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DIFFERENTIAL EQUATIONS IN METRIC SPACES

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Abstract. We give a meaning to derivative of a function $u\colon \mathbb{R} \to X$, where X is a complete metric space. This enables us to investigate differential equations in a metric space. One can prove in particular Gronwall's Lemma, Peano and Picard Existence Theorems, Lyapunov Theorem or Nagumo Theorem in metric spaces.

The main idea is to define the tangent space $\mathcal{T}_x X$ of $x \in X$. Let $u, v\colon [0,1) \to X$, $u(0) = v(0)$ be continuous at zero. Then by the definition u and v are in the same equivalence class if they are tangent at zero, that is if

$$\lim_{h \to 0^+} \frac{d(u(h), v(h))}{h} = 0.$$

By $\mathcal{T}_x X$ we denote the set of all equivalence classes of continuous at zero functions $u\colon [0,1) \to X$, $u(0) = x$, and by $\mathcal{T} X$ the disjoint sum of all $\mathcal{T}_x X$ over $x \in X$.

By $u'(t) \in \mathcal{T}_{u(t)} X$, where $u\colon \mathbb{R} \to X$, we understand the equivalence class of a function $[0,1) \ni h \to u(t+h) \in X$. Given a function $\mathcal{F}\colon X \to \mathcal{T} X$ such that $\mathcal{F}(x) \in \mathcal{T}_x X$ we are now able to investigate solutions to the differential equation $u'(t) = \mathcal{F}(u(t))$.

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Introduction

The motivation for defining and investigating differential equations in metric spaces comes from many sources.

Probably the most important are mutational equations, which appeared during the investigation of the evolution of sets (tubes) in vector spaces. From the introduction to [1], which serves also as our leading idea, we read: "While attempting to give a meaning to a differential equation governing the evolution of tubes, it was observed that no specific property of the Hausdorff distance was used, and that the theorems could be formulated and proved in any metric space". Mutational equations are a particular case of a differential equation in the sense understood in this paper.

Nagumo Theorem (see [2]) gives another motivation. Suppose that A is a closed subset of $\mathbb{R}^n$ and that we are given a differential equation $x'(t) = f(x(t))$. Roughly speaking, Nagumo Theorem says that under the assumption that the vector field f is "tangent" to A for every $a \in A$ there exists a solution which begins in a and stays in A. Looking at this result from the local point of view of the set A we see that although A does not have a vector structure, we may consider the differential equation $x'(t) = f(x(t))$ in A.

Still another motivation comes from the differential equations with discontinuous right hand sides. In this case the solutions may change their speed in a non-continuous way. For such equations it is even nontrivial what we understand by a solution. In some situations the notion of the differential equation in a metric space may help to better understand this problem.

Tangent spaces

To explain what we understand by a differential equation in a metric space, we first have to know what is a differential equation on a manifold M. For the sake of simplicity we discuss here the case of autonomous differential equations. To consider a differential equation on a differential manifold we need a vector field $f\colon M \to TM$, $f(x) \in T_xM$, where TM is the tangent space. A solution $u\colon \mathbb{R}_+ \to M$ of $u'(t) = f(u(t))$ is a function which "locally" moves along the vector field.

The above idea can easily be generalized to complete metric spaces. From now on we assume that X is a complete metric space. We know that the tangent space T_xM at $x \in M$ consists of all equivalence classes of differentiable functions $u\colon [0,1) \to M$, $u(0) = x$, where $u, v\colon [0,1) \to M$ are in the same equivalence class if they are tangent at zero, that is if $\lim_{h\to 0^+} \frac{\|u(h)-v(h)\|}{h} = 0$.

We proceed in a similar way in metric spaces (the difference is that we do not have differentiable functions). By $C_0([0,1), X)$ we denote the space of all continuous at zero functions $u\colon [0,1) \to X$.

Definition. Let $u, v \in C_0([0,1), X)$, $u(0) = v(0) =: x$. Then by the definition u and v are in the same equivalence class if they are *tangent at zero*, that is if

$$\lim_{h\to 0^+} \frac{d(u(h), v(h))}{h} = 0.$$

By $\mathcal{T}_xX$ we denote the set of all equivalence classes of functions $u \in C_0([0,1), X)$, $u(0) = x$ and by $\mathcal{T}X$ the disjoint sum of all $\mathcal{T}_xX$ over $x \in X$.

For $u \in C_0([0,1), X)$, by $[u] \in \mathcal{T}_{u(0)}X$ we denote the equivalence class of u. As in the case of differential manifolds the tangent space $\mathcal{T}_xX$ is a metric space with the (extended) metric defined by

$$d_x([u],[v]) := \limsup_{h\to 0^+} \frac{d(u(h),v(h))}{h},$$

where $u,v \in C_0([0,1),X)$, $u(0)=v(0)=x$.

The main difference between the tangent spaces TM and $\mathcal{T}X$ is that we do not have a topology on $\mathcal{T}X$ which would enable us to reasonably measure the distance between elements from different "tangent spaces".

Now we are ready to define the derivative of a continuous function $u\colon I \to X$, where I is a right open subinterval of $\mathbb{R}$.

Definition. By $u'(t) \in \mathcal{T}_{u(t)}X$ we understand the equivalence class of the function $[0,\delta) \ni h \to u(t+h) \in X$, where $\delta > 0$ is such that $[t,t+\delta) \subset I$.

As we have a tangent space and the derivative we can now obviously investigate differential equations. For simplicity we consider here autonomous differential equations.

Let $\mathcal{F}\colon X \to \mathcal{T}X$ be such that $\mathcal{F}(x) \in \mathcal{T}_xX$ for every $x \in X$. We seek continuous solutions $u\colon I \to X$ of

$$u'(t) = \mathcal{F}(u(t)) \quad \text{for } t \in I.$$

Since the function $\mathcal{F}$ goes into the equivalence classes, an easier way to investigate it is by taking its representative. From now on we assume that $F\colon X \to C_0([0,1),X)$ is such that $F(x)(0) = x$ for every $x \in X$. Then the function $[F]\colon X \ni x \to [F(x)] \in \mathcal{T}X$ satisfies $[F](x) \in \mathcal{T}_xX$ for every $x \in X$. Instead of writing that $u\colon I \to X$ is a solution to $u'(t) = [F](u(t))$ we write for brevity that it is a solution to $u'(t) = F(u(t))$.

Given F we use the abbreviation $x\bullet_F h = F(x)(h)$. Thus a continuous u is a solution to $u'(t) = F(u(t))$ iff $\lim\limits_{h\to 0^+} \frac{d(u(t+h),u(t)\bullet_F h)}{h} = 0 \quad$ for $t \in I$.

For a function $u\colon I \to \mathbb{R}$ we denote by $D^+u(t)$ the right upper Dini derivative of u, that is $D^+u(t) := \limsup\limits_{h\to 0^+} \frac{u(t+h)-u(t)}{h}$.

Example 1. We show that differential equations in metric spaces may have no solutions.

Let $X = \mathbb{R}$ and $F(x)(h) := x + \sqrt{h}$. Suppose that $u\colon I \to \mathbb{R}$ is a solution to $u'(t) = F(u(t))$. Then $D^+u(t) = \infty$ for every $t \in I$, which yields a contradiction. The reason why there are no solutions is that the local "speed" of the "vector field" is infinite at every point.

Let us now investigate another example. Let $F(x)(h) := x + h\sin(1/h)$. Then one can easily check that the differential equation $u'(t) = F(u(t))$ has no solutions. The reason which causes the nonexistence in this case is that according to the "vector field" the points should move simultaneously forward and backward.

As we have seen, there are examples when solutions do not exist. However, every semidynamical system is a solution to itself. Namely, let $U\colon \mathbb{R}_+ \times X \to X$ be a semidynamical system. Then for every $x_0 \in X$, the function $u\colon \mathbb{R}_+ \ni t \to U(t, x_0) \in X$ is a solution to the differential equation

$$u'(t) = F(u(t)) \quad \text{for } t \in \mathbb{R}_+,$$

where $F(x)(h) := U(x, h)$.

Gronwall's lemma

From now on we assume that $\mathcal{F}\colon X \to \mathcal{T}X$ is a function such that $\mathcal{F}(x) \in \mathcal{T}_x X$ for every $x \in X$.

One of the most crucial tools in the differential equations is Gronwall's Lemma. To formulate it we need the notion of approximate solutions.

Definition. Let $\varepsilon \geqslant 0$. We say that a continuous function $u\colon I \to X$ is an *ε-solution* to $u'(t) = \mathcal{F}(u(t))$ if

$$d_{u(t)}(u'(t), \mathcal{F}(u(t))) \leqslant \varepsilon \quad \text{for } t \in I.$$

One can easily notice that a 0-solution is simply a solution.

If $F\colon X \to C_0([0,1), X)$ is a representative of $\mathcal{F}$ then $u\colon I \to X$ is an ε-solution iff $\limsup\limits_{h \to 0^+} \frac{d(u(t+h), u(t) \bullet_F h)}{h} \leqslant \varepsilon$ for $t \in I$.

As we have mentioned before it is not possible to define reasonable topology on $\mathcal{T}X$. However, we can compare elements from different tangent spaces. For $u, v \in \mathcal{T}X$ we define

$$d_X(u, v) := \limsup_{h \to 0^+} \frac{d(u_0(h), v_0(h)) - d(u_0(0), v_0(0))}{h},$$

where $u_0, v_0 \in C_0([0,1), X)$ are such that $[u_0] = u$, $[v_0] = v$. Clearly d_X does not satisfy the axioms of a metric. Let us remark that if $u, v \in \mathcal{T}_x X$ then $d_x(u, v) = d_X(u, v)$.

Gronwall's Lemma. *Let $\mathcal{F}_i\colon X \to \mathcal{T}X$, where $i = 1, 2$, be such that $\mathcal{F}_i(x) \in \mathcal{T}_x X$ for every $x \in X$. Let I be a subinterval of $\mathbb{R}$ and let $\mathcal{L}\colon \mathbb{R}_+ \to \mathbb{R}_+$ be such that*

$$d_X(\mathcal{F}_1(x), \mathcal{F}_2(y)) \leqslant \mathcal{L}(d(x, y)) \quad \text{for } x, y \in X.$$

Let $u_i\colon I \to X$ for $i = 1, 2$ be ε_i-solutions to

$$(u_i)'(t) = \mathcal{F}_i(u_i(t)) \quad \text{for } t \in I.$$

Let $r(t) := d(u_1(t), u_2(t))$. Then r is a continuous function which satisfies the differential inequality

$$D^+ r(t) \leqslant \mathcal{L}(r(t)) + \varepsilon_1 + \varepsilon_2 \quad \text{for } t \in I.$$

Proof. Let $F_i\colon X \to C_0([0,1), X)$ be representatives of $\mathcal{F}_i$. We have

$$\begin{aligned} D^+ r(t) &= \limsup_{h \to 0+} \frac{d(u_1(t+h), u_2(t+h)) - d(u_1(t), u_2(t))}{h} \\ &\leqslant \limsup_{h \to 0^+} \frac{d(u_1(t) \bullet_{F_1} h, u_2(t) \bullet_{F_2} h) - d(u_1(t), u_2(t))}{h} + (\varepsilon_1 + \varepsilon_2) \\ &= d_X(\mathcal{F}_1(u_1(t)), \mathcal{F}_2(u_2(t))) + (\varepsilon_1 + \varepsilon_2) \leqslant \mathcal{L}(r(t)) + (\varepsilon_1 + \varepsilon_2). \end{aligned}$$

□

We say that $\mathcal{F}\colon X \to \mathcal{T}X$ is *Lipschitz* with a constant K if

$$d_X(\mathcal{F}(x), \mathcal{F}(y)) \leqslant K d(x, y) \quad \text{for } x, y \in X.$$

As a direct consequence of Gronwall's Lemma we obtain that solutions to Lipschitz differential equations, if they exist, are unique (Example 1 shows that even if $\mathcal{F}$ is Lipschitz the solutions need not exist).

Corollary 1. *Let $\mathcal{F}$ be Lipschitz with a constant K. Let $x, y \in X$ and let $u_x, u_y\colon [0, \delta) \to X$ be solutions to $u'(t) = \mathcal{F}(u(t))$ such that $u_x(0) = x$, $u_y(0) = y$. Then*

$$d(u_x(t), u_y(t)) \leqslant d(x, y)\mathrm{e}^{Kt} \quad \text{for } t \in [0, \delta).$$

Tangency fields

There appears an obvious question what possible assumption on the function F will enforce the existence of solutions. We are going to give a condition which in the case of classical differential equations follows from the continuity of the vector field.

Definition. We say that F is a *tangency field* if there exists a continuous function $L\colon [0,1)\times X^2\to\mathbb{R}_+$ such that $L(0,x,x)=0$ for $x\in X$ and

$$\text{(1)}\qquad \limsup_{h\to 0^+}\frac{d(x\bullet_F h, y\bullet_F(t+h))-d(x,y\bullet_F t)}{h}\leqslant L(t,x,y)\quad\text{for}\quad \begin{matrix} t\in[0,1),\\ x,y\in X.\end{matrix}$$

If we have a particular L in mind then we say that F is an L-tangency field.

At first the above condition may seem to be artificial, however, this is not the case. It tells us that the "vector fields" of near-by points are near.

In the case of differential equations the above condition follows from the assumption that the right hand side of the differential equation $u'(t)=F(u(t))$ is continuous. Namely, let us consider the differential equation $x'(t)=f(x(t))$ where $f\colon\mathbb{R}^n\to\mathbb{R}^n$ is a continuous function. Then for $t\in[0,1)$, $x,y\in X$

$$\limsup_{h\to 0^+}\frac{\|x+hf(x)-(y+(t+h)f(y))\|-\|x-(y+tf(y))\|}{h}\leqslant\|f(x)-f(y)\|.$$

Thus we may put $L(t,x,y):=\|f(x)-f(y)\|$.

Let us now consider the case when $U\colon\mathbb{R}_+\times X\to X$ is a semidynamical system and $F(x)(h):=U(x,h)$. A natural question is when F is a tangency field. The above reasoning shows that if U is given as a solution to a differential equation this is the case. One can also notice that if $U(t,\cdot)$ is a contraction for every $t\in\mathbb{R}_+$ then this is also the case.

By applying the following proposition it is possible to construct, as in the case of classical differential equations, ε-solutions with arbitrary small $\varepsilon>0$.

Proposition 1. *Let F be an L-tangency field. Fix $x\in X$ and $\varepsilon>0$. Then there exists $\delta>0$ such that the function $u\colon[0,\delta)\ni t\to x\bullet_F t\in X$ is an ε-solution to $u'(t)=F(u(t))$.*

Proof. Let $r>0$ and $\delta>0$ be chosen so that

$$\begin{aligned}&x\bullet_F[0,\delta)\subset B(x,r),\\ &\sup\{L(t,y,z)\colon t\in[0,\delta), y,z\in B(x,2r)\}<\varepsilon.\end{aligned}$$

Let $t\in[0,\delta)$ be arbitrary. Then $u(t)\bullet_F h\subset B(x,2r)$ for sufficiently small h and therefore by (1) (with $x=x\bullet_F t$, $y=x$) we get

$$\begin{aligned}\limsup_{h\to 0^+}\frac{d(u(t+h),u(t)\bullet_F h)}{h}&=\limsup_{h\to 0^+}\frac{d(x\bullet_F(t+h),(x\bullet_F t)\bullet_F h)}{h}\\ &\leqslant L(t,x\bullet_F t,x)+\limsup_{h\to 0^+}\frac{d(x\bullet_F t,x\bullet_F t)}{h}\leqslant\varepsilon.\end{aligned}$$

□

The following lemma is essential in the proof that a limit of a convergent sequence of approximate solutions is a solution.

Lemma. *Let F be a tangency field and let $u\colon I \to X$ be an ε-solution to $u'(t) = F(u(t))$ for $t \in I$. Let $U \subset X$ be such that $u(s)\bullet_F(t-s) \subset U$ for $t, s \in I$, $s \leqslant t$. Let*

$$\mathcal{L} := \sup\{L(t,x,y)\colon\ t \in I, x, y \in U\}.$$

Then

$$d(u(t), u(s)\bullet_F(t-s)) \leqslant (t-s)(\mathcal{L}+\varepsilon) \quad \text{for } s, t \in I, s \leqslant t.$$

P r o o f. Let $r_s(t) = d(u(t), u(s)\bullet_F(t-s))$. Then $r_s(s) = 0$ and applying (1) we get

$$\begin{aligned} D^+ r_s(t) &= \limsup_{h\to 0^+} \frac{d(u(t+h), u(s)\bullet_F(t+h-s)) - d(u(t), u(s)\bullet_F(t-s))}{h} \\ &\leqslant \varepsilon + \limsup_{h\to 0^+} \frac{d(u(t)\bullet_F h, u(s)\bullet_F(t+h-s)) - d(u(t), u(s)\bullet_F(t-s))}{h} \leqslant \varepsilon + \mathcal{L}. \end{aligned}$$

□

Proposition 2. *Let $\mathcal{F}$ be a tangency field and let $u_n\colon [0,T) \to X$ be a sequence of ε_n-solutions, with $\varepsilon_n \to 0$. Let $u\colon [0,T) \to X$ be a continuous function such that*

$$\lim_{n\to\infty} u_n(s) = u(s) \quad \text{for } s \in [0,T).$$

Then u is a solution.

P r o o f. Let $s \in [0,T)$ and $\varepsilon > 0$ be arbitrarily chosen. Applying the fact that F is a tangency field one can get $r > 0$ and $d > 0$ such that

$$\sup\{L(t,x,y)\colon\ x, y \in B(x_0, 2r), t \in [0,\delta)\} < \varepsilon,$$

and $v([0,\delta]) \subset B(x_0, 2r)$ for every ε-solution v such that $v(0) \in B(x_0, r)$. Then by Lemma 1 for every $h \in [0,\delta)$ we have

$$d(u(s+h), u(s)\bullet_F h) = \lim_{n\to\infty} d(u_n(s+h), u_n(s)\bullet_F h) \leqslant \varepsilon h,$$

which yields that u is a solution. □

As a consequence of Proposition 2 and the existence of approximate solutions we can prove

Peano and Picard Theorems. *Let $x_0 \in X$ be arbitrary. Let $r > 0$ and $S > 0$ be chosen so that*

$$K := \sup\{L(t,x,y)\colon\ x,y \in B(x_0,R), s,t \in [0,S]\} < \infty,$$
$$x_0 \bullet_F [0,S] \subset B(x_0,r).$$

Assume that either F is a Lipschitz tangency field on $B(x_0,5r)$ or that $B(x_0,5r)$ is compact.

Then for every $x \in B(x_0,r)$ there exists a solution $v\colon\ [0,\min\{S,\frac{r}{K}\}) \to B(x_0,5r)$ such that $v(0)=x$.

In a similar manner we can consider nonautonomous differential equations. Then we are given a function $F\colon\ X \times \mathbb{R}_+ \to \mathcal{T}X$ such that $F(x,t) \in \mathcal{T}_x X$ for every $x \in X$. All the results of the paper can be easily modified to such a case.

Without much difficulty one can prove in metric spaces a version of the Nagumo Theorem (see [2]) or the Lyapunov Theorem.

We can also investigate functional-differential equations. To do so we need a function $F\colon\ C([-1,0],X) \to \mathcal{T}X$ such that $F(\varphi)[0] \in T_{\varphi(0)}X$. Analogously we can define differential inclusions—then we need a set-valued map $V\colon\ X \multimap \mathcal{T}X$ such that $v \in \mathcal{T}_x X$ for every $x \in X$, $v \in V(x)$.

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