), [,]^{\sigma_q}_q\right), \qquad (13) \\
\partial &\in Der^{\sigma}(A) \longmapsto Q_q(\partial) \in Der^{\sigma_q}(A_q),
\end{aligned}$$

is an isomorphism of vector spaces between the σ-derivations of A and the σ_q-derivations of A_q.

$Der^{\sigma_q}(A_q)$ is a σ_q-symmetric A_q-module,

$$[a\partial_1, \partial_2]_q^{\sigma_q} = a[\partial_1, \partial_2]_q^{\sigma_q} - \sum_{g,h\in G} S_{g,h}^q (h\partial_2(ga))\, g\partial_1 \tag{14}$$

$$[\partial_1, a\partial_2]_q^{\sigma_q} = \sum_{g,h\in G} S_{g,h}^q h(a)\, [g\partial_1, \partial_2]_q^{\sigma_q} + \partial_1(a)\,\partial_2, \tag{15}$$

$\forall a \in A$, $\partial_1, \partial_2 \in Der^{\sigma_q}(A_q)$.

Theorem 6. *$Der^{\sigma_q}(A_q)$ is a σ_q-Lie G-algebra with respect to the $\sigma_q - q$-bracket and the quantized combination, that is the following properties are satisfied,*

$$[Der^{\sigma_q}(A_q), Der^{\sigma_q}(A_q)]_q^{\sigma_q} \subseteq Der^{\sigma_q}(A_q), \tag{i}$$

$$\left[Der_\phi^{\sigma_q}(A_q), Der_\chi^{\sigma_q}(A_q)\right]_q^{\sigma_q} \subseteq Der_{\phi+\chi}^{\sigma_q}(A_q), \tag{i'}$$

$\phi, \chi \in \hat{G}$, skew σ_q-symmetricity,

$$[\partial_1, \partial_2]_q^{\sigma_q} = -\sum_{g,h\in G} S_{gh}^q [h\partial_2, g\partial_1]_q^{\sigma_q}, \tag{ii}$$

and the σ_q-Jacobi identity for σ_q-derivations,

$$\left[\partial_1, [\partial_2, \partial_3]_q^{\sigma_q}\right]_q^{\sigma_q} = \left[[\partial_1, \partial_2]_q^{\sigma_q}, \partial_3\right]_q^{\sigma_q} + \sum_{g,h\in G} S_{gh}^q \left[h\partial_2, [g\partial_1, \partial_3]_q^{\sigma_q}\right]_q^{\sigma_q}, \tag{iii}$$

for $\partial_1, \partial_2, \partial_3 \in Der^{\sigma_q}(A_q)$.

The σ_q-Lie algebra structure of $Der^{\sigma_q}(A_q)$ can be realized within the classical one by dequantization,

$$Q_q^{-1}(\partial) = \partial_c,$$

$\partial \in Der^{\sigma_q}(A_q)$, where we have the following linearity,

$$(\partial_1 + \partial_2)_c = (\partial_1)_c + (\partial_2)_c, \tag{16}$$

$\partial_1, \partial_2 \in Der_{|W|}^{\sigma_q}(A_q)$, A_q-module structure,

$$(a \circ \partial_1)_c = a *_q (\partial_1)_c, \tag{17}$$

the commutator satisfies,

$$([\partial_1, \partial_2]^{\sigma_q})_c = [(\partial_1)_c, (\partial_2)_c]_q^{\sigma_q}, \tag{18}$$

for $\partial_1, \partial_2 \in Der^{\sigma_q}(A_q)$, $a \in A_q$, and the σ_q-Jacobi identity is satisfied and can be found using (18).

4. Braided derivations of G-modules

Let A be a σ-symmetric G-algebra and E be a σ-symmetric G-module over A.

Definition 7. *Let $\partial_A \in Der^{\sigma}(A)$. A σ-derivation ∂ of E over ∂_A is an operator $E \to E$ that satisfies the σ-Leibniz-rule over A,*

$$\partial(ax) = \partial_A(a)\,x + \sum_{g,h\in G} S_{g,h} h(a)(g\circ\partial)(x)$$

$a \in A$, $x \in E$.

Such a pair (∂, ∂_A) is called a σ-derivation of E over A.

The morphism $\pi : (\partial, \partial_A) \to \partial_A$ we call the projection from the σ-derivations of E over A to the σ-derivations of A.

A σ-derivation ∂ of E over ∂_A is said to be of degree $|\partial| \in \hat{G}$ if it is an $(G, |\partial|)$-operator.

The set of σ-derivations of degree χ is denoted by $Der_{\chi}^{(\sigma,A)}(E)$ and the set of all σ-derivations by $Der^{(\sigma,A)}(E)$.

An A-module structure on $Der^{(\sigma,A)}(E)$ is defined by

$$(a\partial)(x) = a(\partial(x)),$$

for $a \in A$, $x \in E$, $\partial \in Der^{(\sigma,A)}(E)$.

The σ-bracket satisfies the A-module conditions (11) and (12) for σ-derivations of E over A, $Der^{(\sigma,A)}(E)$.

Theorem 8. *$Der^{(\sigma,A)}(E)$ is a σ-Lie G-algebra with respect to the σ-bracket, that is the following properties are satisfied for σ-derivations of E over A;*

$$\left[Der^{(\sigma,A)}(E), Der^{(\sigma,A)}(E)\right]^{\sigma} \subseteq Der^{(\sigma,A)}(E), \tag{i}$$

$$\left[Der_{\phi}^{(\sigma,A)}(E), Der_{\chi}^{(\sigma,A)}(E)\right]^{\sigma} \subseteq Der_{\phi+\chi}^{(\sigma,A)}(E), \tag{i'}$$

$\phi, \chi \in \hat{G}$, skew σ-symmetricity, $[,]^{\sigma} = -[,]^{\sigma}\circ\sigma$, and the σ-Jacobi identity, see (ii) and (iii) in theorem 5.

4.1. **Quantization of braided derivations of G-modules.**

Consider derivations of a σ-symmetric G-algebra A and a σ-symmetric A-module E.

Let $(\partial, \partial_A) \in Der^{(\sigma,A)}(E)$. Given a quantizer q define the quantization of ∂ by

$$\partial_q(x) = Q_q(\partial)(x) \stackrel{def}{=} \sum_{g,h\in G} Q_{gh}\, g(\partial)(h(x)), \tag{19}$$

$x \in E$. If $x \in A$, this is the quantization of the derivation of A, ∂_A.

$Q_q(\partial)$ is an operator of the quantized module E_q.

Denote by $Der^{(\sigma_q,A_q)}(E_q)$ the quantization of all σ-derivations of E over A equipped with the quantization of composition.

The operator Q_q,

$$\begin{aligned} Q_q &: \left(Der^{(\sigma,A)}(E),[,]^{\sigma}\right) \to \left(Der^{(\sigma_q,A_q)}(E_q),[,]_q^{\sigma_q}\right), \\ \partial &\in Der^{(\sigma,A)}(E) \longmapsto Q_q(\partial) \in Der^{(\sigma_q,A_q)}(E_q), \end{aligned} \tag{20}$$

is an isomorphism of modules between the σ-derivations of E over A and the σ_q-derivations of E_q over A_q.

$Der^{(\sigma_q,A_q)}(E_q)$ is a σ_q-Lie G-algebra with respect to the $\sigma_q - q$-bracket, that is the following properties are satisfied,

$$[Der^{\sigma_q}(A_q), Der^{\sigma_q}(A_q)]_q^{\sigma_q} \subseteq Der^{\sigma_q}(A_q), \tag{i}$$

$$\left[Der_{\phi}^{\sigma_q}(A_q), Der_{\chi}^{\sigma_q}(A_q)\right]_q^{\sigma_q} \subseteq Der_{\phi+\chi}^{\sigma_q}(A_q), \tag{i'}$$

$\phi, \chi \in \hat{G}$, skew σ_q-symmetricity, $[,]_q^{\sigma_q} = -[,]_q^{\sigma_q} \circ \sigma_q$, and the σ_q-Jacobi identity, see (ii) and (iii) in theorem 6.

The σ_q-Lie algebra of the structure of $\left(Der^{(\sigma_q,A_q)}(E_q),[,]_q^{\sigma_q}\right)$ can be realized within the classical one by dequantization,

$$Q_q^{-1}(\partial) = \partial_c,$$

$\partial \in Der^{(\sigma_q,A_q)}(E_q)$, where we have the linearity (16), the A_q-module structure (17), the commutator satisfies (18) and the braided Jacobi identity is satisfied and can be written in terms of the $\sigma_q - q$-bracket using (18).

5. (G,σ)-connections

Let A be an σ-symmetric algebra and E a σ-commutative A-module.

Definition 9. *The map*

$$\nabla : Der^{\sigma}(A) \to Der^{(\sigma,A)}(E),$$

is a σ-connection in E if

$$\pi \circ \nabla = Id.$$

Furthermore, ∇ is called a (G,σ)-connection in $Der^{(\sigma,A)}(E)$ if the following diagram commutes

$$\begin{tikzcd} Der^{\sigma,A}(E) \arrow[r, "g"] & Der^{\sigma,A}(E) \\ Der^{\sigma}(A) \arrow[r, "g"] \arrow[u, "\nabla"] & Der^{\sigma}(A) \arrow[u, "\nabla"] \end{tikzcd} . \tag{21}$$

Define G-action on ∇ by

$$g(\nabla) = g \circ \nabla \circ g^{-1}. \tag{22}$$

Then a (G,σ)-connection in E is G-invariant,

$$g(\nabla) = \nabla,$$

and ∇ preserves the degree on the σ-derivations,

$$\nabla : Der^{\sigma}_{\chi}(A) \to Der^{\sigma}_{\chi}(E).$$

Assume that ∇ is a (G,σ)-connection.

We say that the σ-connection ∇ is flat if

$$[\nabla\partial_1, \nabla\partial_2]^{\sigma} = \nabla([\partial_1,\partial_2]^{\sigma}), \tag{23}$$

for all $\partial_1, \partial_2 \in Der^{\sigma}(A)$.

Define the σ-*curvature* of ∇,

$$K_{\nabla}(\partial_1,\partial_2) = [\nabla\partial_1, \nabla\partial_2]^{\sigma} - \nabla([\partial_1,\partial_2]^{\sigma}), \tag{24}$$

written in full,

$$\begin{aligned} K_{\nabla}(\partial_1,\partial_2) &= \nabla(\partial_1)\nabla(\partial_2) - \nabla(\partial_1\partial_2) \\ &\quad + \sum_{g,h\in G} S_{g,h}\left(\nabla(h(\partial_2)\,g(\partial_1)) - h(\nabla(\partial_2))\,g(\nabla(\partial_1))\right). \end{aligned}$$

The properties of K_{∇} are the following. K_{∇} satisfies linearity,

$$K_{\nabla}(\partial_1,\partial_2)(ax) = \sum_{g,h\in G} S_{gh}h(a)\,(gK_{\nabla}(\partial_1,\partial_2))(x), \tag{25}$$

is skew σ-symmetric,

$$K_{\nabla}(\partial_1,\partial_2) = -\sum_{g,h\in G} S_{gh}K_{\nabla}(h(\partial_2), g(\partial_1)),$$

and satisfies the A-module homomorphism conditions with respect to σ,

$$K_{\nabla}(a\partial_1,\partial_2) = aK_{\nabla}(\partial_1,\partial_2), \tag{26}$$

$$K_{\nabla}(\partial_1,a\partial_2) = \sum_{g,h\in G} S_{gh}h(a)\,K_{\nabla}(g\partial_1,\partial_2), \tag{27}$$

for $a \in A$, $x \in E$ and braided derivations $\partial_1, \partial_2 \in Der^{\sigma}(A)$.

5.1. **Quantizations of (G,σ)-connections and curvatures.** Let ∇ be a G-invariant (G,σ)-connection.

Given a quantization q, define the quantization of the (G,σ)-connection ∇,

$$\nabla_q : Der^{\sigma_q}\left(A_q\right) \to Der^{\left(A_q,\sigma_q\right)}\left(E_q\right),$$

by

$$\nabla_q = Q_q \circ \nabla \circ Q_q^{-1}.$$

The connection ∇_q is a σ_q-connection in E_q which is easy to see as $\pi_q = Q_q \circ \pi \circ Q_q^{-1}$. Furthermore is the σ_q-connection ∇_q a (G,σ_q)-connection, that is,

$$\begin{tikzcd} Der^{\sigma_q,A_q}(E_q) \arrow[r, "g"] & Der^{\sigma_q,A_q}(E_q) \\ Der^{\sigma}(A_q) \arrow[u, "\nabla_q"] \arrow[r, "g"] & Der^{\sigma}_q(A_q) \arrow[u, "\nabla_q"] \end{tikzcd} \tag{28}$$

commutes, and is G-invariant.

The quantization of the σ-curvature, the $\sigma_q - q$-curvature $K^q_{\nabla_q}$, is defined

$$K^q_{\nabla_q}\left(\partial_1,\partial_2\right) = \left[\nabla_q\partial_1,\nabla_q\partial_2\right]^{\sigma_q}_q - \nabla_q\left(\left[\partial_1,\partial_2\right]^{\sigma_q}_q\right),$$

$\partial_1,\partial_2 \in Der^{\sigma_q}\left(A_q\right)$.

The $\sigma_q - q$-curvature satisfies linearity,

$$K^q_{\nabla_q}\left(\partial_1,\partial_2\right)(ax) = \sum_{g,h\in G} S^q_{gh} h\left(a\right)\left(gK^q_{\nabla_q}\left(\partial_1,\partial_2\right)\right)(x), \tag{29}$$

is skew σ_q-symmetric,

$$K^q_{\nabla_q}\left(\partial_1,\partial_2\right) = -\sum_{g,h\in G} S^q_{gh} K^q_{\nabla_q}\left(h\left(\partial_2\right),g\left(\partial_1\right)\right), \tag{30}$$

and satisfies the A_q-module homomorphism conditions with respect to σ_q,

$$K^q_{\nabla_q}\left(a\partial_1,\partial_2\right) = aK^q_{\nabla_q}\left(\partial_1,\partial_2\right), \tag{31}$$

$$K^q_{\nabla_q}\left(\partial_1,a\partial_2\right) = \sum_{g,h\in G} S^q_{gh} h\left(a\right) K^q_{\nabla_q}\left(g\partial_1,\partial_2\right), \tag{32}$$

for $a \in A_q$, $x \in E_q$ and braided derivations $\partial_1,\partial_2 \in Der^{\sigma_q}\left(A_q\right)$.

We have the following picture for dequantizations of braided derivations.

Theorem 10. *The $\sigma_q - q$-curvature K_{∇}^{q} of the σ-connection ∇ of E defined by*

$$K_{\nabla}^{q}\left(\left(\partial_1\right)_c,\left(\partial_2\right)_c\right)=\left[\nabla\left(\partial_1\right)_c,\nabla\left(\partial_2\right)_c\right]_q^{\sigma_q}-\nabla\left(\left[\left(\partial_1\right)_c,\left(\partial_2\right)_c\right]_q^{\sigma_q}\right), \tag{33}$$

and the σ_q-curvature of the σ_q-connection ∇_q defined by,

$$K_{\nabla_q}\left(\partial_1,\partial_2\right)=\left[\nabla_q\partial_1,\nabla_q\partial_2\right]^{\sigma_q}-\nabla_q\left(\left[\partial_1,\partial_2\right]^{\sigma_q}\right), \tag{34}$$

are related as follows

$$\left(K_{\nabla_q}\left(\partial_1,\partial_2\right)\right)_c=K_{\nabla}^{q}\left(\left(\partial_1\right)_c,\left(\partial_2\right)_c\right), \tag{35}$$

$\partial_1,\partial_2\in Der^{\sigma_q}\left(A_q\right)$.

6. Braided differential operators

We shall see how the picture is for braided differential operators. Let G be a finite abelian group.

6.1. **Braided differential operators in G-algebras.** Let A be a σ-symmetric G-algebra.

Define a σ-differential operator f of order at most k as the linear map

$$f:A\to A,$$

such that

$$\left[a_0,\left[a_1,\ldots\left[a_k,f\right]^{\sigma}\cdots\right]^{\sigma}\right]^{\sigma}=0, \tag{36}$$

$\forall a_0,\ldots,a_k\in A$.

A σ-differential operator f of order at most k is of degree $|f|\in\hat{G}$ if it is an $(G,|f|)$-operator $A\to A$, that is

$$f\circ g=|f|\left(g\right)g\circ f, \tag{37}$$

$g\in G$.

Denote by $Diff_{k,\chi}^{\sigma}\left(A,A\right)$ the σ-differential operators of order at most k and degree $\chi\in\hat{G}$ and by $Diff_{k}^{\sigma}\left(A,A\right)$ the set of all of order at most k.

Let's consider $Diff^{\sigma}\left(A,A\right)=\cup Diff_{k}^{\sigma}\left(A,A\right)$.

From [16] we have the two following results. The σ-commutator of two σ-differential operators $f_1\in Diff_{i}^{\sigma}\left(A,A\right)$ and $f_2\in Diff_{j}^{\sigma}\left(A,A\right)$ is a σ-differential operator of order at most $i+j-1$,

$$\left[f_1,f_2\right]^{\sigma}\in Diff_{i+j-1}^{\sigma}\left(A,A\right).$$

and $Diff^{\sigma}\left(A,A\right)$ is a σ-Lie G-algebra. Clearly,

$$\left[f_1,f_2\right]^{\sigma}\in Diff_{i+j-1,|f_1|+|f_2|}^{\sigma}\left(A,A\right),$$

for homogeneous $f_1\in Diff_{i,|f_1|}^{\sigma}\left(A,A\right)$ and $f_2\in Diff_{j,|f_2|}^{\sigma}\left(A,A\right)$.

We consider $A-A$-module structure on $Diff^{\sigma}(A,A)$, [9].

Proposition 11. *There is an $A-A$-module structure on $Diff^{\sigma}(A,A)$ defined by*

$$\begin{aligned}\nu^{l}(a\otimes f)(b) &= af(b),\\ \nu^{r}(f\otimes a)(b) &= f(ab),\end{aligned}$$

$a,b\in A$, $f\in Diff^{\sigma}(A,A)$ and

$$af, fa\in Diff_{k}^{\sigma}(A,A),$$

for $f\in Diff_{k}^{\sigma}(A,A)$.

Consider the symbol of the differential operators which is the leading part with respect to derivatives,

$$Smbl_{k}^{\sigma}(A,A)=Diff_{k}^{\sigma}(A,A)/Diff_{k-1}^{\sigma}(A,A),$$

then we have the $\mathbb{Z}$-graded object

$$Smbl^{\sigma}(A,A)=\sum_{k\in\mathbb{Z}}Smbl_{k}^{\sigma}(A,A).$$

The class of $[f_1,f_2]^{\sigma}\in Diff_{i+j-1}^{\sigma}(A,A)$,

$$\overline{[f_1,f_2]^{\sigma}}\in Smbl_{i+j-1}^{\sigma}(A,A),$$

depends on the class of the two σ-differential operators $f_1\in Diff_{i}^{\sigma}(A,A)$ and $f_2\in Diff_{j}^{\sigma}(A,A)$, and there is a σ-Poisson structure on the braided symbol algebra, see [16].

6.2. **Braided differential operators in G-modules.** Let A be a σ-symmetric G-algebra and let E be a σ-symmetric A-module.

Define a σ-differential operator f of E of order at most k as the linear map

$$f:E\to E,$$

such that

$$[x,[a_0,\dots[a_{k-1},f]^{\sigma}\cdots]^{\sigma}]^{\sigma}=0, \tag{38}$$

$\forall a_0,\dots,a_{k-1}\in A$, $x\in E$.

A σ-differential operator f of E of order at most k is of degree $|f|\in G$ if it is an $(G,|f|)$-operator $A\to A$, that is

$$f\circ g=|f|(g)\,g\circ f, \tag{39}$$

$g\in G$.

Denote by $Diff_{k,\chi}^{\sigma}(E,E)$ the σ-differential operators of order at most k and degree $\chi\in\hat{G}$, the σ-differential operators in order at most k of E by $Diff_{k}^{(\sigma,A)}(E,E)$ and we consider $Diff^{(\sigma,A)}(E,E)=\cup Diff_{k}^{(\sigma,A)}(E,E)$.

From [16] we have the two following results. The σ-commutator of two σ-differential operators $f_1 \in Diff_i^\sigma(E,E)$ and $f_2 \in Diff_j^\sigma(E,E)$ is a σ-differential operator of order at most $i+j$,

$$[f_1, f_2]^\sigma \in Diff_{i+j}^\sigma(E,E).$$

and $Diff^\sigma(E,E)$ is a σ-Lie G-algebra. Furthermore,

$$[f_1, f_2]^\sigma \in Diff_{i+j,|f_1|+|f_2|}^\sigma(E,E),$$

for homogeneous $f_2 \in Diff_{i,|f_1|}^\sigma(E,E)$ and $f_2 \in Diff_{j,|f_2|}^\sigma(E,E)$.

Proposition 12. *There is an $A-A$-module structure on $Diff^\sigma(E,E)$ defined by*

$$\begin{aligned} \nu^l(a \otimes f)(x) &= af(x), \\ \nu^r(f \otimes a)(x) &= f(ax), \end{aligned}$$

and

$$af, fa \in Diff_k^\sigma(E,E),$$

for $a \in A$ and $f \in Diff_k^\sigma(E,E)$.

Consider the symbol of the differential operators which is the leading part with respect to derivatives,

$$Smbl_k^\sigma(E,E) = Diff_k^\sigma(E,E)/Diff_{k-1}^\sigma(E,E),$$

then we have the $\mathbb{Z}$-graded object

$$Smbl^\sigma(E,E) = \sum_{k\in\mathbb{Z}} Smbl_k^\sigma(E,E).$$

The class of $[f_1, f_2]^\sigma \in Diff_{i+j}^\sigma(E,E)$,

$$\overline{[f_1, f_2]^\sigma} \in Smbl_{i+j}^\sigma(E,E),$$

depends on the class of the two σ-differential operators $f_1 \in Diff_i^\sigma(E,E)$ and $f_2 \in Diff_j^\sigma(E,E)$, and there is a σ-Poisson structure on the braided symbol algebra, see [16].

6.3. **Quantizations of braided differential operators in G-algebras.** We can define quantization of σ-differential operators in G-algebras. Let A be a σ-commutative G-algebra.

Definition 13. *Given a quantization q and $f \in Diff^\sigma(A,A)$ define the quantization of f by*

$$Q_q(f)(a) = f_q(a) \overset{def}{=} \sum_{g,h\in G} Q_{gh}\, g(f)(h(a)),$$

for $a \in A$.

The operator $Q_q(f)$ is an operator of the quantized G-algebra A_q. Let us denote by $Diff^{\sigma_q}(A_q, A_q)$ the quantization of all σ-differential operators of A equipped with the quantization of composition.

From [15] we have the following result. Given a braiding σ, let σ_q be the quantization of σ. The operator

$$\begin{aligned} Q_q &: \left(Diff^{\sigma}(A,A), [,]^{\sigma}\right) \to \left(Diff^{\sigma_q}(A_q,A_q), [,]_q^{\sigma_q}\right), \qquad (40) \\ f &\in Diff^{\sigma}(A,A) \longmapsto Q_q(f) \in Diff^{\sigma_q}(A_q,A_q), \end{aligned}$$

is an isomorphism of modules. The symbol of Q_q is an isomorphism of modules

$$\begin{aligned} Smbl(Q_q) &: \left(Smbl^{\sigma}(A,A), [,]^{\sigma}\right) \to \left(Smbl^{\sigma_q}(A_q,A_q), [,]_q^{\sigma_q}\right), \qquad (41) \\ f &\in Smbl^{\sigma}(A,A) \longmapsto Smbl(Q_q)(f) \in Smbl^{\sigma_q}(A_q,A_q). \end{aligned}$$

By proposition 11 is $Diff^{\sigma_q}(A_q, A_q)$ a σ_q-symmetric module and a σ_q-Lie G-algebra with respect to the $\sigma_q - q$-bracket and the quantized composition.

Furthermore, there is a σ_q-Poisson structure on the quantized braided symbol algebra.

The σ_q-Lie algebra structure of $Diff^{\sigma_q}(A_q, A_q)$ can be realized within the classical one by dequantization, where we have the conditions for linearity (16), the A_q-module structure (17) and the commutator (18), with Der replaced by $Diff$.

6.4. Quantizations of braided differential operators in G-modules.

Let A be a σ-symmetric G-algebra and let E be a σ-symmetric A-module.

Definition 14. *Given a quantization q and $f \in Diff^{(\sigma,A)}(E,E)$ define the quantization of f by*

$$Q_q(f)(x) = f_q(x) \overset{def}{=} \sum_{g,h\in G} Q_{gh}\, g(f)(h(x)),$$

where $x \in E$.

The operator $Q_q(f)$ is an operator of the quantized G-module E_q. Denote by $Diff^{(\sigma_q,A_q)}(E_q, E_q)$ the quantization of all σ-differential operators of E equipped with the quantization of composition.

Given a braiding σ, let σ_q be the quantization of σ. The operator

$$Q_q : \left(Diff^{(\sigma,A)}(E,E), [,]^{\sigma}\right) \to \left(Diff^{(\sigma_q,A_q)}(E_q,E_q), [,]_q^{\sigma_q}\right), \qquad (42)$$

$$f \in Diff^{(\sigma,A)}(E,E) \longmapsto Q_q(f) \in Diff^{(\sigma_q,A_q)}(E_q,E_q),$$

is an isomorphism of modules. The symbol of Q_q is an isomorphism of modules

$$Smbl\left(Q_q\right):\left(Smbl^{(\sigma,A)}\left(E,E\right),[,]^{\sigma}\right)\rightarrow\left(Smbl^{(\sigma_q,A_q)}\left(E_q,E_q\right),[,]_q^{\sigma_q}\right),\tag{43}$$

$$f\in Smbl^{(\sigma,A)}\left(E,E\right)\longmapsto Smbl\left(Q_q\right)\left(f\right)\in Smbl^{(\sigma_q,A_q)}\left(E_q,E_q\right).$$

$Diff^{(\sigma_q,A_q)}\left(E_q,E_q\right)$ is a σ_q-symmetric module and a σ_q-Lie G-algebra with respect to the σ_q-q-bracket and the quantized composition.

Furthermore, there is a σ_q-Poisson structure on the quantized braided symbol algebra, $Smbl^{(\sigma_q,A_q)}\left(E_q,E_q\right)$.

The σ_q-Lie algebra structure of $Diff^{(\sigma_q,A_q)}\left(E_q,E_q\right)$ can be realized within the classical one by dequantization, where we have the conditions for linearity (16), the A_q-module structure (17) and the commutator (18), with Der replaced by $Diff$.

6.5. **Braided symbol and $G\oplus S^1$-modules.** Any $\hat{G}$-graded differential operator has a fibration by $\mathbb{Z}$. However, the symbol of the braided differential operators of $\hat{G}$-graded algebras A and modules E, $Smbl^{\hat{\sigma}}\left(A,A\right)$ and $Smbl^{(\hat{\sigma},A)}\left(E,E\right)$, is a graded by $\hat{G}\oplus\mathbb{Z}$. Hence there is an action of the dual of $\hat{G}\oplus\mathbb{Z}$, $G\oplus S^1$ on the symbol of the braided differential operators of G-algebras A and G-modules E, $Smbl^{\sigma}\left(A,A\right)$ and $Smbl^{(\sigma,A)}\left(E,E\right)$.

In [11] and [10] we consider quantizations and braidings with respect to $\hat{G}\oplus\mathbb{Z}$-grading. Here, instead of considering quantizations and braidings with respect to the G-action, we consider such with respect to $G\oplus S^1$-action.

Let $\bar{G}=\hat{G}\oplus\mathbb{Z}$, denote its elements by $\bar{\chi}=\left(\chi,\chi_{\mathbb{Z}}\right)$ and $\widetilde{G}=G\oplus S^1$ and denote elements by $\widetilde{g}=\left(g,g_S\right)$.

Any symmetry in the monoidal category of $\widetilde{G}=G\oplus S^1$-modules is of the form

$$\widetilde{\sigma}=\widetilde{\tau}\circ\left(\frac{1}{|G|^2}\sum_{\substack{\widetilde{g},\widetilde{h}\in\widetilde{G}\\ \bar{\chi},\bar{\phi}\in\bar{G}}}\bar{\sigma}\left(\bar{\phi},\bar{\chi}\right)\bar{\phi}\left(\widetilde{g}\right)\bar{\chi}\left(\widetilde{h}\right)\delta_{\widetilde{g}}\otimes\delta_{\widetilde{h}}\right),\tag{44}$$

where

$$\bar{\sigma}:\left(\hat{G}\oplus\mathbb{Z}\right)\times\left(\hat{G}\oplus\mathbb{Z}\right)\rightarrow U\left(\mathbb{C}\right)$$

is a symmetry of the monoidal category of $\bar{G}=\hat{G}\oplus\mathbb{Z}$-graded modules defined by

$$\bar{\sigma}\left(\bar{\chi},\bar{\chi}'\right)=\sigma\left(\chi,\chi'\right)\tau\left(\chi_{\mathbb{Z}},\chi'_{\mathbb{Z}}\right)\gamma\left(\chi_,\chi'_{\mathbb{Z}}\right)\gamma^{-1}\left(\chi',\chi_{\mathbb{Z}}\right),\tag{45}$$

$\bar{\chi}, \bar{\chi}' \in \bar{G}$, for which

$$\bar{\sigma}|_{(\hat{G}\oplus\{0\})\times(\hat{G}\oplus\{0\})} = \sigma : \hat{G} \times \hat{G} \to U(R),$$

is a symmetry of $\bar{G}$-graded modules,

$$\bar{\sigma}|_{(\{0\}\oplus\mathbb{Z})\times(\{0\}\oplus\mathbb{Z})} = \tau : \mathbb{Z} \times \mathbb{Z} \to U(R),$$

is a symmetry of $\mathbb{Z}$-graded modules and

$$\bar{\sigma}|_{(\hat{G}\oplus\{0\})\times(\{0\}\oplus\mathbb{Z})} = \gamma : \hat{G} \times \mathbb{Z} \to U(R),$$

is a bihomomorphism.

Denote by

$$\widetilde{\sigma} = \sum_{\widetilde{g},\widetilde{h}\in\widetilde{G}} \bar{S}_{\widetilde{g}\widetilde{h}} \delta_{\widetilde{g}} \otimes \delta_{\widetilde{h}}.$$

Any quantization in the monoidal category of $\widetilde{G} = G \oplus S^1$-modules is of the form

$$\widetilde{q} = \frac{1}{|G|^2} \sum_{\substack{\widetilde{g},\widetilde{h}\in\widetilde{G} \\ \bar{\chi},\bar{\phi}\in\bar{G}}} \bar{q}\left(\bar{\phi}, \bar{\chi}\right) \bar{\phi}\left(\widetilde{g}\right) \bar{\chi}\left(\widetilde{h}\right) \delta_{\widetilde{g}} \otimes \delta_{\widetilde{h}}, \tag{46}$$

where $\bar{q}$ is quantization in the monoidal category of $\bar{G} = \hat{G} \oplus \mathbb{Z}$-graded modules,

$$\bar{q}\left(\bar{\chi}, \bar{\chi}'\right) = q\left(\chi, \chi'\right) \varkappa\left(\chi, \chi'_{\mathbb{Z}}\right) \varkappa^{-1}\left(\chi', \chi_{\mathbb{Z}}\right), \tag{47}$$

where $\bar{\chi}, \bar{\chi}' \in \bar{G}$,

$$\bar{q}|_{(\hat{G}\oplus\{0\})\times(\{0\}\oplus\mathbb{Z})} = \varkappa : \hat{G} \times \mathbb{Z} \to U(R)$$

is a bihomomorphism and

$$\bar{q}|_{(\hat{G}\oplus\{0\})\times(\hat{G}\oplus\{0\})} = q : \hat{G} \times \hat{G} \to U(R),$$

is quantization of $\hat{G}$-graded modules.

Denote by

$$\widetilde{q} = \sum_{\widetilde{g},\widetilde{h}\in\widetilde{G}} \bar{Q}_{\widetilde{g}\widetilde{h}} \delta_{\widetilde{g}} \otimes \delta_{\widetilde{h}}.$$

Considering $Smbl^{\sigma}(A, A)$ and $Smbl^{(\sigma,A)}(E, E)$ as $\widetilde{G}$-modules they are equipped with a symmetry $\widetilde{\sigma}$ of $\widetilde{G}$ given by $\bar{\sigma}$ where $\bar{\sigma}|_{(\{0\}\oplus\mathbb{Z})\times(\{0\}\oplus\mathbb{Z})} = \tau$ and $\bar{\sigma}|_{(\hat{G}\oplus\{0\})\times(\{0\}\oplus\mathbb{Z})} = \gamma$ trivial, that is $\bar{\sigma} = \hat{\sigma}$.

Remark 15. *If we quantize $Smbl^{\sigma}(A, A)$ and $Smbl^{(\sigma,A)}(E, E)$ by the quantizer $\widetilde{q}$ given by $\gamma = \bar{\sigma}|_{(\hat{G}\oplus\{0\})\times(\{0\}\oplus\mathbb{Z})}$ then the resulting algebra is*

$Smbl^{\widetilde{\sigma}}(A,A)$ and $Smbl^{(\widetilde{\sigma},A)}(E,E)$, which are $\widetilde{\sigma}$-Poisson algebras with respect to the braiding $\widetilde{\sigma}$ given by the braiding

$$\bar{\sigma}(\bar{\chi},\bar{\chi}')=\sigma(\chi,\chi')\,\gamma\left(\chi,\chi'_{\mathbb{Z}}\right)\gamma^{-1}(\chi',\chi_{\mathbb{Z}})$$

in the category of $\bar{G}$-graded modules.

We show the quantized braided Poisson structure for the quantization of $Smbl^{\widetilde{\sigma}}(A,A)$ and $Smbl^{(\widetilde{\sigma},A)}(E,E)$ for a general braiding $\widetilde{\sigma}$ in theorems 16 and 17.

Note that for any braiding $\widetilde{\sigma}$, which is given by some

$$\bar{\sigma},\bar{\sigma}|_{(\{0\}\oplus\mathbb{Z})\times(\{0\}\oplus\mathbb{Z})}=\tau$$

always will be trivial since the structure arises from (quantizations of) $Smbl^{\sigma}(A,A)$ and $Smbl^{(\sigma,A)}(E,E)$.

Assume we have a braided symbols $Smbl^{\widetilde{\sigma}}(A,A)$ and $Smbl^{(\widetilde{\sigma},A)}(E,E)$ with respect to a symmetry $\widetilde{\sigma}$, given by a symmetry $\bar{\sigma}$ in the category of $\bar{G}$-graded modules, $\bar{\sigma}|_{(\{0\}\oplus\mathbb{Z})\times(\{0\}\oplus\mathbb{Z})}=\tau=1$.

A quantization of $f\in Smbl_k^{\widetilde{\sigma}}(A,A)$ or $f\in Smbl_k^{(\widetilde{\sigma},A)}(E,E)$ is

$$Q_{\widetilde{q}}(f)(x)=f_{\widetilde{q}}(x)\overset{def}{=}\sum_{\widetilde{g},\widetilde{h}\in\widetilde{G}}\bar{Q}_{\widetilde{g}\widetilde{h}}\widetilde{g}f\left(\widetilde{h}x\right).$$

where x is in either $E_{\widetilde{q}}$ or $A_{\widetilde{q}}$.

The operator $Q_{\widetilde{q}}$ is an isomorphism of modules

$$\begin{aligned}Smbl(Q_{\widetilde{q}})\ &:\ \left(Smbl^{\widetilde{\sigma}}(A,A),[,]^{\widetilde{\sigma}}\right)\to\left(Smbl^{\widetilde{\sigma}_{\widetilde{q}}}(A_{\widetilde{q}},A_{\widetilde{q}}),[,]_{\widetilde{q}}^{\widetilde{\sigma}_{\widetilde{q}}}\right),\\ f\ &\in\ Smbl^{\widetilde{\sigma}}(A,A)\longmapsto Smbl(Q_{\widetilde{q}})(f)\in Smbl^{\widetilde{\sigma}_{\widetilde{q}}}(A_{\widetilde{q}},A_{\widetilde{q}}).\end{aligned}$$

and

$$Smbl(Q_{\widetilde{q}}):\left(Smbl^{(\widetilde{\sigma},A)}(E,E),[,]^{\widetilde{\sigma}}\right)\to\left(Smbl^{\left(\widetilde{\sigma}_{\widetilde{q}},A_{\widetilde{q}}\right)}(E_{\widetilde{q}},E_{\widetilde{q}}),[,]_{\widetilde{q}}^{\widetilde{\sigma}_{\widetilde{q}}}\right),$$

$$f\in Smbl^{(\widetilde{\sigma},A)}(E,E)\longmapsto Smbl(Q_{\widetilde{q}})(f)\in Smbl^{\left(\widetilde{\sigma}_{\widetilde{q}},A_{\widetilde{q}}\right)}(E_{\widetilde{q}},E_{\widetilde{q}}).$$

We obtain the following properties for the quantization of $Smbl^{\widetilde{\sigma}}(A,A)$.

Theorem 16. *$Smbl^{\widetilde{\sigma}_{\widetilde{q}}}(A_{\widetilde{q}},A_{\widetilde{q}})$ is a $\widetilde{\sigma}_{\widetilde{q}}$-Poisson algebra with respect to the $\widetilde{\sigma}_{\widetilde{q}}-\widetilde{q}$-bracket, that is, the following properties are satisfied*

$$\left[Smbl^{\widetilde{\sigma}_{\widetilde{q}}}(A_{\widetilde{q}},A_{\widetilde{q}}),Smbl^{\widetilde{\sigma}_{\widetilde{q}}}(A_{\widetilde{q}},A_{\widetilde{q}})\right]_{\widetilde{q}}^{\widetilde{\sigma}_q}\subseteq Smbl^{\widetilde{\sigma}_{\widetilde{q}}}(A_{\widetilde{q}},A_{\widetilde{q}}),\tag{i}$$

$$\left[Smbl_i^{\widetilde{\sigma}_{\widetilde{q}}}(A_{\widetilde{q}},A_{\widetilde{q}}),Smbl_j^{\widetilde{\sigma}_{\widetilde{q}}}(A_{\widetilde{q}},A_{\widetilde{q}})\right]_{\widetilde{q}}^{\widetilde{\sigma}_{\widetilde{q}}}\subseteq Smbl_{i+j-1}^{\widetilde{\sigma}_{\widetilde{q}}}(A_{\widetilde{q}},A_{\widetilde{q}}),\tag{i'}$$

skew $\widetilde{\sigma}_{\widetilde{q}}$-symmetricity,

$$[f_1, f_2]_{\widetilde{q}}^{\widetilde{\sigma}_{\widetilde{q}}} = -\sum_{\widetilde{g},\widetilde{h}\in\widetilde{G}} \bar{S}_{\widetilde{g}\widetilde{h}} \left[\widetilde{h}f_2, \widetilde{g}f_1\right]_{\widetilde{q}}^{\widetilde{\sigma}_{\widetilde{q}}}, \tag{ii}$$

the $\widetilde{\sigma}_{\widetilde{q}}$-Jacobi identity,

$$\left[f_1, [f_2, f_3]_{\widetilde{q}}^{\widetilde{\sigma}_{\widetilde{q}}}\right]_{\widetilde{q}}^{\widetilde{\sigma}_{\widetilde{q}}} = \left[[f_1, f_2]_{\widetilde{q}}^{\widetilde{\sigma}_{\widetilde{q}}}, f_3\right]_{\widetilde{q}}^{\widetilde{\sigma}_{\widetilde{q}}} + \sum_{\widetilde{g},\widetilde{h}\in\widetilde{G}} \bar{S}_{\widetilde{g}\widetilde{h}} \left[\widetilde{h}f_2, [\widetilde{g}f_1, f_3]_{\widetilde{q}}^{\widetilde{\sigma}_{\widetilde{q}}}\right]_{\widetilde{q}}^{\widetilde{\sigma}_{\widetilde{q}}}, \tag{iii}$$

and

$$[f_1, f_2f_3]_{\widetilde{q}}^{\widetilde{\sigma}_{\widetilde{q}}} = [f_1, f_2]_{\widetilde{q}}^{\widetilde{\sigma}_{\widetilde{q}}} h + \sum_{\widetilde{g},\widetilde{h}\in\widetilde{G}} \bar{S}_{\widetilde{g}\widetilde{h}}\widetilde{h}f_2 \, [\widetilde{g}f_1, f_3]_{\widetilde{q}}^{\widetilde{\sigma}_{\widetilde{q}}}, \tag{iv}$$

for $f_1 \in Smbl_i^{\widetilde{\sigma}_{\widetilde{q}}}(A_{\widetilde{q}}, A_{\widetilde{q}})$, $f_2 \in Smbl_j^{\widetilde{\sigma}_{\widetilde{q}}}(A_{\widetilde{q}}, A_{\widetilde{q}})$, $f_3 \in Smbl_k^{\widetilde{\sigma}_{\widetilde{q}}}(A_{\widetilde{q}}, A_{\widetilde{q}})$.

Except for (i') we obtain the same properties for the quantization of $Smbl^{(\widetilde{\sigma},A)}(E, E)$.

Theorem 17. *$Smbl^{\left(\widetilde{\sigma}_{\widetilde{q}},A_{\widetilde{q}}\right)}(E_{\widetilde{q}}, E_{\widetilde{q}})$ is a $\bar{G} = \hat{G} \oplus \mathbb{Z}$-graded $\widetilde{\sigma}_{\widetilde{q}}$-Poisson algebra with respect to the $\widetilde{\sigma}_{\widetilde{q}} - \widetilde{q}$-bracket, that is, the properties (i), (ii), (iii) and (iv) of theorem 16 are satisfied when replacing $A_{\widetilde{q}}$ by $E_{\widetilde{q}}$ and $Smbl^{\widetilde{\sigma}_{\widetilde{q}}}$ by $Smbl^{\left(\widetilde{\sigma}_{\widetilde{q}},A_{\widetilde{q}}\right)}$, and*

$$\left[Smbl_i^{\left(\widetilde{\sigma}_{\widetilde{q}},A_{\widetilde{q}}\right)}(E_{\widetilde{q}}, E_{\widetilde{q}}), Smbl_j^{\left(\widetilde{\sigma}_{\widetilde{q}},A_{\widetilde{q}}\right)}(E_{\widetilde{q}}, E_{\widetilde{q}})\right]_{\widetilde{q}}^{\widetilde{\sigma}_{\widetilde{q}}} \subseteq Smbl_{i+j}^{\left(\widetilde{\sigma}_{\widetilde{q}},A_{\widetilde{q}}\right)}(E_{\widetilde{q}}, E_{\widetilde{q}}). \tag{i'}$$

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