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ON THE EXISTENCE OF NONNEGATIVE SOLUTIONS OF THE PERIODIC BOUNDARY VALUE PROBLEM FOR A SYSTEM OF LINEAR GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

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Consider the ω-periodic boundary value problem

$$dx(t) = dA(t)\cdot\mathcal{P}(t)x(t) + dq(t), \tag{1}$$

$$x(0) = x(\omega), \tag{2}$$

where ω is a positive number, $A = (a_{ik}(t))_{i,k=1}^{n}$, $a_{ik}(t) \equiv a_{ik}^{(1)}(t) - a_{ik}^{(2)}(t)$, $a_{ik}^{(\sigma)} : R \to R$ $(\sigma = 1,2)$ are functions nondecreasing on $[0,\omega]$, $A^{(\sigma)} = (a_{ik}^{(\sigma)})_{i,k=1}^{n} \in \mathrm{BV}_{\omega}^{n\times n}$ $(\sigma = 1,2)$, $q = (q_k)_{k=1}^{n} \in \mathrm{BV}_{\omega}^{n}$ and $\mathcal{P} = (p_{ik})_{i,k=1}^{n} \in \bigcap\limits_{\sigma=1}^{2} L([0,\omega], R^{n\times n}; A^{(\sigma)})$ is such that $\int\limits_0^t dA(\tau)\cdot\mathcal{P}(\tau) \in \mathrm{BV}_{\omega}^{n\times n}$.

In this note, sufficient conditions are given guaranteeing both the unique solvability of the problem (1),(2) and the nonnegativeness of the solution.

The following notation and definitions will be used: $R =]-\infty,+\infty[$, $R_+ = [0,+\infty[$, $[a,b]$ $(a,b \in R)$ is a closed segment, $R^{n\times m}$ is the space of all real $n\times m$-matrices $X = (x_{ik})_{i,k=1}^{n,m}$ with the norm $\|X\| = \max\limits_{k=1,\dots,m}\sum\limits_{i=1}^{n}|x_{ik}|$; if $X \in R^{n\times n}$, then $\det(X)$ is the determinant of X, I_n is the identity $n\times n$-matrix; δ_{ij} is the Kroneker symbol, i.e., $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$; $R^n = R^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^{n}$.

$\mathrm{BV}([a,b], R^{n\times m})$ is the set of all matrix-functions $X = (x_{ik})_{i,k=1}^{n,m} : [a,b] \to R^{n\times m}$ such that every its component x_{ik} has bounded total variation on $[a,b]$.

$s_k : \mathrm{BV}([a,b], R) \to \mathrm{BV}([a,b], R)$ $(k = 0,1,2)$ are the operators defined by $s_1(x)(a) = s_2(x)(a) = 0$,

$$s_1(x)(t) = \sum_{a<\tau\le t} d_1x(\tau) \quad\text{and}\quad s_2(x)(t) = \sum_{a\le\tau<t} d_2x(\tau) \quad\text{for}\quad t\in]a,b],$$

$$s_0(x)(t) \equiv x(t) - s_1(x)(t) - s_2(x)(t).$$

$\mathrm{BV}_{\omega}^{n\times m}$ is the set of all matrix-functions $X : R \to R^{n\times m}$ such that $X(t+\omega) = X(t) + X(\omega)$ for $t \in R$, and its restriction on $[0,\omega]$ belongs to $\mathrm{BV}([0,\omega], R^{n\times m})$; $X(t-)$ and $X(t+)$ are the left and the right limits of X at the point $t \in R$; $d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

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If $x \in \mathrm{BV}([0,\omega],R)$, $x(0)=0$, $1+(-1)^j d_j x(t) \neq 0$ for $t \in [0,\omega]$ $(j=1,2)$ and $\lambda(x)(\omega) \neq 1$, where

$$\lambda(x)(t) = \exp\big(s_0(x)(t)\big) \prod_{0\le\tau<t} \big(1+d_2x(\tau)\big) \Big/ \prod_{0<\tau\le t} \big(1-d_1x(\tau)\big),$$

then $g_0(x)(t,\tau) = (1-\lambda(x)(\omega))^{-1}\lambda(x)(\omega)\lambda(x)(t)\lambda^{-1}(x)(\tau)$ for $0 \le t \le \tau \le \omega$ and $g_0(x)(t,\tau) = (1-\lambda(x)(t)\lambda^{-1}(x)(\tau)$ for $0 \le \tau < t \le \omega$; $g_k(x)(t,\tau) = (1+(-1)^k d_k x(\tau))^{-1}$ $g_0(x)(t,\tau)$ for $t \neq \tau$ $(k=1,2)$, $g_1(x)(t,t) = (1-\lambda(x)(\omega))^{-1}(1-d_1x(t))^{-1}$ and $g_2(x)(t,t) = (1-\lambda(x)(\omega))^{-1}(1+d_2x(t))^{-1}\lambda(x)(\omega)$.

If $g : R \to R$ is nondecreasing on the interval $I \subset R$, $x : R \to R$ and $s < t$ $(s,t \in I)$, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_{]s,t[} x(\tau)\,dg(\tau) + x(t)d_1g(t) + x(s)d_2g(s),$$

where $\int_{]s,t[} x(\tau)\,dg(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure μ_g corresponding to the function g (if $s=t$, then $\int_s^t x(\tau)dg(\tau) = 0$); $L([a,b],R;g)$ is the set of all μ_g-measurable functions $x : [0,\omega] \to R$ such that $\int_a^b |x(t)|dg(t) < +\infty$.

A matrix-function is said to be nondecreasing if every of its components are such.

If $G = (g_{ik})_{i,k=1}^{l,n} : R \to R^{l\times n}$ is a matrix-function nondecreasing on the interval $I \subset R$ and $X = (x_{kj})_{k,j=1}^{n,m} : R \to R^{n\times m}$, then

$$\int_s^t dG(\tau)\cdot X(\tau) = \Big(\sum_{k=1}^n \int_s^t x_{kj}(\tau)\,dg_{ik}(\tau)\Big)_{i,j=1}^{l,m} \quad \text{for} \quad s \le t \ \ (s,t \in I);$$

$L([a,b],R^{n\times m};G)$ is the set of all matrix-functions $X = (x_{ik})_{k,j=1}^{n,m} : [a,b] \to R^{n\times m}$ such that $x_{kj} \in L([a,b],R;g_{ik})$ $(i=1,\dots,l;\ k=1,\dots,n;\ j=1,\dots,m)$.

If $G^{(\sigma)} : R \to R^{l\times n}$ $(\sigma = 1,2)$ are matrix-functions nondecreasing on the interval $I \subset R$, $G = G^{(1)} - G^{(2)}$ and $X : R \to R^{n\times m}$, then

$$\int_s^t dG(\tau)\cdot X(\tau) = \int_s^t dG^{(1)}(\tau)\cdot X(\tau) - \int_s^t dG^{(2)}(\tau)\cdot X(\tau) \quad \text{for} \quad s \le t \ \ (s,t \in I).$$

A vector-function $x : R \to R^n$ is said to be a solution of the system (1) (of the system $dx(t) \le dA(t)\cdot\mathcal{P}(t)x(t) + dq(t)$) if its restriction on $[s,t]$ belongs to $\mathrm{BV}([s,t],R^n)$ and

$$x(t) - x(s) - \int_s^t dA(\tau)\cdot\mathcal{P}(\tau)x(\tau) - q(t) + q(s) = 0 \ \ (<0) \quad \text{for} \quad s < t \ \ (s,t \in R).$$

Definition. Let $\sigma_i \in \{-1,1\}$ $(i=1,\dots,n)$. We say that the matrix-function $C = (c_{il})_{i,l=1}^n : R \to R^{n\times n}$ belongs to the set $U_\omega^{\sigma_1,\dots,\sigma_n}$ if $c_{il} \in \mathrm{BV}_\omega$ $(i,l=1,\dots,n)$, the functions c_{il} $(i \neq l;\ i,l = 1,\dots,n)$ are nondecreasing on $[0,\omega]$ and continuous at the point $t_i = \frac{1-\sigma_i}{2}\omega$,

$$d_j c_{ii}(t_i) \geq 0 \quad (j=1,2;\ \ i=1,\dots,n), \tag{3}$$

$$\|d_j C(t)\| < 1 \quad \text{for} \quad t \in [0,\omega] \ \ (j=1,2) \tag{4}$$

and the system of differential inequalities

$$\sigma_i dy_i(t) \leq \sum_{l=1}^{n} y_l(t) dc_{il}(t) \quad (i=1,\dots,n)$$

has no nontrivial nonnegative ω-periodic solution.

Let $I_\omega^{\sigma_i} =]0,\omega[\cup \{\frac{1+\sigma_i}{2}\omega\}$ and $j_i = \frac{3+\sigma_i}{2}$ for $i \in \{1,\dots,n\}$.

Theorem 1. *Let*

$$(-1)^{\sigma+1}\sigma_i p_{ki}(t) \leq p_{iki}^{(\sigma)}(t) \quad \textit{and} \quad 0 \leq (-1)^{\sigma+1}\sigma_i p_{kl}(t) \leq p_{ikl}^{(\sigma)}(t)$$
$$\textit{for } \mu_{a_{ik}^{(\sigma)}} - \textit{almost everywhere } t \in I_\omega^{\sigma_i} \ \ (i \neq l;\ \ i,k,l = 1,\dots,n), \tag{5}$$

$$0 \leq (-1)^{\sigma+1}\sigma_i p_{kl}(t_i) d_{j_i} a_{ik}^{(\sigma)}(t_i) \leq \alpha_{ikl}^{(\sigma)} d_{j_i} a_{ik}^{(\sigma)}(t_i) \quad (i,k,l=1,\dots,n) \tag{6}$$

and

$$\Big| \sum_{k=1}^{n} p_{ki}(t) d_j a_{ik}(t) \Big| \leq |d_j c_{ii}(t)| \quad \textit{for} \quad t \in I_\omega^{\sigma_i} \ \ (j=1,2;\ \ i=1,\dots,n) \tag{7}$$

for every $\sigma \in \{1,2\}$, where $\sigma_i \in \{-1,1\}$, $\alpha_{ikl}^{(\sigma)} \in R_+$, $\mathcal{P}_i^{(\sigma)} = (p_{ikl}^{(\sigma)})_{k,l=1}^n \in L([0,\omega], R^{n\times n}; A^{(\sigma)})$. Let, moreover, the functions $\sigma_i q_i$ $(i=1,\dots,n)$ be nondecreasing on R,

$$\sum_{\sigma=1}^{2}\sum_{k=1}^{n}\int_s^t p_{ikl}^{(\sigma)}(\tau) da_{ik}^{(\sigma)}(\tau) \leq c_{il}(t) - c_{il}(s) \ \ \textit{for} \ \ s<t;\ \ s,t \in I_\omega^{\sigma_i} \ \ (i,l=1,\dots,n) \tag{8}$$

and

$$\sum_{\sigma=1}^{2}\sum_{k=1}^{n} \alpha_{ikl}^{(\sigma)} d_{j_i} a_{ik}^{(\sigma)}(t_i) \leq \delta_{il} d_{j_i} c_{il}(t_i) \ \ (i,l=1,\dots,n), \tag{9}$$

where

$$C = (c_{il})_{i,l=1}^n \in U_\omega^{\sigma_1,\dots,\sigma_n}.$$

Then the problem (1),(2) *has a unique solution which is nonnegative.*

Corollary 1. *Let the conditions* (3)–(9) *and*

$$c_{ii}(\omega) - \sigma_i \sum_{0<\tau\leq\omega} \Big[\ln(1-\sigma_i d_1 c_{ii}(\tau)) + \sigma_i d_1 c_{ii}(\tau)\Big] +$$
$$+\sigma_i \sum_{0\leq\tau<\omega} \Big[\ln(1+\sigma_i d_2 c_{ii}(\tau)) - \sigma_i d_2 c_{ii}(\tau)\Big] < 0 \quad (i=1,\dots,n)$$

hold for every $\sigma \in \{1,2\}$, where $\sigma_i \in \{1,2\}$, $\alpha_{ikl}^{(\sigma)} \in R_+$, $\mathcal{P}_i^{(\sigma)} = (p_{ikl}^{(\sigma)})_{k,l=1}^n \in L([0,\omega], R^{n\times n}; A^{(\sigma)})$, $c_{il} \in \mathrm{BV}_\omega$, c_{il} $(i \neq l)$ be nondecreasing on $[0,\omega]$ and continuous at the point t_i. Let, moreover, the functions $\sigma_i q_i$ $(i=1,\dots,n)$ be nondecreasing on

R and the modulus of every characteristic value of the matrix $(s_{il})_{i,l=1}^n$,

$$s_{ii}=0,\quad s_{il}=\sup\left\{\sum_{j=0}^{2}\int_0^\omega \sigma_i g_j(\sigma_i c_{ii})(t,\tau)ds_j(c_{il})(\tau): t\in[0,\omega]\right\}\ \ (i\neq l),$$

be less than 1*. Then the conclusion of Theorem* 1 *is true.*

Corollary 2. *Let the conditions* (5)–(9),

$$c_{il}(t)=\eta_{il}\alpha_i(t)\quad for\quad t\in R\quad (i,l=1,\dots,n)$$

and

$$d_j\alpha_i(t)<\left(|\eta_{ii}|+\sum_{l\neq i;\,l=1}^{n}\eta_{il}\right)d_j\alpha_i(t)<1\quad for\quad t\in[0,\omega]\ \ (j=1,2;\ \ i=1,\dots,n)$$

hold for every $\sigma\in\{-1,1\}$, *where* $\sigma_i\in\{1,2\}$, $\alpha_{ikl}^{(\sigma)}\in R_+$, $\eta_{ii}\in R$, $\eta_{il}\in R_+$ $(i\neq l)$, $\mathcal{P}_i^{(\sigma)}=(p_{ikl}^{(\sigma)})_{k,l=1}^n\in L([0,\omega],R^{n\times n};A^{(\sigma)})$, $\alpha_i\in \mathrm{BV}_\omega$ *be nondecreasing on* $[0,\omega]$ *and continuous at the point* t_i, $\alpha_i(\omega)\neq 0$. *Let, moreover, the functions* $\sigma_i q_i$ $(i=1,\dots,n)$ *be nondecreasing on* R *and the real part of every characteristic value of the matrix* $(\eta_{il})_{i,l=1}^n$ *be nonnegative. Then the conclusion of Theorem* 1 *is true.*

Corollary 3. *Let the conditions* (3)–(9) *hold for every* $\sigma\in\{-1,1\}$, *where* $\sigma_1=\sigma_2=\dots=\sigma_n=\sigma_0\in\{1,2\}$, $\alpha_{ikl}^{(\sigma)}\in R$, $\mathcal{P}_i^{(\sigma)}=(p_{ikl}^{(\sigma)})_{k,l=1}^n\in L([0,\omega],R^{n\times n};A^{(\sigma)})$, $c_{il}\in\mathrm{BV}_\omega$, c_{il} $(i\neq l)$ *be nondecreasing on* $[0,\omega]$ *and continuous at the point* t_i. *Let, moreover, the functions* $\sigma_i q_i$ $(i=1,\dots,n)$ *be nondecreasing on* R *and the modulus of every multiplicator of the system*

$$dy(t)=dC_{\sigma_0}(t)\cdot y(t)$$

be less than 1, *where* $C_{\sigma_0}(t)=\sigma_0 C(\sigma_0 t+\frac{1-\sigma_0}{2}\omega)$. *Then the conclusion of Theorem* 1 *is true.*

The analogous question has been considered in [1] for a system of linear ordinary differential equations.

References

1. I. T. Kiguradze, Boundary value problems for systems of ordinary differential equations. (Russian) *Current problems in Mathematics. Newest results, vol.* 30, 3–103; *Itogi Nauki i Tekhniki Akad. Nauk SSSR, Vsesoyuzn. Inst. Nauch. i Tekhn. Inform., Moscow,* 1087.

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