POSITIVE SOLUTIONS OF INEQUALITY WITH p-LAPLACIAN IN EXTERIOR DOMAINS

Robert Mařík, Brno

(Received March 8, 2001)

Abstract. In the paper the differential inequality

$$\Delta_p u + B(x,u) \leqslant 0,$$

where $\Delta_p u = \operatorname{div}(\|\nabla u\|^{p-2}\nabla u)$, $p > 1$, $B(x,u) \in C(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ is studied. Sufficient conditions on the function $B(x,u)$ are established, which guarantee nonexistence of an eventually positive solution. The generalized Riccati transformation is the main tool.

Keywords: p-Laplacian, oscillation criteria

MSC 2000: 35B05

1. Introduction

In the paper we study positive solutions of the partial differential inequality

$$\Delta_p u + B(x,u) \leqslant 0, \tag{1}$$

where $\Delta_p u = \operatorname{div}(\|\nabla u\|^{p-2}\nabla u)$ is the p-Laplace operator, $p > 1$, $B(x,u)\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\|\cdot\|$ is the usual Euclidean norm in $\mathbb{R}^n$. Inequality (1) covers several equations and inequalities studied in literature. If $p = 2$ then (1) reduces to the semilinear Schrödinger inequality

$$\Delta u + B(x,u) \leqslant 0, \tag{2}$$

Supported by the grant No. 923/2000 of Foundation for Development of Czech Universities.

studied in [6], [7]. Another important special case of (1) is the half-linear differential equation

$$\Delta_p u + c(x)|u|^{p-1} \operatorname{sgn} u = 0, \tag{3}$$

studied in [2], [3]. For important applications of equations with p-Laplacian see [1].

The aim of this paper is to introduce sufficient conditions for nonexistence of a solution which would be eventually positive (i.e., positive outside of some ball in $\mathbb{R}^n$). Remark that in a similar way one can study also negative solutions of the inequality

$$\Delta_p u + B(x,u) \geqslant 0,$$

and a combination of these results produces criteria for nonexistence of a solution of the inequality

$$u[\Delta_p u + B(x,u)] \leqslant 0 \tag{4}$$

which would have no zero outside of some ball in $\mathbb{R}^n$, the so called weak oscillation criteria. A simple version of this procedure is used in Corollary 6. A more elaborated version of this procedure can be found in [6].

The following notation is used throughout the paper: $\langle \cdot, \cdot \rangle$ denotes the scalar product, $q = \frac{p}{p-1}$ is the conjugate number to the number p,

$$\begin{aligned} \Omega(a,b) &= \{x \in \mathbb{R}^n : \ a \leqslant \|x\| \leqslant b\}, \\ \Omega_a &= \Omega(a,\infty) = \{x \in \mathbb{R}^n : \ a \leqslant \|x\|\}, \\ S_a &= \partial\Omega_a = \{x \in \mathbb{R}^n : \ \|x\| = a\}, \end{aligned}$$

and $\omega_1 = \int_{S_1} 1 \, \mathrm{d}\sigma$ is the measure of the n-dimensional unit sphere in $\mathbb{R}^n$.

2. Riccati transformation

The main tool used for the study of positive solutions is the generalized Riccati transformation. The special case of this transformation has been used in [6], where inequality (2) is studied. A simple version of this transformation, convenient for the half-linear equation, has been introduced in [2].

Our approach combines both these methods. We use the transformation

$$\vec{w}(x) = -\alpha(\|x\|) \frac{\|\nabla u(x)\|^{p-2} \nabla u(x)}{\varphi(u(x))} \tag{5}$$

$\alpha \in C^1([a_0,\infty),\mathbb{R}^+)$, $\varphi \in C^1(\mathbb{R}^+,\mathbb{R}^+)$ which maps a positive C^1 function $u(x)$ into an n-vector function $\vec{w}(x)$.

Lemma 1. *Let u be a positive solution of* (1) *on Ω_{a_0}. The n-vector function $\vec{w}(x)$ is well-defined by* (5) *and satisfies the Riccati-type inequality*

$$\begin{aligned}\operatorname{div}\vec{w}(x) \geqslant \frac{\alpha(\|x\|)B(x,u(x))}{\varphi(u(x))} &+ \frac{\alpha'(\|x\|)}{\alpha(\|x\|)}\langle\vec{\nu}(x),\vec{w}(x)\rangle \\ &+ \alpha^{1-q}(\|x\|)\varphi^{q-2}(u(x))\varphi'(u(x))\|\vec{w}(x)\|^q\end{aligned} \tag{6}$$

on Ω_{a_0}, where $\vec{\nu}(x)=\frac{x}{\|x\|}$ is the outward unit normal vector to the sphere $S_{\|x\|}$.

Proof. Let $u(x)\geqslant 0$ be a solution of (1) on Ω_{a_0} and let $\vec{w}(x)$ be defined by (5). From (5) it follows that

$$\operatorname{div}\vec{w} = \frac{\alpha}{\varphi(u)}\Delta_p u - \|\nabla u\|^{p-2}\Big\langle \nabla u, \nabla\Big(\frac{\alpha}{\varphi(u)}\Big)\Big\rangle$$

and in view of (1)

$$\operatorname{div}\vec{w} \geqslant \frac{\alpha B(x,u)}{\varphi(u)} - \frac{\alpha'\|\nabla u\|^{p-2}}{\varphi(u)}\langle\nabla u,\vec{\nu}\rangle + \frac{\alpha\varphi'(u)}{\varphi^2(u)}\|\nabla u\|^p$$

holds (the dependence on $x\in\Omega_{a_0}$ is suppressed in the notation). In view of (5) this inequality is equivalent to (6). $\square$

3. Nonexistence of positive solution

The main result of the paper is the following

Theorem 1. *Let $a_0 \geqslant 0$. Suppose that there exist functions*

$$\alpha \in C^1([a_0,\infty),\mathbb{R}^+), \quad \varphi\in C^1(\mathbb{R}^+,\mathbb{R}^+), \quad c \in C(\mathbb{R}^n,\mathbb{R}),$$

and numbers k, l, $k>0$, $l>1$, such that
(i) $B(x,u)\geqslant c(x)\varphi(u)$ *for* $x\in\mathbb{R}^n$, $u>0$,
(ii) $\varphi'(u)\varphi^{q-2}(u)\geqslant k$ *for* $u>0$,
(iii) $\lim\limits_{r\to\infty}\int_{\Omega(a_0,r)}\Big[\alpha(\|x\|)c(x)-\frac{1}{p}\Big(\frac{l}{kq}\Big)^{p-1}\big|\alpha'(\|x\|)\big|^p\alpha^{1-p}(\|x\|)\Big]\,\mathrm{d}x=+\infty$,
(iv) $\lim\limits_{r\to\infty}\int_{a_0}^{r}\alpha^{\frac{1}{1-p}}(r)\,r^{\frac{1-n}{p-1}}\,\mathrm{d}r=+\infty$.

Then (1) *has no positive solution on Ω_a for arbitrary $a>0$.*

P r o o f. Suppose, by contradiction, that u is a solution of (1) positive on Ω_a for some $a > a_0$. Lemma 1 and the assumptions (i), (ii) imply

$$\begin{aligned}\operatorname{div}\vec{w} &\geqslant \alpha c + \frac{\alpha'}{\alpha}\langle\vec{\nu},\vec{w}\rangle + \alpha^{1-q}k\|\vec{w}\|^q\\ &= \alpha c + \alpha^{1-q}\frac{kq}{l}\Big[\frac{\|w\|^q}{q} + \Big\langle\vec{w}\frac{l\alpha^{q-2}\alpha'}{kq}\vec{\nu}\Big\rangle\Big] + \alpha^{1-q}\frac{k}{l^*}\|\vec{w}\|^q,\end{aligned}$$

where $l^* = \frac{l}{l-1}$ is the conjugate number to the number l. The Young inequality implies

$$\frac{\|\vec{w}\|^q}{q} + \Big\langle\vec{w}\frac{l\alpha^{q-2}\alpha'}{qk}\vec{\nu}\Big\rangle + \frac{1}{p}\Big(\frac{l\alpha^{q-2}|\alpha'|}{qk}\Big)^p \geqslant 0.$$

Combining both these inequalities we obtain

$$\begin{aligned}\operatorname{div}\vec{w} &\geqslant \alpha c - \alpha^{1-q}\frac{kq}{lp}\Big(\frac{l\alpha^{q-2}|\alpha'|}{qk}\Big)^p + \alpha^{1-q}\frac{k}{l^*}\|\vec{w}\|^q\\ &= \alpha c - \frac{1}{p}\Big(\frac{l}{qk}\Big)^{p-1}|\alpha'|^p\alpha^{1-p} + \alpha^{1-q}\frac{k}{l^*}\|\vec{w}\|^q.\end{aligned}$$

Integration of the last inequality over $\Omega(a,r)$ and the Gauss-Ostrogradski divergence theorem gives

$$\begin{aligned}\int_{S_r}\langle\vec{w},\vec{\nu}\rangle\,\mathrm{d}\sigma - \int_{S_a}\langle\vec{w},\vec{\nu}\rangle\,\mathrm{d}\sigma &\geqslant \frac{k}{l^*}\int_{\Omega(a,r)}\alpha^{1-q}\|\vec{w}\|^q\,\mathrm{d}x\\ &\quad + \int_{\Omega(a,r)}\Big[\alpha c - \frac{1}{p}\Big(\frac{l}{qk}\Big)^{p-1}p|\alpha'|^p\alpha^{1-p}\Big]\,\mathrm{d}x.\end{aligned}$$

By assumption (iii) there exists r_0, $r_0 > a$, such that

$$\int_{\Omega(a,r)}\Big[\alpha c - \frac{1}{p}\Big(\frac{l}{qk}\Big)^{p-1}|\alpha'|^p\alpha^{1-p}\Big]\,\mathrm{d}x + \int_{S_a}\langle\vec{w},\vec{\nu}\rangle\,\mathrm{d}\sigma \geqslant 0 \quad \text{for } r > r_0.$$

Hence

$$\int_{S_r}\langle\vec{w},\vec{\nu}\rangle\,\mathrm{d}\sigma \geqslant \frac{k}{l^*}g(r) \tag{7}$$

holds for $r > r_0$, where

$$g(r) = \int_{\Omega(a,r)}\alpha^{1-q}(\|x\|)\|\vec{w}(x)\|^q\,\mathrm{d}x.$$

The Hölder inequality gives

$$\int_{S_r}\langle\vec{w},\vec{\nu}\rangle\,\mathrm{d}\sigma \leqslant \Big(\int_{S_r}\|w\|^q\,\mathrm{d}\sigma\Big)^{\frac{1}{q}}\Big(\int_{S_r}1\,\mathrm{d}\sigma\Big)^{\frac{1}{p}} = \alpha^{\frac{1}{p}}(r)(g'(r))^{\frac{1}{q}}\omega_1^{\frac{1}{p}}r^{\frac{n-1}{p}}. \tag{8}$$

From (7) and (8) we obtain

$$(g'(r))^{\frac{1}{q}}\alpha^{\frac{1}{p}}(r)\omega_1^{\frac{1}{p}}r^{\frac{n-1}{p}} \geqslant \frac{k}{l^*}g(r) \qquad \text{for} \quad r \geqslant r_0$$

and equivalently

$$\frac{g'(r)}{g^q(r)}\omega_1^{\frac{q}{p}} \geqslant \Big(\frac{k}{l^*}\Big)^q \alpha^{-\frac{q}{p}}(r)r^{(1-n)\frac{q}{p}} = \Big(\frac{k}{l^*}\Big)^q \alpha^{\frac{1}{1-p}}(r)r^{\frac{1-n}{p-1}} \qquad \text{for} \quad r \geqslant r_0.$$

Integration of this inequality over the interval (r_0,∞) gives a convergent integral on the left-hand side and a divergent integral on the right-hand side of this inequality, by virtue of the assumption (iv). This contradiction completes the proof. □

R e m a r k 1. For $\varphi(u) = u^{p-1}$ we have $\varphi'(u)\varphi^{q-2}(u) = p-1$ and the assumption (ii) holds with $k = p-1$. Conversely, $\varphi(u) \geqslant \big(\frac{k}{p-1}\big)^{p-1}u^{p-1}$ is necessary for (ii) to be satisfied. Remark also that neither sign restrictions, nor radial symmetry, are supposed for the function $c(x)$ in (i).

Corollary 2 (Leighton type criterion)**.** *Let $p \geqslant n$. Suppose that there exists a continuous function $c(x)$ such that*

$$B(x,u) \geqslant c(x)u^{p-1} \qquad \textit{for } u > 0 \tag{9}$$

and

$$\lim_{r\to\infty}\int_{\Omega(1,r)} c(x)\,\mathrm{d}x = +\infty. \tag{10}$$

Then (1) *has no positive solution on Ω_a for arbitrary $a > 0$.*

P r o o f. Follows from Theorem 1 for $\alpha(r) \equiv 1$ and $\varphi(u) = u^{p-1}$. □

R e m a r k 2. Remark that (10) is known to be a sufficient condition for oscillation of (3) provided $p \geqslant n$, see [2]. It is also known that the condition $p \geqslant n$ in this criterion cannot be omitted.

Corollary 3. *Suppose that* (9) *holds and there exists $m > 1$ such that*

$$\lim_{r\to\infty}\int_{\Omega(1,r)} \Big[\|x\|^{p-n}c(x) - m\Big|\frac{p-n}{p}\Big|^p\frac{1}{\|x\|^n}\Big]\,\mathrm{d}x = +\infty. \tag{11}$$

Then (1) *has no positive solution on Ω_a for arbitrary $a > 0$.*

P r o o f. Follows from Theorem 1 for $\alpha(r) = r^{p-n}$ and $\varphi(u) = u^{p-1}$, $m = l^{p-1}$. □

R e m a r k 3. If the limit $\lim\limits_{r\to\infty} \frac{1}{\ln r} \int_{\Omega(1,r)} \|x\|^{p-n} c(x)\,\mathrm{d}x$ exists, or if this limit equals $+\infty$, then (11) is equivalent to the condition

$$\lim_{r\to\infty} \frac{1}{\ln r} \int_{\Omega(1,r)} \|x\|^{p-n} c(x)\,\mathrm{d}x > \omega_1 \Big|\frac{p-n}{p}\Big|^p.$$

This condition is very close to the criterion for oscillation of the half-linear equation [5, Corollary 2.1], which contains "lim sup" instead of "lim" and one additional condition

$$\liminf_{r\to\infty} \Big[r^{p-1}\Big(C_0 - \int_{\Omega(1,r)} \|x\|^{1-n} c(x)\,\mathrm{d}x\Big)\Big] > -\infty,$$

where

$$C_0 = \lim_{r\to\infty} \frac{p-1}{r^{p-1}} \int_1^r t^{p-2} \int_{\Omega(1,t)} \|x\|^{1-n} c(x)\,\mathrm{d}x\,\mathrm{d}t.$$

Among other, the constant $\big|\frac{p-n}{p}\big|^p$ is here shown to be optimal.

Corollary 4. *Let $p \geqslant n$, $p > 2$, (9) and*

$$\lim_{r\to\infty} \int_{\Omega(\cdot,r)} \ln(\|x\|) c(x)\,\mathrm{d}x = +\infty.$$

Then (1) has no positive solution on Ω_a for arbitrary $a > 0$.

P r o o f. Let $a > e$, $p \geqslant n$, $p > 2$, $\alpha(r) = \ln r$. Since

$$\lim_{r\to\infty} \frac{\alpha^{\frac{1}{1-p}}(r) r^{\frac{1-n}{p-1}}}{\frac{1}{r\ln r}} = \lim_{r\to\infty} r^{\frac{p-n}{p-1}} \ln^{\frac{p-2}{p-1}} r \geqslant 1,$$

the condition (iv) of Theorem 1 holds. Further,

$$\begin{aligned}\int_{\Omega(e,r)} |\alpha'(\|x\|)|^p \alpha^{1-p}(\|x\|)\,\mathrm{d}x &= \omega_1 \int_e^r \xi^{n-1-p} \ln^{1-p} \xi\,\mathrm{d}\xi \\ &\leqslant \omega_1 \int_e^r \xi^{-1} \ln^{1-p} \xi\,\mathrm{d}\xi = \omega_1 \frac{1}{p-2}[1 - \ln^{2-p} r].\end{aligned}$$

Hence $\lim\limits_{r\to\infty} \int_{\Omega(e,r)} |\alpha'(\|x\|)|^p \alpha^{1-p}(\|x\|)\,\mathrm{d}x$ exists and (12) is equivalent to the condition (iii) of Theorem 1. Now Theorem 1 implies the conclusion. □

The choice $\alpha(r) = \ln^\beta r$ leads to

Corollary 5. *Let $p \geqslant n$, let (9) hold and suppose that there exists β, $\beta \in (0, p-1)$ such that*

$$\lim_{r\to\infty} \int_{\Omega(\cdot,r)} \ln^\beta(\|x\|) c(x)\,\mathrm{d}x = +\infty.$$

Then (1) has no positive solution on Ω_a for arbitrary $a > 0$.

P r o o f. The proof is a complete analogue of the proof of Corollary 4. □

Following terminology in [6], a function $f\colon \Omega \to \mathbb{R}$ is called *weakly oscillatory* if and only if $f(x)$ has a zero in $\Omega \cap \Omega_a$ for every $a > 0$. The inequality (4) is called *weakly oscillatory* in Ω whenever every solution u of the inequality is oscillatory in Ω.

Corollary 6. *Let $B(x,u)\colon \mathbb{R}^{n+1} \to \mathbb{R}$ be a continuous function which is odd with respect to the variable u, i.e. let $B(x,-u) = -B(x,u)$. Let the assumptions of Theorem 1 be satisfied. Then inequality (4) is weakly oscillatory in $\mathbb{R}^n$.*

Proof. Suppose that there exists $a > 0$ such that inequality (4) has a solution u without zeros on Ω_a. If u is a positive function, then Theorem 1 yields a contradiction. Further, if u is a negative solution on Ω_a, then $v(x) := -u(x)$ is a positive solution of (4) on Ω_a and the same argument as in the first part of this proof leads to a contradiction. □

3.1. Perturbed half-linear differential inequality. Let us consider a perturbed half-linear differential inequality

$$\Delta_p u + c(x)|u|^{p-1} \operatorname{sgn} u + \sum_{i=1}^{m} q_i(x)\psi_i(u) \leqslant 0, \tag{13}$$

where $c(x)$, $q_i(x)$ are continuous functions, $\psi_i(u)$ are continuously differentiable, positive and nondecreasing for $u > 0$. Define

$$q(x) = \min\{c(x), q_1(x), q_2(x), \ldots, q_m(x)\}$$

and

$$\varphi(u) = u^{p-1} + \sum_{i=1}^{m} \psi_i(u).$$

Then

$$c(x)|u|^{p-1} \operatorname{sgn} u + \sum_{i=1}^{m} q_i(x)\psi_i(u) \geqslant q(x)\varphi(u), \qquad \varphi'(u)\varphi^{q-2}(u) \geqslant p-1$$

and hence Theorem 1 can be applied. Remark that since q_i may change sign, a standard argument based on the Sturmian majorant and a comparison with half-linear differential equation (3) cannot be applied (as has been explained for $p = 2$ already in [6]).

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Author's address: *Robert Mařík*, Dept. of Mathematics, Mendel University, Zemědělská 3, 613 00 Brno, Czech Republic, e-mail: `marik@mendelu.cz`.