# MODULAR INEQUALITIES FOR THE HARDY AVERAGING OPERATOR

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*Dedicated to Professor Alois Kufner on the occasion of his 65th birthday*

*Abstract.* If $P$ is the Hardy averaging operator—or some of its generalizations, then weighted modular inequalities of the form

$$\int u\varphi(Pf) \leqslant C \int v\varphi(f)$$

are established for a general class of functions $\varphi$. Modular inequalities for the two- and higher dimensional Hardy averaging operator are also given.

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## 1. Introduction

Let $P$ denote the Hardy averaging operator

$$(Pf)(x) = \frac{1}{x}\int_0^x f, \qquad x > 0,\ f \geqslant 0,$$

then the classical Hardy inequality asserts that for $1 < p < \infty$, the inequality

$$\int_0^\infty (Pf)(x)^p \,\mathrm{d}x \leqslant \Big(\frac{p}{p-1}\Big)^p \int_0^\infty f(x)^p \,\mathrm{d}x \tag{1.1}$$

holds.

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Here and in the sequel, functions are assumed to be measurable, weight functions, denoted by $u, v$ and $w$, are locally integrable, and the left sides of inequalities exist if the right sides do. Also, constants are positive and denoted (sometimes with subscripts) by $C$.

Extensive generalizations and applications of (1.1) exist and we refer to [9] and the bibliography cited there.

The generalizations considered here are the modular forms of (1.1), obtained by replacing the power function $\varphi(x) = x^p$, $p > 1$ by a general function $\varphi$. To explain some results in this direction, the notion of $N$-function is required.

A function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is an $N$-function if it has the form

$$\varphi(x) = \int_0^x p(t)\,\mathrm{d}t, \qquad x > 0,$$

where $p$ is non-decreasing, right continuous on $\mathbb{R}_+$ with $p(0+) = 0$, $p(\infty) = \infty$, $p(t) > 0$ if $t > 0$, and

$$\lim_{x\to 0+} \frac{\varphi(x)}{x} = \lim_{x\to\infty} \frac{x}{\varphi(x)} = 0.$$

The complementary function $\widetilde{\varphi}$ of $\varphi$ is given by $\widetilde{\varphi}(y) = \sup\limits_{x>0}\{xy - \varphi(x)\}$.

Recall also that a function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the $\Delta_2$ condition ($\varphi \in \Delta_2$), if there is a constant $C > 1$, such that $\varphi(2x) \leqslant C\varphi(x)$ for all $x > 0$.

Let $v$ be a weight function, $V(x) = \int_0^x v$, such that $V(\infty) = \infty$, then (see [6]) for any $N$-function $\varphi$

$$\int_0^\infty \varphi\left[\frac{1}{V(x)} \int_0^x fv\right] v(x)\,\mathrm{d}x \leqslant \int_0^\infty \varphi[Cf(x)]v(x)\,\mathrm{d}x \tag{1.2}$$

holds, if and only if $\widetilde{\varphi} \in \Delta_2$. (For other weighted generalizations of this type see [7]).

If $\varphi$ is not an $N$-function, then modular inequalities of the form (1.2) (with $v \equiv 1$) still hold. For example, if $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ and $\varphi(0+) = 0$ then (see [1])

$$\int_0^\infty \varphi[(Pf)(x)]\,\mathrm{d}x \leqslant C_1 \int_0^\infty \varphi(C_2 f(x))\,\mathrm{d}x \tag{1.3}$$

is satisfied, if and only if, for all $t > 0$

$$t\int_0^t \frac{\varphi(y)}{y^2}\,\mathrm{d}y \leqslant 2C_1\varphi(2C_2 t)$$

holds.

In this and the previous result the functions $\varphi$ are increasing and satisfy $\varphi(0+) = 0$. However, it is well known that (1.1) holds also for the negative index $p$. In fact, the function $\varphi(x)=x^\alpha$, $\alpha<0$ is neither an $N$-function, nor does it satisfy the condition of the previous result, yet (1.3) holds for this function. This follows from a general result of Levinson ([8]), who proved in 1964 that if $\varphi\colon \mathbb{R}_+\to\mathbb{R}_+$ satisfies

$$\varphi(x)\varphi''(x)\geqslant\Big(1-\frac{1}{p}\Big)[\varphi'(x)]^2,\qquad x>0 \tag{1.4}$$

for some $p>1$, then modular inequalities of the form (1.3) with $C_1=(p/(p-1))^p$ and $C_2=1$ hold.

The object of this paper is to extend Levinson's results to modular inequalities involving weights and to consider operators more general than the averaging operator. Specifically we prove (1.2) where $\varphi$ is not necessarily an $N$-function but satisfies a condition similar, but more general than (1.4).

Recall that the limiting case $p\to\infty$ in (1.1) yields the inequality

$$\int_0^\infty \exp((Pg)(x))\,\mathrm{d}x\leqslant \mathrm{e}\int_0^\infty \exp(g(x))\,\mathrm{d}x \tag{1.5}$$

due to Knopp ([4, Thm. 2.50]) (actually the inequality is in the form $\exp g(x)=f(x)$), and various weighted generalizations of (1.5) are known (cf. [5], [10]). Here we shall show via a simple adaptation of the method in [8] that the function $\varphi(x)=\exp x$ in the weighted analogue of (1.5) can be replaced by any convex function $\varphi$ satisfying (1.4) with $p=\infty$.

If $0<p\leqslant 1$ then it is known that (1.1) fails unless the $L^p$-spaces on the right are replaced by the Hardy spaces $H_p$. If however the range of integration is finite, then such inequalities hold, where the constant depends on the length of the interval.

In the next section, we begin with a characterization of functions for which such "finite" Hardy inequalities hold. This result is an adaptation of Kolmolgorov's principle ([2]) and seems to be new. Following this, weighted extension of Levinson's modular inequalities are given. The final section contains weighted modular inequalities for the two-dimensional and $n$-dimensional case.

## 2. A modular characterization and weighted extensions of Levinson's modular inequality

The first result is a characterization for the averaging operator over finite intervals $(0,\ell)$ $\ell>0$, via an adaptation of Kolmolgorov's principle ([2]).

**Theorem 2.1.** *Suppose $\varphi\colon \mathbb{R}_+\to\mathbb{R}_+$ is increasing, differentiable, and satisfies $\varphi(0+)=0$, $\lim\limits_{x\to\infty}\frac{\varphi(x)}{x}=0$. Then*

$$\int_0^\ell \varphi(Pf(x))\,\mathrm{d}x\leqslant C\ell\psi\Big(\frac{1}{\ell}\int_0^\ell f\Big),\qquad f\geqslant 0 \tag{2.1}$$

*is satisfied for all $\ell \in (0,\infty)$, if and only if, for each $t > 0$*

$$t\int_t^\infty \frac{\varphi(y)}{y^2}\,\mathrm{d}y \leqslant C\psi(t). \tag{2.2}$$

Here the constants in (2.1) and (2.2) are the same.

P r o o f. If $\chi_{(0,\ell)}$ denotes the characteristic function of the interval $(0,\ell)$ then the distribution function of $\chi_{(0,\ell)}Pf$ is

$$\lambda_{\chi_{(0,\ell)}Pf}(y) = \left|\left\{x \in (0,\infty)\colon\ \frac{\chi_{(0,\ell)}(x)}{x}\int_0^x f > y\right\}\right| \leqslant \min\Bigl(\ell, \frac{\|f\|_1}{y}\Bigr),$$

where $|E|$ denotes Lebesgue measure of $E$ and $\|f\|_1 = \int_0^\ell f$. Applying Fubini's theorem, then for any $s > 0$

$$\begin{aligned}\int_0^\ell \varphi((Pf)(x))\,\mathrm{d}x &= \int_0^\infty \varphi'(y)\lambda_{\chi_{(0,\ell)}Pf}(y)\,\mathrm{d}y\\ &\leqslant \ell\int_0^s \varphi'(y)\,\mathrm{d}y + \|f\|_1\int_s^\infty \frac{\varphi'(y)}{y}\,\mathrm{d}y.\end{aligned}$$

But as a function of $s$, the minimum of the right side occurs at $s = \|f\|_1/\ell$, so that

$$\begin{aligned}\int_0^\ell \varphi((Pf)(x))\,\mathrm{d}x &\leqslant \ell\varphi\Bigl(\frac{\|f\|_1}{\ell}\Bigr) + \|f\|_1\left[\frac{\varphi(y)}{y}\Big|_{\|f\|_1/\ell}^{\infty} + \int_{\|f\|_1/\ell}^\infty \frac{\varphi(y)}{y^2}\,\mathrm{d}y\right]\\ &= \ell\,\frac{\|f\|_1}{\ell}\int_{\|f\|_1/\ell}^\infty \frac{\varphi(y)}{y^2}\,\mathrm{d}y \leqslant C\ell\psi\Bigl(\frac{\|f\|_1}{\ell}\Bigr).\end{aligned}$$

Here (2.2) was applied with $t = \|f\|_1/\ell$. Therefore (2.2) implies (2.1).

Conversely, let $f(y) = \ell\chi_{(0,t)}(y)$, $t > 0$ fixed, in (2.1) then $(Pf)(x) = \frac{\ell}{x}\min(x,t)$, so that for $t < \ell$

$$\int_0^t \varphi(\ell)\,\mathrm{d}x + \int_t^\ell \varphi\Bigl(\frac{\ell t}{x}\Bigr)\,\mathrm{d}x \leqslant C\ell\psi(t)$$

or via the change of variable $t\ell/x = y$

$$t\varphi(\ell) + \ell t\int_t^\ell \frac{\varphi(y)}{y^2}\,\mathrm{d}y = t\ell\left[\frac{\varphi(\ell)}{\ell} + \int_t^\ell \frac{\varphi(y)}{y^2}\,\mathrm{d}y\right] \leqslant C\ell\psi(t).$$

But since (2.1) holds for all $\ell > 0$, then the condition $\frac{\varphi(\ell)}{\ell} \to 0$ as $\ell \to \infty$, shows that this implies (2.2). $\square$

Note that $\varphi(t) = t^\alpha$, $0 < \alpha < 1$, satisfies the conditions of the theorem with $\psi(t) = t^\alpha$ and $C = 1/(1-\alpha)$, so that (2.1) holds in this case.

Another example is obtained by taking $\varphi(t) = \log(1+t)$, $t > 0$, then (2.1) holds with $C = 1$ and $\psi(t) = \log(1+t) + t\log\left(\frac{1+t}{t}\right)$.

To give the weighted extensions of Levinson's modular inequality we require the following:

**Definition 2.1.** A function $\varphi\colon (a,b) \to \mathbb{R}_+$, $0 \leqslant a < b \leqslant \infty$, belongs to the class $\Phi_p$, $p > 1$, if

$$\varphi(x)\varphi''(x) \geqslant (1-1/p)[\varphi'(x)]^2$$

holds for all $x > 0$. If $p = \infty$ we write $\Phi_\infty = \Phi$.

Clearly $\Phi \subset \Phi_p$, and since $\varphi(x) = x^q$, $q \geqslant p$ is in $\Phi_p$ but not in $\Phi$, the inclusion is strict.

Note that $\varphi(t) = t^{-a}$, $a > 0$, $\varphi(t) = \mathrm{e}^{-t^a}$, $0 < a \leqslant 1$, $\varphi(t) = \mathrm{e}^{t^b}$, $b \geqslant 1$ are in the class $\Phi$ and hence in $\Phi_p$ for any $p > 1$. However, if in the last two examples $a > 1$, respectively, $0 < b < 1$ then neither of these functions is in $\Phi_p$ for any $p > 1$.

**Lemma 2.1.**

(i) *If $\varphi \in \Phi_p$, $p > 1$, then $\psi = \varphi^{1/p}$ is convex.*

(ii) *If $\varphi \in \Phi$, then $\psi = \log\varphi$ is convex.*

Proof. We only need to show that $\psi'' \geqslant 0$. Since for (i) $p(p-1)\psi^{2p-2}(\psi')^2 + p\psi^{2p-1}\psi'' = \varphi\varphi'' \geqslant (1-1/p)p^2\psi^{2p-2}(\psi')^2$ and for (ii), $\psi'' = \frac{\varphi\varphi''-(\varphi')^2}{(\varphi)^2}$ if $\varphi \in \Phi$, it follows in each case that $\psi'' \geqslant 0$. □

In what follows, we shall always assume that domain of $\varphi$ is so chosen that it contains the range of $f \geqslant 0$.

The weighted extension of Levinson's result [8] is the following:

**Theorem 2.2.**

(i) *If $\varphi \in \Phi_p$, $p > 1$, and $\sup\limits_{r>0}\left(\int_r^\infty \frac{u(t)}{t^p}\,\mathrm{d}t\right)^{1/p}\left(\int_0^r v(t)^{1-p'}\,\mathrm{d}t\right)^{1/p'} < \infty$, then*

$$\int_0^\infty u\varphi(Pf) \leqslant C\int_0^\infty v\varphi(f). \tag{2.3}$$

(ii) *If $\varphi \in \Phi$, and $v(x) = x^\alpha\int_x^\infty \frac{u(t)}{t^{\alpha+1}}\,\mathrm{d}t$, $\alpha > 0$ then (2.3) holds with $C = \mathrm{e}^\alpha$.*

P r o o f. (i) By Lemma 2.1 (i), $\psi=\varphi^{1/p}$ is convex, hence by Jensen's inequality and the well known weight conditions of Muckenhoupt (cf. [9]) we obtain

$$\begin{aligned}\int_0^\infty u(x)\varphi\Big(\frac{1}{x}\int_0^x f\Big)\,\mathrm{d}x &\leqslant \int_0^\infty u(x)\Big[\frac{1}{x}\int_0^x \psi(f(t))\,\mathrm{d}y\Big]^p\,\mathrm{d}x\\ &\leqslant C\int_0^\infty v(x)\psi(f(x))^p\,\mathrm{d}x\\ &= C\int_0^\infty v(x)\varphi(f(x))\,\mathrm{d}x,\end{aligned}$$

which proves the first part.

For the second part we apply Lemma 2.1 (ii), Jensen's inequality and the fact that

$$-\alpha=\int_0^1 \log y^\alpha\,\mathrm{d}y,\qquad \alpha>0.$$

Interchanging the order of integration twice (Fubini) and making obvious changes of variables yields

$$\begin{aligned}\int_0^\infty u\varphi(Pf) &= \int_0^\infty u(x)\exp\Big[\psi\Big(\frac{1}{x}\int_0^x f\Big)\Big]\,\mathrm{d}x\\ &\leqslant \int_0^\infty u(x)\exp\Big[\frac{1}{x}\int_0^x \psi(f(t))\,\mathrm{d}t\Big]\,\mathrm{d}x\\ &= \int_0^\infty u(x)\exp\Big[\frac{1}{x}\int_0^x \log\varphi(f(t))\,\mathrm{d}t\Big]\,\mathrm{d}x\\ &= \int_0^\infty u(x)\exp\Big[\int_0^1 \log\varphi(f(xy))\,\mathrm{d}y\Big]\,\mathrm{d}x\\ &= \mathrm{e}^\alpha\int_0^\infty \exp\Big[\int_0^1 \log[y^\alpha\varphi(f(xy))]\,\mathrm{d}y\Big]\,\mathrm{d}x\\ &\leqslant \mathrm{e}^\alpha\int_0^\infty u(x)\int_0^1 y^\alpha\varphi(f(xy))\,\mathrm{d}y\,\mathrm{d}x\\ &= \mathrm{e}^\alpha\int_0^1 y^\alpha\int_0^\infty u(x)\varphi(f(xy))\,\mathrm{d}x\,\mathrm{d}y\\ &= \mathrm{e}^\alpha\int_0^1 y^{\alpha-1}\int_0^\infty u\Big(\frac{t}{y}\Big)\varphi(f(t))\,\mathrm{d}t\,\mathrm{d}y\\ &= \mathrm{e}^\alpha\int_0^\infty \varphi(f(t))\int_0^1 y^{\alpha-1}u\Big(\frac{t}{y}\Big)\,\mathrm{d}y\,\mathrm{d}t\\ &= \mathrm{e}^\alpha\int_0^\infty \varphi(f(t))\Big[t^\alpha\int_t^\infty \frac{u(x)}{x^{\alpha+1}}\,\mathrm{d}x\Big]\,\mathrm{d}t\end{aligned}$$

from which the result follows. □

If $u=v=1$ in the first part of the theorem, then the constant in (2.1) is $[p/(p-1)]^p$ and we obtain the result in [8]. If $u(x)=1$ then $v(x)=\frac{1}{\alpha}$, and since $\min \alpha^{-1}\mathrm{e}^{\alpha}=\mathrm{e}$, then with $\varphi(x)=\mathrm{e}^x$ we obtain from the second part of the theorem a result of Knopp [4, Thm. 2.50].

Note that if $\varphi$ is monotone, so that $\varphi^{-1}$ exists, then we obtained

$$\int_0^\infty u(x)\varphi\Big(\frac{1}{x}\int_0^x \varphi^{-1}(f)\Big)\,\mathrm{d}x \leqslant \mathrm{e}^{\alpha}\int_0^\infty v(x)f(x)\,\mathrm{d}x.$$

Such inequalities when $\varphi(x)=\mathrm{e}^x$ were studied in [5] and [10] where weight characterizations were also given.

It is clear that the argument of Theorem 2.2 carries over to more general positive integral operators of the form

$$(Kf)(x)=\int_0^\infty k(x,y)f(y)\,\mathrm{d}y \qquad f\geqslant 0$$

where the kernel satisfies certain natural conditions.

**Corollary 2.1.** *Suppose $\varphi\in\Phi_p$, $p>1$ and the kernel $k$ is homogeneous of degree $-1$. If*

$$\int_0^\infty k(x,y)\,\mathrm{d}y=1 \quad \text{and} \quad \int_0^\infty k(1,t)t^{-1/p}\,\mathrm{d}t=C<\infty,$$

*then*

$$\int_0^\infty \varphi[(Kf)(x)]\,\mathrm{d}x \leqslant C^p\int_0^\infty \varphi(f(x))\,\mathrm{d}x.$$

Proof. Since $\psi=\varphi^{1/p}$ is convex by Lemma 2.1, then Jensen's inequality yields

$$\psi[(Kf)(x)]\leqslant \int_0^\infty k(x,y)\psi(f(y))\,\mathrm{d}y=\int_0^\infty k(1,t)\psi(f(xt))\,\mathrm{d}t.$$

Integrating and applying Minkowski's inequality shows that

$$\begin{aligned}\int_0^\infty \psi[(Kf)(x)]\,\mathrm{d}x &= \int_0^\infty [\psi((Kf)(x))]^p\,\mathrm{d}x \leqslant \int_0^\infty \Big[\int_0^\infty k(1,t)\psi(f(xt))\,\mathrm{d}t\Big]^p\,\mathrm{d}x\\ &\leqslant \Big\{\int_0^\infty k(1,t)\Big(\int_0^\infty \psi(f(xt))^p\,\mathrm{d}x\Big)^{1/p}\mathrm{d}t\Big\}^p\\ &= \Big\{\int_0^\infty t^{-1/p}k(1,t)\Big(\int_0^\infty \psi(f(y))^p\,\mathrm{d}y\Big)^{1/p}\mathrm{d}t\Big\}^p\\ &= C^p\int_0^\infty \varphi(f(y))\,\mathrm{d}y.\end{aligned}$$

If $\alpha>0$, then the kernel $k(x,y)=\alpha x^{-\alpha}(x-y)^{\alpha-1}\chi_{(0,x)}(y)$ satisfies the conditions of the Corollary, and hence the result holds in this case.

A modification of the conditions on the kernel provides also results for $\varphi\in\Phi$. □

It is reasonable to assume that Levinson's method carries over to other than $L^p$-mappings of operators. This is indeed possible if one generalizes the $\Phi_p$ classes of Definition 2.1. To do this, note that if $P$ is an $N$-function, then $P$ is convex and hence $P'' \geqslant 0$ exists a.e. We require however that for an $N$-function $P''$ exists everywhere.

**Definition 2.2.** Let $P$ be an $N$-function, such that $P''(x) \geqslant 0$, $x \in \mathbb{R}_+$, and that $\widetilde{P} \in \Delta_2$. We say that the function

$$\varphi\colon\ (a,b) \to \mathbb{R}_+ \qquad 0 \leqslant a < b \leqslant \infty,$$

belongs to $\Phi_\gamma^N$ if

$$\varphi(x)\varphi'' \geqslant \gamma[\varphi'(x)]^2 \tag{2.4}$$

is satisfied, where $\gamma = \sup\limits_{0<x} \frac{P''(x)P(x)}{[P'(x)]^2}$.

If $\gamma = 1$ we write $\varphi \in \Phi^N$.

Note that if $P(x) = x^p/p$, $p > 1$, then $\widetilde{P}(x) = x^{p'}/p' \in \Delta_2$ and

$$P''(x)P(x)/[P'(x)]^2 = 1 - \tfrac{1}{p},$$

so this generalized class contains the class of Definition 2.1. Also the $N$-function given by

$$P(x) = \int_0^x (\mathrm{e}^t - 1)\,\mathrm{d}t$$

satisfies Definition 2.2. In fact a straightforward calculation shows that the complementary function $\widetilde{P}$ of $P$ is

$$\widetilde{P}(x) = \int_0^x \ln(y+1)\,\mathrm{d}y = (x+1)\ln(x+1) - x,$$

which is clearly in $\Delta_2$. Moreover, since $\sup\limits_{x>0} \frac{P''(x)P(x)}{[P'(x)]^2} = 1$ it follows that $P \in \Phi^N$.

A generalization of Theorem 2.2 (i) is the following:

**Theorem 2.3.** *Suppose $\varphi \in \Phi_\gamma^N$ and $v$ a weight function and $V(x) = \int_0^x v$ where $V(\infty) = \infty$. Then*

$$\int_0^\infty \varphi\left[\frac{1}{V(x)}\int_0^x fv\right] v(x)\,\mathrm{d}x \leqslant \int_0^\infty \varphi(Cf(x))v(x)\,\mathrm{d}x.$$

P r o o f. First we show that $\psi = P^{-1}(\varphi)$ is convex. Since $\varphi = P(\psi)$, $\varphi' = P'(\psi)\psi'$; $\varphi'' = P''(\psi)(\psi')^2 + P'(\psi)(\psi'')$, so (2.4) is satisfied if and only if

$$P(\psi)\left[(\psi')^2 P''(\psi) + P'(\psi)\psi''\right] \geqslant \gamma\left[P'(\psi)\psi'\right]^2$$

is satisfied. But this is equivalent to

$$P'(\psi)\psi'' \geqslant \frac{P'(\psi)^2(\psi')^2}{P(\psi)}\Big[\gamma - \frac{P''(\psi)P(\psi)}{[P'(\psi)]^2}\Big].$$

But since the right side is non-negative it follows that $\psi'' \geqslant 0$.

Now since $\psi$ is convex, Jensen's inequality yields

$$\begin{aligned}\int_0^\infty \varphi\left[\frac{1}{V(x)}\int_0^x fv\right]v(x)\,\mathrm{d}x &= \int_0^\infty P\left[\psi\Big(\frac{1}{V(x)}\int_0^x fv\Big)\right]v(x)\,\mathrm{d}x \\ &\leqslant \int_0^\infty P\left[\frac{1}{V(x)}\int_0^x \psi(f(t))v(t)\,\mathrm{d}t\right]\mathrm{d}x \leqslant \int_0^\infty P[C\psi(x)]v(x)\,\mathrm{d}x \\ &\leqslant \int_0^\infty P\psi(Cf(x))v(x)\,\mathrm{d}x = \int_0^\infty \varphi(Cf(x))v(x)\,\mathrm{d}x.\end{aligned}$$

Since $C > 1$ (w.l.g.) the last inequality follows from the convexity of $\psi$ and the second to last inequality is (1.2) of [6]. (Note that the characterization of (1.2) requires $V(\infty) = \infty$ only in the necessity part.) □

## 3. Higher dimensional cases

Higher dimensional analogues of the previous results can be given. Here we consider first the two dimensional case, where the two dimensional averaging operator is defined by

$$(P_2 f)(x,y) = \frac{1}{xy}\int_0^x \int_0^y f(s,t)\,\mathrm{d}t\,\mathrm{d}s \qquad x,y > 0,\ f \geqslant 0.$$

Before proving the analogue of Theorem 2.2, the following special case of a theorem of Sawyer [11] is required.

**Theorem 3.1.** *Let $u$ and $v$ be weight functions on $\mathbb{R}^2_+ = (0,\infty)\times(0,\infty)$. Let $1 < p < \infty$, then*

$$\int_{\mathbb{R}^2_+} u[(P_2 f)]^p \leqslant C \int_{\mathbb{R}^2_+} v f^p \tag{3.1}$$

*is satisfied, if and only if, for all* $a > 0$, $b > 0$

$$\left(\int_a^\infty \int_b^\infty \frac{u(x,y)}{(xy)^p}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/p}\left(\int_0^a \int_0^b v(x,y)^{1-p'}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/p'} \leqslant A < \infty \tag{3.2}$$

$$\int_0^a \int_0^b \left(\int_0^x \int_0^y v^{1-p'}\right)^p \frac{u(x,y)}{(xy)^p}\,\mathrm{d}y\,\mathrm{d}x \leqslant A \int_0^a \int_0^b v^{1-p'} \tag{3.3}$$

*and*

(3.4)

$$\int_a^\infty \int_b^\infty \left(\int_x^\infty \int_y^\infty \frac{u(s,t)}{(st)^p}\,\mathrm{d}t\,\mathrm{d}s\right)^{p'} v(x,y)^{1-p'}\,\mathrm{d}y\,\mathrm{d}x \leqslant A \int_a^\infty \int_b^\infty \frac{u(x,y)}{(xy)^p}\,\mathrm{d}y\,\mathrm{d}x$$

*are satisfied.*

The corresponding modular inequality is now the following:

**Theorem 3.2.**

(i) *If* $\varphi \in \Phi_p$, $p > 1$, *and* (3.2), (3.3), (3.4) *hold, then*

$$\int_{\mathbb{R}_+^2} u\varphi(P_2 f) \leqslant C \int_{\mathbb{R}_+^2} v\varphi(f) \tag{3.5}$$

*is satisfied.*

(ii) *If* $\varphi \in \Phi$, *then* (3.5) *holds with* $C = \mathrm{e}^{\alpha+\beta}$, $\alpha > 0$, $\beta > 0$ *and*

$$v(x,y) = x^\alpha y^\beta \int_x^\infty \int_y^\infty \frac{u(s,t)\,\mathrm{d}t\,\mathrm{d}s}{s^{\alpha+1}t^{\beta+1}}.$$

P r o o f. The proof of (i) follows as in Theorem 2.2 (i) only now Theorem 3.1 is applied. We omit the details.

The proof of (ii) is also as in the proof of Theorem 2.2 (ii) and is given here for completeness.

By Lemma 2.1, $\psi = \log\varphi$ is convex. Hence by Jensen's inequality and changes of variables

$$\begin{aligned}\int_{\mathbb{R}_+^2} u\varphi(P_2 f) &= \int_0^\infty \int_0^\infty u(x,y)\exp[\psi(P_2 f)(x,y)]\,\mathrm{d}y\,\mathrm{d}x \\ &\leqslant \int_0^\infty \int_0^\infty u(x,y)\exp\left[\frac{1}{xy}\int_0^x \int_0^y (\psi f)(s,t)\,\mathrm{d}t\,\mathrm{d}s\right]\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^\infty \int_0^\infty u(x,y)\exp\left[\int_0^1 \int_0^1 (\psi f)(x\sigma,y\tau)\,\mathrm{d}\tau\,\mathrm{d}\sigma\right]\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

But since $-\delta = \int_0^1 \log t^\delta \, \mathrm{d}y$, $\delta > 0$, then with $\delta = \alpha$ and $\delta = \beta$ the last integral is equal to

$$\begin{aligned}
&\mathrm{e}^{\alpha+\beta} \int_0^\infty \int_0^\infty u(x,y) \exp\left[\int_0^1 \int_0^1 \log[\sigma^\alpha \tau^\beta (\varphi f)(x\sigma, y\tau)] \, \mathrm{d}\tau \, \mathrm{d}\sigma\right] \mathrm{d}y \, \mathrm{d}x \\
&\quad \leqslant \mathrm{e}^{\alpha+\beta} \int_0^\infty \int_0^\infty u(x,y) \left[\int_0^1 \int_0^1 \sigma^\alpha t^\beta (\varphi f)(x\sigma, y\tau) \, \mathrm{d}\tau \, \mathrm{d}\sigma\right] \mathrm{d}y \, \mathrm{d}x \\
&\quad = \mathrm{e}^{\alpha+\beta} \int_0^1 \int_0^1 \sigma^\alpha \tau^\beta \left[\int_0^\infty \int_0^\infty u(x,y)(\varphi f)(x\sigma, y\tau) \, \mathrm{d}y \, \mathrm{d}x\right] \mathrm{d}\tau \, \mathrm{d}\sigma \\
&\quad = \mathrm{e}^{\alpha+\beta} \int_0^1 \int_0^1 \sigma^{\alpha-1} \tau^{\beta-1} \left[\int_0^\infty \int_0^\infty u\Big(\frac{s}{\sigma}, \frac{t}{\tau}\Big)(\varphi f)(s,t) \, \mathrm{d}t \, \mathrm{d}s\right] \mathrm{d}\tau \, \mathrm{d}\sigma \\
&\quad = \mathrm{e}^{\alpha+\beta} \int_0^\infty \int_0^\infty (\varphi f)(s,t) \left[\int_0^1 \int_0^1 \sigma^{\alpha-1} t^{\beta-1} u\Big(\frac{s}{\sigma}, \frac{t}{\tau}\Big) \, \mathrm{d}\tau \, \mathrm{d}\sigma\right] \mathrm{d}t \, \mathrm{d}s \\
&\quad = \mathrm{e}^{\alpha+\beta} \int_0^\infty \int_0^\infty (\varphi f)(s,t) \left[s^\alpha t^\beta \int_s^\infty \int_t^\infty \frac{u(x,y)}{x^{\alpha+1} y^{\beta+1}} \, \mathrm{d}y \, \mathrm{d}x\right] \mathrm{d}t \, \mathrm{d}s,
\end{aligned}$$

and the result follows. □

Observe that if $\varphi$ is monotone then $\varphi^{-1}$ exists and writing $\varphi(f) = g$ the inequality may be written as

$$\int_0^\infty \int_0^\infty u(x,y) \varphi\left[\frac{1}{xy} \int_0^x \int_0^y (\varphi^{-1} g)(st) \, \mathrm{d}t \, \mathrm{d}s\right] \leqslant \mathrm{e}^{\alpha+\beta} \int_0^\infty \int_0^\infty v(x,y) g(x,y) \, \mathrm{d}y \, \mathrm{d}x.$$

Specifically, if $\varphi(t) = \mathrm{e}^t$, this generalizes a result of [5] and if $u = v = 1$ one obtains the two dimensional analogue of [4, Thm. 250].

If $n > 2$, a corresponding weighted inequality as in Theorem 3.1 is not available. However, if one defines in the $n$-dimensional setting the averaging operator $P$ by

$$(Pf)(x) = \frac{1}{|B(|x|)|} \int_{B(|x|)} f(y) \, \mathrm{d}y, \qquad x \in \mathbb{R}^n, \ f \geqslant 0,$$

where $B(|x|)$ is the $n$-ball in $\mathbb{R}^n$, centred at zero with radius $|x|$ and $|B(|x|)|$ its volume, then modular inequalities are possible. Note that

$$|B(|x|)| = \int_{|y| \leqslant |x|} \mathrm{d}y = \frac{|x|^n |\Sigma_{n-1}|}{n}$$

where $\Sigma_{n-1}$ is the surface of the unit ball in $\mathbb{R}^n$ and $|\Sigma_{n-1}|$ its area.

**Theorem 3.3.**

(i) *If* $\varphi \in \Phi_p$, $p > 1$ *and the weight functions* $U$, $V$ *in* $\mathbb{R}^n$ *satisfy*

$$\sup_{a>0}\left\{\int_{|x|\geqslant a}\frac{U(x)\,\mathrm{d}x}{|B(|x|)|^p}\right\}^{1/p}\left\{\int_{|x|\leqslant a}V(x)^{1-p'}\,\mathrm{d}x\right\}^{1/p'}<\infty$$

*then*

$$\int_{\mathbb{R}^n}U\varphi(Pf)\leqslant C\int_{\mathbb{R}^n}V\varphi(f) \tag{3.6}$$

*holds.*

(ii) *If* $\varphi \in \Phi$ *and* $V(x) = |x|^\alpha \int_{|y|\geqslant|x|}\frac{U(y)\,\mathrm{d}y}{y^{n+\alpha}}$, $\alpha > 0$, *then* (3.6) *holds with* $C = \frac{\mathrm{e}^{\alpha/n}n}{|\Sigma_{n-1}|}$.

Proof. Since part (i) follows as in the proof of Theorem 2.2(i), (but now applying [3, Theorem 2.1]) we only prove part (ii).

By Lemma 2.1 $\psi = \log\varphi$ is convex, so by Jensen's inequality

$$\begin{aligned}\int_{\mathbb{R}^n}U\varphi(Pf)&=\int_{\mathbb{R}^n}U(x)\exp[\psi P(f)(x)]\,\mathrm{d}x\\&\leqslant\int_{\mathbb{R}^n}U(x)\exp\left[\frac{1}{|B(|x|)|}\int_{B(|x|)}(\psi f)(y)\,\mathrm{d}y\right]\mathrm{d}x.\end{aligned}$$

Let $\overline{U}(t) = \int_{\Sigma_{n-1}}U(t\tau)d\tau$, then, since $\psi = \log\varphi$ a change to polar coordinates shows that the last integral is

$$\begin{aligned}&\int_0^\infty\overline{U}(t)t^{n-1}\exp\left[\frac{n}{t^n|\Sigma_{n-1}|}\int_0^t s^{n-1}\int_{\Sigma_{n-1}}\log(\varphi f)(s\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s\right]\mathrm{d}t\\&\quad=\int_0^\infty\overline{U}(t)t^{n-1}\exp\left[\frac{n}{|\Sigma_{n-1}|}\int_0^1 v^{n-1}\int_{\Sigma_{n-1}}\log(\varphi f)(tv\sigma)\,\mathrm{d}\sigma\,\mathrm{d}v\right]\mathrm{d}t\quad(s=tv).\end{aligned}$$

But if $\alpha > 0$, then

$$\mathrm{e}^{-\alpha/n}=\exp\left[\frac{n}{|\Sigma_{n-1}|}\int_0^1 v^{n-1}\int_{\Sigma_{n-1}}\log v^\alpha\,\mathrm{d}\sigma\,\mathrm{d}v\right]$$

so that the previous integral is equal to

$$
\begin{aligned}
&\mathrm{e}^{\alpha/n}\int_0^\infty \overline{U}(t)t^{n-1}\exp\Big[\frac{n}{|\Sigma_{n-1}|}\int_0^1 v^{n-1}\int_{\Sigma_{n-1}}\log[v^\alpha(\varphi f)(tv\sigma)\,\mathrm{d}\sigma\,\mathrm{d}v\Big]\,\mathrm{d}t\\
&\quad\leqslant \frac{\mathrm{e}^{\alpha/n}n}{|\Sigma_{n-1}|}\int_0^\infty \overline{U}(t)t^{n-1}\int_0^1 v^{n-1}\int_{\Sigma_{n-1}} v^\alpha(\varphi f)(tv\sigma)\,\mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}t\\
&\quad= \frac{\mathrm{e}^{\alpha/n}n}{|\Sigma_{n-1}|}\int_0^1 v^{n+\alpha-1}\int_0^\infty t^{n-1}\overline{U}(t)\int_{\Sigma_{n-1}}(\varphi f)(tv\sigma)\,\mathrm{d}\sigma\,\mathrm{d}t\,\mathrm{d}v\\
&\quad= \frac{\mathrm{e}^{\alpha/n}n}{|\Sigma_{n-1}|}\int_0^1 v^{\alpha-1}\int_0^\infty s^{n-1}\overline{U}\Big(\frac{s}{v}\Big)\int_{\Sigma_{n-1}}(\varphi f)(s\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s\,\mathrm{d}v \quad (tv=s)\\
&\quad= \frac{\mathrm{e}^{\alpha/n}n}{|\Sigma_{n-1}|}\int_0^\infty s^{n-1}\Big(\int_{\Sigma_{n-1}}(\varphi f)(s\sigma)\,\mathrm{d}\sigma\Big)\Big(\int_0^1 v^{\alpha-1}\overline{U}\Big(\frac{s}{v}\Big)\,\mathrm{d}v\Big)\,\mathrm{d}s\\
&\quad= \frac{\mathrm{e}^{\alpha/n}n}{|\Sigma_{n-1}|}\int_0^\infty s^{n-1}\Big(\int_{\Sigma_{n-1}}(\varphi f)(s\sigma)\,\mathrm{d}\sigma\Big)\int_s^\infty \frac{s^\alpha}{t^{\alpha+1}}\Big(\int_{\Sigma_{n-1}}U(t\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}t\,\mathrm{d}s\\
&\quad= \frac{\mathrm{e}^{\alpha/n}n}{|\Sigma_{n-1}|}\int_{\mathbb{R}^n}\varphi(f(x))V(x)\,\mathrm{d}x.
\end{aligned}
$$

Here we applied Jensen's inequality again and interchanged the order of integration twice, with obvious changes of variables. □

Again, if $\varphi$ is monotone, so that $\varphi^{-1}$ exist, then with $\varphi(f)=g$ the inequality takes the form

$$
\int_{\mathbb{R}^n} U(x)\varphi(P(\varphi^{-1}g)(x))\,\mathrm{d}x \leqslant \frac{\mathrm{e}^{\alpha/n}n}{|\Sigma_{n-1}|}\int_{\mathbb{R}^n} V(x)g(x)\,\mathrm{d}x
$$

which was obtained in [3, Thm. 4.1] in the case $\varphi(x)=\mathrm{e}^x$.

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