THE HENSTOCK-KURZWEIL APPROACH TO YOUNG INTEGRALS WITH INTEGRATORS IN BV_φ

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Dedicated to Prof. J. Kurzweil on the occasion of his 80th birthday

Abstract. In 1938, L. C. Young proved that the Moore-Pollard-Stieltjes integral $\int_a^b f\,\mathrm{d}g$ exists if $f \in \mathrm{BV}_\varphi[a,b]$, $g \in \mathrm{BV}_\psi[a,b]$ and $\sum\limits_{n=1}^{\infty} \varphi^{-1}(1/n)\psi^{-1}(1/n) < \infty$. In this note we use the Henstock-Kurzweil approach to handle the above integral defined by Young.

Keywords: Henstock integral, Stieltjes integral, Young integral, φ-variation

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1. Introduction

In 1936, L. C. Young proved that the Riemann-Stieltjes integral $\int_a^b f\,\mathrm{d}g$ exists, if $f \in \mathrm{BV}_p[a,b]$, $g \in \mathrm{BV}_q[a,b]$, $1/p + 1/q > 1$ and f, g do not have common discontinuous points, see [7], [11]. Two years later, he was able to drop the condition on common discontinuity for his new integral (called Young integral), see [12]. The Young integral is defined by the Moore-Pollard approach, see [2, pp. 23–27, pp. 113–138] and [3], [8], [9]. In other words, the integral is defined by way of refinements of partitions and the integral is the Moore-Smith limit of the Riemann-Stieltjes sums using the directed set of partitions. However, modified Riemann-Stieltjes sums involving $g(x+)$ and $g(x-)$ are used in Young integrals. Furthermore, he generalized his result and proved that the Young integral $\int_a^b f\,\mathrm{d}g$ exists if the following Young's condition holds:

$$f \in \mathrm{BV}_\varphi[a,b], \quad g \in \mathrm{BV}_\psi[a,b]$$

and

$$\sum_{n=1}^{\infty} \varphi^{-1}\Bigl(\frac{1}{n}\Bigr)\psi^{-1}\Bigl(\frac{1}{n}\Bigr) < \infty,$$

where $\mathrm{BV}_\varphi[a,b]$ is the space of functions of bounded φ-variation on [a,b].

The Young integral with an integrator in BV_p using the Henstock-Kurzweil approach is given in [1]. In this note we will again use the Henstock-Kurzweil approach to handle the Young integral with an integrator in BV_φ.

Now we shall introduce Henstock-Kurzweil integrals, see [4].

Let $P = \{[u_i, v_i]\}_{i=1}^n$ be a finite collection of non-overlapping subintervals of $[a,b]$, then P is said to be a *partial partition* of $[a,b]$. If, in addition, $\bigcup_{i=1}^n [u_i, v_i] = [a,b]$, then P is said to be a *partition* of $[a,b]$.

Let δ be a positive function on $[a,b]$, $[u,v] \subseteq [a,b]$ and $\xi \in [a,b]$. Then an interval-point pair $(\xi, [u,v])$ is said to be *δ-fine* if $\xi \in [u,v] \subseteq (\xi - \delta(\xi), \xi + \delta(\xi))$. Let $D = \{(\xi_i, [u_i, v_i])\}_{i=1}^n$ be a finite collection of interval-point pairs. Then D is said to be a *δ-fine partial division of* $[a,b]$ if $\{[u_i, v_i]\}_{i=1}^n$ is a partial partition of $[a,b]$ and for each i, $(\xi_i, [u_i, v_i])$ is δ-fine. In addition, if $\{[u_i, v_i]\}_{i=1}^n$ is a partition of $[a,b]$, then D is said to be a *δ-fine division* of $[a,b]$.

In this note, $\mathbb{R}$ denotes the set of real numbers.

Now, we shall define integrals of Stieltjes type by the Henstock-Kurzweil approach.

Definition 1.1. Let $f, g\colon [a,b] \to \mathbb{R}$. Then f is said to be Henstock-Kurzweil integrable (or HK-integrable) to a real number A on $[a,b]$ with respect to g if for every $\varepsilon > 0$ there exists a positive function δ defined on $[a,b]$ such that for every δ-fine division $D = \{(\xi_i, [t_i, t_{i+1}])\}_{i=1}^n$ of $[a,b]$, we have

$$|S(f,\delta,D) - A| \leqslant \varepsilon,$$

where

$$S(f,\delta,D) = \sum_{i=1}^n f(\xi_i)(g(t_{i+1}) - g(t_i)).$$

A is denoted by $\int_a^b f \,\mathrm{d}g$.

It is known that if $f \in \mathrm{BV}_p[a,b], g \in \mathrm{BV}_q[a,b], 1/p + 1/q > 1$, then f is HK-integrable with respect to g on $[a,b]$, see [1].

In this note we follow ideas of Young to show that if $f \in \mathrm{BV}_\varphi[a,b], g \in \mathrm{BV}_\psi[a,b]$ and $\sum_{n=1}^{\infty} \varphi^{-1}(1/n)\psi^{-1}(1/n) < \infty$, then f is HK-integrable with respect to g on $[a,b]$.

The above series is called Young's series. We shall present some properties of Young's series. Results and proofs are known, see [5], [6], [12]. We give proofs here for easy reference.

In this section, let λ, μ be strictly increasing continuous non-negative functions on $[0,\infty)$ with $\lambda(0)=\mu(0)=0$ and let ω, κ be increasing functions on $[\alpha,\beta]$ with

$$\omega(\beta)-\omega(\alpha)\leqslant A \quad \text{and} \quad \kappa(\beta)-\kappa(\alpha)\leqslant B.$$

Lemma 2.1. *For $p=0,1,2,\ldots$, there exists $E_p=\{x_1,x_2,\ldots,x_{n_p}\}\subset[\alpha,\beta]$ such that for any $\xi,\eta\in(x_i,x_{i+1})$, $i=1,2,\ldots,n_p-1$, we have*

$$|\omega(\eta)-\omega(\xi)|\leqslant A2^{-p}$$

and

$$|\kappa(\eta)-\kappa(\xi)|\leqslant B2^{-p}.$$

Furthermore, $E_q\supseteq E_p$ if $p\leqslant q$, $\#(E_p)\leqslant 2^{p+1}$ and $\#(E_{p+1}\setminus E_p)\leqslant 2^{p+1}$, where $\#(E_p)$ denotes the number of elements in the set E_p.

Proof. Denote $|\omega(\xi)-\omega(\eta)|$, $|\kappa(\xi)-\kappa(\eta)|$ by $\omega(\xi,\eta)$, $\kappa(\xi,\eta)$ respectively.

Let $E_0^\omega=\{x_1^{(0)},x_2^{(0)}\}$, where $x_1^{(0)}=\alpha$, $x_2^{(0)}=\beta$. Then for any $\xi,\eta\in(x_1^{(0)},x_2^{(0)})$, we can see that

$$\omega(\xi,\eta)\leqslant A.$$

Let $x_{1'}^{(0)}=\sup\{x\in[x_1^{(0)},x_2^{(0)}];\ \omega(\xi,\eta)\leqslant A2^{-1}$ for any $\xi,\eta\in(x_1^{(0)},x)\}$ and let

$$E_1^\omega=\{x_1^{(0)},x_{1'}^{(0)},x_2^{(0)}\}.$$

It is possible that $x_{1'}^{(0)}=x_2^{(0)}$, i.e., $E_1^\omega=E_0^\omega$. We may assume that the above supremum is well-defined, otherwise we use $(x,x_2^{(0)})$ instead of $(x_1^{(0)},x)$. We will rename points in E_1^ω according to their order using the notation

$$E_1^\omega=\{x_1^{(1)},x_2^{(1)},x_3^{(1)}\}.$$

We claim that for any $\xi,\eta\in(x_2^{(1)},x_3^{(1)})$, $\omega(\xi,\eta)\leqslant A2^{-1}$.

Suppose that there exist $\xi,\eta\in(x_2^{(1)},x_3^{(1)})$ such that $\omega(\xi,\eta)>A2^{-1}$. Since $\xi>x_2^{(1)}$, there exists a point $\beta\in(x_1^{(1)},x_2^{(1)})$ such that $\omega(\beta,\xi)>A2^{-1}$. Since $\xi>x_2^{(1)}$,

$$\omega(\beta,\eta)=\omega(\beta,\xi)+\omega(\xi,\eta)>A2^{-1}+A2^{-1}=A.$$

It contradicts the definition of E_0^ω. Hence, for any $\xi, \eta \in (x_2^{(1)}, x_3^{(1)})$,

$$\omega(\xi, \eta) \leqslant A2^{-1}.$$

That is, for any $\xi, \eta \in (x_i^{(1)}, x_{i+1}^{(1)})$, $i = 1, 2$, we have

$$\omega(\xi, \eta) \leqslant A2^{-1}.$$

Let $x_{1'}^{(1)} = \sup\{x \in [x_1^{(1)}, x_2^{(1)}];\ \omega(\xi, \eta) \leqslant A2^{-2}$ for every $\xi, \eta \in (x_1^{(1)}, x)\}$, $x_{2'}^{(1)} = \sup\{x \in [x_2^{(1)}, x_3^{(1)}];\ \omega(\xi, \eta) \leqslant A2^{-2}$ for every $\xi, \eta \in (x_2^{(1)}, x)\}$ and

$$E_2^\omega = \{x_1^{(1)}, x_{1'}^{(1)}, x_2^{(1)}, x_{2'}^{(1)}, x_3^{(1)}\}.$$

It is still possible that $x_{1'}^{(1)} = x_2^{(1)}$ or $x_{2'}^{(1)} = x_3^{(1)}$. We again rename E_2^ω according to their order by

$$E_2^\omega = \{x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, x_4^{(2)}, x_5^{(2)}\}.$$

Using the same argument as above, we also have for any $\xi, \eta \in (x_i^{(2)}, x_{i+1}^{(2)})$ and for every $i = 1, 2, 3, 4$,

$$\omega(\xi, \eta) \leqslant A2^{-2}.$$

Using this method, we can have $E_p^\omega = \{x_1^{(p)}, x_2^{(p)}, \ldots, x_{n_p}^{(p)}\}$, $p = 0, 1, 2, \ldots$, such that for any $\xi, \eta \in (x_i^{(p)}, x_{i+1}^{(p)})$, $i = 1, 2, \ldots, n_p - 1$, we have

$$\omega(\xi, \eta) \leqslant A2^{-p}.$$

We can also see that $E_q^\omega \subseteq E_p^\omega$ whenever $q \leqslant p$, the number of elements in E_p^ω is at most $2^p + 1$ and the number of elements in $E_{p+1}^\omega \setminus E_p^\omega$ is at most 2^p.

Using the same argument, we also can define $E_p^\kappa = \{y_1^{(p)}, y_2^{(p)}, \ldots, y_{m_p}^{(p)}\}$ for $p = 0, 1, 2, \ldots$, so that for any $\xi, \eta \in (y_i^{(p)}, y_{i+1}^{(p)})$, $i = 1, 2, \ldots, m_p - 1$, we have

$$\kappa(\xi, \eta) \leqslant B2^{-p}.$$

Furthermore, $E_q^\kappa \subseteq E_p^\kappa$ whenever $q \leqslant p$, the number of elements in E_p^κ is at most $2^p + 1$ and the number of element in $E_{p+1}^\kappa \setminus E_p^\kappa$ is at most 2^p.

Now, let $E_p = E_p^\omega \cup E_p^\kappa = \{z_1^{(p)}, z_2^{(p)}, \ldots, z_{r_p}^{(p)}\}$. Then for every $\xi, \eta \in (z_i^{(p)}, z_{i+1}^{(p)})$, $i = 1, 2, \ldots, r_p - 1$,

$$\omega(\xi, \eta) \leqslant A2^{-p} \text{ and } \kappa(\xi, \eta) \leqslant B2^{-p}.$$

Furthermore, $E_q \subseteq E_p$ whenever $q \leqslant p$, the number of elements in $E_{p+1} \setminus E_p$ is at most $2 \cdot 2^p = 2^{p+1}$ and number of elements in E_p is at most $2(2^p + 1) - 2 = 2^{p+1}$, since $\alpha, \beta \in E_p^\omega \cap E_p^\kappa$. □

Lemma 2.2. (i) *For any positive integer v, the following inequalities hold:*

$$\sum_{n=0}^{\infty} 2^{n+v}\lambda(A2^{-(n+v)})\mu(B2^{-(n+v)}) \leqslant 2 \sum_{n=2^{v-1}}^{\infty} \lambda\Big(\frac{A}{n}\Big)\mu\Big(\frac{B}{n}\Big)$$

and

$$\sum_{n=1}^{\infty} \lambda\Big(\frac{A2^{-v}}{n}\Big)\mu\Big(\frac{B2^{-v}}{n}\Big) \leqslant \frac{1}{2^{v-1}} \sum_{n=2^{v-1}}^{\infty} \lambda\Big(\frac{A}{n}\Big)\mu\Big(\frac{B}{n}\Big),$$

(ii)

$$\sum_{n=0}^{\infty} 2^{n}\lambda(A2^{-n})\mu(B2^{-n}) \leqslant 3 \sum_{n=1}^{\infty} \lambda\Big(\frac{A}{n}\Big)\mu\Big(\frac{B}{n}\Big).$$

P r o o f. (i)

$$\sum_{n=0}^{\infty} 2^{n+v}\lambda(A2^{-(n+v)})\mu(B2^{-(n+v)}) = \sum_{k=v}^{\infty} 2^{k}\lambda(A2^{-k})\mu(B2^{-k})$$
$$\leqslant 2 \sum_{k=v}^{\infty} \sum_{n=2^{k-1}+1}^{2^k} \lambda\Big(\frac{A}{n}\Big)\mu\Big(\frac{B}{n}\Big) = 2 \sum_{n=2^{v-1}+1}^{\infty} \lambda\Big(\frac{A}{n}\Big)\mu\Big(\frac{B}{n}\Big)$$
$$\leqslant 2 \sum_{n=2^{v-1}}^{\infty} \lambda\Big(\frac{A}{n}\Big)\mu\Big(\frac{B}{n}\Big).$$

Similarly, we have

$$\sum_{n=1}^{\infty} \lambda\Big(\frac{A2^{-v}}{n}\Big)\mu\Big(\frac{B2^{-v}}{n}\Big) = \sum_{n=0}^{\infty} \sum_{k=2^n}^{2^{n+1}-1} \lambda\Big(\frac{A2^{-v}}{k}\Big)\mu\Big(\frac{B2^{-v}}{k}\Big)$$
$$\leqslant \sum_{n=0}^{\infty} 2^{n}\lambda(A2^{-(v+n)})\mu(B2^{-(v+n)}) = \frac{1}{2^v} \sum_{n=0}^{\infty} 2^{n+v}\lambda(A2^{-(n+v)})\mu(B2^{-(n+v)})$$
$$\leqslant \frac{1}{2^{v-1}} \sum_{n=2^{v-1}}^{\infty} \lambda\Big(\frac{A}{n}\Big)\mu\Big(\frac{B}{n}\Big).$$

(ii) As in the first part of (i),

$$\sum_{n=0}^{\infty} 2^{n}\lambda(A2^{-n})\mu(B2^{-n}) = \lambda(A)\mu(B) + \sum_{k=1}^{\infty} 2^{k}\lambda(A2^{-k})\mu(B2^{-k})$$
$$\leqslant \lambda(A)\mu(B) + 2\sum_{n=1}^{\infty} \lambda\Big(\frac{A}{n}\Big)\mu\Big(\frac{B}{n}\Big) \leqslant 3\sum_{n=1}^{\infty} \lambda\Big(\frac{A}{n}\Big)\mu\Big(\frac{B}{n}\Big).$$

□

Lemma 2.3.

$$\sum_{n=1}^{\infty} \lambda\Bigl(\frac{1}{n}\Bigr)\mu\Bigl(\frac{1}{n}\Bigr) < \infty \text{ if and only if } \sum_{n=1}^{\infty} \lambda\Bigl(\frac{A}{n}\Bigr)\mu\Bigl(\frac{B}{n}\Bigr) < \infty.$$

P r o o f. Suppose $\sum\limits_{n=1}^{\infty} \lambda(1/n)\mu(1/n) < \infty$. Let m be a positive integer such that $A \leqslant m$ and $B \leqslant m$. Then

$$\begin{aligned}\sum_{n=1}^{\infty} \lambda\Bigl(\frac{A}{n}\Bigr)\mu\Bigl(\frac{B}{n}\Bigr) &= \sum_{n=1}^{m-1} \lambda\Bigl(\frac{A}{n}\Bigr)\mu\Bigl(\frac{B}{n}\Bigr) + \sum_{n=m}^{\infty} \lambda\Bigl(\frac{A}{n}\Bigr)\mu\Bigl(\frac{B}{n}\Bigr)\\ &= \sum_{n=1}^{m-1} \lambda\Bigl(\frac{A}{n}\Bigr)\mu\Bigl(\frac{B}{n}\Bigr) + \sum_{k=1}^{\infty}\sum_{n=km}^{(k+1)m} \lambda\Bigl(\frac{A}{n}\Bigr)\mu\Bigl(\frac{B}{n}\Bigr)\\ &\leqslant \sum_{n=1}^{m-1} \lambda\Bigl(\frac{A}{n}\Bigr)\mu\Bigl(\frac{B}{n}\Bigr) + m\sum_{k=1}^{\infty} \lambda\Bigl(\frac{1}{k}\Bigr)\mu\Bigl(\frac{1}{k}\Bigr) < \infty.\end{aligned}$$

Conversely, suppose $\sum\limits_{n=1}^{\infty} \lambda(A/n)\mu(B/n) < \infty$. Let $\lambda'(x) = \lambda(Ax), \mu'(x) = \mu(Bx)$. Then $\sum\limits_{n=1}^{\infty} \lambda'(1/n)\mu'(1/n) < \infty$. Therefore, $\sum\limits_{n=1}^{\infty} \lambda'(1/(An))\mu'(1/(Bn)) < \infty$. Consequently, $\sum\limits_{n=1}^{\infty} \lambda(1/n)\mu(1/n) < \infty$. □

3. Integrals of step functions

In this section we shall present Young's results on integrals of step functions, see [12]. Let g be a regulated function on $[\alpha, \beta]$ and s a step function on $[\alpha, \beta]$ with

$$s(x) = \sum_{i=1}^{q} c_i \chi_{(t_i, t_{i+1})}(x) + \sum_{i=1}^{q+1} d_i \chi_{\{t_i\}}(x),$$

where χ_G is the characteristic function of G, and $\alpha = t_1 < t_2 < \ldots < t_{q+1} = \beta$.

It is known, see [1], that

$$\int_{\alpha}^{\beta} s \,\mathrm{d}g = \sum_{i=1}^{q} c_i(g(t_{i+1}-) - g(t_i+)) + \sum_{i=1}^{q+1} d_i(g(t_i+) - g(t_i-)).$$

Furthermore, we always assume that the following conditions hold:

$$\left\{\begin{aligned} &|s(\xi) - s(\eta)| \leqslant \lambda(\omega(\xi) - \omega(\eta)),\\ &|g(\xi) - g(\eta)| \leqslant \mu(\kappa(\xi) - \kappa(\eta))\end{aligned}\right. \tag{1}$$

for any $\xi, \eta \in [\alpha, \beta]$ with $\xi > \eta$, where $\lambda, \mu, \omega, \kappa$ are given in Section 1. In this section we always assume that $\sum\limits_{n=1}^{\infty} \lambda(A/n)\mu(B/n) < \infty$. Recall that $\omega(\beta) - \omega(\alpha) \leqslant A$ and $\kappa(\beta) - \kappa(\alpha) \leqslant B$.

Definition 3.1. Let s be a step function defined on $[\alpha, \beta]$ and E_p the finite set as defined in Lemma 2.1. Let $E = \{x_i\colon\ i = 1, 2, \ldots, m+1\}$ be any fixed finite set containing E_0. We define s_E to be the step function induced by s and E as follows:

$$s_E(x) = \sum_{i=1}^{m} s(x_i+)\chi_{(x_i, x_{i+1})}(x) + \sum_{i=1}^{m+1} s(x_i)\chi_{\{x_i\}}(x).$$

We have, by the formula for the value of the integral of a step function with respect to g presented above,

$$\int_\alpha^\beta s_E \,\mathrm{d}g = \sum_{i=1}^{m} s(x_i+)(g(x_{i+1}-) - g(x_i+)) + \sum_{i=1}^{m+1} s(x_i)(g(x_i+) - g(x_i-)).$$

We remark that if E contains all points of discontinuity of s, then $s_E = s$ and $\int_\alpha^\beta s_E \,\mathrm{d}g = \int_\alpha^\beta s \,\mathrm{d}g$.

Lemma 3.2. *Let* $E \supseteq E_0$. *Then*

$$\left| \int_\alpha^\beta (s_{E\cup E_p} - s_{E_p}) \,\mathrm{d}g \right| \leqslant N_p \lambda\Big(\frac{A}{2^p}\Big)\mu\Big(\frac{B}{2^p}\Big),$$

where $N_p = \#(E \setminus E_p)$. *Furthermore,*

$$\lim_{p\to\infty} \left| \int_\alpha^\beta (s_{E\cup E_p} - s_{E_p}) \,\mathrm{d}g \right| = 0.$$

P r o o f. Let N_p denote $\#(E \setminus E_p)$. Let s' denote the step function $s_{E\cup E_p} - s_{E_p}$. Suppose s' is induced by a partition $\{[y_i, y_{i+1}]\}_{i=1}^m$ of $[\alpha, \beta]$. If $y_i \in E_p$, then s' has zero values over a half-open subinterval $[y_i, y_{i+1})$. Therefore, the number of subintervals where s' has nonzero value is at most N_p. Then

$$\left| \int_\alpha^\beta (s_{E\cup E_p} - s_{E_p}) \,\mathrm{d}g \right| \leqslant N_p \lambda(A2^{-p})\mu(B2^{-p}) \leqslant N_0 \lambda(A2^{-p})\mu(B2^{-p}).$$

Hence, for any fixed finite set E,

$$\lim_{p\to\infty} \left| \int_\alpha^\beta (s_{E\cup E_p} - s_{E_p}) \,\mathrm{d}g \right| \leqslant \lim_{p\to\infty} N_0 \lambda(A2^{-p})\mu(B2^{-p}) = 0.$$

In the above, we use the fact that λ, μ are continuous at 0 and $\lambda(0) = \mu(0) = 0$. $\square$

Theorem 3.3. *Let s be a step function and E_0 as above. Then*

$$\Big|\int_\alpha^\beta (s - s_{E_0})\,\mathrm{d}g\Big| \leqslant 6\sum_{n=1}^{\infty}\lambda\Big(\frac{A}{n}\Big)\mu\Big(\frac{B}{n}\Big).$$

Proof. From Lemma 3.2, $E_{p+1} = E_{p+1}\cup E_p$ and $\#(E_{p+1}-E_p)\leqslant 2^{p+1}$, we have

$$\begin{aligned}\Big|\int_\alpha^\beta (s_{E_{p+1}} - s_{E_p})\,\mathrm{d}g\Big| &= \Big|\int_\alpha^\beta (s_{E_{p+1}\cup E_p} - s_{E_p})\,\mathrm{d}g\Big|\\ &\leqslant 2^{p+1}\lambda(A2^{-p})\mu(A2^{-p}) = 2\cdot 2^p\lambda(A2^{-p})\mu(A2^{-p}).\end{aligned}$$

Now, let E^* be a finite set containing E_0 and all points of discontinuity of s, then $\int_\alpha^\beta s\,\mathrm{d}g = \int_\alpha^\beta s_{E^*}\,\mathrm{d}g = \int_\alpha^\beta s_{E^*\cup E_v}\,\mathrm{d}g$ for all $v = 0,1,2,\ldots$. Hence we have

$$\begin{aligned}\Big|\int_\alpha^\beta (s - s_{E_p})\,\mathrm{d}g\Big| &\leqslant \lim_{q\to\infty}\Big(\Big|\int_\alpha^\beta (s_{E^*} - s_{E_{p+q}})\,\mathrm{d}g\Big| + \Big|\int_\alpha^\beta (s_{E_{p+q}} - s_{E_{p+q-1}})\,\mathrm{d}g\Big|\\ &\qquad + \ldots + \Big|\int_\alpha^\beta (s_{E_{p+1}} - s_{E_p})\,\mathrm{d}g\Big|\Big)\\ &= \lim_{q\to\infty}\Big(\Big|\int_\alpha^\beta (s_{E^*\cup E_{p+q}} - s_{E_{p+q}})\,\mathrm{d}g\Big| + \Big|\int_\alpha^\beta (s_{E_{p+q}} - s_{E_{p+q-1}})\,\mathrm{d}g\Big|\\ &\qquad + \ldots + \Big|\int_\alpha^\beta (s_{E_{p+1}} - s_{E_p})\,\mathrm{d}g\Big|\Big)\\ &\leqslant 0 + \lim_{q\to\infty}\sum_{m=0}^{q-1} 2\cdot 2^{p+m}\lambda(A2^{-(p+m)})\mu(A2^{-(p+m)})\\ &\leqslant 4\sum_{n=2^{p-1}}^{\infty}\lambda\Big(\frac{A}{n}\Big)\mu\Big(\frac{B}{n}\Big)\quad\text{for } p = 1,2,\ldots.\end{aligned}$$

The last inequality holds by Lemma 2.2 (i).

When $p = 0$, by Lemma 2.2 (ii) we get

$$\Big|\int_\alpha^\beta (s - s_{E_0})\,\mathrm{d}g\Big| \leqslant 6\sum_{n=1}^{\infty}\lambda\Big(\frac{A}{n}\Big)\mu\Big(\frac{B}{n}\Big).$$

□

Corollary 3.4. *Suppose that* $s_1(x) = \sum_{i=1}^{n} d_i\chi_{[u_i,u_{i+1})}(x) + d_n\chi_{\{u_{n+1}\}}(x)$, $s_2(x) = \sum_{i=1}^{m} e_i\chi_{[v_i,v_{i+1})}(x) + e_m\chi_{\{v_{m+1}\}}(x)$ *are step functions defined on* $[\alpha,\beta]$. *Let* (1) *hold with* $s = s_1, s_2$ *and* $|d_1 - e_1| \leqslant \lambda(A)$. *Then*

$$\Bigl|\sum_{i=1}^{n} d_i(g(u_{i+1}) - g(u_i)) - \sum_{i=1}^{m} e_i(g(v_{i+1}) - g(v_i))\Bigr| \leqslant 13\sum_{n=1}^{\infty} \lambda\Bigl(\frac{A}{n}\Bigr)\mu\Bigl(\frac{B}{n}\Bigr).$$

P r o o f. First we shall prove that the following inequality holds:

$$\Bigl|\sum_{i=1}^{n} d_i(g(u_{i+1}) - g(u_i)) - d_1(g(\beta) - g(\alpha))\Bigr| \leqslant 6\sum_{n=1}^{\infty} \lambda\Bigl(\frac{A}{n}\Bigr)\mu\Bigl(\frac{B}{n}\Bigr). \tag{2}$$

Let $g^*(u_i) = g(u_i)$, $g^*(t) = g(u_i)$ for those t close to u_i from the left, otherwise, let $g^*(t) = g(t)$. Then $g^*(u_i-) = g^*(u_i)$ and g^* also satisfies $|g^*(\xi) - g^*(\eta)| \leqslant \mu(|\kappa(\xi) - \kappa(\eta)|)$ for any $\xi, \eta \in [\alpha,\beta]$. Then

$$\begin{aligned}\int_\alpha^\beta s_1\,\mathrm{d}g^* &= \sum_{i=1}^{n} d_i(g^*(u_{i+1}-) - g^*(u_i)) + d_n(g^*(u_{n+1}) - g^*(u_{n+1}-))\\ &= \sum_{i=1}^{n} d_i(g^*(u_{i+1}-) - g^*(u_i)) = \sum_{i=1}^{n} d_i(g(u_{i+1}) - g(u_i)).\end{aligned}$$

Applying Theorem 3.3 to $s = s_1$ and $g = g^*$, $\int_\alpha^\beta s_{E_0}\,\mathrm{d}g^* = d_1(g^*(\beta-) - g^*(\alpha)) + d_n(g^*(\beta) - g^*(\beta+)) = d_1(g(\beta) - g(\alpha))$, we get the inequality (2).

Thus

$$\begin{aligned}&\Bigl|\sum_{i=1}^{n} d_i(g(u_{i+1}) - g(u_i)) - \sum_{i=1}^{m} e_i(g(v_{i+1}) - g(v_i))\Bigr|\\ &\quad\leqslant 12\sum_{n=1}^{\infty} \lambda\Bigl(\frac{A}{n}\Bigr)\mu\Bigl(\frac{B}{n}\Bigr) + |d_1(g(\beta_1) - g(\alpha_1)) - e_1(g(\beta_1) - g(\alpha_1))|\\ &\quad\leqslant 12\sum_{n=1}^{\infty} \lambda\Bigl(\frac{A}{n}\Bigr)\mu\Bigl(\frac{B}{n}\Bigr) + |d_1 - e_1||g(\beta_1) - g(\alpha_1)|\\ &\quad\leqslant 12\sum_{n=1}^{\infty} \lambda\Bigl(\frac{A}{n}\Bigr)\mu\Bigl(\frac{B}{n}\Bigr) + \lambda(A)\mu(B) = 13\sum_{n=1}^{\infty} \lambda\Bigl(\frac{A}{n}\Bigr)\mu\Bigl(\frac{B}{n}\Bigr).\end{aligned}$$

□

Now we shall introduce $\mathrm{BV}_\varphi[a,b]$, which is a generalization of $\mathrm{BV}[a,b]$, the space of functions of bounded variation on $[a,b]$, and prove an existence theorem (Theorem 4.6) in the Henstock-Kurzweil setting.

Definition 4.1. A function $\varphi\colon [0,\infty) \to \mathbb{R}$ is said to be an N-function if

1. $\varphi(0) = 0$;
2. φ is continuous on $[0,\infty)$;
3. φ is strictly increasing and
4. $\varphi(u) \to \infty$ as $u \to \infty$.

Examples of N-functions are $\varphi_1(u) = u^p$, $p \geqslant 1$ and $\varphi_2(u) = \mathrm{e}^u - 1$.

Definition 4.2. Let $\varphi\colon [0,\infty) \to \mathbb{R}$ be an N-function and $f\colon [a,b] \to \mathbb{R}$. We define

$$V_\varphi(f;[a,b]) = \sup \sum_{i=1}^{n} \varphi(|f(x_{i+1}) - f(x_i)|),$$

where supremum is taken over all partitions $\{[x_i,x_{i+1}]\}_{i=1}^n$ of $[a,b]$. The number $V_\varphi(f;[a,b])$ is called the φ-variation of f on $[a,b]$. Let $\mathrm{BV}_\varphi[a,b]$ denote the collection of all functions $f\colon [a,b] \to \mathbb{R}$ satisfying $V_\varphi(f;[a,b]) < \infty$, see [5], [6], [12]. Such functions are said to be of bounded φ-variation. When it is clear that we are considering the interval $[a,b]$, we shall denote $V_\varphi(f;[a,b])$ by $V_\varphi(f)$.

For example, where $\varphi(u) = u^p$, $p \geqslant 1$, $\mathrm{BV}_\varphi[a,b]$ is the space of functions of bounded p-variation on $[a,b]$.

The following lemma and its proof are known.

Lemma 4.3. *If $f \in \mathrm{BV}_\varphi[a,b]$, then f is bounded on $[a,b]$ and f is a regulated function.*

Proof. Suppose f is unbounded. Let M be any positive real number. Then there exists $x \in [a,b]$ such that $M \leqslant |f(x) - f(a)|$. Hence

$$\begin{aligned} M \leqslant |f(x) - f(a)| &= \varphi^{-1}(\varphi(|f(x) - f(a)|)) \\ &\leqslant \varphi^{-1}(\varphi(|f(x) - f(a)|) + \varphi(|f(b) - f(x)|)) \leqslant \varphi^{-1}(V_\varphi(f;[a,b])). \end{aligned}$$

Therefore

$$\varphi(M) \leqslant V_\varphi(f;[a,b]) \text{ for all } M > 0.$$

Since $\varphi(M) \to \infty$ as $M \to \infty$, we have $V_\varphi(f;[a,b]) = \infty$. This leads to a contradiction.

The proof that f is regulated is standard. □

Lemma 4.4. *Let $g \in \mathrm{BV}_\psi[a,b]$, $E = \{x_1, x_2, \ldots, x_n\} \supseteq E_0$, and $\varepsilon > 0$, where E_0 is given in Lemma 2.1. Then there exists a constant $\delta > 0$ such that for any finite collection of disjoint subintervals $\{[u_i, v_i]\}_{i=1}^n$ with $[u_i, v_i] \subset (x_i, x_i + \delta)$ or $[u_i, v_i] \subset (x_i - \delta, x_i)$ for each i, we have*

$$\sum_{i=1}^{n} |g(v_i) - g(u_i)| \leqslant \varepsilon.$$

Proof. Let $\varepsilon > 0$ be given. First, since g is a regulated function, there exists a constant $\delta > 0$ such that

$$|g(t) - g(x_i-)| \leqslant \frac{\varepsilon}{2n} \quad \text{whenever} \quad 0 < x_i - t < \delta$$

and

$$|g(x_i+) - g(t)| \leqslant \frac{\varepsilon}{2n} \quad \text{whenever} \quad 0 < t - x_i < \delta$$

for each i. Therefore, we get the required result. □

Next, we shall prove Lemma 4.5 using Lemma 2.2 and Corollary 3.4. We need the following notation.

Let $A \geqslant V_\varphi(f)$ and $B \geqslant V_\psi(g)$. Define $\omega(x) = V_\varphi(f; [a,x])$ and $\kappa(x) = V_\psi(g; [a,x])$. Let $\lambda = \varphi^{-1}$, $\mu = \psi^{-1}$. Hence, for any $\xi, \eta \in [a,b]$ with $\eta > \xi$,

$$\lambda(\omega(\eta) - \omega(\xi)) = \varphi^{-1}(\omega(\eta) - \omega(\xi)) \geqslant |f(\eta) - f(\xi)|.$$

Similarly, $\mu(\kappa(\eta) - \kappa(\xi)) \geqslant |g(\eta) - g(\xi)|$.

Let $E_v = \{x_1, x_2, \ldots, x_{n_v}\}$ be given as in Lemma 2.1 with $v \geqslant 1$ and $[\alpha, \beta] = [a,b]$. Then $\#(E_v) \leqslant 2^{v+1}$. Furthermore,

$$|f(\eta) - f(\xi)| \leqslant \lambda(\omega(\eta) - \omega(\xi)) \leqslant \lambda(A2^{-v}) = \varphi^{-1}(A2^{-v})$$

and

$$|g(\eta) - g(\xi)| \leqslant \mu(\kappa(\eta) - \kappa(\xi)) \leqslant \mu(B2^{-v}) = \psi^{-1}(B2^{-v})$$

for any $\eta, \xi \in (x_k, x_{k+1})$ with $\eta > \xi$, $k = 1, 2, \ldots, n_v - 1$. The above is equivalent to (1) mentioned before Definition 3.1.

From now onwards, a division $D = \{(\xi_i, [u_i, v_i])\}_{i=1}^n$ is always denoted by $D = \{(\xi, [u,v])\}$.

Lemma 4.5. *Let $f \in \mathrm{BV}_\varphi[a,b]$ and $g \in \mathrm{BV}_\psi[a,b]$ with $\sum\limits_{n=1}^{\infty} \varphi^{-1}(1/n)\psi^{-1}(1/n) < \infty$. Let $v \geqslant 1$ be fixed and $E_v = \{x_1, x_2, \ldots, x_{n_v}\}$ given as above. Suppose $D = \{(\xi, [u,v])\}$ and $D' = \{(\xi', [u',v'])\}$ are two partial divisions of $[a,b]$ such that $\bigcup[u,v] = \bigcup[u',v']$ and $[u,v] \subset (x_k, x_{k+1})$, $[u',v'] \subset (x_k, x_{k+1})$. Then for any $\xi \in [u,v]$, $\xi' \in [u',v']$, we have*

$$\Big|(D)\sum f(\xi)(g(v)-g(u)) - (D')\sum f(\xi')(g(v')-g(u'))\Big| \leqslant 52 \sum_{n=2^{v-1}}^{\infty} \varphi^{-1}\Big(\frac{A}{n}\Big)\psi^{-1}\Big(\frac{B}{n}\Big),$$

where $v \geqslant 1$.

P r o o f. Let $D_k = \{(\xi, [u,v]) \in D;\ [u,v] \subseteq (x_k, x_{k+1})\}$ and $D'_k = \{(\xi', [u',v']) \in D';\ [u',v'] \subseteq (x_k, x_{k+1})\}$, $k = 1, 2, \ldots, n_v - 1$. It is clear that $|f(\xi) - f(\xi')| < \varphi^{-1}(A2^{-v})$. Note that $\bigcup\{[u,v];\ [u,v] \subset (x_k, x_{k+1})\} = \bigcup\{[u',v']\colon [u',v'] \subset (x_k, x_{k+1})\} =: [\alpha, \beta]$. Applying Corollary 3.4, for any ξ, ξ' we have

$$\Big|(D_k)\sum f(\xi)(g(v)-g(u)) - (D'_k)\sum f(\xi')(g(v')-g(u'))\Big| \leqslant 13 \sum_{n=1}^{\infty} \varphi^{-1}\Big(\frac{A2^{-v}}{n}\Big)\psi^{-1}\Big(\frac{B2^{-v}}{n}\Big)$$

for $k = 1, 2, \ldots, n_v - 1$.

Note that $D = \bigcup\limits_{k=1}^{n_v-1} D_k$ and $n_v \leqslant 2^{v+1}$. Hence, by Lemma 2.2 (i),

$$\begin{aligned}\Big|(D)\sum f(\xi)(g(v)-g(u)) &- (D')\sum f(\xi')(g(v')-g(u'))\Big| \\ &\leqslant 13(2^{v+1}-1)\sum_{n=1}^{\infty} \varphi^{-1}\Big(\frac{A2^{-v}}{n}\Big)\psi^{-1}\Big(\frac{B2^{-v}}{n}\Big) \\ &\leqslant 13(2^{v+1}-1)\frac{1}{2^{v-1}} \sum_{n=2^{v-1}}^{\infty} \varphi^{-1}\Big(\frac{A}{n}\Big)\psi^{-1}\Big(\frac{B}{n}\Big) \\ &\leqslant 52 \sum_{n=2^{v-1}}^{\infty} \varphi^{-1}\Big(\frac{A}{n}\Big)\psi^{-1}\Big(\frac{B}{n}\Big).\end{aligned}$$

The following existence theorem is proved in [12] by the Moore-Pollard approach. Now we will prove it by the Henstock-Kurzweil approach.

Theorem 4.6 (Existence Theorem). *Let $f \in \mathrm{BV}_\varphi[a,b]$ and $g \in \mathrm{BV}_\psi[a,b]$. Suppose that $\sum_{n=1}^{\infty} \varphi^{-1}(1/n)\psi^{-1}(1/n) < \infty$. Then $\int_a^b f\,\mathrm{d}g$ exists.*

Proof. First, let $A \geqslant V_\varphi(f;[a,b])$ and $B \geqslant V_\psi(f;[a,b])$. Then by Lemma 2.3, $\sum_{n=1}^{\infty} \varphi^{-1}(A/n)\psi^{-1}(B/n) < \infty$.

Let $\varepsilon > 0$, choose v such that $\sum_{n=2^{v-1}}^{\infty} \lambda(A/n)\mu(B/n) \leqslant \varepsilon/52$.

Let $E_v = \{x_1, x_2, \ldots, x_{n_v}\}$ be given as in Lemma 2.1. Let δ' be given as in Lemma 4.4 with $E = E_v$. Let δ be a positive function defined on $[a,b]$ with $\delta(x) < \delta'$ for all $x \in [a,b]$ such that if $D = \{(\xi,[u,v])\}$ is a δ-fine division of $[a,b]$, then $[u,v] \subset (\xi-\delta', \xi+\delta')$ and $\xi \in E_v$, $[u,v] \subset (x_k, x_{k+1})$ and $\xi \in (x_k, x_{k+1})$, $k = 1,2,\ldots,n_v-1$. Now let $D = \{(\xi,[u,v])\}$ and $D' = \{(\xi',[u',v'])\}$ be two δ-fine divisions of $[a,b]$. Let $D = D_1 \cup D_2$, $D' = D_1' \cup D_2'$ where $D_1 = \{(\xi,[u,v]) \in D;\ \xi \in E_v\}$, $D_1' = \{(\xi',[u',v']) \in D;\ \xi' \in E_v\}$, $D_2 = D \setminus D_1$ and $D_2' = D' \setminus D_1'$. Suppose $(\xi,[u,v]) \in D_2$ and $x_i + \delta(x_i) \in [u,v]$ (or $x_i - \delta(x_i) \in [u,v]$). Then we divide $[u,v]$ into two parts $[u, x_i+\delta(x_i)], [x_i+\delta(x_i), v]$ ($[u, x_i-\delta(x_i)], [x_i-\delta(x_i), v]$, respectively).

Let $\overline{D}_1$ be the union of D_1 and $(\xi,[u,x_i+\delta(x_i)]), (\xi,[x_i-\delta(x_i),v])$. Let $\overline{D}_2 = D \setminus \overline{D}_1$. Similarly, we construct $\bar{D}'_1$ and $\bar{D}'_2 = D' \setminus \bar{D}'_1$.

Then, by Lemmas 4.4 and 4.5, we get

$$\begin{aligned}
&\Big|(D)\sum f(\xi)(g(v)-g(u)) - (D')\sum f(\xi')(g(v')-g(u'))\Big| \\
&\quad\leqslant \Big|(\overline{D}_1)\sum f(\xi)(g(v)-g(u)) - (\bar{D}'_1)\sum f(\xi)(g(v)-g(u))\Big| \\
&\qquad + \Big|(\overline{D}_2)\sum f(\xi)(g(v)-g(u)) - (\overline{D}_2')\sum f(\xi')(g(v')-g(u'))\Big| \\
&\quad\leqslant 8\|f\|_\infty\varepsilon + \varepsilon,
\end{aligned}$$

where $\|f\|_\infty = \sup\{f(x);\ x \in [a,b]\}$. Thus $\int_a^b f\,\mathrm{d}g$ exists. □

5. Approximation

In this section we show that $\int_a^b f\,\mathrm{d}g$ can be approximated by $\int_a^b s\,\mathrm{d}g$, where s is a step function. This approximation theorem can be found in [12].

Theorem 5.1. *Let $f \in \mathrm{BV}_\varphi[a,b]$, $g \in \mathrm{BV}_\psi[a,b]$ and $\sum_{n=1}^{\infty} \varphi^{-1}(1/n)\psi^{-1}(1/n) < \infty$. Then, given any $\varepsilon > 0$, there exists a step function s on $[a,b]$ such that $|\int_a^b (f-s)\,\mathrm{d}g| \leqslant \varepsilon$.*

P r o o f. First, let $A \geqslant V_\varphi(f;[a,b])$ and $B \geqslant V_\psi(f;[a,b])$. Then, by Lemma 2.3, $\sum\limits_{n=1}^{\infty} \varphi^{-1}(A/n)\psi^{-1}(B/n) < \infty$.

Let $E_v = \{x_1, x_2, \ldots, x_{n_v}\}$ be given as in Lemma 4.5 with $v \geqslant 1$. Define

$$s(x) = \sum_{k=1}^{n_v} f(x_k)\chi_{\{x_k\}}(x) + \sum_{k=1}^{n_v-1} f(x_k+)\chi_{(x_k,x_{k+1})}(x).$$

Then $(f-s)(x_k) = 0$ for all $x_k \in E_v$. By Theorem 4.6, $\int_a^b f\,\mathrm{d}g - \int_a^b s\,\mathrm{d}g = \int_a^b (f-s)\,\mathrm{d}g$ exists. Let $\varepsilon > 0$; there exists a positive function δ on $[a,b]$ such that whenever $D = \{(\xi,[u,v])\}$ is a δ-fine division of $[a,b]$, x_i is a tag for every $i = 1,2,\ldots,n_v$, and

$$\Bigl|\int_a^b (f-s)\,\mathrm{d}g - (D)\sum (f-s)(\xi)(g(v)-g(u))\Bigr| \leqslant \frac{\varepsilon}{2}.$$

By Lemma 4.5, and all x_i being tags, we have

$$\Bigl|(D)\sum (f-s)(\xi)(g(v)-g(u))\Bigr| = \Bigl|(D)\sum_{\xi\notin E_v} (f-s)(\xi)(g(v)-g(u))\Bigr| \leqslant 52\sum_{n=2^{v-1}}^{\infty} \lambda\Bigl(\frac{A}{n}\Bigr)\mu\Bigl(\frac{B}{n}\Bigr) \text{ for } v \geqslant 1.$$

Therefore

$$\Bigl|\int_a^b (f-s)\,\mathrm{d}g\Bigr| \leqslant \frac{\varepsilon}{2} + 52\sum_{n=2^{v-1}}^{\infty} \lambda\Bigl(\frac{A}{n}\Bigr)\mu\Bigl(\frac{B}{n}\Bigr).$$

Choosing v big enough, we get the required result. □

Corollary 5.2. *Let $f \in \mathrm{BV}_\varphi[a,b], g \in \mathrm{BV}_\psi[a,b]$ and $\sum\limits_{n=1}^{\infty} \varphi^{-1}(1/n)\psi^{-1}(1/n) < \infty$. Then*

$$\Bigl|\int_a^b f\,\mathrm{d}g\Bigr| \leqslant 6\sum_{n=1}^{\infty} \varphi^{-1}\Bigl(\frac{V_\varphi(f)}{n}\Bigr)\psi^{-1}\Bigl(\frac{V_\psi(g)}{n}\Bigr) + f(a)(g(a+)-g(a)) + f(a+)(g(b-)-g(a+)) + f(b)(g(b)-g(b-)).$$

P r o o f. Let $\varepsilon > 0$. By Theorem 5.1 there exists a step function s on $[a,b]$ such that $|\int_a^b (f-s)\,\mathrm{d}g| \leqslant \varepsilon$. Hence

$$\Bigl|\int_a^b f\,\mathrm{d}g\Bigr| \leqslant \varepsilon + \Bigl|\int_a^b s\,\mathrm{d}g\Bigr|.$$

By Theorem 3.3,

$$\left|\int_a^b s\,\mathrm{d}g\right| \leqslant 6\sum_{n=1}^{\infty}\lambda\Big(\frac{V_\varphi(f)}{n}\Big)\mu\Big(\frac{V_\psi(g)}{n}\Big)+\left|\int_a^b s_{E_0}\,\mathrm{d}g\right|.$$

Note that $\int_a^b s_{E_0}\,\mathrm{d}g = f(a)(g(a+)-g(a))+f(a+)(g(b-)-g(a+))+f(b)(g(b)-g(b-))$. Hence we get the required result. □

6. Integration by parts

A general result for integration by parts in the setting of Henstock-Kurzweil integrals of Stieltjes type can be found in [10]. In this section, we will prove this result in more concrete forms.

For any partial division $D=\{(\xi,[u,v])\}$ on $[a,b]$, define

$$S_-(f,g,D)=(D)\sum(f(\xi)-f(u))(g(\xi)-g(u)),$$
$$S_+(f,g,D)=(D)\sum(f(v)-f(\xi))(g(v)-g(\xi))$$

and

$$S(f,g,D)=S_-(f,g,D)-S_+(f,g,D).$$

We say that $S_-(f,g)$ exists if there exists $S^{(1)}$ such that for every $\varepsilon>0$ there exists a positive function δ on $[a,b]$ such that when D is a δ-fine division of $[a,b]$, we have

$$|S_-(f,g,D)-S^{(1)}|\leqslant\varepsilon.$$

We then denote $S^{(1)}$ by $S_-(f,g)$. Similarly, we can define $S_+(f,g)$ and $S(f,g)$. Clearly, if two of $S_-(f,g), S_+(f,g)$ and $S(f,g)$ exist, then the third exists and

$$S(f,g)=S_-(f,g)-S_+(f,g).$$

Lemma 6.1. *Let $f\in \mathrm{BV}_\varphi[a,b]$ and $g\in\mathrm{BV}_\psi[a,b]$ with $\sum\limits_{n=1}^{\infty}\varphi^{-1}(1/n)\psi^{-1}(1/n)<\infty$. Then*

$$S_+(f,g)=\sum_{i=1}^{\infty}(f(t_i+)-f(t_i))(g(t_i+)-g(t_i))$$

and

$$S_-(f,g)=\sum_{i=1}^{\infty}(f(t_i)-f(t_i-))(g(t_i)-g(t_i-)),$$

where t_i are the common points of discontinuity of f and g and the above series converge absolutely.

Proof. Let $\varepsilon > 0$, let v be a positive integer such that $\sum_{n=2^{v-1}}^{\infty} \varphi^{-1}(A/n)\psi^{-1}(B/n) \leqslant \varepsilon/52$. Let $E_v = \{x_1, x_2, \ldots, x_{n_v}\}$ be given as in Lemma 2.1. E_v may contain some points of $\{t_i\}_{i=1}^{\infty}$. We may assume that there exists a positive integer N such that $t_j \notin E_v$ whenever $j \geqslant N$. Now take any two positive integers $m, n \geqslant N$ with $m < n$. Let δ be a positive number such that for every $i = m, m+1, \ldots, n$, if $t_i \in (x_j, x_{j+1})$ for some j, then $(t_i, t_i + \delta) \subset (x_j, x_{j+1})$ and $\{(t_i, t_i + \delta)\}_{i=m}^{n}$ are non-overlapping intervals. Let $\eta_i \in (t_i, t_i + \delta)$ for all $i = m, m+1, \ldots, n$, and $D = \{t_i, [t_i, \eta_i]\}_{i=m}^{n}$ and $D' = \{\eta_i, [t_i, \eta_i]\}_{i=m}^{n}$. From the definition of D and D' it is clear that D and D' are partial divisions of $[a, b]$ and satisfy the condition of Theorem 4.5. Hence, we have

$$\begin{aligned}\Bigl|\sum_{i=m}^{n}&(f(\eta_i) - f(t_i))(g(\eta_i) - g(t_i))\Bigr| \\ &= \Bigl|(D')\sum f(\eta_i)(g(\eta_i) - g(t_i)) - (D)\sum f(t_i)(g(\eta_i) - g(t_i))\Bigr| \\ &\leqslant 52 \sum_{n=2^{v-1}}^{\infty} \varphi^{-1}\Bigl(\frac{A}{n}\Bigr)\psi^{-1}\Bigl(\frac{B}{n}\Bigr) \leqslant 52\frac{\varepsilon}{52} = \varepsilon.\end{aligned}$$

Then

$$\Bigl|\sum_{i=m}^{n}(f(t_i+) - f(t_i))(g(t_i+) - g(t_i))\Bigr| \leqslant \varepsilon.$$

Observe that D and D' are partial divisions. Therefore

$$\sum_{i=m}^{n} |(f(t_i+) - f(t_i))(g(t_i+) - g(t_i))| \leqslant 2\varepsilon,$$

where $m, n \geqslant N$.

Hence $S_+(f, g)$ converges absolutely. Similarly, $S_-(f, g)$ converges absolutely. $\square$

Lemma 6.2. *Let $f \in \mathrm{BV}_{\varphi}[a, b]$ and $g \in \mathrm{BV}_{\psi}[a, b]$, $E = \{x_1, x_2, \ldots, x_n\} \supseteq E_0$, and $\varepsilon > 0$. Then there exists a constant $\delta' > 0$ such that for any finite collection of disjoint subintervals $\{[u_i, v_i]\}_{i=1}^{n}$ with $[u_i, v_i] \subset (x_i, x_i + \delta')$ for each i or $[u_i, v_i] \subset (x_i - \delta', x_i)$ for each i, we have*

$$\sum_{i=1}^{n} |f(v_i) - f(u_i)||g(v_i) - g(u_i)| \leqslant \varepsilon,$$

$$\sum_{i=1}^{n} |f(v_i) - f(u_i)| \leqslant \frac{\varepsilon}{V_{\psi}(B)}$$

and

$$\sum_{i=1}^{n} |g(v_i) - g(u_i)| \leqslant \frac{\varepsilon}{V_\psi(A)}.$$

P r o o f. The proof is similar to that of Lemma 4.4. Let $\varepsilon > 0$ be given. First, observe that f and g are regulated functions. Therefore, there exists a constant $\delta' > 0$ such that

$$|g(t) - g(x_i-)| \leqslant \min\left\{\left[\frac{\varepsilon}{2n}\right]^{\frac{1}{2}}, \frac{\varepsilon}{2nV_\varphi(A)}\right\} \quad \text{whenever } 0 < x_i - t < \delta',$$

$$|g(x_i+) - g(t)| \leqslant \min\left\{\left[\frac{\varepsilon}{2n}\right]^{\frac{1}{2}}, \frac{\varepsilon}{2nV_\varphi(A)}\right\} \quad \text{whenever } 0 < t - x_i < \delta',$$

$$|f(t) - f(x_i-)| \leqslant \min\left\{\left[\frac{\varepsilon}{2n}\right]^{\frac{1}{2}}, \frac{\varepsilon}{2nV_\psi(B)}\right\} \quad \text{whenever } 0 < x_i - t < \delta'$$

and

$$|f(x_i+) - f(t)| \leqslant \min\left\{\left[\frac{\varepsilon}{2n}\right]^{\frac{1}{2}}, \frac{\varepsilon}{2nV_\psi(B)}\right\} \quad \text{whenever } 0 < t - x_i < \delta'$$

for each i. Therefore, the required result follows. □

Lemma 6.3. *Let $f \in \mathrm{BV}_\varphi[a,b]$ and $g \in \mathrm{BV}_\psi[a,b]$ with $\sum\limits_{n=1}^{\infty} \varphi^{-1}(1/n)\psi^{-1}(1/n) < \infty$. Let $\varepsilon > 0$. If $E_v = \{x_1, x_2, \ldots, x_{n_v}\}$ is the set given in Lemma 2.1 and $\{t_j\}_{j=1}^m \subseteq E_v$, where $\{t_j\}_{j=1}^m$ are all common points of discontinuity of f and g such that $\sum\limits_{n=2^{v-1}}^{\infty} \varphi^{-1}(A/n)\psi^{-1}(B/n) < \varepsilon/312$ and $\sum\limits_{j=m+1}^{\infty} |(f(t_j) - f(t_j-))(g(t_j) - g(t_j-))| \leqslant \varepsilon/6$, then there exists a positive real number δ' such that for any δ'-fine partial division $D = \{(x_i, [u_i, v_i])\}_{i=1}^{n_v}$ of $[a,b]$ we have*

$$|S(f,g,D) - (S_+(f,g) - S_-(f,g))| \leqslant \frac{2}{3}\varepsilon.$$

P r o o f. Applying Lemma 6.2 to $\varepsilon/18$ and $E = E_v$, we get a positive constant δ'. Let $D = \{(x_i, [u_i, v_i])\}_{i=1}^{n_v}$, then

$$S_-(f,g,D) = \sum_{i=1}^{n_v} (f(x_i) - f(u_i))(g(x_i) - g(u_i))$$
$$= \sum_{i=1}^{n_v} [(f(x_i) - f(x_i-)) + (f(x_i-) - f(u_i))][(g(x_i) - g(x_i-)) + (g(x_i-) - g(u_i))]$$

$$= \sum_{i=1}^{n_v}(f(x_i) - f(x_i-))(g(x_i) - g(x_i-)) + \sum_{i=1}^{n_v}(f(x_i) - f(x_i-))(g(x_i-) - g(u_i))$$
$$+ \sum_{i=1}^{n_v}(f(x_i-) - f(u_i))(g(x_i) - g(x_i-)) + \sum_{i=1}^{n_v}(f(x_i-) - f(u_i))(g(x_i-) - g(u_i)).$$

Let $F = E_v \setminus \{t_j\}_{j=1}^m$, then F is the set of points in E_v which are not common points of discontinuity of f and g. Hence

$$\sum_{x_i \in F}(f(x_i) - f(x_i-))(g(x_i) - g(x_i-)) = 0.$$

Consider

$$\sum_{i=1}^{n_v}(f(x_i) - f(x_i-))(g(x_i) - g(x_i-))$$
$$= \sum_{x_i \in F}(f(x_i) - f(x_i-))(g(x_i) - g(x_i-)) + \sum_{x_i \notin F}(f(x_i) - f(x_i-))(g(x_i) - g(x_i-))$$
$$= 0 + \sum_{x_i \notin F}(f(x_i) - f(x_i-))(g(x_i) - g(x_i-))$$
$$= \sum_{j=1}^{m}(f(t_j) - f(t_j-))(g(t_j) - g(t_j-)).$$

Then

$$S_-(f, g, D) = \sum_{i=1}^{n_v}(f(x_i) - f(u_i))(g(x_i) - g(u_i))$$
$$= \sum_{j=1}^{m}(f(t_j) - f(t_j-))(g(t_j) - g(t_j-)) + \sum_{i=1}^{n_v}(f(x_i) - f(x_i-))(g(x_i-) - g(u_i))$$
$$+ \sum_{i=1}^{n_v}(f(x_i-) - f(u_i))(g(x_i) - g(x_i-)) + \sum_{i=1}^{n_v}(f(x_i-) - f(u_i))(g(x_i-) - g(u_i)).$$

Therefore

$$|S_-(f, g, D) - S_-(f, g)|$$
$$= \left| \sum_{i=1}^{n_v}(f(x_i) - f(u_i))(g(x_i) - g(u_i)) - \sum_{j=1}^{\infty}(f(t_j) - f(t_j-))(g(t_j) - g(t_j-)) \right|$$
$$\leqslant \left| \sum_{i=1}^{n_v}(f(x_i) - f(u_i))(g(x_i) - g(u_i)) - \sum_{j=1}^{m}(f(t_j) - f(t_j-))(g(t_j) - g(t_j-)) \right|$$
$$+ \left| \sum_{j=m+1}^{\infty}(f(t_j) - f(t_j-))(g(t_j) - g(t_j-)) \right|$$

$$= \left| \sum_{i=1}^{n_v} (f(x_i) - f(x_i-))(g(x_i-) - g(u_i)) \right| + \left| \sum_{i=1}^{n_v} (f(x_i-) - f(u_i))(g(x_i) - g(x_i-)) \right|$$
$$+ \left| \sum_{i=1}^{n_v} (f(x_i-) - f(u_i))(g(x_i-) - g(u_i)) \right| + \frac{\varepsilon}{6}.$$

By Lemma 6.2 we have

$$\left| \sum_{i=1}^{n_v} (f(x_i-) - f(u_i))(g(x_i-) - g(u_i)) \right| \leqslant \frac{\varepsilon}{18},$$
$$\left| \sum_{i=1}^{n_v} (f(x_i) - f(x_i-))(g(x_i-) - g(x_i)) \right| \leqslant \frac{\varepsilon}{18 V_\varphi(A)} V_\varphi(A) = \frac{\varepsilon}{18}$$

and

$$\left| \sum_{i=1}^{n_v} (f(x_i) - f(x_i-))(g(x_i) - g(x_i-)) \right| \leqslant \frac{\varepsilon}{18 V_\psi(B)} V_\psi(B) = \frac{\varepsilon}{18}.$$

Thus

$$|S_-(f,g,D) - S_-(f,g)| \leqslant \frac{\varepsilon}{18} + \frac{\varepsilon}{18} + \frac{\varepsilon}{18} + \frac{\varepsilon}{6} \leqslant \frac{\varepsilon}{3}.$$

Similarly,

$$|S_+(f,g,D) - S_+(f,g)| \leqslant \frac{\varepsilon}{3}.$$

Hence

$$|S(f,g,D) - (S_+(f,g) - S_-(f,g))|$$
$$\leqslant |S_-(f,g,D) - S_-(f,g)| + |S_+(f,g,D) - S_+(f,g)| \leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2}{3}\varepsilon.$$

□

Lemma 6.4. *Let $f \in \mathrm{BV}_\varphi[a,b]$ and $g \in \mathrm{BV}_\psi[a,b]$ with $\sum\limits_{n=1}^{\infty} \varphi^{-1}(1/n)\psi^{-1}(1/n) < \infty$. Then for any given $\varepsilon > 0$ there exists a positive function δ such that for any δ-fine division D of $[a,b]$ we have*

$$|S(f,g,D) - (S_+(f,g) - S_-(f,g))| \leqslant \varepsilon.$$

Proof. Let $\varepsilon > 0$, choose v such that $\sum\limits_{n=2^{v-1}}^{\infty} \varphi^{-1}(A/n)\psi^{-1}(B/n) \leqslant \varepsilon/312$. Let $E_v = \{x_1, x_2, \ldots, x_{n_v}\}$ be given as in Lemma 2.1. Applying Lemma 6.3 to $E = E_v$, we get a positive constant δ'. Let δ be a positive function defined on $[a,b]$ with $\delta(x) < \delta'$ for all $x \in [a,b]$ such that if $D = \{(\xi, [u,v])\}$ is a δ-fine division of $[a,b]$, then $[u,v] \subset (\xi-\delta', \xi+\delta')$ when $\xi \in E_v$ and $[u,v] \subset (x_k, x_{k+1})$ when $\xi \in (x_k, x_{k+1}), k = 1, 2, \ldots, n_v - 1$. Now let $D = \{(\xi, [u,v])\}$ be a δ-fine division of $[a,b]$. Let $D = D_1 \cup D_2$, where $D_1 = \{(\xi, [u,v]) \in D\colon \xi \in E_v\}$, $D_2 = D \setminus D_1$. Hence, by Lemma 6.3,

$$\begin{aligned}
&|S(f,g,D) - (S_+(f,g) - S_-(f,g))| \\
&\quad \leqslant |S_-(f,g,D_2)| + |S_+(f,g,D_2)| + |S(f,g,D_1) - (S_+(f,g) - S_-(f,g))| \\
&\quad \leqslant |S_-(f,g,D_2)| + |S_+(f,g,D_2)| + \tfrac{2}{3}\varepsilon.
\end{aligned}$$

By Lemma 4.5, we have

$$\begin{aligned}
|S_-(f,g,D_2)| &= \Big|(D_2)\sum (f(\xi) - f(u))(g(\xi) - g(u))\Big| \\
&= \Big|(D_2)\sum f(\xi)(g(\xi) - g(u)) - (D_2)\sum f(u)(g(\xi) - g(u))\Big| \\
&\leqslant 52 \sum_{n=2^{v-1}}^{\infty} \varphi^{-1}\Big(\frac{A}{n}\Big)\psi^{-1}\Big(\frac{B}{n}\Big) \leqslant \frac{\varepsilon}{6}
\end{aligned}$$

and

$$\begin{aligned}
|S_+(f,g,D_2)| &= \Big|(D_2)\sum (f(v) - f(\xi))(g(v) - g(\xi))\Big| \\
&\leqslant 52 \sum_{n=2^{v-1}}^{\infty} \varphi^{-1}\Big(\frac{A}{n}\Big)\psi^{-1}\Big(\frac{B}{n}\Big) \leqslant \frac{\varepsilon}{6}.
\end{aligned}$$

Hence,

$$\begin{aligned}
|S(f,g,D) - (S_+(f,g) - S_-(f,g))| &\leqslant |S_-(f,g,D_2)| + |S_+(f,g,D_2)| + \tfrac{2}{3}\varepsilon \\
&\leqslant \tfrac{1}{6}\varepsilon + \tfrac{1}{6}\varepsilon + \tfrac{2}{3}\varepsilon = \varepsilon.
\end{aligned}$$

□

We can verify that $S_-(f,g)$, $S_+(f,g)$ exist and $S_-(f,g) = \sum\limits_{i=1}^{\infty}(f(t_i+) - f(t_i))(g(t_i+) - g(t_i)), S_+(f,g) = \sum\limits_{i=1}^{\infty}(f(t_i) - f(t_i-))(g(t_i) - g(t_i-))$.

Theorem 6.5. *Let $f \in \mathrm{BV}_\varphi[a,b]$ and $g \in \mathrm{BV}_\psi[a,b]$ with $\sum\limits_{m=1}^{\infty}\varphi^{-1}(1/m)\psi^{-1}(1/m) < \infty$. Then*

$$\int_a^b f\,\mathrm{d}g + \int_a^b g\,\mathrm{d}f = f(b)g(b) - f(a)g(a) + S(f,g),$$

where $S(f,g) = \sum\limits_{i=1}^{\infty}(f(t_i+)-f(t_i))(g(t_i+)-g(t_i)) - \sum\limits_{i=1}^{\infty}(f(t_i)-f(t_i-))(g(t_i)-g(t_i-))$ *and* $\{t_i\}$ *are all common points of discontinuity of* f *and* g.

Proof. Since $S(f,g) = S_+(f,g) - S_-(f,g)$, by Lemma 6.1, $S(f,g)$ exists. Let $\varepsilon > 0$ be given and let $f, g\colon [a,b] \to \mathbb{R}$. Then there exists a positive function δ_1 on $[a,b]$ such that for any δ_1-fine partial division $D' = \{([u_i, v_i], \xi_i)\}$ of $[a,b]$,

$$|S(f,g,D') - S(f,g)| \leqslant \frac{\varepsilon}{2}.$$

Since f is integrable with respect to g, there exists a positive function δ_2 on $[a,b]$ such for any δ_2-fine division $D'' = \{([t_i, t_{i+1}], \xi_i)\}$ of $[a,b]$, we have

$$\Bigl|(D'')\sum f(\xi_i)(g(t_{i+1}) - g(t_i)) - \int_a^b f\,\mathrm{d}g\Bigr| \leqslant \frac{\varepsilon}{2}.$$

Choose $\delta(\xi) = \min\{\delta_1(\xi), \delta_2(\xi)\}$. Let $D = \{([t_i, t_{i+1}], \xi_i)\}$ be a δ-fine partial division of $[a,b]$. We can see that

$$\begin{aligned}
&\Bigl|\Bigl((D)\sum g(\xi_i)(f(t_{i+1}) - f(t_i))\Bigr) - \Bigl(f(b)g(b) - f(a)g(a) + S(f,g) - \int_a^b f\,\mathrm{d}g\Bigr)\Bigr| \\
&= \Bigl|(D)\sum \Bigl(g(\xi_i)(f(t_{i+1}) - f(t_i)) - f(t_{i+1})g(t_{i+1}) + f(t_i)g(t_i) \\
&\qquad + \int_{t_i}^{t_{i+1}} f\,\mathrm{d}g\Bigr) - S(f,g)\Bigr| \\
&= \Bigl|(D)\sum -f(\xi_i)(g(t_{i+1}) - g(t_i)) + (f(\xi_i) - f(t_i))(g(\xi_i) - g(t_i)) \\
&\quad - (f(t_{i+1}) - f(\xi_i))(g(t_{i+1}) - g(\xi_i)) + \int_{t_i}^{t_{i+1}} f\,\mathrm{d}g - S(f,g)\Bigr| \\
&\leqslant \Bigl|(D)\sum f(\xi_i)(g(t_{i+1}) - g(t_i)) - \int_{t_i}^{t_{i+1}} f\,\mathrm{d}g\Bigr| + |S(f,g,D) - S(f,g)| \\
&\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus, we can conclude that g is integrable to $f(b)g(b) - f(a)g(a) + S(f,g) - \int_a^b f\,\mathrm{d}g$ on $[a,b]$ with respect to f.

Hence, we have

$$\int_a^b f\,\mathrm{d}g + \int_a^b g\,\mathrm{d}f = f(b)g(b) - f(a)g(a) + S(f,g).$$

□

7. Convergence theorem

In this section we will use Young's idea, see [11], [12], to prove some convergence theorems for our setting.

Definition 7.1 (Two-norm convergence). A sequence $\{f^{(n)}\}$ of functions in $\mathrm{BV}_\varphi[a,b]$ is said to be two-norm convergent to f if

(i) $f^{(n)}$ is uniformly convergent to f on $[a,b]$, and

(ii) $V_\varphi(f^{(n)}) \leqslant A$ for every $n = 1, 2, \ldots$.

In symbols, we denote the two-norm convergence by $f^{(n)} \twoheadrightarrow f$.

It is clear that $\mathrm{BV}_\varphi[a,b]$ is complete under two-norm convergence, i.e., if $f^{(n)} \in \mathrm{BV}_\varphi[a,b]$, $n = 1, 2, \ldots$, and $f^{(n)} \twoheadrightarrow f$, then $f \in \mathrm{BV}_\varphi[a,b]$.

We need the following two lemmas.

Lemma 7.2. *Let ϑ be a strictly decreasing continuous function on $(0,\infty)$ with $\lim\limits_{x\to\infty} \vartheta(x) = 0$ and let $\int_1^\infty \vartheta(x)\,\mathrm{d}x$ exist. Then there exists a strictly increasing continuous function ϱ on $[0,\infty)$ with $\lim\limits_{x\to\infty} \varrho(x) = \infty$ such that*

$$\lim_{x\to\infty} \frac{\varrho(x)}{x} = \infty \quad \textit{and} \quad \int_1^\infty \vartheta(x)\,\mathrm{d}\varrho(x) \textit{ exists.}$$

Proof. Since $\int_1^\infty \vartheta(x)\,\mathrm{d}x$ exists, there exists a positive function on $[0,\infty)$ $\iota(x) \geqslant 0$ with $\lim\limits_{x\to\infty} \iota(x) = \infty$ and $\iota(x) = 0$ for $x \leqslant 1$, such that $\int_1^\infty \vartheta(x)\iota(x)\,\mathrm{d}x$ and $\int_0^x \iota(t)\,\mathrm{d}t$ exist for every $x \in (0,\infty)$. Let

$$\varrho(x) = x + \int_0^x \iota(t)\,\mathrm{d}t.$$

Then ϱ is a strictly increasing function with $\lim\limits_{x\to\infty} \varrho(x) = \infty$. Therefore,

$$\int_1^\infty \vartheta(x)\,\mathrm{d}\varrho(x) = \int_1^\infty \vartheta(x)[1 + \iota(x)]\,\mathrm{d}x < \infty.$$

Now we shall prove that $\lim\limits_{x\to\infty} \varrho(x)/x = \infty$. Let $x > 2n$. Then $(x-n)/x > \frac{1}{2}$. By Mean-Value Theorem for integral, there exists $y \in (n,x)$ such that

$$\frac{1}{x-n}\int_n^x \iota(x)\,\mathrm{d}x = \iota(y).$$

Hence

$$\frac{\varrho(x)}{x}=1+\frac{1}{x}\int_0^x \iota(x)\,\mathrm{d}x \geqslant \frac{x-n}{x}\Big[\frac{1}{x-n}\int_n^x \iota(x)\,\mathrm{d}x\Big] \geqslant \frac{1}{2}\iota(y) \geqslant \frac{1}{2}\iota(n).$$

Since $\iota(n)\to\infty$ as $n\to\infty$, we have

$$\lim_{x\to\infty}\frac{\varrho(x)}{x}=\infty.$$

□

Corollary 7.3. *Let ϑ be a strictly decreasing continuous function on $(0,\infty)$ with $\lim\limits_{x\to\infty}\vartheta(x)=0$ and let $\int_1^\infty \vartheta(x)\,\mathrm{d}x$ exist. Then there exists a strictly increasing continuous function ς on $(0,\infty)$ with $\lim\limits_{x\to\infty}\varsigma(x)=\infty$, such that*

$$\lim_{x\to\infty}\frac{\varsigma(x)}{x}=0 \quad \textit{and} \quad \int_1^\infty \vartheta(\varsigma(x))\,\mathrm{d}x \quad \textit{exists.}$$

Proof. Let $\varsigma=\varrho^{-1}$, where ϱ is given in Lemma 7.2. Thus we get the required result. □

Lemma 7.4. *Suppose $\sum\limits_{n=1}^{\infty}\varphi^{-1}(1/n)\psi^{-1}(1/n)<\infty$. Then there exist two N-functions φ^*,ψ^* such that $\varphi^*(u)\leqslant\overline{\pi}(u)\varphi(u)$ and $\psi^*(u)\leqslant\overline{\gamma}(u)\psi(u)$, where $\overline{\pi},\overline{\gamma}$ are increasing and $\lim\limits_{x\to0}\overline{\pi}(x)=\lim\limits_{x\to0}\overline{\gamma}(x)=0$, with*

$$\sum_{n=1}^{\infty}(\varphi^*)^{-1}\Big(\frac{1}{n}\Big)(\psi^*)^{-1}\Big(\frac{1}{n}\Big)<\infty.$$

Proof. Given φ,ψ and $\sum\limits_{n=1}^{\infty}\varphi^{-1}(1/n)\psi^{-1}(1/n)<\infty$, we want to construct φ^*,ψ^* such that $\varphi^*(u)\leqslant\overline{\pi}(u)\varphi(u)$, $\psi^*(u)\leqslant\overline{\gamma}(u)\psi(u)$, where $\overline{\pi},\overline{\gamma}$ are increasing functions with $\lim\limits_{x\to0}\overline{\pi}(x)=\lim\limits_{x\to0}\overline{\gamma}(x)=0$ and

$$\sum_{n=1}^{\infty}(\varphi^*)^{-1}\Big(\frac{1}{n}\Big)(\psi^*)^{-1}\Big(\frac{1}{n}\Big)<\infty.$$

Let $\vartheta(u)=\varphi^{-1}(1/u)\psi^{-1}(1/u)$ for $u\in(0,\infty)$. Then ϑ satisfies the conditions of Corollary 7.3. Hence there exists a strictly increasing continuous function ς on $[0,\infty)$ with $\lim\limits_{x\to\infty}\varsigma(x)=\infty$, such that

$$\lim_{x\to\infty}\frac{\varsigma(x)}{x}=0 \quad \text{and} \quad \int_1^\infty \vartheta(\varsigma(x))\,\mathrm{d}x \quad \text{exists.}$$

Let $\theta(u) = u\varsigma(u^{-1})$ for $u \in (0,\infty)$ and $\theta(0) = 0$. Then $\lim_{u\to 0}\theta(u) = \lim_{u\to 0}\varsigma(u^{-1})/u^{-1} = 0$ and $u/\theta(u) = 1/\varsigma(u^{-1})$ is a strictly increasing continuous function on $(0,\infty)$.

Let $\Phi(u) = \varphi^{-1}(u/\theta(u))$, $\Psi(u) = \psi^{-1}(u/\theta(u))$, $\Phi(0) = 0$ and $\Psi(0) = 0$. Then Φ and Ψ are strictly increasing continuous functions on $[0,\infty)$. Furthermore, let $\varphi^* = (\Phi)^{-1}$ and $\psi^* = (\Psi)^{-1}$. Then

$$\begin{aligned}\sum_{n=1}^{\infty}(\varphi^*)^{-1}\Big(\frac{1}{n}\Big)(\psi^*)^{-1}\Big(\frac{1}{n}\Big) &= \sum_{n=1}^{\infty}\varphi^{-1}\Big(\frac{n^{-1}}{\theta(n^{-1})}\Big)\psi^{-1}\Big(\frac{n^{-1}}{\theta(n^{-1})}\Big)\\ &= \sum_{n=1}^{\infty}\varphi^{-1}\Big(\frac{1}{\varsigma(n)}\Big)\psi^{-1}\Big(\frac{1}{\varsigma(n)}\Big)\\ &= \sum_{n=1}^{\infty}\vartheta(\varsigma(n)) \leqslant \int_1^{\infty}\vartheta(\varsigma(x))\,\mathrm{d}x < \infty,\end{aligned}$$

since $\vartheta(\varsigma(x))$ is non-negative.

If $t = (\varphi^*)^{-1}(u^*) = \varphi^{-1}(u^*/\theta(u^*))$, then $\varphi^*(t) = u^*$. On the other hand, if $t = \varphi^{-1}(u)$, then $\varphi(t) = u$. Hence $u = u^*/\theta(u^*)$ and

$$\frac{\varphi^*(t)}{\varphi(t)} = \frac{u^*}{u} = \theta(u^*) = \theta(\varphi^*(t)) =: \pi(t);$$

clearly $\lim_{t\to 0}\pi(t) = \lim_{t\to 0}\theta(\varphi^*(t)) = 0$. Similarly, we have

$$\frac{\psi^*(t)}{\psi(t)} = \frac{u^*}{u} = \theta(u^*) = \theta(\psi^*(t)) =: \gamma(t),$$

and $\lim_{t\to 0}\pi(t) = \lim_{t\to 0}\theta(\varphi^*(t)) = 0$. Denoting by $\overline{\pi}(t), \overline{\gamma}(t)$ the upper bounds of $\pi(u), \gamma(u)$ for $0 < u \leqslant t$ we see that $\overline{\pi}, \overline{\gamma}$ are increasing functions. Then

$$\varphi^*(t) \leqslant \overline{\pi}(t)\varphi(t)$$

and

$$\psi^*(t) \leqslant \overline{\gamma}(t)\psi(t).$$

□

Let $D = \{[u,v]\}$ be a partition of an interval $[\alpha,\beta]$. By Lemma 7.4, we have

$$\begin{aligned}(D)\sum\varphi^*(|f(v)-f(u)|) &= (D)\sum\overline{\pi}(|f(v)-f(u)|)\varphi(|f(v)-f(u)|)\\ &\leqslant \overline{\pi}(2\|f\|_\infty)(D)\sum\varphi(|f(v)-f(u)|).\end{aligned}$$

Hence, if A and A^* are the φ-variation and φ^*-variation of f, respectively, on $[\alpha,\beta]$, we have

$$A^* \leqslant A\overline{\pi}(2\|f\|_\infty) \leqslant A\overline{\pi}(\varphi^{-1}(A)).$$

Theorem 7.5. *If $g \in \mathrm{BV}_\psi[a,b]$ and $\{f^{(n)}\}$ is two-norm convergent to f in $\mathrm{BV}_\varphi[a,b]$ with $\sum\limits_{m=1}^{\infty} \varphi^{-1}(1/m)\psi^{-1}(1/m) < \infty$, then $\int_a^b f\,\mathrm{d}g$ exists and*

$$\lim_{n\to\infty} \int_a^b f^{(n)}\,\mathrm{d}g = \int_a^b f\,\mathrm{d}g.$$

Proof. Let $\varepsilon > 0$ be given. Let $\{f^{(n)}\}$ be two-norm convergent to f in $\mathrm{BV}_\varphi[a,b]$ and $g \in \mathrm{BV}_\psi[a,b]$. By the convexity of $\mathrm{BV}_\varphi[a,b]$, $\frac{1}{2}(f^{(n)} - f) \in \mathrm{BV}_\varphi[a,b]$. Hence, $\int_a^b (f^{(n)} - f)\,\mathrm{d}g$ exists. Thus, there is a positive function δ_n such that for every δ_n-fine division $D = \{([t_i, t_{i+1}], \xi_i)\}$ of $[a,b]$,

$$\left|\left(\int_a^b (f^{(n)} - f)\,\mathrm{d}g\right) - (D)\sum (f^{(n)}(\xi_i) - f(\xi_i))(g(t_{i+1}) - g(t_i))\right| \leqslant \varepsilon.$$

Let

$$V_\varphi(\tfrac{1}{2}(f^{(n)} - f)) \leqslant V_\varphi(f^{(n)}) + V_\varphi(f) \leqslant A \text{ for every } n \text{ and } V_\psi(g) = B.$$

By Lemma 7.4, there exist two N-function φ^* and ψ^* such that $\varphi^*(u) \leqslant \overline{\pi}(u)\varphi(u)$ and $\psi^*(u) \leqslant \overline{\gamma}(u)\psi(u)$, where $\overline{\pi}, \overline{\gamma}$ are increasing and $\lim\limits_{x\to 0} \overline{\pi}(x) = \lim\limits_{x\to 0} \overline{\gamma}(x) = 0$, with

$$\sum_{n=1}^{\infty} (\varphi^*)^{-1}\Big(\frac{1}{n}\Big)(\psi^*)^{-1}\Big(\frac{1}{n}\Big) < \infty.$$

By Lemma 2.3, there exists a positive integer v such that

$$\sum_{n=v+1}^{\infty} (\varphi^*)^{-1}\Big(\frac{A\overline{\pi}(\varphi^{-1}(A))}{n}\Big)(\psi^*)^{-1}\Big(\frac{B\overline{\gamma}(\psi^{-1}(B))}{n}\Big) < \varepsilon.$$

For this v, choose $\tau > 0$ such that

$$(\varphi^*)^{-1}(A\overline{\pi}(\tau)) \leqslant \frac{\varepsilon}{v(\psi^*)^{-1}\Big(\dfrac{B\overline{\pi}(\psi^{-1}(B))}{v}\Big)}.$$

Hence for $n = 1, 2, \ldots, v$,

$$(\varphi^*)^{-1}\Big(\frac{A\overline{\pi}(\tau)}{n}\Big) \leqslant \frac{\varepsilon}{v(\psi^*)^{-1}\Big(\dfrac{B\overline{\pi}(\psi^{-1}(B))}{n}\Big)}.$$

Since $f^{(n)}$ converge to f uniformly on $[a,b]$, there is a positive integer N such that for every $n \geqslant N$, we have

$$\sup_{t\in[a,b]} \tfrac{1}{2}(|f^{(n)}(t) - f(t)|) = \|\tfrac{1}{2}(f^{(n)} - f)\|_\infty < \min\{\varepsilon, \tfrac{1}{2}\tau\}.$$

We may assume that when $n \geqslant N$, then $|(f^{(n)} - f)(a)(g(a+) - g(a)) + (f^{(n)} - f)(a+)(g(b-) - g(a+)) + (f^{(n)} - f)(b)(g(b) - g(b-))| \leqslant \varepsilon$.

Hence for $n \geqslant N$, applying Corollary 5.2 to $\frac{1}{2}(f^{(n)} - f)$, we get

$$\begin{aligned}
\Big|\int_a^b f^{(n)}\,\mathrm{d}g - \int_a^b f\,\mathrm{d}g\Big| &= 2\Big|\int_a^b \frac{f^{(n)} - f}{2}\,\mathrm{d}g\Big| \\
&\leqslant 2\cdot 6\sum_{n=1}^{\infty}(\varphi^*)^{-1}\Big(\frac{V_{\varphi^*}(\frac{1}{2}(f^{(n)}-f))}{n}\Big)(\psi^*)^{-1}\Big(\frac{V_{\psi^*}(g)}{n}\Big) + \varepsilon \\
&\leqslant 12\sum_{n=1}^{v}(\varphi^*)^{-1}\Big(\frac{V_{\varphi^*}(\frac{1}{2}(f^{(n)}-f))}{n}\Big)(\psi^*)^{-1}\Big(\frac{V_{\psi^*}(g)}{n}\Big) \\
&\quad + 12\sum_{n=v+1}^{\infty}(\varphi^*)^{-1}\Big(\frac{V_{\varphi^*}(\frac{1}{2}(f^{(n)}-f))}{n}\Big)(\psi^*)^{-1}\Big(\frac{V_{\psi^*}(g)}{n}\Big) + \varepsilon \\
&\leqslant 12\sum_{n=1}^{v}(\varphi^*)^{-1}\Big(\frac{A\overline{\pi}(2\|\frac{1}{2}(f^{(n)}-f)\|_\infty)}{n}\Big)(\psi^*)^{-1}\Big(\frac{B\overline{\gamma}(\psi^{-1}(B))}{n}\Big) \\
&\quad + 12\sum_{n=v+1}^{\infty}(\varphi^*)^{-1}\Big(\frac{A\overline{\pi}(\varphi^{-1}(A))}{n}\Big)(\psi^*)^{-1}\Big(\frac{B\overline{\gamma}(\psi^{-1}(B))}{n}\Big) + \varepsilon \\
&\leqslant 12\sum_{n=1}^{v}(\varphi^*)^{-1}\Big(\frac{A\overline{\pi}(2\|\frac{1}{2}(f^{(n)}-f)\|_\infty)}{n}\Big)(\psi^*)^{-1}\Big(\frac{B\overline{\gamma}(\psi^{-1}(B))}{n}\Big) + 13\varepsilon \\
&\leqslant 12\sum_{n=1}^{v}(\varphi^*)^{-1}\Big(\frac{A\overline{\pi}(\tau)}{n}\Big)(\psi^*)^{-1}\Big(\frac{B\overline{\gamma}(\psi^{-1}(B))}{n}\Big) + 13\varepsilon \\
&\leqslant 12\varepsilon + 13\varepsilon = 25\varepsilon.
\end{aligned}$$

Hence, $\lim\limits_{n\to\infty} \int_a^b f^{(n)}\,\mathrm{d}g = \int_a^b f\,\mathrm{d}g$. □

Theorem 7.6. *If $f \in \mathrm{BV}_\varphi[a,b]$ and $\{g^{(n)}\}$ is two-norm convergent to g in $\mathrm{BV}_\varphi[a,b]$ with $\sum\limits_{m=1}^{\infty} \varphi^{-1}(1/m)\psi^{-1}(1/m) < \infty$, then $\int_a^b f\,\mathrm{d}g$ exists and*

$$\lim_{n\to\infty}\int_a^b f\,\mathrm{d}g^{(n)} = \int_a^b f\,\mathrm{d}g.$$

Proof. Since $g^{(n)}$ converge to g uniformly, there exists a positive integer N such that for every $n > N_1$ we have

$$|S(f, g^{(n)}) - S(f, g)| \leqslant \tfrac{1}{4}\varepsilon,$$
$$|(f^{(n)}(b) - f(b))g(b)| \leqslant \tfrac{1}{4}\varepsilon$$

and

$$|(f^{(n)}(a) - f(a))g(a)| \leqslant \tfrac{1}{4}\varepsilon.$$

By Theorem 7.5 there exists a positive integer $N > N_1$ such that for any $n > N$ we have

$$\left|\int_a^b g^{(n)}\,\mathrm{d}f - \int_a^b g\,\mathrm{d}f\right| \leqslant \frac{\varepsilon}{4}.$$

Hence

$$\left|\int_a^b f\,\mathrm{d}g^{(n)} - \int_a^b f\,\mathrm{d}g\right| \leqslant \left|\int_a^b g^{(n)}\,\mathrm{d}f - \int_a^b g\,\mathrm{d}f\right| + |(f^{(n)}(b) - f(b))g(b)| + |(f^{(n)}(a) - f(a))g(a)| + |S(f, g^{(n)}) - S(f, g)| \leqslant \varepsilon.$$

Hence, $\lim\limits_{n\to\infty} \int_a^b f\,\mathrm{d}g^{(n)} = \int_a^b f\,\mathrm{d}g$. □

Hence, we also have the following theorem.

Theorem 7.7. *If* $\{f^{(n)}\}$ *and* $\{g^{(n)}\}$ *are two-norm convergent to* f *and* g *in* $\mathrm{BV}_\varphi[a,b]$ *and* $\mathrm{BV}_\psi[a,b]$, *respectively, with* $\sum\limits_{m=1}^{\infty} \varphi^{-1}(1/m)\psi^{-1}(1/m) < \infty$, *then* $\int_a^b f\,\mathrm{d}g$ *exists and*

$$\lim_{n\to\infty} \int_a^b f^{(n)}\,\mathrm{d}g^{(n)} = \int_a^b f\,\mathrm{d}g.$$

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