ON SIGNED EDGE DOMINATION NUMBERS OF TREES

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(Received April 18, 2000)

Abstract. The signed edge domination number of a graph is an edge variant of the signed domination number. The closed neighbourhood $N_G[e]$ of an edge e in a graph G is the set consisting of e and of all edges having a common end vertex with e. Let f be a mapping of the edge set $E(G)$ of G into the set $\{-1,1\}$. If $\sum\limits_{x\in N[e]} f(x) \geqslant 1$ for each $e \in E(G)$, then f is called a signed edge dominating function on G. The minimum of the values $\sum\limits_{x\in E(G)} f(x)$, taken over all signed edge dominating function f on G, is called the signed edge domination number of G and is denoted by $\gamma'_s(G)$. If instead of the closed neighbourhood $N_G[e]$ we use the open neighbourhood $N_G(e) = N_G[e] - \{e\}$, we obtain the definition of the signed edge total domination number $\gamma'_{st}(G)$ of G. In this paper these concepts are studied for trees.

The number $\gamma'_s(T)$ is determined for T being a star of a path or a caterpillar. Moreover, also $\gamma'_s(C_n)$ for a circuit of length n is determined. For a tree satisfying a certain condition the inequality $\gamma'_s(T) \geqslant \gamma'(T)$ is stated. An existence theorem for a tree T with a given number of edges and given signed edge domination number is proved.

At the end similar results are obtained for $\gamma'_{st}(T)$.

Keywords: tree, signed edge domination number, signed edge total domination number

MSC 2000: 05C69, 05C05

We consider finite undirected graphs without loops and multiple edges. The edge set of a graph G is denoted by $E(G)$, its vertex set by $V(G)$. Two edges e_1, e_2 of G are called adjacent if they are distinct and have a common end vertex. The open neighbourhood $N_G(e)$ of an edge $e \in E(G)$ is the set of all edges adjacent to e. Its closed neighbourhood $N_G[e] = N_G(e) \vee \{e\}$.

If we consider a mapping $f\colon E(G) \to \{-1,1\}$ and $s \subseteq E(G)$, then we denote $f(s) = \sum\limits_{x\in s} f(x)$.

A mapping $f\colon E(G) \to \{-1,1\}$ is called a signed edge dominating function (or signed edge total dominating function) on G, if $f(N_G[e]) \geqslant 1$ (or $f(N_G(e)) \geqslant 1$

respectively) for each edge $e \in E(G)$. The minimum of the values $f(E(G))$, taken over all signed edge dominating (or all signed edge total dominating) functions f on G, is called the signed edge domination number (or signed edge total domination number respectively) of G. The signed edge domination number was introduced by B. Xu in [1] and is denoted by $\gamma'_s(G)$. The signed edge total domination number of G is denoted by $\gamma'_{st}(G)$.

A signed edge dominating function will be shortly called SEDF, a signed edge total domination function will be called SETDF. The number $\gamma'_s(G)$ is an edge variant of the signed domination number [2].

Remember another numerical invariant of a graph which concerns domination. A subset D of the edge set $F(G)$ of a graph G is called edge dominating in G if each edge of G either is in D, or is adjacent to an edge of D. The minimum number of edges of an edge dominating set in G is called the edge domination number of G and denoted by $\gamma'(G)$.

We shall study $\gamma'_s(G)$ and $\gamma'_{st}(G)$ in the case when G is a tree.

Proposition 1. *Let G be a graph with m edges. Then*

$$\gamma'_s(G) \equiv m \pmod 2.$$

P r o o f. Let f be a SFDF of G such that $\gamma'_s(G) = f(E(G))$. Let m^+ (or m^-) be the number of edges e of G such that $f(e) = 1$ (or $f(e) = -1$ respectively). We have $m = m^+ + m^-$, $\gamma'_s(G) = m^+ - m^-$ and hence $\gamma'_s(G) = m - 2m^-$. This implies the assertion. □

Proposition 2. *Let u, v, w be three vertices of a tree T such that u is a pendant vertex of T and v is adjacent to exactly two vertices u, w. Let f be a* SFDF *on T. Then*

$$f(uv) = f(vw) = 1.$$

P r o o f. We have $N[uv] = \{uv, vw\}$ and $f(N[uv]) = f(uv) + f(vw)$. This implies the assertion. □

Proposition 3. *Let T be a star with m edges. If m is odd, then $\gamma'_s(T) = 1$. If m is even, then $\gamma'_s(T) = 2$.*

P r o o f. In a star all edges are pairwise adjacent and thus $N_T[e] = E(T)$ for each $e \in E(T)$. If f is a SEDF, then $f(E(T)) = f(N_T[e]) \geqslant 1$ and thus $\gamma'_s(T) \geqslant 1$. Let m^- be the number of edges e of T such that $f(e) = -1$; then $f(E(T)) = m - 2m^-$. If m is odd, we may choose a function f such that $m^- = \frac{1}{2}(m-1)$ and then $f(E(T)) = \gamma'_s(T) - 1$. If m is even, the value $m - 2m^-$ is always even; we may choose f such that $m^- = \frac{1}{2}(m-2)$ and then $F(E(T)) = \gamma'_s(T) = 2$. □

Let $e \in E(T)$. The neighbourhood subtree $T_N[e]$ of T is the subtree of T whose edge set is $N_T[e]$ and whose vertex set is the set of all end vertices of the edges of $N_T[e]$. If e is a pendant edge of T, then $T_N[e]$ is the star whose central vertex is the vertex of e having the degree greater than 1; this is the maximal (with respect to subtree inclusion) subtree of T of diameter 2 containing e. In the opposite case $T_N[e]$ is the maximal subtree of T of diameter 3 whose central edge is e. The set of all subtrees $T_N[e]$ for $e \in E(T)$ will be denoted by $\mathcal{T}_N$.

Theorem 1. *Let T be a tree having the property that there exists a subset $\mathcal{T}_0$ of $\mathcal{T}_N$ consisting of edge-disjoint trees whose union is T. Then*

$$\gamma'(T) \leqslant \gamma'_s(T).$$

P r o o f. Let E_0 be the set of edges e such that $T_N[e] \in \mathcal{T}_0$. For each $e \in F_0$ the set $N_T[e]$ is the set of neighbours of e and the union of all these sets is $E(T)$. Thus F_0 is an edge dominating set in T. Therefore $|E_0| \geqslant \gamma'(T)$.

Let $f\colon E(T) \to \{-1, 1\}$ be an SEDF of T such that $f(E(T)) = \gamma'_s(T)$. As the trees from $\mathcal{T}_0$ are pairwise edge-disjoint, we have

$$\gamma'_s(T) = f(E(T)) = \sum_{T' \in \mathcal{T}_0} f(E(T')) = \sum_{e \in \mathcal{E}_0} f(N_T[e]) \geqslant \sum_{e \in E_0} 1 = |E_0| \geqslant \gamma'(T).$$

□

As $\gamma'(T) \geqslant 1$ for every tree T, we have a corollary.

Corollary 1. *Let T have the property from Theorem* 1. *Then*

$$\gamma'_s(T) \geqslant 1.$$

Conjecture. For every tree T we have $\gamma'_s(T) \geqslant 1$.

By the symbol P_m we denote the path of length m, i.e. with m edges and $m+1$ vertices. By C_m we denote the circuit of length m.

Theorem 2. *For the signed edge domination number on a path P_m with $m \geqslant 2$ we have*

$$\begin{aligned}
\gamma'_s(P_m) &= \frac{1}{3}m + 2 && \text{for } m \equiv 0 \pmod 3,\\
\gamma'_s(P_m) &= \frac{1}{3}(m+2) + 2 && \text{for } m \equiv 1 \pmod 3,\\
\gamma'_s(P_m) &= \frac{1}{3}(m+1) + 1 && \text{for } m \equiv 2 \pmod 3.
\end{aligned}$$

P r o o f. Let f be an SEDF on P such that $f(E(P_m)) = \gamma'_s(P_m)$. Denote $E+ = \{e \in E(P_m);\ f(e) = 1\}$, $E^- = \{e \in EP_m;\ f(e) = -1\}$. Evidently each edge of E^- must be adjacent to at least two edges of E^+ and each edge of F^+ is adjacent to at most one edge of E'. By Proposition 2 between an edge of E^- and an end vertex of P_m there are at least two edges of E^+ and also between two edges of E^- there are at least two edges of E^+. Hence $|E'| \leqslant \lfloor \frac{1}{3}(m-2) \rfloor$ and $f(E(P_m)) = |E| - 2|F^-| \geqslant m - 2\lfloor \frac{1}{2}(m-2) \rfloor$. If we choose one end vertex of P_m and number the edges of P_m starting at it, we may choose a function f such that $f(a) = -1$ if and only if the number of e is divisible by 3 and less than $m-1$. The $f(E(P_m)) = m - 2\lfloor \frac{1}{2}(m-2) \rfloor$ and this is $\gamma'_s(P_m)$. And this number treted separately for particular congruence classes modulo 3 can be expressed as in the text of the theorem. □

As an aside, we state an assertion on circuits; its proof is quite analogous to the proof of Theorem 2.

Theorem 3. *For the signed edge domination number of a circuit C_m we have*

$$\gamma'_s(C_m) = \frac{1}{3}m \quad \text{for } m \equiv 0 \pmod 3,$$
$$\gamma'_s(C_m) = \frac{1}{3}(m+2) \quad \text{for } m \equiv 1 \pmod 3,$$
$$\gamma'_s(C_m) = \frac{1}{3}(m+1) + 1 \quad \text{for } m \equiv 2 \pmod 3.$$

Now we shall investigate caterpillars. A caterpillar is a tree C with the property that upon deleting all pendant edges from it a path is obtained: this path is called the body of the caterpillar. Particular cases of caterpillars include stars and paths.

Let the vertices of the body of C be $u_1, \ldots, u_k$ and edges $u_i u_{i+1}$ for $i = 1, \ldots, k-1$. For $i = 1, \ldots, k$ let p_i be the number of pendant edges incident to u_i. The finite sequence $(p_i)_{i=1}^k$ determines the caterpillar uniquely. From the definition it is clear that $p_1 \geqslant 1$ and $p_k \geqslant 1$. If $k = 1$, then such a caterpillar is a star. If $p_1 = p_k = 1$, $p_i = 0$ for $i = 2, \ldots, k-1$, then it is a path.

Theorem 4. *Let $(p_i)_{i=1}^k$ be a finite sequence of integers such that $p_1 \geqslant 2$, $p_k \geqslant 2$, $p_i \geqslant 1$ for $2 \leqslant i \leqslant k-1$. Let k_0 be the number of even numbers among the numbers $p_1 - 1$, $p_2, \ldots, p_{k-1}$, $p_k - 1$. Let C be the caterpillar determined by this sequence. Then $\gamma'_s(C) = k_0 + 1$.*

P r o o f. The assumption of the theorem implies that each vertex of the body of C is incident to at least one pendant edge. For $i = 1, \ldots, k$ let M_i be the set of all edges incident to p_i. Let p_i be a vertex of the body of C and let e be a pendant edge incident to it. We have $N[e] = M_i$.

We have $\bigcup_{i=1}^{k} M_i = E(C)$, $M_i \cap M_{i+1} = \{u_i u_{i+1}\}$, $M_i \cap M_j = \emptyset$ for $|j - i| \geqslant 2$. Hence $f(E(C)) = \sum_{i=1}^{k} f(M_i) - \sum_{i=1}^{k-1} f(\{u_i, u_{i+1}\})$. The function f may be described in the following way. If $i = 1$ or $i = k$, then $f(e) = -1$ for exactly $\frac{1}{2}p_i$ pendant edges from M_i if p_i is even and for exactly $\frac{1}{2}(p_i - 1)$ ones if p_i is odd. If $2 \leqslant i \leqslant k - 1$, then $f(e) = -1$ for exactly $\frac{1}{2}p_i$ pendant edges e from M_i if p is even and for exactly $\frac{1}{2}(p_i + 1)$ ones if p_i is odd. For an edge e from the body of C always $f(e) = 1$. If $i = 1$ or $i = k$, then $f(M_i) = 1$ for p_i even and $f(M_i) = 2$ for p_i odd. If $2 \leqslant i \leqslant k-1$, then $f(M_i) = 1$ for p_i odd and $f(M_i) = 2$ for p_i even. We have $\sum_{i=1}^{k} f(M_i) = k + k_0$, $\sum_{i=1}^{k-1} f(u_i u_{i+1}) = k - 1$, which implies the assertion. □

Our considerations concerning $\gamma_s'(T)$ will be finished by an existence theorem.

Theorem 5. *Let m, g be integers, $1 \leqslant g \leqslant m$, $g \equiv m \pmod 2$. Let $g \neq m$ for m odd and $g \neq m - 2$ for m even. Then there exists a tree T with m edges such that $\gamma_s'(T) = g$.*

Proof. Consider the following tree $T(p, q)$ for a positive integer p and a non-negative integer q. Take a vertex v and p paths of length 2 having a common terminal vertex v and no other common vertex. Denote the set of edges of all these paths by E_1. Further add q edges with a common end vertex v; they form the set E_2. Let f be a SEDF on $T(p, q)$ such that $f(E(T(p, q))) = \gamma_s'(T(p, q))$. We have $f(e) = 1$ for each $e \in E_1$ by Proposition 2. If $q < p$, then $f(e) = -1$ for each $e \in F_2$ and $\gamma_s'(T(p, q)) = 2p - q$. If $q \geqslant p$, then for our purpose it suffices to consider the case when $p + q$ is odd. Then $f(e) = -1$ for $\frac{1}{2}(p + q - 1)$ edges of E_2 and $f(e) = 1$ for the remaining edges. Hence $\gamma_s'(T(p, q)) = p + 1$. Further let $T'(p, q)$ be the tree obtained from $T(p, q)$ by adding a path Q of length 7 with the terminal vertex in v. If $q \leqslant p + 1$, then exactly two edges of Q have the value of a SEDF f equal to -1. Again let f be such a SEDF that $f(T'(p, q)) = \gamma_s'(T'(p, q))$. Further $f(e) = -1$ for all edges $e \in E_2$. Then $\gamma_s'(T'(p, q)) = 2p - q + 3$.

Now return to the numbers m, g and consider particular cases:

Case $3g \leqslant m$: Put $p = g - 1$, $q = m - 2g + 2$. We have $q > p$ and thus $\gamma_s'(T(p, q) = p + 1 = g$. The tree $T(p, q)$ has evidently m edges. The sum $p + q = m - g + 1$ is odd, because $m \equiv g \pmod 2$.

Case $3g > m$, $m + g \equiv 0 \pmod 4$: Put $p = \frac{1}{4}(m + g)$, $q = \frac{1}{2}(m - g)$. Now $q < p$. Again $T(p, q)$ has m edges and $\gamma_s'(T(p, q)) = g$.

C a s e $3g > m$, $m + g \equiv 2 \pmod 4$: Put $p = \frac{1}{4}(m + g - 2) - 2$, $q = \frac{1}{2}(m - g) - 2$. Evidently $q \geqslant 0$ if and only if $g < m - 4$; this is fulfilled if m is even and $g \neq m - 2$ or if m is odd and $g \neq m$. The tree $T'(p, q)$ has m edges and $\gamma'_s(T'(p, q)) = g$. □

Now we shall consider the signed edge total domination number $\gamma'_{st}(T)$ of a tree T. Note that $\gamma'_s(G)$ is well-defined for every graph G with $E(G) \neq \emptyset$; for each edge $e \in E(G)$ we have $N[e] \neq \emptyset$, because $e \in N[e]$. On the contrary if there is a connected component of G isomorphic to K_2 (the complete graph with two vertices) and e is its edge, then $N(e) = \emptyset$ and there exists no SETDF on G. Therefore $\gamma'_{st}(G)$ is defined only for graphs G which have no connected component isomorphic to K_2. If we restrict our considerations to trees, we must suppose that the considered tree T has at least two edges.

Proposition 4. *Let G be a graph with m edges and without a connected component isomorphic to K_2. Then*

$$\gamma'_{st}(G) \equiv m \pmod 2.$$

The proof is quite similar to the proof of Proposition 1.

Proposition 5. *Let G be a graph without a connected component isomorphic to K_2. Let $|N(e)| \leqslant 2$ for some edge $e \in E(G)$. Then $f(x) = 1$ for each $x \in N(e)$.*

The proof is straightforward.

This proposition implies two corollaries.

Corollary 2. *Let P_m be a path of length $m \geqslant 2$. Then $\gamma'_{st}(P_m) = m$.*

Corollary 3. *Let C_m be a circuit of length m. Then $\gamma'_{st}(C_m) = m$.*

Namely, in both cases the unique SETDF is the constant equal to 1.

Theorem 6. *Let T be a star with $m \geqslant 2$ edges. If m is odd, then $\gamma'_{st}(T) = 3$. If m is even, then $\gamma'_{st}(T) = 2$.*

P r o o f. Let f be a SETDF such that $f(E(T)) = \gamma'_{st}(T)$. Evidently there exists at least one edge $e \in E(T)$ such that $f(e) = 1$. We have $E(T) = N(e) \cup \{e\}$ and $\gamma'_{st}(T) = f(E(T)) = f(N(e)) + f(e) \geqslant 1 + 1 = 2$. If m is even, the value 2 can be attained by constructing a SETDF f such that $f(e) = 1$ for $\frac{1}{2}m + 1$ edges e and $f(e) = -1$ for $\frac{1}{2}m - 1$ edges. If m is odd, then, according to Proposition 4, we have $\gamma'_{st}(T) \geqslant 3$. We may construct a SETDF f such that $f(e) = 1$ for $\frac{1}{2}(m + 3)$ edges e and $f(e) = -1$ for $\frac{1}{2}(m - 3)$ edges e. □

We finish again by an existence theorem.

Theorem 7. *Let m, g be integers, $2 \leqslant g \leqslant m$, $g \equiv m \pmod 2$. Then there exists a tree T with m edges such that $\gamma'_{st}(T) = g$.*

Proof. Let Ω be a path of length $g-1$. Let S be a star with $m-g+1$ edges. Let these two trees be disjoint. Identify a terminal vertex of Q with the center v of S: the tree thus obtained will be denoted by T. Let f be a SETDF such that $f(E(T)) = \gamma'_{st}(T)$. By Proposition 5 we have $f(e) = 1$ for each edge e of Q. For each edge e of S the set $N(e)$ consists of $E(S) - \{e\}$ and one edge of Q. We have $f(N(e)) = 1$ if and only if $f(e) = -1$ for exactly $\frac{1}{2}(m-g)$ edges e of S. Then we have $f(E(T)) = \gamma'_{st}(T) = g$. □

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