127 (2002) MATHEMATICA BOHEMICA No. 3, 437–448

A GALOIS CONNECTION BETWEEN DISTANCE FUNCTIONS AND INEQUALITY RELATIONS

Árpád Száz, Debrecen

(Received October 17, 2000)

Abstract. Following the ideas of R. DeMarr, we establish a Galois connection between distance functions on a set S and inequality relations on $X_S = S \times \mathbb{R}$. Moreover, we also investigate a relationship between the functions of S and X_S.

Keywords: distance functions and inequality relations, closure operators and Galois connections, Lipschitz and monotone functions, fixed points

MSC 2000: 54E25, 06A06, 47H10, 06A15

Introduction

Extending and supplementing some of the results of R. DeMarr [6] we establish a few consequences of the following definitions.

Let S be a nonvoid set, and denote by $\mathcal{D}_S$ the family of all functions d on S^2 such that $0 \leqslant d(p,q) \leqslant +\infty$ for all $p, q \in S$.

Moreover, let $X_S = S \times \mathbb{R}$, and denote by $\mathcal{E}_S$ the family of all relations $\leqslant$ on X_S such that $(p,\lambda) \leqslant (q,\mu)$ implies $\lambda \leqslant \mu$.

If $d \in \mathcal{D}_S$, then for all (p,λ), $(q,\mu) \in X_S$ we define

$$(p,\lambda) \leqslant_d (q,\mu) \iff d(p,q) \leqslant \mu - \lambda.$$

While, if $\leqslant \in \mathcal{E}_S$, then for all $p, q \in S$ we define

$$d_{\leqslant}(p,q) = \inf\{\mu - \lambda \colon\ (p,\lambda) \leqslant (q,\mu)\}.$$

The research of the author has been supported by the grants OTKA T-030082 and FKFP 0310/1997.

Moreover, if f is a function of S into S and $\alpha \in \mathbb{R}$, then for all $(p,\lambda) \in X_S$ we define

$$F(p,\lambda) = (f(p), \alpha\lambda).$$

Concerning the above definitions, for instance, we prove the following statements.

Theorem 1. *The mappings*

$$d \longmapsto \leqslant_d \qquad \text{and} \qquad \leqslant \longmapsto d_{\leqslant}$$

establish a Galois conection between the posets $\mathcal{D}_S$ and $\mathcal{E}_S$ such that every element of $\mathcal{D}_S$ is closed.

Theorem 2. *The family $\mathcal{E}_S^-$ of all closed elements of $\mathcal{E}_S$ consists of all relations $\leqslant \in \mathcal{E}_S$ such that for all (p,λ), $(q,\mu) \in X_S$*
(1) *$(p,\lambda) \leqslant (q,\mu)$ implies $(p,\lambda+\omega) \leqslant (q,\mu+\omega)$ for all $\omega \in \mathbb{R}$;*
(2) *$(p,\lambda) \leqslant (q,\mu)$ if and only if $(p,\lambda) \leqslant (q,\mu+\varepsilon)$ for all $\varepsilon > 0$.*

Theorem 3. *If $d \in \mathcal{D}_S$, then $\leqslant_d$ is a partial order on X_S if and only if d is a quasi-metric on S in the sense that*
(1) *$d(p,p) = 0$ for all $p \in S$;*
(2) *$d(p,q) = 0$ and $d(q,p) = 0$ imply $p = q$;*
(3) *$d(p,r) \leqslant d(p,q) + d(q,r)$ for all $p,q,r \in S$.*

Theorem 4. *For the families of all fixed points of f and F we have*

$$\operatorname{Fix}(F) = \operatorname{Fix}(f) \times \mathbb{R} \quad \textit{if} \quad \alpha = 1 \quad \textit{and} \quad \operatorname{Fix}(F) = \operatorname{Fix}(f) \times \{0\} \quad \textit{if} \quad \alpha \neq 1.$$

Theorem 5. *If $\alpha > 0$ and $d \in \mathcal{D}_S$,then the following assertions are equivalent:*
(1) *$d(f(p), f(q)) \leqslant \alpha d(p,q)$ for all $p,q \in S$;*
(2) *$(p,\lambda) \leqslant_d (q,\mu)$ implies $F(p,\lambda) \leqslant_d F(q,\mu)$.*

Theorem 6. *If $0 < \alpha < 1$ and $d \in \mathcal{D}_S$ is such that d is finite valued, then for any $p,q \in S$ there exist $\lambda_0, \mu_0 \in \mathbb{R}$ with $\lambda_0 \leqslant 0 \leqslant \mu_0$ such that*

$$(p,\lambda) \leqslant_d F(p,\lambda) \leqslant_d F(q,\mu) \leqslant_d (q,\mu)$$

for all $\lambda, \mu \in \mathbb{R}$ with $\lambda \leqslant \lambda_0$ and $\mu_0 \leqslant \mu$.

Remark. From Theorems 3, 5 and 6, by writing $d_{\leqslant}$ instead of d, we can get some similar assertions for the relations $\leqslant \in \mathcal{E}_S^-$. Namely, by Theorem 2, we have $\leqslant = \leqslant_{d_{\leqslant}}$ for all $\leqslant \in \mathcal{E}_S^-$.

The only prerequisites for reading this paper is a knowledge of some basic facts on posets which will be briefly laid out in the next two preparatory sections. The proofs of most of those facts can be found in [10].

1. Closure operations on posets

If $\leqslant$ is a reflexive, antisymmetric and transitive relation on a nonvoid set X, then the relation $\leqslant$ is called a partial order on X, and the ordered pair $X(\leqslant) = (X, \leqslant)$ is called a poset (partially ordered set).

If A is a subset of a poset X, then $\inf_X(A)$ and $\sup_X(A)$ will denote the greatest lower bound and the least upper bound of A in X, respectively. Further, the poset X is called complete if $\inf(A)$ and $\sup(A)$ exist for all $A \subset X$.

The following useful characterization of infimum was already observed by Rennie [9]. However, despite this, it is not included in the standard textbooks.

Lemma 1.1. *If X is a poset, and moreover $A \subset X$ and $\alpha \in X$, then the following assertions are equivalent:*
(1) $\alpha = \inf(A)$*;*
(2) *for each $u \in X$ we have $u \leqslant \alpha$ if and only if $u \leqslant x$ for all $x \in A$.*

Concerning the completeness of posets, according to Birkhoff [1, p. 112] we can at once state

Theorem 1.2. *If X is a poset, then the following assertions are equivalent:*
(1) *X is complete;*
(2) $\inf(A)$ *exists for all $A \subset X$.*

Remark 1.3. To obtain the corresponding results for supremum, one can observe that if $X(\leqslant)$ is a partial ordered set, then its dual $X(\geqslant)$ is also a partial ordered set. Moreover, we have $\inf_{X(\geqslant)}(A) = \sup_{X(\leqslant)}(A)$ for all $A \subset X$.

Definition 1.4. If $-$ is a function of a poset $X(\leqslant)$ into itself such that
(1) $x \leqslant y$ implies $x^- \leqslant y^-$ for all $x, y \in X$,
(2) $x \leqslant x^-$; and (3) $x^- = x^{--}$ for all $x \in X$,
then the function $-$ is called a closure operation on $X(\leqslant)$, and the ordered triple $X(\leqslant, -) = (X, \leqslant, -)$ is called a closure space.

Remark 1.5. Note that the expansivity property (2) already implies that $x^- \leqslant x^{--}$ for all $x \in X$. Therefore, instead of the idempotency property (3), it suffices to assume only that $x^{--} \leqslant x^-$ for all $x \in X$.

The following useful characterization of closure operations was already observed by Everett [3]. However, despite this, it is not included in the standard textbooks.

Lemma 1.6. *If $-$ is a function of a poset X into itself, then the following assertions are equivalent:*
(1) *the function $-$ is a closure operation on X;*
(2) *for all $x, y \in X$ we have $x \leqslant y^-$ if and only if $x^- \leqslant y^-$.*

If X is a closure space, then the members of the family $X^- = \{x^- \colon x \in X\}$ may be called the closed elements of X. Namely, we have

Theorem 1.7. *If X is a closure space and $x \in X$, then the following assertions are equivalent:*
(1) $x^- \leqslant x$;
(2) $x = x^-$;
(3) $x \in X^-$.

Remark 1.8. Note that if X is a closure space, then we have $x^- = \inf\{y \in X^- \colon x \leqslant y\}$ for all $x \in X$. Therefore, the closed elements of X uniquely determine the closure operation of X.

A closure space will be called complete if it is complete as a poset. Concerning the closed elements of complete closure spaces, according to Birkhoff [1, p. 112] we can also state

Theorem 1.9. *If X is a complete closure space, then X^- is a complete poset.*

Remark 1.10. Note that if $A \subset X^-$, then we have $\inf\limits_{X^-}(A) = \inf\limits_{X}(A)$ and $\sup\limits_{X^-}(A) = (\sup\limits_{X}(A))^-$.

2. Galois connections between posets

Definition 2.1. If X and Y are posets and $*$ and $\#$ are functions of X and Y into Y and X, respectively, such that
(1) $x_1 \leqslant x_2$ implies $x_2^* \leqslant x_1^*$ for all $x_1, x_2 \in X$,
(2) $y_1 \leqslant y_2$ implies $y_2^\# \leqslant y_1^\#$ for all $y_1, y_2 \in Y$,
(3) $x \leqslant x^{*\#}$ for all $x \in X$,
(4) $y \leqslant y^{\#*}$ for all $y \in Y$,
then we say that the functions $*$ and $\#$ establish a Galois connection between the posets X and Y.

R e m a r k 2.2. Galois connections between posets were first investigated by Ore [7] and Everett [3].

The following useful characterization of Galois connections was already observed by J. Schmidt [1, p. 124]. However, despite this, it is not included in the standard textbooks.

Lemma 2.3. *If X and Y are posets and $*$ and $\#$ are functions of X and Y into Y and X, respectively, then the following assertions are equivalent:*
(1) *the functions $*$ and $\#$ establish a Galois connection between X and Y;*
(2) *for all $x \in X$ and $y \in Y$ we have $x \leqslant y^{\#}$ if and only if $y \leqslant x^{*}$.*

The following basic theorem has already been established by Ore [7] and Everett [3]. $\blacksquare$

Theorem 2.4. *If the functions $*$ and $\#$ establish a Galois connection between the posets X and Y, then*
(1) $x^{*} = x^{*\#*}$ *for all* $x \in X$ *and* $y^{\#} = y^{\#*\#}$ *for all* $y \in Y$;
(2) *the functions $*\#$ and $\#*$ are closure operations on X and Y, respectively, such that $Y^{\#} = X^{*\#}$ and $X^{*} = Y^{\#*}$;*
(3) *the restrictions of the functions $*$ and $\#$ to $Y^{\#}$ and X^{*}, respectively, are injective, and they are inverses of each other.*

R e m a r k 2.5. Note that actually $A = Y^{\#}$ is the largest subset of X such that the restriction of the function $*$ to A is injective and $A^{*\#} \subset A$.

Definition 2.6. A Galois connection between posets X and Y established by the functions $*$ and $\#$ will be called lower (upper) semiperfect if $x = x^{*\#}$ for all $x \in X$ ($y = y^{\#*}$ for all $y \in Y$).

R e m a r k 2.7. Note that by Definition 2.1 we always have $x \leqslant x^{*\#}$ for all $x \in X$. Therefore, to define the lower semiperfectness of the above Galois connection it suffices to assume the reverse inequality.

The above definition and the following theorem are again due to Ore [7].

Theorem 2.8. *A Galois connection between posets X and Y established by the functions $*$ and $\#$ is lower semiperfect if and only if $X = Y^{\#}$, or equivalently the function $*$ is injective.*

R e m a r k 2.9. Note that if X is a poset, then the Galois connection between the posets $\mathcal{P}(X)$ and $\mathcal{P}(X)$, established by the mappings

$$A \longmapsto \mathrm{lb}\,(A) \qquad \text{and} \qquad A \longmapsto \mathrm{ub}\,(A),$$

where $\mathrm{lb}\,(A)$ and $\mathrm{ub}\,(A)$ are the families of all lower and upper bounds of the set A in X, respectively, is not, in general, lower or upper semiperfect.

The importance of this Galois connection lies mainly in the Dedekind-McNeille completion of the poset X by the cuts $\mathrm{lb}\,(\mathrm{ub}\,(A))$ where $A \subset X$. (See, for instance, [1, p. 126].)

3. A Galois connection between distance functions and inequality relations

Definition 3.1. Let S be a nonvoid set, and denote by $\mathcal{D}_S$ the family of all functions d on S^2 such that $0 \leqslant d(p,q) \leqslant +\infty$ for all $p, q \in S$.

Moreover, let $X_S = S \times \mathbb{R}$, and denote by $\mathcal{E}_S$ the family of all relations $\leqslant$ on X_S such that $(p,\lambda) \leqslant (q,\mu)$ implies $\lambda \leqslant \mu$ for all $(p,\lambda), (q,\mu) \in X_S$.

Remark 3.2. The members of the families $\mathcal{D}_S$ and $\mathcal{E}_S$ will be called distance functions and inequality relations on S and X_S, respectively.

The following theorems do not actually need the nonnegativity of distance functions on S and the corresponding property of inequality relations on X_S.

Theorem 3.3. *The families $\mathcal{D}_S$ and $\mathcal{E}_S$, equipped with the pointwise inequality and the ordinary set inclusion, respectively, are complete posets.*

Hint. If $\mathcal{D} \subset \mathcal{D}_S$, then by defining $d_*(p,q) = \inf_{d \in \mathcal{D}} d(p,q)$ for all $p, q \in S$ we can see that $d_* = \inf(\mathcal{D})$.

On the other hand, if $\mathcal{E} \subset \mathcal{E}_S$, then by defining $\leqslant_* = \bigcap \mathcal{E}$ if $\mathcal{E} \neq \emptyset$ and $\leqslant_* = \bigcup \mathcal{E}_S$ if $\mathcal{E} = \emptyset$ we can see that $\leqslant_* = \inf(\mathcal{E})$. □

Definition 3.4. If $d \in \mathcal{D}_S$, then for all $(p,\lambda), (q,\mu) \in X_S$ we define

$$(p,\lambda) \leqslant_d (q,\mu) \iff d(p,q) \leqslant \mu - \lambda,$$

while if $\leqslant \in \mathcal{E}_S$, then for all $p, q \in S$ we define

$$d_{\leqslant}(p,q) = \inf\{\mu - \lambda\colon\ (p,\lambda) \leqslant (q,\mu)\}.$$

Remark 3.5. The relation $\leqslant_d$, for an ordinary metric d, has formerly been studied by DeMaar [6].

However, the function $d_{\leqslant}$ and the following theorem seem to be completely new.

Theorem 3.6. *The mappings*

$$d \longmapsto \leqslant_d \qquad \text{and} \qquad \leqslant \longmapsto d_{\leqslant}$$

establish a lower semiperfect Galois connection between the posets $\mathcal{D}_S$ and $\mathcal{E}_S$.

Proof. If $d \in \mathcal{D}_S$ and $\leqslant \in \mathcal{E}_S$, then by the corresponding definitions it is clear that $\leqslant_d \in \mathcal{E}_S$ and $d_\leqslant \in \mathcal{D}_S$. Therefore, by Lemma 2.3 and Remark 2.7, it suffices to prove only that $d \leqslant d_\leqslant$ if and only if $\leqslant \subset \leqslant_d$, and moreover $d_{\leqslant_d} \leqslant d$.

If $(p,\lambda),(q,\mu) \in X_S$ are such that $(p,\lambda) \leqslant (q,\mu)$, then by the definition of $d_\leqslant$ we have $d_\leqslant(p,q) \leqslant \mu - \lambda$. Hence, if the inequality $d \leqslant d_\leqslant$ holds, we can infer that $d(p,q) \leqslant \mu - \lambda$. Thus, by the definition of $\leqslant_d$, we also have $(p,\lambda) \leqslant_d (q,\mu)$. Therefore, the inclusion $\leqslant \subset \leqslant_d$ is also true.

Further, if $p,q \in S$ and $\beta \in \mathbb{R}$ are such that $d_\leqslant(p,q) < \beta$, then by the definition of $d_\leqslant$ there exist $\lambda,\mu \in \mathbb{R}$ such that $(p,\lambda) \leqslant (q,\mu)$ and $\mu - \lambda < \beta$. Hence, if the inclusion $\leqslant \subset \leqslant_d$ holds, we can infer that $(p,\lambda) \leqslant_d (q,\mu)$. Thus, by the definition of $\leqslant_d$, we also have $d(p,q) \leqslant \mu - \lambda < \beta$. Hence, letting $\beta \to d_\leqslant(p,q)$, we can infer that $d(p,q) \leqslant d_\leqslant(p,q)$. Therefore, the inequality $d \leqslant d_\leqslant$ is also true.

Finally, if $p,q \in S$ and $\beta \in \mathbb{R}$ are such that $d(p,q) < \beta$, then by the definition of $\leqslant_d$ we have $(p,0) \leqslant_d (q,\beta)$. Hence, by the definition of $d_{\leqslant_d}$, it follows that $d_{\leqslant_d}(p,q) \leqslant \beta$. Hence, letting $\beta \to d(p,q)$, we can infer that $d_{\leqslant_d}(p,q) \leqslant d(p,q)$. Therefore, the inequality $d_{\leqslant_d} \leqslant d$ is also true. □

Remark 3.7. Note that, by Theorem 3.6 and Definition 2.6, we actually have $d = d_{\leqslant_d}$ for all $d \in \mathcal{D}_S$. Therefore, the mapping $\leqslant \longmapsto d_\leqslant$ is onto $\mathcal{D}_S$. Moreover, the mapping $d \longmapsto \leqslant_d$ is injective.

To briefly describe the range of the mapping $d \longmapsto \leqslant_d$ or that of the closure operation $\leqslant \longmapsto \leqslant_{d_\leqslant}$, we shall need the following

Definition 3.8. Denote by $\mathcal{E}_S^-$ the family of all relations $\leqslant \in \mathcal{E}_S$ such that for all $(p,\lambda),(q,\mu) \in X_S$

(1) $(p,\lambda) \leqslant (q,\mu)$ implies $(p,\lambda+\omega) \leqslant (q,\mu+\omega)$ for all $\omega \in \mathbb{R}$;

(2) $(p,\lambda) \leqslant (q,\mu)$ if and only if $(p,\lambda) \leqslant (q,\mu+\varepsilon)$ for all $\varepsilon > 0$.

The appropriateness of the above definition is apparent from

Theorem 3.9. *If $\leqslant \in \mathcal{E}_S$, then the following assertions are equivalent;*

(1) $\leqslant \in \mathcal{E}_S^-$;

(2) $\leqslant = \leqslant_{d_\leqslant}$;

(3) $\leqslant = \leqslant_d$ *for some* $d \in \mathcal{D}_S$.

Proof. Suppose that the assertion (1) holds, and $(p,\lambda),(q,\mu) \in X_S$ are such that $(p,\lambda) \leqslant_{d_\leqslant} (q,\mu)$. Then, by the definition of $\leqslant_{d_\leqslant}$, we have $d_\leqslant(p,q) \leqslant \mu - \lambda$. Therefore, by the definition of $d_\leqslant$, for each $\varepsilon > 0$ there exist $\omega,\tau \in \mathbb{R}$ such that $(p,\omega) \leqslant (q,\tau)$ and $\tau - \omega < \mu - \lambda + \varepsilon$. Hence, by the property 3.8 (2), it follows that $(p, \omega) \leqslant (q, \mu - \lambda + \varepsilon + \omega)$. However, by the property 3.8 (1), this is equivalent to $(p, \lambda) \leqslant (q, \mu + \varepsilon)$. Hence, again by the property 3.8 (2), it follows that $(p, \lambda) \leqslant (q, \mu)$. Therefore, $\leqslant_{d_{\leqslant}} \subset \leqslant$. And now, since the converse inclusion is automatic by Theorem 3.6, the assertion (2) also holds.

Now, since the implication (2)$\Longrightarrow$(3) trivially holds, and the implication (3)$\Longrightarrow$(1) follows immediately from the definition of $\leqslant_d$, the proof is complete. $\square$

Remark 3.10. By Theorem 3.9, it is clear that the Galois connection established in Theorem 3.6 is not upper semiperfect, and the mapping $d \longmapsto \leqslant_d$ is only a partial inverse of the mapping $\leqslant \longmapsto d_{\leqslant}$.

4. Some further properties of the relations $\leqslant_d$ and $d_{\leqslant}$

By using the definition of the relation $\leqslant_d$ we can easily prove the following theorems.

Theorem 4.1. *If $d \in \mathcal{D}_S$, then the following assertions are equivalent:*
(1) *$\leqslant_d$ is reflexive on X_S;*
(2) *$d(p,p) = 0$ for all $p \in S$.*

Remark 4.2. More generally, we can also easily see that a relation $\leqslant \in \mathcal{E}_S$ is reflexive on X_S if and only if $d_{\leqslant}(p,p) = 0$ for all $p \in S$.

Theorem 4.3. *If $d \in \mathcal{D}_S$, then the following assertions are equivalent:*
(1) *$\leqslant_d$ is antisymmetric;*
(2) *$d(p,q) = 0$ and $d(q,p) = 0$ imply $p = q$.*

Hint. If $(p, \lambda) \leqslant_d (q, \mu)$ and $(q, \mu) \leqslant_d (p, \lambda)$, then by the definition of $\leqslant_d$ we have $d(p,q) \leqslant \mu - \lambda$ and $d(q,p) \leqslant \lambda - \mu$. Hence, by using the nonnegativity of d, we can infer that $\lambda = \mu$. Therefore, we actually have $d(p,q) = 0$ and $d(q,p) = 0$. Hence, if the assertion (2) holds, we can infer that $p = q$. Therefore, $(p, \lambda) = (q, \mu)$, and thus the assertion (1) also holds. $\square$

Remark 4.4. Note that the relation $\leqslant_d$ is reflexive (antisymmetric) if and only if its restriction to $S \times \{0\}$ is reflexive (antisymmetric).

Theorem 4.5. *If $d \in \mathcal{D}_S$, then the following assertions are equivalent:*
(1) *$\leqslant_d$ is transitive;*
(2) *$d(p,r) \leqslant d(p,q) + d(q,r)$ for all $p, q, r \in S$.*

H i n t. If $d(p,q) < +\infty$ and $d(q,r) < +\infty$, then by the definition of $\leqslant_d$ we have

$$(p,\,0) \leqslant_d (q, d(p,q)) \quad \text{and} \quad (q, d(p,q)) \leqslant_d (r, d(p,q) + d(q,r)).$$

Hence, if the assertion (1) holds, we can infer that

$$(p,0) \leqslant_d (r, d(p,q) + d(q,r)).$$

Therefore, by the definition of $\leqslant_d$, we also have $d(p,r) \leqslant d(p,q) + d(q,r)$, and thus the assertion (2) also holds. □

R e m a r k 4.6. Now, by using a reasonable modification of the usual definition of quasi-metrics [4, p. 3], we can also state that a function $d \in \mathcal{D}_S$ is a quasi-metric on S if and only if the relation $\leqslant_d$ is a partial order on X_S.

Theorem 4.7. *If $d \in \mathcal{D}_S$, then the following assertions are equivalent:*
(1) *$d(p,q) = d(q,p)$ for all $p, q \in S$;*
(2) *$(p,\lambda) \leqslant_d (q,\mu)$ implies $(q,\lambda) \leqslant_d (p,\mu)$.*

H i n t. If $d(p,q) < +\infty$, then by the definition of $\leqslant_d$ we have

$$(p,0) \leqslant_d (q, d(p,q)).$$

Hence, if the assertion (2) holds, we can infer that $(q,0) \leqslant_d (p, d(p,q))$. Therefore, by the definition of $\leqslant_d$, we also have $d(q,p) \leqslant d(p,q)$. Hence, by changing the roles of p and q, we can see that the converse inequality is also true. Therefore, the assertion (1) also holds. □

R e m a r k 4.8. The latter theorem shows that symmetry is a less natural property of distance functions than the properties considered in the previous three theorems. This may be another reason why quasi-pseudo-metrics are more natural objects than pseudo-metrics.

Note that if d is only an extended real-valued quasi-pseudo-metric on S, then by identifying p with $(p,0)$ for all $p \in S$ we can already get a natural preorder $\leqslant_d$ on S such that for all $p, q \in S$ we have $p \leqslant_d q$ if and only if $d(p,q) = 0$.

Theorem 4.9. *If $d \in \mathcal{D}_S$, then the following assertions are equivalent:*
(1) *$\leqslant_d$ is symmetric;*
(2) *$d(p,q) = +\infty$ for all $p, q \in S$.*

H i n t. If $p, q \in S$ are such that $d(p,q) < +\infty$, then by defining $\mu = d(p,q) + 1$ we have $(p,0) \leqslant_d (q,\mu)$. Hence, if the assertion (1) holds we can infer that $(q,\mu) \leqslant_d (p,0)$. Therefore, we also have $d(q,p) \leqslant -\mu$. Hence, by using the nonnegativity of d, we can infer that $0 < -1$. Therefore, the implication (1)$\Longrightarrow$(2) is true. □

R e m a r k 4.10. Hence, it is clear that the relation $\leqslant_d$ is symmetric if and only if $\leqslant_d = \emptyset$.

5. A relationship between the functions of S and X_S

Definition 5.1. Let f be a function of S into itself, $\alpha \in \mathbb{R}$, and

$$F(p, \lambda) = (f(p), \alpha\lambda)$$

for all $(p, \lambda) \in X_S$.

R e m a r k 5.2. The relationships between the functions f and F have formerly been studied by DeMarr [6].

The following theorems will only extend and supplement some of the observations of the above mentioned author.

Theorem 5.3. *For the families of all fixed points of f and F we have*

$$\operatorname{Fix}(F) = \operatorname{Fix}(f) \times \mathbb{R} \quad \textit{if} \quad \alpha = 1 \quad \textit{and} \quad \operatorname{Fix}(F) = \operatorname{Fix}(f) \times \{0\} \quad \textit{if} \quad \alpha \neq 1.$$

P r o o f. By the corresponding definitions, for any $(p, \lambda) \in X_S$ we have

$$\begin{aligned}(p, \lambda) \in \operatorname{Fix}(F) &\iff F(p, \lambda) = (p, \lambda) \iff (f(p), \alpha\lambda) = (p, \lambda) \iff \\ &\iff f(p) = p \text{ and } \alpha\lambda = \lambda \iff p \in \operatorname{Fix}(f) \text{ and } (\alpha - 1)\lambda = 0.\end{aligned}$$

Consequently, the assertions of the theorem are immediate. □

Under the notation of Definition 5.1, we can also easily prove the following theorems.

Theorem 5.4. *If $\alpha > 0$ and $d \in \mathcal{D}_S$, then the following assertions are equivalent:*
(1) *$d(f(p), f(q)) \leqslant \alpha d(p, q)$ for all $p, q \in S$;*
(2) *$(p, \lambda) \leqslant_d (q, \mu)$ implies $F(p, \lambda) \leqslant_d F(q, \mu)$.*

P r o o f. If $(p, \lambda), (q, \mu) \in X_S$ are such that $(p, \lambda) \leqslant_d (q, \mu)$, then by the definition of $\leqslant_d$ we have $d(p, q) \leqslant \mu - \lambda$. Hence, if the assertion (1) holds, we can infer that $d(f(p), f(q)) \leqslant \alpha\mu - \alpha\lambda$. Therefore, by the definition $\leqslant_d$, we also have $(f(p), \alpha\lambda) \leqslant_d (f(q), \alpha\mu)$. Hence, by the definition of F, it follows that $F(p, \lambda) \leqslant_d F(q, \mu)$. Therefore, the assertion (2) also holds.

On the other hand, if $p, q \in S$ are such that $d(p,q) < +\infty$, then by the definition of $\leqslant_d$ we have $(p,0) \leqslant_d (q, d(p,q))$. Hence, if the assertion (2) holds, we can infer that $F(p,0) \leqslant_d F(q, d(p,q))$. Therefore, by the definition of F, we also have $(f(p),0) \leqslant_d (f(q), \alpha d(p,q))$. Hence, again by the definition of $\leqslant_d$, it follows that $d(f(p), f(q)) \leqslant \alpha d(p,q)$. Therefore, the assertion (1) also holds. □

Theorem 5.5. *If $0 \leqslant \alpha \leqslant 1$ and $d \in \mathcal{D}_S$ is such that $d(p,p) = 0$ for all $p \in S$, then*

$$(p,\lambda) \leqslant_d F(p,\lambda) \leqslant_d F(p,\mu) \leqslant_d (p,\mu)$$

for all $p \in \operatorname{Fix}(f)$ and $\lambda, \mu \in \mathbb{R}$ with $\lambda \leqslant 0 \leqslant \mu$.

Proof. Under the above conditions, we have

$$d(p, f(p)) \leqslant \alpha\lambda - \lambda; \quad d(f(p), f(p)) \leqslant \alpha\mu - \alpha\lambda; \quad d(f(p), p) \leqslant \mu - \alpha\mu.$$

Hence, by the definition of $\leqslant_d$, it follows that

$$(p,\lambda) \leqslant_d (f(p), \alpha\lambda) \leqslant_d (f(p), \alpha\mu) \leqslant_d (p,\mu).$$

Therefore, by the definition of F, the required equalities are also true. □

Theorem 5.6. *If $0 < \alpha < 1$ and $d \in \mathcal{D}_S$ is such that d is finite valued, then for any $p, q \in S$ there exist $\lambda_0, \mu_0 \in \mathbb{R}$ with $\lambda_0 \leqslant 0 \leqslant \mu_0$ such that*

$$(p,\lambda) \leqslant_d F(p,\lambda) \leqslant_d F(q,\mu) \leqslant_d (q,\mu)$$

for all $\lambda, \mu \in \mathbb{R}$ with $\lambda \leqslant \lambda_0$ and $\mu_0 \leqslant \mu$.

Proof. Let $p, q \in S$, and define

$$\lambda_0 = \frac{d(p, f(p))}{(\alpha - 1)} \qquad \text{and} \qquad \mu_0 = \max\left\{\frac{d(f(p), f(q))}{\alpha}, \frac{d(f(q), q)}{(1-\alpha)}\right\}.$$

Then, by our assumptions on d and α, it is clear that $\lambda_0, \mu_0 \in \mathbb{R}$ are such that $\lambda_0 \leqslant 0 \leqslant \mu_0$. Moreover, we can easily see that, for all $\lambda, \mu \in \mathbb{R}$ with $\lambda \leqslant \lambda_0$ and $\mu_0 \leqslant \mu$, we have

$$d(p, f(p)) \leqslant \alpha\lambda - \lambda; \quad d(f(p), f(q)) \leqslant \alpha\mu - \alpha\lambda; \quad d(f(q), q) \leqslant \mu - \alpha\mu.$$

Hence, by the definitions of $\leqslant_d$ and F, it is clear that the required inequalities are also true. □

Theorem 5.7. *If $\alpha > 1$, $d \in \mathcal{D}_S$ and $(p,\lambda), (q,\mu) \in X_S$ are such that*

$$(p,\lambda) \leqslant_d F(p,\lambda) \leqslant_d F(q,\mu) \leqslant_d (q,\mu),$$

then $\lambda = \mu = d(p, f(p)) = d(f(p), f(q)) = d(f(q), q) = 0$.

P r o o f. Again by the definitions of F and $\leqslant_d$, it is clear that

$$d(p, f(p)) \leqslant \alpha\lambda - \lambda; \quad d(f(p), f(q)) \leqslant \alpha\mu - \alpha\lambda; \quad d(f(q), q) \leqslant \mu - \alpha\mu.$$

Hence, by using our assumptions on d and α, we can easily see that

$$0 \leqslant \frac{d(p, f(p))}{(\alpha - 1)} \leqslant \lambda \leqslant \mu \leqslant \frac{d(f(q), q)}{(1 - \alpha)} \leqslant 0.$$

Therefore, $\lambda = \mu = 0$, and thus the required equalities are also true. □

R e m a r k 5.8. Note that, by writing $d_{\leqslant}$ instead of d in the results of Sections 4 and 5, we can get some similar assertions for the relations $\leqslant \in \mathcal{E}_S^-$. Namely, by Theorem 3.9 we have $\leqslant = \leqslant_{d_{\leqslant}}$ for all $\leqslant \in \mathcal{E}_S^-$.

A c k n o w l e d g e m e n t. The author is indebted to the referee for suggesting some improvements in the presentation and the omission of several obvious corollaries.

References

[1] *Birkhoff, G.*: Lattice Theory. Amer. Math. Soc. Colloq. Publ. 25, Providence, Rhode Island, 1967.
[2] *Ekeland, I.*: On the variational principle. J. Math. Anal. Appl. *47* (1974), 324–353.
[3] *Everett, C. J.*: Closure operators and Galois theory in lattices. Trans. Amer. Math. Soc. *55* (1944), 514–525.
[4] *Fletcher, P., Lindgren, W. F.*: Quasi-Uniform Spaces. Marcel Dekker, New York, 1982.
[5] *Jachymski, J. R.*: Fixed point theorems in metric and uniform spaces via the Knaster-Tarski principle. Nonlinear Anal. *32* (1998), 225–233.
[6] *DeMarr, R.*: Partially ordered spaces and metric spaces. Amer. Math. Monthly *72* (1965), 628–631.
[7] *Ore, O.*: Galois connexions. Trans. Amer. Math. Soc. *55* (1944), 493–513.
[8] *Pataki, G.*: On the extensions, refinements and modifications of relators. To appear in Math. Balk.
[9] *Rennie, B. C.*: Lattices. Proc. London Math. Soc. *52* (1951), 386–400.
[10] *Szász, G.*: Introduction to Lattice Theory. Academic Press, New York, 1963.

Author's address: *Árpád Száz*, Institute of Mathematics and Informatics, University of Debrecen, H-4010 Debrecen, Pf. 12, Hungary, e-mail: `szaz@math.klte.hu`.