ON REFLEXIVITY AND HYPERREFLEXIVITY OF SOME SPACES OF INTERTWINING OPERATORS

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(Received July 24, 2006)

Abstract. Let T, T' be weak contractions (in the sense of Sz.-Nagy and Foiaş), m, m' the minimal functions of their C_0 parts and let d be the greatest common inner divisor of m, m'. It is proved that the space $I(T, T')$ of all operators intertwining T, T' is reflexive if and only if the model operator $S(d)$ is reflexive. Here $S(d)$ means the compression of the unilateral shift onto the space $H^2 \ominus dH^2$. In particular, in finite-dimensional spaces the space $I(T, T')$ is reflexive if and only if all roots of the greatest common divisor of minimal polynomials of T, T' are simple. The paper is concluded by an example showing that quasisimilarity does not preserve hyperreflexivity of $I(T, T')$.

Keywords: intertwining operator, reflexivity, C_0 contraction, weak contraction, hyperreflexivity

MSC 2000: 47A10, 47A15

1. Introduction

Let H, H' be complex separable Hilbert spaces, let $\mathcal{B}(H, H')$ denote the space of all bounded linear operators $H \to H'$. If $H = H'$ then $\mathcal{B}(H, H) = \mathcal{B}(H)$ is the algebra of all bounded linear operators on H. By a subspace we mean a closed linear subspace. For a subset $A \subset H$, we denote by $\bigvee A$ the closed linear span of A. A subspace $L \subset H$ is called invariant for $T \in \mathcal{B}(H)$ if $TL \subset L$. As usual, $T|L$ means the restriction of the operator T to L. If $\mathcal{A} \subset \mathcal{B}(H)$ then $\operatorname{Alg} \mathcal{A}$ denotes the smallest weakly closed subalgebra of $\mathcal{B}(H)$ containing $\mathcal{A}$ and the identity. $\operatorname{Lat} \mathcal{A}$ denotes the set of all subspaces of H that are invariant for each $A \in \mathcal{A}$. If $\mathcal{L}$ is a set of subspaces

The author was supported by the grant G-1/3025/06 of MŠ SR and project No. SK79/CZ-89 of bilateral research cooperation between Czech and Slovak Republics.

of H, then $\operatorname{Alg}\mathcal{L} = \{T \in \mathcal{B}(H)\colon\ \mathcal{L} \subset \operatorname{Lat} T\}$. A (unital weakly closed) subalgebra $\mathcal{A} \subset \mathcal{B}(H)$ is called reflexive if $\mathcal{A} = \operatorname{Alg}\operatorname{Lat}\mathcal{A}$. An operator $T \in \mathcal{B}(H)$ is called reflexive if $\operatorname{Alg}\{T\}$ is reflexive.

H. Bercovici, C. Foiaş and B. Sz.-Nagy [3] studied reflexivity of C_0 contractions and their commutants. They showed also that if the commutant of a C_0 contraction T is reflexive then T is also reflexive. Generally, the reflexivity of $\{T\}'$ does not imply the reflexivity of the operator T [6].

The reflexivity of subalgebras was studied for the first time in [12]. The notion of reflexivity of algebras of operators was generalized to subspaces of operators by V. S. Shul'man [13]:

Definition 1.1. Let $\mathcal{M}$ be a subset of $\mathcal{B}(H,H')$. Then the reflexive closure of $\mathcal{M}$ is

$$\operatorname{ref}\mathcal{M} = \bigcap_{x\in H}\Big\{T \in \mathcal{B}(H,H')\colon\ Tx \in \bigvee\{Mx\colon\ M \in \mathcal{M}\}\Big\}.$$

A (closed linear) subspace $\mathcal{M} \subset \mathcal{B}(H,H')$ is called *reflexive* if $\mathcal{M} = \operatorname{ref}\mathcal{M}$.

Clearly, in Definition 1.1 the Hilbert spaces H, H' can be replaced by arbitrary Banach spaces. A stronger concept of hyperreflexivity was introduced for algebras in [1] and extended to subspaces in [10].

Definition 1.2. Let X, X' be complex Banach spaces and let $\mathcal{M}$ be a norm-closed subspace of $\mathcal{B}(X,X')$. $\mathcal{M}$ is called *hyperreflexive* if there exists $c > 0$ such that for all $T \in \mathcal{B}(X,X')$

$$\operatorname{dist}(T,\mathcal{M}) \leqslant c\alpha(T,\mathcal{M}), \text{ where } \alpha(T,M) = \sup\{\operatorname{dist}(Tx,\mathcal{M}x)\colon\ x \in H, \|x\| = 1\}.$$

$\inf\big\{c > 0\colon\ \operatorname{dist}(T,\mathcal{M}) \leqslant c\alpha(T,\mathcal{M})\big\}$ is called the hyperreflexivity constant of $\mathcal{M}$.

Note that if $\mathcal{M}$ is hyperreflexive then it is reflexive. It is well-known that if both H and H' are finite-dimensional then reflexivity and hyperreflexivity coincide. In [11, Theorem 2.5] V. Müller and M. Ptak have shown that if X, X' are arbitrary Banach spaces and $\mathcal{M}$ is a finite dimensional subspace of $\mathcal{B}(X,X')$ then $\mathcal{M}$ is reflexive if and only if it is hyperreflexive. Clearly, if $\mathcal{M}$ is a subalgebra of $\mathcal{B}(H)$ then $\operatorname{ref}\mathcal{M} = \operatorname{Alg}\operatorname{Lat} M$.

In [13] reflexivity of the space

$$I(T,T') = \{A \in \mathcal{B}(H,H')\colon\ AT = T'A\}$$

of operators intertwining $T \in \mathcal{B}(H)$ and $T' \in \mathcal{B}(H')$ was studied and a characterization of reflexive spaces $I(T,T')$ was given in the case of isometries T, T'. Moreover, it was stated that if $\dim H < \infty$, $\dim H' < \infty$ then $I(T,T')$ is reflexive if T or T'

is similar to a normal operator. In [5] $\mathrm{Alg}\{T\}'$ was described if $\dim H < \infty$ and this showed that $\{T\}'$ is reflexive if and only if T is similar to a normal operator or equivalently, if all roots of the minimal polynomial of T are simple.

In [20] we described (using the Jordan forms of $T \in \mathcal{B}(H)$, $T' \in \mathcal{B}(H')$) $I(T,T')$ and $\operatorname{ref} I(T,T')$ in finite-dimensional spaces and we showed that $I(T,T')$ is reflexive if all roots of the greatest common divisor of the minimal polynomials of T and T' are simple. The purpose of this paper is to extend this result to pairs of weak contractions. To prove our results we use the fact that quasi-similarity preserves reflexivity of $I(T,T')$. We give an example showing that quasi-similarity does not preserve hyperreflexivity of $I(T,T')$.

2. Compressions of the unilateral shift

We will use the terminology and results of Sz.-Nagy-Foiaş dilation theory [14]. In particular, H^2, H^∞ mean the Hardy spaces of analytic functions in the unit disc, $S(\Theta)$ means the compression of the unilateral shift S onto the space $H(\Theta) = H^2 \ominus \Theta H^2$. For $f, g \in H^\infty$ we write $f \mid g$ (f divides g) if there exists $\varphi \in H^\infty$ such that $g = \varphi f$. The orthogonal projection onto a subspace K of a Hilbert space H is denoted by P_K. For $f_1, f_2 \in H^\infty$ we denote by $f_1 \wedge f_2$ the greatest common inner divisor of f_1 and f_2.

The following result is an easy consequence of [2, Theorem III.1.16].

Theorem 2.1. *Let v_1, v_2, d be inner functions, $v_1 \wedge v_2 = 1$. Put $\Theta_1 = v_1 d$, $\Theta_2 = v_2 d$. Then*

(i) *$X \in I(S(\Theta_1), S(\Theta_2))$ if and only if there exists a function $\varphi \in H^\infty$ such that*

$$X = P_{H(\Theta_2)} u(S)|H(\Theta_1), \quad \text{where} \quad u = v_2\varphi.$$

Moreover, $X = 0$ if and only if $d \mid \varphi$.

(ii) *An operator $A \in \operatorname{ref} I(S(\Theta_1), S(\Theta_2))$ if and only if*

$$\begin{aligned} &A|H^2 \ominus dH^2 \in \operatorname{ref} I\big(S(d), S(\Theta_2)|v_2(H^2 \ominus dH^2)\big), \\ \text{and} \quad &A|d(H^2 \ominus v_1 H^2) = 0. \end{aligned}$$

(iii) *$I(S(\Theta_1), S(\Theta_2))$ is reflexive if and only if $S(d)$ is reflexive.*

Proof. (i) According to [2, Theorem III.1.16], $X \in I\big(S(\Theta_1), S(\Theta_2)\big)$ if and only if there exists an inner function u such that $X = P_{H(\Theta_2)} u(S)\big|H(\Theta_1)$ and $\Theta_2 \mid u\Theta_1$. Since $v_1 \wedge v_2 = 1$, we have $v_2 d \mid u v_1 d \iff v_2 \mid u$ and consequently there exists

$\varphi \in H^\infty$ such that $u = \varphi v_2$. Moreover, $X = 0$ if and only if $\Theta_2 \mid u$, i.e. if and only if $d \mid \varphi$.

(ii) $H(\Theta_1)$ and $H(\Theta_2)$ can be written as orthogonal sums

$$H(\Theta_1) = (H^2 \ominus dH^2) \oplus d(H^2 \ominus v_1 H^2), \quad H(\Theta_2) = (H_2 \ominus v_2 H^2) \oplus v_2(H^2 \ominus dH^2).$$

It is well-known that $v_2(H^2 \ominus dH^2) \in \operatorname{Lat} S(\Theta_2)$. Using (i) we obtain for any $f \in H(\Theta_1)$

$$\bigvee_{X \in I(S(\Theta_1), S(\Theta_2))} Xf = \bigvee_{\varphi \in H^\infty} P_{H(\Theta_2)} v_2 \varphi f \subset v_2(H^2 \ominus dH^2).$$

If $f \in d(H^2 \ominus v_1 H^2)$ then $v_2 \varphi f \in dv_2 H^2 \perp H(\Theta_2)$, consequently

$$\bigvee_{X \in I(S(\Theta_1), S(\Theta_2))} Xf = 0.$$

Herefrom (ii) follows easily.

(iii) $S(\Theta_2)|v_2 H(d)$ is unitarily equivalent to $S(d)$. So the reflexivity of $I(S(\Theta_1), S(\Theta_2))$ implies that the commutant of $S(d)$ is reflexive. Since $\{S(d)\}' = \operatorname{Alg} S(d)$, this proves (iii). $\square$

3. General C_0 contractions

To prove a characterization of pairs T, T' of C_0 contractions having reflexive $I(T, T')$ we need two simple lemmas.

Lemma 3.1. *Let $T, X \in \mathcal{B}(H)$, $T', Y \in \mathcal{B}(H')$ and $TX = XT$, $T'Y = YT'$. Put $T_X = T|(XH)^-$, $T'_Y = T'|(YH')^-$.*

If $I(T, T')$ is reflexive then $I(T_X, T'_Y)$ is reflexive as well.

Proof. Suppose that $A \in \operatorname{ref} I(T_X, T'_Y)$. If $B \in I(T_X, T'_Y)$ then $BX \in I(T, T')$. Therefore for all $h \in H$ we have

$$AXh \in \bigvee_{B \in I(T_X, T'_Y)} BXh \subset \bigvee_{C \in I(T, T')} Ch, \quad \text{i.e.} \quad AX \in \operatorname{ref} I(T, T')$$

and so $ATX = AXT = T'AX = T'_Y AX$, i.e. $A \in I(T_X, T'_Y)$. $\square$

Lemma 3.2. *Let $\vartheta_1, \Theta_1, \vartheta_2, \Theta_2$ be inner functions such that $\vartheta_1 \mid \Theta_1$ and $\vartheta_2 \mid \Theta_2$. If $I(S(\Theta_1), S(\Theta_2))$ is reflexive then $I(S(\vartheta_1), S(\vartheta_2))$ is reflexive as well.*

Proof. Put $\varphi_k = \Theta_k/\vartheta_k$, $k = 1, 2$. Since $S(\vartheta_k)$ is unitarily equivalent to $S(\Theta_i)|\bigl(\varphi_k(S(\Theta_k))H(\Theta_k)\bigr)^{-}$, Lemma 3.2 is a consequence of Lemma 3.1. $\square$

Now we are ready to state one of our main results.

Theorem 3.3. *Let $T \in \mathcal{B}(H)$, $T' \in \mathcal{B}(H')$ be C_0 contractions having minimal functions m, m', respectively. Let $d = m \wedge m'$. Then $I(T, T')$ is reflexive if and only if the operator $S(d)$ is reflexive.*

Proof. If $T_1 \in \mathcal{B}(H_1)$ and $T_1' \in \mathcal{B}(H_1')$ are quasisimilar to $T_2 \in \mathcal{B}(H_2)$ and $T_2' \in \mathcal{B}(H_2')$, respectively, then $I(T_1, T_1')$ is reflexive if and only if $I(T_2, T_2')$ is reflexive. This was first stated (without proof which is easy) in [13, Proposition 1]. Since any C_0 contraction is quasisimilar to its Jordan model it is enough to prove the theorem for Jordan models

$$T = \bigoplus_{\alpha} S(m_\alpha), \qquad T' = \bigoplus_{\beta} S(m'_\beta),$$

where $\oplus$ means the direct orthogonal sum. According to [13, Proposition 2], $I(T, T')$ is reflexive if and only if each of the spaces $I(S(m_\alpha), S(m'_\beta))$ is reflexive. For all indices α, β, we have $m_\alpha \mid m$, $m'_\beta \mid m'$. Therefore, by Lemma 3.2, $I(T, T')$ is reflexive if and only if $I(S(m), S(m'))$ is reflexive. According to assertion (iii) of Theorem 2.1 this completes the proof. $\square$

Theorem 3.3 generalizes [3, Theorem B]. In finite-dimensional spaces we obtain the following corollary (a generalization of [5, Theorem 3]).

Corollary 3.4. *Let H, H' be finite-dimensional. Then $I(T, T')$ is reflexive if and only if all roots of the greatest common divisor of the minimal polynomials m_T and $m_{T'}$ of T and T', respectively, are simple.*

Proof. Replacing T and T' by $\|T\|^{-1}T$ and $\|T'\|^{-1}T'$ we obtain a pair of contractions the minimal functions m, m' of which are finite Blaschke products whose numerators are m_T and $m_{T'}$, respectively. Then d is also a finite Blaschke product and its numerator is the greatest common inner divisor of the minimal polynomials m_T and $m_{T'}$. It is well-known (see e.g. [7]) that then $S(d)$ is reflexive if and only if all zeroes of d are simple. $\square$

Note that in [20] Corollary 3.4 was proved more directly by describing $I(T, T')$ and $\operatorname{ref} I(T, T')$ for nilpotent T and T'. In the case $T = T'$ this was done in [5].

4. Weak contractions

Now, let $T \in \mathcal{B}(H)$, $T' \in \mathcal{B}(H')$ be weak contractions. (For the definition of weak contractions and basic results we refer to [14, Chapter VIII]). It is well-known (see, e.g., [18]) that T and T' can be splitted into orthogonal sums $T = T_{ac} \oplus T_{su}$, $T' = T'_{ac} \oplus T'_{su}$ of their absolutely continuous and singular unitary parts and that

$$I(T, T') = I(T_{ac}, T'_{ac}) \oplus I(T_{su}, T'_{su}).$$

It follows that

$$\operatorname{ref} I(T, T') = \operatorname{ref} I(T_{ac}, T'_{ac}) \oplus \operatorname{ref} I(T_{su}, T'_{su}).$$

Since for normal operators A, B the space $I(A, B)$ is reflexive [13], $I(T, T')$ is reflexive if and only if so is $I(T_{ac}, T'_{ac})$. According to [17, Lemma 3] any absolutely continuous weak contraction S is similar to a completely non-unitary (c.n.u.) weak contraction S' and, moreover, the C_0 parts of S and S' coincide. Since similarity (even quasi-similarity [13, Proposition 1]) preserves reflexivity of $I(T, T')$, it does not restrict generality if we suppose that T, T' are c.n.u.

Theorem 4.1. *Let $T \in \mathcal{B}(H)$, $T' \in \mathcal{B}(H')$ be c.n.u. weak contractions and let $T_0 \in \mathcal{B}(H_0)$, $T'_0 \in \mathcal{B}(H'_0)$ be their C_0 parts and $T_1 \in \mathcal{B}(H_1)$, $T'_1 \in \mathcal{B}(H'_1)$ their C_{11} parts. Then*

(i) *if $X \in I(T, T')$ then $XH_0 \subset H'_0$ and $XH_1 \subset H'_1$;*

(ii) *if $A \in \operatorname{ref} I(T, T')$ then its restrictions to subspaces H_0, H_1 satisfy $A_0 = A|H_0 \in \operatorname{ref} I(T_0, T'_0)$, $A_1 = A|H_1 \in \operatorname{ref} I(T_1, T'_1)$;*

(iii) *$I(T, T')$ is reflexive if and only if $I(T_0, T'_0)$ is reflexive.*

Proof. (i) According to [14, Chapters II.4 and VIII.2]

$$H_0 = \{h \in H\colon\ T^n h \to 0\}, \qquad H'_0 = \{h' \in H'\colon\ T'^n h' \to 0\}$$
$$\text{and}\quad H_1^\perp = \{h \in H\colon\ T^{*n} h \to 0\}, \quad H_1'^\perp = \{h' \in H'\colon\ T'^{*n} h' \to 0\}.$$

$XT = T'X$ implies $XT^n = T'^n X$ for all positive integers n. Therefore $h_0 \in H_0 \Longrightarrow \lim T'^n X h_0 = \lim XT^n h_0 = 0$, i.e. $Xh_0 \in H'_0$. By taking adjoints we obtain $XT = T'X \Longrightarrow T^* X^* = X^* T'^*$ and so $X^* H_1'^\perp \subset H_1^\perp$, which is equivalent to $XH_1 \subset H'_1$.

(ii) This is an obvious consequence of (i).

(iii) There are operators $R, S \in \{T\}''$, $R', S' \in \{T'\}''$ such that

$$H_0 = \ker R = (SH)^-, \qquad H_1 = (RH)^- = \ker S,$$
$$H'_0 = \ker R' = (S'H)^-, \qquad H_1 = (R'H)^- = \ker S'$$

([14], [15], [16, Theorem 1]). Suppose that $I(T,T')$ is reflexive. Then, by Lemma 3.1, $I(T_0,T_0')$ is reflexive. Conversely, if $I(T_0,T_0')$ is reflexive and $A \in \operatorname{ref} I(T,T')$ then by (ii) $A|H_0 \in \operatorname{ref} I(T_0,T_0')$ and $A|H_1 \in \operatorname{ref} I(T_1,T_1')$. The operators T_1, T_1' are quasi-similar to unitary operators and so $I(T_1,T_1')$ is reflexive. Therefore $A|H_0 \in I(T_0,T_0')$ and $A|H_1 \in I(T_1,T_1')$. Since $H_0 \vee H_1 = H$, this shows that $I(T,T')$ is reflexive. □

Theorem 4.2. *Let T, T' be weak contractions and let their C_0 parts T_0, T_0' have minimal functions m, m', respectively. Let $d = m \wedge m'$ be the greatest common inner divisor of m, m'. Then the space $I(T,T')$ is reflexive if and only if the operator $S(d)$ is reflexive.*

P r o o f. This is an obvious consequence of Theorems 3.3 and 4.1. □

R e m a r k s.

1. Theorems 4.1 and 4.2 are generalizations of [19, Theorem 5.1].
2. Inner functions m for which $S(m)$ is a reflexive operator were characterized in [7, Theorem 3.1].

5. Quasisimilarity does not preserve hyperreflexivity

First, let us recall the definition of quasisimilarity:

Definition 5.1. $T \in B(H)$, $S \in B(K)$ are *quasi-similar* (we write $T \overset{\text{q.s.}}{\sim} S$) if there are quasi-affinities (injective operators with dense range) $X \in I(T,S)$, $Y \in I(S,T)$.

E x a m p l e 5.2. Put $H_n = H_n' = C^2$, $H = H' = \bigoplus_{n=1}^{\infty} H_n$,

$$T_n = \frac{1}{n}\begin{pmatrix} 2n & n \\ 0 & 2n+1 \end{pmatrix}, \quad T_n' = \frac{1}{n}\begin{pmatrix} 2n & 0 \\ -n & 2n+1 \end{pmatrix},$$

$$S_n = S_n' = \frac{1}{n}\begin{pmatrix} 2n+1 & 0 \\ 0 & 2n \end{pmatrix},$$

$$T = \bigoplus_{n=1}^{\infty} T_n, \quad T' = \bigoplus_{n=1}^{\infty} T_n', \quad S = S' = \bigoplus_{n=1}^{\infty} S_n.$$

Then, obviously, $T \in \mathcal{B}(H)$, $T' \in \mathcal{B}(H')$, $S = S' \in \mathcal{B}(H)$.

The following assertions hold.

(a) $T \overset{\text{q.s.}}{\sim} S = S' \overset{\text{q.s.}}{\sim} T'$,

(b) all $I(T_m,T_n')$ are hyperreflexive,

(c) $I(T,T')$ is not hyperreflexive,

(d) $I(S,S')$ is hyperreflexive.

P r o o f. The common minimal polynomial $(\lambda - 2n)(\lambda - 2n - 1)$ of T_n, T_n', S_n has simple roots, which implies that all $I(T_m, T_n')$ are reflexive. In finite dimension this implies that they are also hyperreflexive and this proves (b).

Putting $A_n = \begin{pmatrix} 0 & n \\ 0 & 1 \end{pmatrix}$, $B_n = \begin{pmatrix} 0 & 0 \\ -n & 1 \end{pmatrix}$, $C_n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ we obtain

$$T_n = \frac{1}{n}(2nI + A_n),\ T_n' = \frac{1}{n}(2nI + B_n),\ S_n = S_n' = \frac{1}{n}(2nI + C_n)$$

and if $P_n = \begin{pmatrix} n & 1 \\ 1 & 0 \end{pmatrix}$ then $P_n^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -n \end{pmatrix}$ and $A_n = P_n C_n P_n^{-1}$.

Hence $A_n P_n = P_n C_n$, $P_n^{-1} A_n = C_n P_n^{-1}$ and after perturbation by $2nI$, $T_n P_n = P_n S_n$, $P_n^{-1} T_n = S_n P_n^{-1}$.

Now, it is easy to compute $\|P_n\|$ and $\|P_n^{-1}\|$:

$$P_n^{\top} = P_n \Longrightarrow \|P_n\| = \varrho(P_n) = \frac{n + \sqrt{n^2 + 4}}{2} = \varrho(P_n^{-1}) = \|P_n^{-1}\|.$$

Putting $Y = \bigoplus\limits_{n=1}^{\infty} n^{-1} P_n$, $X = \bigoplus\limits_{n=1}^{\infty} n^{-1} P_n^{-1}$ we obtain quasiaffinities $X \in I(T, S)$, $Y \in I(S, T)$, i.e., $T \overset{\text{q.s.}}{\sim} S$. Similarly, it can be proved that $T' \overset{\text{q.s.}}{\sim} S$. This completes the proof of (a).

(c): $m \neq n \Longrightarrow I(T_n, T_m') = \{0\}$ because their minimal polynomials are relatively prime. Therefore $I(T, T') = \bigoplus\limits_{n=1}^{\infty} I(T_n, T_n')$ and similarly $I(S, S') = \bigoplus\limits_{n=1}^{\infty} I(S_n, S_n')$ By a simple direct computation we obtain $X_n \in I(T_n, T_n') = I(A_n, B_n)$ if and only if $X_n = \begin{pmatrix} 0 & \alpha \\ \beta & -n(\alpha + \beta) \end{pmatrix}$ for some $\alpha, \beta \in C$. So $I(T_n, T_n') = \mathcal{S}_n$ from an example due to Kraus and Larson [9] (see also [4, Example 58.9]) who proved that $\mathcal{S}_n$ is hyperreflexive with $\kappa_{\mathcal{S}_n} \geqslant \frac{1}{3} n$. So $I(T, T') = \bigoplus\limits_{n=1}^{\infty} \mathcal{S}_n$ is not hyperreflexive.

(d): Observe that $I(S_n, S_n) = I\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$ for all n, i.e. its hyperreflexivity constant does not depend on n. Using a recent result of K. Kliś and M. Ptak [8, Theorem 5.1] we obtain that $I(S, S')$ is hyperreflexive. □

It easy to show that if $T = \bigoplus\limits_{n=1}^{\infty} T_n$, $T' = \bigoplus\limits_{n=1}^{\infty} T_n'$ and $I(T, T')$ is hyperreflexive, then all $I(T_n, T_m')$ are hyperreflexive. From Example 5.2 it follows that the converse implication does not hold.

References

[1] *W. B. Arveson*: Ten Lectures in Operator Algebras. C.B.M.S. Regional Conf. Ser. in Math, Amer. Math. Soc., Providence, 1984. zbl

[2] *H. Bercovici*: Operator Theory and Arithmetic in H^∞. Mathematical surveys and monographs 26, A.M.S. Providence, Rhode Island, 1988. zbl

[3] *H. Bercovici, C. Foiaş, B. Sz.-Nagy*: Reflexive and hyper-reflexive operators of class C_0. Acta Sci. Math. (Szeged) *43* (1981), 5–13. zbl

[4] *J. B. Conway*: A Course in Operator Theory. American Mathematical Society, Providence RI, 2000. zbl

[5] *Š. Drahovský, M. Zajac*: Hyperreflexive operators on finite dimensional Hilbert spaces. Math. Bohem. *118* (1993), 249–254. zbl

[6] *Š. Drahovský, M. Zajac*: Hyperinvariant subspaces of operators on Hilbert spaces. Functional Analysis and Operator Theory, Banach Center Publications, vol 30. Institute of Mathematics, Warszawa, 1994, pp. 117–126. zbl

[7] *V. V. Kapustin*: Reflexivity of operators: general methods and a criterion for almost isometric contractions. Algebra i Analiz *4* (1992), 141–160 (In Russian.) zbl; English transl.: St.Petersburg Math. J. *4* (1993), 319–335. zbl

[8] *K. Kliś, M. Ptak*: k-hyperreflexive subspaces. Houston J. Math. *l32* (2006), 299–313. zbl

[9] *J. Kraus, D. Larson*: Some applications of technique for constructing reflexive operator algebras. J. Operator Theory *13* (1985), 227–236. zbl

[10] *J. Kraus, D. Larson*: Reflexivity and distance formulae. Proc. London Math. Soc. *53* (1986), 340–356. zbl

[11] *V. Müller, M. Ptak*: Hyperreflexivity of finite-dimensional subspaces. J. Funct. Anal. *218* (2005), 395–408. zbl

[12] *D. Sarason*: Invariant subspaces and unstarred operator algebras. Pacific J. Math. *17* (1966), 511–517. zbl

[13] *V. S. Shul'man*: The Fuglede-Putnam theorem and reflexivity. Dokl. Akad. Nauk SSSR *210* (1973), 543–544 (In Russian.); English transl.: Soviet Math. Dokl. *14* (1973), 783–786.

[14] *B. Sz.-Nagy, C. Foiaş*: Harmonic Analysis of Operators on Hilbert Space. North-Holland, Akadémiai kiadó, Budapest, 1970. zbl

[15] *P. Y. Wu*: Hyperinvariant subspaces of weak contractions. Acta Sci. Math. (Szeged) *41* (1979), 259–266. zbl

[16] *M. Zajac*: Hyperinvariant subspaces of weak contractions. Operator Theory: Advances and Applications. vol. 14, Birkhäuser, Basel, 1984, pp. 291–299. zbl

[17] *M. Zajac*: Hyperinvariant subspaces of weak contractions. II. Operator Theory: Advances and Applications. vol. 28, Birkhäuser, Basel, 1988, pp. 317–322. zbl

[18] *M. Zajac*: On the singular unitary part of a contraction. Revue Roum. Math. Pures Appl. *35* (1990), 379–384. zbl

[19] *M. Zajac*: Hyper-reflexivity of isometries and weak contractions. J. Operator Theory *25* (1991), 43–51. zbl

[20] *M. Zajac*: Reflexivity of intertwining operators in finite dimensional spaces. Proc. 2nd Workshop Functional Analysis and its Applications, Sept. 16–18, 1999, Nemecká, Slovakia, 75–78.

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