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PROBLEM OF OPTIMAL CONTROL WITH ONE-SIDED MIXED RESTRICTIONS FOR CONTROLLED OBJECTS DESCRIBED BY INTEGRAL EQUATIONS WITH MEASURE

Abstract. For extremal problems in B-spaces, a modified version of the method of joint covering is considered and a necessary condition of criticality is proved. In the space of regular functions with values in a B-space, the properties of a bilinear integral with measure are studied; a theorem on the existence of a solution of a nonhomogeneous integral equation with measure is proved; for controlled objects described both by integral equations with measure and by one-sided mixed restrictions, the necessary conditions of optimality are derived from those of criticality.

Mathematics Subject Classification. 42K22.

Key words and phrases. Method of joint covering, differential of a convex set, necessary conditions of criticality, regular functions, integral calculus with measure, integral equation with measure, optimal control, mixed restrictions, the necessary conditions of optimality.

რეზიუმე. B-სივრცეებში ექსტრემალური ამოცანებისათვის განხილულია ერთობლივი დაფარვის მეთოდის მოდიფიცირებული ვარიანტი და დამტკიცებულია კრიტიკულობის აუცილებელი პირობა. B-სივრცულ მნიშვნელობებიან რეგულარულ ფუნქციათა სივრცეში შესწავლილია ზომიანი ინტეგრალის თვისებები; დამტკიცებულია თეორემა არაერთგვაროვანი ზომიანი ინტეგრალური განტოლების ამონახსნის პირობების შესახებ; ზომიანი ინტეგრალური განტოლებებითა და ცალხმრივი შერეული შეზღუდვებით აღწერილი მართვადი ობიექტებისათვის კრიტიკულობის აუცილებელი პირობებიდან გამოყვანილია ოპტიმალურობის აუცილებელი პირობები.

§ 1. Introduction

The present paper is devoted to deepening and generalization of the apparatus of investigation of optimal problems with mixed restrictions considered in the paper [4]. The proposed apparatus is also convenient for controlled objects which are described by equations with measure. Controls are chosen from the space of regular functions, i.e., from the Banach space obtained by completion of the set of step-functions with respect to the uniform norm. This space has been investigated in detail in [4] and [6].

The paper consists of five sections. In the second section, we consider (with certain modifications) a way of investigation of extremal problems named in literature as "the method of joint covering" [12]. Presentation of the method is principally the same as in [3]. One of the first variants of the above-mentioned method has been elaborated in [1].

The principal difference of the method suggested in the present paper from the well-known schemes [1–2] consists in that the restrictions prescribed in terms of differential equations are understood just as any other restrictions. This permits us, without principal difficulties, to consider the restrictions given by more complicated equations. A survey of the methods of joint covering is given in [12].

In the third section, in a standard manner we determine a bilinear integral of regular functions by means of finitely additive functions of sets generated by functions of bounded variation. This integral is a generalized Stieltjes integral. A number of basic properties of the integral calculus is also given.

In the fourth section, we study nonhomogeneous Volterra–Stieltjes equations with measure

$$x(t) = w(t) + \int_{t_0}^{t} g(s, x(s))d\sigma(s), \tag{1.1}$$

where σ is a function of bounded variation and w is a regular function. It turnes out that the equation (1.1) describes a great many physical processes [18]. Equations of this type with various integrals, in particular, with Stieltjes, Lebesgue-Stieltjes, Young and Dushnik integrals [6–10], have been considered in various works. Moreover, these works differ in that the integration in these equations takes place over closed, half-closed and open intervals on which w is taken from the class of functions of bounded variation or is a constant, and the solution is sought in the same class of functions of bounded variation. In the linear case, the equation (1.1) for the integral suggested in §3 has been studied in [5]. In the present work, we consider the equation (1.1) in the case where w is a regular function and a solution is sought in a class of regular functions. Existence and uniqueness theorems are proved.

In the fifth section, we study general problem of optimal control with mixed one-sided restrictions. Necessary conditions of optimality are derived in the class of regular controls, without additional assumptions of such type as the piecewise constancy of the set of active indices and the conditions of generality of position [16].

§ 2. Extremal Problems Connected with the Use of Vector Semi-Ordered Spaces

A Banach space X is said to be a partially ordered B-space if a convex cone $\mathcal{K}$ with a vertex at zero is selected in it possessing the property: if $x \in \mathcal{K}$, $y \in \mathcal{K}$ and $x + y = 0$, then $x = y = 0$. Elements of $\mathcal{K}$ are called positive elements of the space X; $\mathcal{K}$ is called the cone of positive elements of the space X; the relation $x \in \mathcal{K}$ is written either as $x \geq 0$ or by $0 \leq x$; by definition, the inequality $x \geq y$ implies $x - y \geq 0$. As for the space X, we say that it is ordered by means of the cone $\mathcal{K}$. We denote the cone of positive elements of the partially ordered B-space X by X_+. A set X_+ is said to be reproducing if every element $x \in X$ is representable as a difference of two positive elements.

A partially ordered B-space X is said to be Krein's or a BK-space if the set X_+ contains at least one inner point. The fact that x is an inner point of the set X_+ we write either as $x \gg 0$ or $0 \ll x$; by definition, the inequality $x \gg 0$ means that $-x \ll 0$. Note that if X is a Krein's space, then X_+ is a reproducing set [17].

If X is a B-space, partially ordered by the cone $\mathcal{K}$, then we can introduce in the conjugate space X^* the notion of positiveness by assuming that the functional x^* is positive if $x^*x \geq 0$ for all $x \in \mathcal{K}$. The set of all positive functionals forms a convex cone which we denote by $\mathcal{K}^*$. If $\mathcal{K}$ is a reproducing cone, then X^* can be partially ordered by the cone $\mathcal{K}^*$. In what follows, X^* is assumed to be closed.

Consider the following general extremal problem. Let $\mathcal{A}$ be a set in a metric space X, and let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be B-spaces. Note that $\mathcal{F}$ and $\mathcal{H}$ are partially ordered by the cones $\mathcal{F}_+$ and $\mathcal{H}_+$, respectively. Let f, g, h be the mappings of $\mathcal{A}$ into $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$, respectively. A point $x_0 \in \mathcal{A}$ is said to be the minimum point f on $\mathcal{A}$ under the restrictions

$$\begin{cases} g(x) = 0, \\ h(x) \leq 0, \end{cases} \tag{2.1}$$

if $g(x_0) = 0$, $h(x_0) \leq 0$, and for any, satisfying the same conditions, point $x \in \mathcal{A}$ for which $f(x)$ is comparable with $f(x_0)$ the inequality $f(x_0) \leq f(x)$ is fulfilled.

The maximum point is defined analogously.

Now the general extremal problem can be formulated as that of finding minimum (or maximum) points of the function f on the set $\mathcal{A}$ under the restrictions (2.1).

When studying the problem under consideration, it has been found more convenient to consider only one function instead of several functions involved non-symmetrically in the conditions. Thus, just as in [3] and [4], we introduce the notion of a critical point of mapping and show that the above formulated extremal problem can be reduced to the problem of finding critical points of some, specially constructed, mapping.

Definition 2.1. Let M be a set from a metric space $\mathcal{Y}$, $\mathcal{W}$ be a B-space and let $\mathcal{P} : M \to \mathcal{W}$ be some mapping. The point $z_0 \in M$ is said to be the critical point of the mapping $\mathcal{P}$ if the point $w_0 = p(z_0)$ is the boundary point of the set $\mathcal{P}(M)$.

It can be easily seen that the point $z_0 \in M$ is critical for the mapping $p : M \to \mathcal{W}$ if and only if for any positive number ε there exists a point $w \in \mathcal{W}$, $|w| < \varepsilon$, such that the image of the set M under the mapping $z \to p(z) + w$, $z \in M$, does not contain the point $w_0 = p(z_0)$.

Let

$$\begin{cases} \mathcal{Y} = X \times \mathcal{F} \times \mathcal{H}, \\ M = \mathcal{A} \times \mathcal{F}_+ \times \mathcal{H}_+, \\ W = \mathcal{F} \times \mathcal{G} \times \mathcal{H}, \end{cases}$$

and let $p : M \to \mathcal{W}$ be the mapping defined by

$$\begin{cases} p_1(x,k,y) = f(x) + k, \\ p_2(x,k,y) = g(x) \\ p_3(x,k,y) = h(x) + y, \end{cases}$$

$$z = (x,k,y), \quad w = p(z) = (p_1(z), p_2(z), p_3(z)).$$

Lemma 2.1. *If $x_0 \in \mathcal{A}$ is a minimum point of the general extremal problem, then $z_0(x_0, 0, -h(x_0)) \in Ms$ a critical point of the mapping p [3], [4].*

In [4] we describe the method of joint covering for the case where M is a convex closed cone with the vertex at zero, containing inner points. These results, with slight modifications, remain also valid for the case where M is a convex closed subset of the Banach space. For the sake of convenience, zero will be assumed to belong to M.

Let $\mathcal{D} \subset \mathcal{Y}$ be a convex set. Denote by $\operatorname{cone} \mathcal{D}$ the least convex cone with the vertex at zero, containing $\mathcal{D}$ and zero. It can be easily verified that

$$\operatorname{cone} \mathcal{D} = \{\lambda z | \lambda \geq 0, \ z \in \mathcal{D}\}.$$

Denote a minimal linear manifold or a subspace containing $\mathcal{D}$ by $\operatorname{sp} \mathcal{D}$. As is seen,

$$\operatorname{sp} \mathcal{D} = \operatorname{cone} \mathcal{D} - \operatorname{cone} \mathcal{D}.$$

Lemma 2.2. *Let $\mathcal{D} \subset \mathcal{Y}$ be a convex set containing zero. Then the equality*

$$\operatorname{sp} \mathcal{D} = \operatorname{cone}(\mathcal{D} - \mathcal{D}).$$

holds.

It is not difficult to prove this lemma.

Lemma 2.3. *Let $\mathcal{D} \subseteq \mathcal{Y}$ be a convex set and let $z_0 \in \mathcal{D}$ be some point. Then for any $\varepsilon > 0$ the equality*

$$\operatorname{sp} \mathcal{D} = \operatorname{sp}(\mathcal{B}(z_0, \varepsilon) \cap \mathcal{D})$$

is valid.

Proof. Obviously, $\operatorname{sp} \mathcal{D} \supseteq (\mathcal{B}(z_0, \varepsilon) \cap \mathcal{D})$. Prove the inverse inclusion. Let on the contrary $\operatorname{sp} \mathcal{D} \backslash (\mathcal{B}(z_0, \varepsilon) \cap \mathcal{D}) \neq \mathcal{D}$ for some number $\varepsilon > 0$. This implies that $\operatorname{cone} \mathcal{D} \operatorname{sp}(\mathcal{B}(z_0, \varepsilon) \cap \mathcal{D}) \neq \varnothing$. Hence there exists a point $\overline{z}$ such that $\overline{z} \in D$ and $\overline{z} \notin \operatorname{sp}\ B(z_0, \varepsilon) \cap D$. As far as D is convex, for any $\lambda \in [0, 1]$ the point $(1 - \lambda)z_0 + \lambda\overline{z}$ belongs to the set D. Let $a = \min\{\varepsilon/|\overline{z} - z_0|, 1\}$, $\lambda_1 \in (0, a)$ and assume $z_1 = (1 - \lambda)z_0 + \lambda_1\overline{z}$. We have $z_1 - z_0 = \lambda(\overline{z} - z_0)$ and $|z_1 - z_0| = \lambda_1|\overline{z} - z_0| \leq \varepsilon/|\overline{z} - z_0| \cdot |\overline{z} - z_0| = \varepsilon$, whence $z_1 \in B(z_0, \varepsilon) \cap \mathcal{D}$. But then

$$\overline{z} = \frac{1}{\lambda_1} z_1 - \frac{1 - \lambda_1}{\lambda_1} z_0 \in \operatorname{sp}(B(z_0, \varepsilon) \cap D).$$

The obtained contradiction proves the lemma. ■

The following lemma substitutes the principle of the openness of the mapping.

Lemma 2.4. *Let J, W be a B-space, $M \subseteq J$, and M be a convex closed set containing the origin, and let $M_\gamma = B(0, \gamma) \cap M$, $\gamma > 0$, $\mathcal{K} = \operatorname{cone} M_\gamma$, $T : J \to W$ be a bounded linear operator such that $T(\mathcal{K}) = W$. Then there exists $k > 0$ such that for any $w \in W$ there is $z \in \frac{|w|}{k} \cdot M\gamma$ satisfying the condition $Tz = w$.*

Proof. Obviously, $\cup_{N=1}^{\infty} nM_\gamma = \mathcal{K}$. Hence $\cup_{N=1}^{\infty} T(nM_\gamma) = T(\mathcal{K}) = W$ and consequently W is representable in the form of a countable union of closed sets $\overline{T(nM_\gamma)}$, $n = 1, 2, \ldots$. By the Baire theorem on the categories, one of these sets contains an inner point. Then, as is easily seen, $\overline{T(M_\gamma)}$ contains an inner point as well.

If zero is not an inner point of $\overline{T(M_\gamma)}$, then by the Hahn-Banach theorem, there exists a nonzero continuous linear functional w which is supporting the convex closed set $\overline{T(M_\gamma)}$ at zero, i.e., such that $w^*Tz \leq 0$ for all $z \in M_\gamma$. But then $w^*Tz \leq 0$ for all $z \in \mathcal{K}$ which contradicts the condition $T(\mathcal{K}) = w$.

Thus the set $\overline{T(M\gamma)}$ contains a sphere with the center at zero. If k is the radius of the sphere, then the above said means that for any $w \in W$ and any $\varepsilon > 0$ there exists a point $z \in \frac{1}{k} \cdot |w| \cdot M\gamma$ such that $|Tz - w| < \varepsilon$.

Let now w be an arbitrary vector from w. Suppose $w_1 = w$ and denote by z_1 the vector belonging to $\frac{1}{k} \cdot |w_1| \cdot M_{gm}$ for which $|Tz_1 - w_1| < \frac{1}{2}|w_1|$.

By induction we define $w_{n=1} = w_n - Tz_n$, and find that $z_{n+1} \in \frac{1}{k} \cdot |w_{n+1}| \cdot M_\gamma$ satisfies the condition $|Tz_{n+1} - w_{n+1}| < \frac{1}{2}|w_{n+1}|$. We have $w_{n+1} = |w_n - Tz_n| < \frac{1}{2} \cdot |w_n|$, whence $|z_{n+1}| < \frac{1}{2^n} \cdot |w_1|$.

Next,

$$z_{n+1} \in \frac{1}{k} \cdot |w_{n+1}| \cdot M_\gamma \subset \frac{1}{2} \cdot \frac{1}{2^n} \cdot |w_1| \cdot M\gamma,$$

whence $|z_{n+1}| \le \frac{|w_1|}{k} \cdot \frac{1}{2^n}$ and the series $z_1 + z_2 + \cdots$ is absolutely convergent. If we denote by z the sum of this series, then $|z| \le \frac{1}{k} \cdot |w_1|$, $z \in \frac{1}{k}|w_1| \cdot M_\gamma$. Summing up the equalities $w_1 = w$, $w_2 = w_1 - Tz_1, \dots,$ $w_{n+1} = w_n - Tz_n$, we obtain $w_{n+1} = w - Tz_1 - \cdots - Tz_n$.

Passing in the above equality to limit as $n \to \infty$ and taking into account that $w_n \to 0$, we arrive at $w = Tz$. ■

In what follows, we denote by $\varkappa(T, M_\gamma)$ the least upper bound of the set of all numbers k satisfying the conditions of Lemma 2.4.

Let M be a convex set in the B-space J, and let $0 \in M$, $L = \operatorname{sp} M$, $p : M \to W$ be a mapping taking the values from the B-space W.

Definition 2.2. We say that the mapping p is differentiable at the point $z_0 \in M$ if there exists a linear operator $T : L \to W$ satisfying:

$$\lim_{\substack{z \to z_0 \\ z \in M}} \frac{|p(z) - p(z_0) - T(z - z_0)|}{|z - z_0|} = 0. \tag{2.2}$$

We call the operator T the differential of p at the point z_0 and denote it by the symbol $T = Dp(z_0)$.

If p is differentiable in the above indicated sense, then the differential $Dp(z_0)$ is defined on L uniquely.

Indeed, let T_1 and T_2 be two operators satisfying (2.2). For simplicity we assume that $z_0 = 0$. Then for any $z \in M$ we have

$$|T_1 z - T_2 z| \le |p(z) - p(z_0) - T_2(z - z_0) - p(z) + p(z_0) + T_1(z - z_0)| \le \gamma(z) \cdot |z|,$$

where $\gamma(z) \to 0$ together with $|z|$. Substituting instead of z the value αz, $\alpha > 0$, we get

$$\alpha|T_1 z - T_2 z| \le \alpha \cdot \gamma(\alpha \cdot z) \cdot |z|, \quad \text{i.e.,}$$
$$|T_1 z - T_2 z| \le \gamma(\alpha \cdot z) \cdot |z|.$$

Since the left-hand side of the above equality does not depend on α and the right-hand side tends to zero as $\alpha \to 0$, we obtain $T_1 z = T_2 z$ for all $z \in M$. This implies that $T_1 z = T_2 z$ for all $z \in L$.

Everywhere below, unless otherwise stated, L is assumed to be dense in J.

Let M be a convex set in J and let p be differentiable at every point of M. Then for any z_1, $z_2 \in M$ the linear operator $T = Dp(z_1) - Dp(z_2)$ is defined on L. Since we have nowhere stated the continuity of the corresponding mappings, the operator T may happen to be unbounded. If, however, it is bounded, then by continuity it can be extended to the whole space J. In this case $T \in B(J, W)$.

Definition 2.3. The mapping p is said to be continuously differentiable on M if p is differentiable at all points of M, for every pair of points z_1, $z_2 \in M$ the operator $Dp(z_1) - Dp(z_2)$ is bounded and the mapping $(z_1, z_2) \to Dp(z_1) - Dp(z_2)$ is continuous as an operator $M \times M \to B(J, W)$.

Lemma 2.5. *If a mapping p is continuously differentiable in the vicinity of the point $z_0 \in M$, $T = Dp(z_0)$, then for any $\delta > 0$ there exists a neighborhood V of the point z_0 such that for all z_1, $z_2 \in V \cap M$ the inequality*

$$|p(z_1) - p(z_2) - T(z_1 - z_2)| < \delta \cdot |z_1 - z_2|.$$

is fulfilled.

The proof of this lemma can be found in [3] and [4]. Now we are able to formulate a generalization of the Graves lemma [11].

Lemma 2.6. *Let J and W be B-spaces, M be a convex closed set in J, and let $p : M \to W$ be a continuous mapping. Let, moreover, $\mathcal{K} = \mathrm{cone}(M - z_0)$ and T be a bounded linear operator defined on $\mathrm{sp}\,\mathcal{K}$ and satisfying the following conditions:*

(a) T(K)=W;

(b) there exists $\gamma > 0$, such that for any $z_1, z_2 \in B(z_0, \gamma) \cap M$, the inequality

$$|p(z_1) - p(z_2) - T(z_1 - z_2)| < \delta |z_1 - z_2|$$

is fulfilled; moreover, $\delta < \alpha = \varkappa(T, M_\gamma)$.

Then the equation $w = p(z)$ has a solution $z \in B(z_0, \gamma) \cap M$ for all $w \in B(p(z_0), \rho)$, where $\rho = \gamma \cdot (\alpha - \delta)$.

The proof of this lemma, which is similar to that of the Graves lemma, is given in [2] for the case where M is a cone with the vertex at zero but this proof is also fit for proving Lemma 2.6.

Comparing this result with Lemma 2.5, we convince ourselves that the following lemma is valid.

Lemma 2.7. *Let J and W be B-spaces, $M \subseteq J$ be a convex closed set, let $p : M \to W$ be a continuously differentiable mapping in the vicinity of a point $z_0 \in M$, and let the linear operator $T = \mathcal{D}p(z_0)$ map $\mathrm{cone}(m - z_0)$ onto the whole space W. Then there exists a neighborhood V of the point Z_0 such that the set $P(M \cap V)$ contains a neighborhood of the point $p(z_0)$ in W.*

From this lemma we immediately get the following

Corollary 2.8. *If $z_0 \in M$ is a critical point of the mapping p and $T = \mathcal{D}p(z_0)$ is a linear bounded operator, $T : \mathrm{sp}(M - z_0) \to W$, satisfying for some γ the condition (b) of Lemma 2.6, then T cannot satisfy the condition (a) of the same lemma.*

Before we proceed to formulating the final result, note that $\mathcal{K}_1 = T(\mathcal{K})$, where $\mathcal{K} = \mathrm{cone}(M - z_0)$ is a convex cone in the space W. Suppose that it either is closed or contains inner points. Then the condition $\mathcal{K} \neq W$ means that we can separate this cone from zero, i.e., there exists a different from zero functional $w^* \in W^*$ such that for all $z \in \mathcal{K}$ we have $w^* T z \leq 0$.

Theorem 2.9 (A necessary condition of criticality). *Let J and W be B-spaces, $M \subseteq J$, $z_0 \in M$, be a convex closed set, and let $p : M \to W$ be a continuous mapping continuously differentiable in the vicinity of the point $T = \mathcal{D}p(z_0)$, $\mathcal{K} = \mathrm{cone}(M - z_0)$. Then if $z_0 \in M$ is a critical point of the mapping p and $T(\mathcal{K})$ either is closed or contains inner points, then there exists a non-zero continuous linear functional $w^* \in W^*$ separating this cone from zero, i.e., such that*

$$w^* T \delta z \leq 0 \quad \textit{for all} \quad \delta z \in \mathcal{K}.$$

In other words,

$$w^* T \delta z \leq 0 \quad \textit{for all} \quad \delta z : z_0 + \delta z \in M.$$

Since in the applications the space W is often a product of B-spaces, we will give in conclusion a simple criterion for $\mathrm{cone}\, T(\mathcal{K}) t_0$ t_0 contain inner points of the product.

Lemma 2.10. *Let J, W_1, W_2 be B-spaces, $W = W_1 \times W_2$, $\mathcal{K} \subseteq J$ be a convex closed cone with the vertex at zero, and let $T_1 : \mathrm{sp}\,\mathcal{K} \to W_1$ and $T_2 : \mathrm{sp}\,\mathcal{K} \to W_2$ be linear bounded operators, $T_1 \mathcal{K} = W$. Denote by $N(T_1)$ the set of those $z \in \mathcal{K}$ for which $T_1 z = 0$. If $T_2(N(T_1))$ is open, then $T_1(\mathcal{K}) \times T_2(\mathcal{K})$ is also open in W.*

§ 3. Integral Calculus in the Space of Regular Functions

Let $\mathcal{J} = [0,1]$ and let X be a Banach space. For every $E \subseteq \mathcal{J}$ we denote by χ_E the characteristic function of the set E. If $E = \{t\}$, $t \in \mathcal{J}$, then its characteristic function will be denoted by χ_t.

Definition 3.1. The function $x : \mathcal{J} \to X$ is said to be a step-function if there exists a partition of the segment $\mathcal{J} : 0 = t_0 < t_1 < \cdots < t_n = 1$ such that the function x is constant on every open interval (t_{i-1}, t_i), $i = 0, 1, \ldots, n$.

By Σ we denote the algebra generated in $\mathcal{J} = [0,1]$ by the sets $[0,t]$ and $[t,1]$, $0 \leq t \leq 1$. Obviously, the sets $\{a\}$, $[a,b]$, (a,b), $[a,b)$, $(a,b]$, where $0 \leq a \leq b \leq 1$, belong to the algebra Σ. It is easily seen that for any step-function $x : \mathcal{J} \to X$ there exist the sets $E_1, \ldots, E_n$ from Σ and the vectors $\xi_1, \ldots, \xi_n$ from X such that

$$x(t) = \sum_{i=1}^{n} \chi_{E_i}(t)\xi_i \quad \text{for all} \quad t = \mathcal{J},$$

and vice versa.

Definition 3.2. A function $x : \mathcal{J} \to X$ is said to be regular if for every $t \in [0,1)$ it has a limit from the right, $x(t+)$, and for every $t \in (0,1]$ it x has a limit from the left, $x(t-)$. By definition we assume that $x(0-) = x(0)$ and $x(1+) = x(1)$.

Clearly, the step-function is regular.

Theorem 3.1. *For a function $x : \mathcal{J} \to X$ to be regular, it is necessary and sufficient that x be the limit of a uniformly convergent sequence of step-functions* [4], [6], [13].

We denote the space of regular functions which map a segment $\mathcal{J}$ into a B-space X by $NCN(\mathcal{J}, X)$. It is easily verified that $NCN(\mathcal{J}, X)$ is a B-space with respect to the uniform norm

$$|x|_{NCN} = \sup_{t \in \mathcal{J}} |x(t)|.$$

Denote by $CN(1, \mathcal{J}X)$ the closed subspace of the B-space $NCN(\mathcal{J}, X)$ generated by continuous from the left regular functions; by definition, the functions from $CN(\mathcal{J}, X)$ are assumed to be continuous at zero.

Definition 3.3. Let $\sigma : \mathcal{J} \to X$ be a function. The complete variation of the function σ on $\mathcal{J}$ is defined as

$$v(\sigma, \mathcal{J}) = \sup \sum_{i=1}^{n} |\sigma(t_i) - \sigma(t_{i-1})|,$$

where the upper bound is taken over all finite partitions of the segment $\mathcal{J} : 0 \leq t_0 < t_1 < t_n \leq 1$. If $v(\sigma, \mathcal{J}) < \infty$, then σ is called a function of bounded variation on $\mathcal{J}$.

It is not difficult to verify that any function of bounded variation is regular.

Denote by $BV(\mathcal{J}, X)$ the space of all functions of bounded variation which map $\mathcal{J}$ into X. This is a B-space with respect to the norm

$$|\sigma|_{BV} = |\sigma(\tau_0)| + V(\sigma, \mathcal{J}),$$

where τ_0 is some fixed point from $\mathcal{J}$. Denote by $BV_0(\mathcal{J}, X)$ the closed subspace of the B-space $BV(\mathcal{J}, X)$ generated by the functions $\sigma \in BV(\mathcal{J}, X)$ satisfying the condition $\sigma(\tau_0) = 0$.

Let $\sigma \in BV(\mathcal{J}, X)$. Denote by λ_σ the finitely additive set function which is defined on the algebra Σ by the relation $\lambda_\sigma([a,b]) = \sigma(b) - \sigma(a)$ and takes the values from the B-space X [4].

Regular functions were studied in detail in [4] and [6].

Definition 3.4. A bilinear triple (BT) is a set of three B-spaces X, Y, Z with a bilinear continuous mapping $T : X \times Y \to Z$. We will write $x \cdot y = T(x, y)$ and denote BT by $(X, Y, Z)_T$, or simply by (X, Y, Z) always assuming that $|T| \leq 1$.

Examples: Let E, F, G be B-spaces.
(a) $X = B(E, F)$, $Y = E$, $Z = F$ and $T(u, y) = u(y)$;
(b) $X = B(E, F)$, $Y = B(G, E)$, $Z = B(G, F)$ and $T(u, v) = u \circ v$;
(c) $X = E$, $Y = E^*$, $Z = \mathbb{R}^1$ and $T(e, e^*) = e^* e$;
(d) $X = Z = F$, $Y = \mathbb{R}^1$ and $T(f, \lambda) = \lambda \cdot f$.

Obviously, Examples (a), (c) and (d) are particular cases of Example (b).

Define now a bilinear integral. Let (X, Y, Z) be a bilinear triple, $E \in \Sigma$, $\sigma \in BV(\mathcal{J}, Y)$ and $x \in NCN(\mathcal{J}, X)$ is a step-function defined by the equality

$$x(t) = \Sigma \chi_{E_i}(t) \quad \text{for all} \quad t \in \mathcal{J},$$

where $E_1, \ldots, E_n$ is a family of mutually disjoint sets from the algebra Σ, and $\xi_1, \ldots, \xi_n$ are the vectors from X. By definition we assume that

$$\int_E x(t) \cdot \lambda_\sigma(dt) = \sum_{i=1}^n \xi_i \lambda_\sigma(EE_i).$$

Clearly, integration of step-functions with respect to a set is a linear operation, and

$$\left| \int_E x(t) \cdot \lambda_\sigma(dt) \right| \leq \sup_{t \in E} |x(t)| \cdot v(\lambda_\sigma, E).$$

Let $x \in NCN(\mathcal{J}, X)$ be an arbitrary function and let $\{x_n\}$ be a sequence of step-functions uniformly convergent to x. Then, as is easily seen, the sequence $\int_E x_n(t) \cdot \lambda_\sigma(dt)$, $n = 1, 2, \ldots$, converges for every $E \in \Sigma$ in the norm Z. The limit of this sequence of integrals is called, by definition, a bilinear integral of x with respect to the finitely additive set function of the set λ_σ, taken over the set $E \in Z$. We denote it by $\int_E x(t) \cdot \lambda_\sigma(dt)$, i.e.,

$$\int_E x(t) \cdot \lambda_\sigma(dt) = \lim_{n \to \infty} \int_E x_n(t) \cdot \lambda_\sigma(dt).$$

We can easily see that the above defined integral preserves almost all basic properties of the common integral [4], [14]. In the sequel, we will denote it in some other way,

$$\int_E *x(t) \cdot d\sigma(dt).$$

If $x \in NCN(\mathcal{J}, X)$, then for every $t \in \mathcal{J}$ denote by $x_\nu(t)$ (respectively, by $x_\nu^-(t)$) the right (the left) jump of the function x at the point t, i.e., $x_\nu(t) = x(t+) - x(t)$ (respectively, $x_\nu^-(t) = x(t) - x(t-)$). By definition we assume that $x_\nu^-(0) = x_\nu(1) = 0$.

Lemma 3.2. *Let $x \in NCN(\mathcal{J}, X)$ and let $M \subseteq X$ be a subset. Next, let F_1, F_2 and G_1, G_2 be the sets defined by the relations*

$$F_1\{t \in \mathcal{J} | x(t-) \in \overline{M} \vee x(t) \in \overline{M} \vee x(t+) \in \overline{M}\},$$
$$F_2\{t \in \mathcal{J} | x(t-) \in \overline{M} \vee x(t+) \in \overline{M}\},$$
$$G_1\{t \in \mathcal{J} | x(t-) \in X \backslash \overline{M} \wedge x(t) \in X \backslash \overline{M} \wedge x(t+) \in X \backslash \overline{M}\},$$
$$G_2\{t \in \mathcal{J} | x(t-) \in X \backslash \overline{M} \wedge x(t+) \in X \backslash \overline{M}\}.$$

Then the sets F_1, F_2 are closed, while G_1, G_2 are open in $\mathcal{J}$.

Proof. Obviously, $G_1 = \mathcal{J} \backslash F_1$ and $G_2 = \mathcal{J} \backslash F_2$. If we prove that the sets F_1 and F_2 are closed, then the openness of the sets G_1, G_2 will be proved as of the complements of closed ones. Prove the closeness of the set F_1. Let $\{t_n\}$ be a sequence convergent to some point $t \in \mathcal{J}$. There exists a finite subsequence $\{t_{n_k}\}$ such that either $x(t_{n_k}-) \in \overline{M}$ for all $k = 1, 2, \ldots$, or $x(t_{n_k}) \in \overline{M}$ for all $k = 1, 2, \ldots$, or $x(t_{n_k}+) \in \overline{M}$ for all $k = 1, 2, \ldots$. Without restriction of generality, we may assume that $t_{n_k} \leq t$, $k = 1, 2, \ldots$, or $t_{n_k} \geq t$ $k = 1, 2, \ldots$. Consider the case where $t_{n_k} \leq t$, $k = 1, 2, \ldots$. Then

$$\lim_{k\to\infty} x(t_{n_k}-) = x(t-), \quad \lim_{k\to\infty} x(t_{n_k}) = x(t-) \quad \lim_{k\to\infty} x(t_{n_k}+) = x(t-).$$

Hence $x(t-) \in \overline{M}$ and thus $t \in F_1$. The case where $t_{n_k} \geq t$, $k = 1, 2, \ldots$, is considered in a similar manner. It is not difficult to verify that F_2 is a closed subset of the set F_1. ■

Corollary 3.3. *Let $x \in NCN(\mathcal{J}, X)$ and let $\mathcal{K} \subseteq X$ be a subset. Then the following statements are valid:*

(*a*) *the set $\{t \in \mathcal{J} | x(t-) \in \partial\mathcal{K} \vee x(t) \in \partial\mathcal{K} \vee x(t+) \in \partial\mathcal{K}\}$ is closed in $\mathcal{J}$;*

(*b*) *the set $\{t \in \mathcal{J} | x(t-) \in \operatorname{int}\mathcal{K} \wedge x(t) \in \operatorname{int}\mathcal{K} \wedge x(t+) \in \operatorname{int}\mathcal{K}\}$ is open in $\mathcal{J}$;*

(*c*) *the set $\{t \in \mathcal{J} | x(t-) \neq \overline{\mathcal{K}} \wedge x(t) \neq \overline{\mathcal{K}} \vee x(t+) \neq \overline{\mathcal{K}}\}$ is open in $\mathcal{J}$.*

Proof. It follows from Lemma 3.2 that: (*a*) holds if we put $M = \partial\mathcal{K}$; (*b*) holds if we put $M = X \backslash \operatorname{int}\mathcal{K}$; (*c*) holds if we put $M = \overline{\mathcal{K}}$. ■

Lemma 3.4. *Let X_1, X_2, X be B-spaces and let (X_1, X_2, X) be a bilinear triple. Next, let $t_0 \in \mathcal{J}$, $\alpha \in BV(\mathcal{J}X_1)$, $x \in NCN(\mathcal{J}, X_2)$, and let a function α_1 be defined by the equality*

$$\alpha_1(t) = \alpha_1(t_0) + \int_{t_0}^{t} * \, d\alpha(s)s(s), \quad \textit{for all} \quad t \in \mathcal{J}.$$

Then $\alpha_1 \in BV(\mathcal{J}, X)$, $v(\alpha_1, \mathcal{J}) \leq v(\alpha, \mathcal{J}) \cdot |x|_{NCN}$ and for every $t \in \mathcal{J}$ the equalities

$$\alpha_{1\nu}(t) = \alpha_\nu(t)x(t+), \quad \alpha_{1\nu}^-(t) = \alpha_\nu^-(t)x(t-).$$

are valid.

Proof. Let $0 \leq s_0 < s_1 < \cdots < s_n \leq 1$ be an arbitrary partition of the segment $\mathcal{J}$. Then we have

$$\sum_{i=1}^{n} |\alpha_1(s_i) - \alpha_1(s_{i-1})| = \sum_{i=1}^{n} \left| \int_{s_{i-1}}^{s_i} *d\alpha(t)x(t) \right| \leq$$

$$\leq \sum_{i=1}^{n} v(\alpha, [s_{i-1}, s_i]) \cdot \sup_{t \in [s_{i-1}, s_i]} |x(t)| \leq$$

$$\leq \sum_{i=1}^{n} v(\alpha, [s_{i-1}, s_i]) \cdot |x|_{NCN} = v(\alpha, \mathcal{J}) \cdot |x|_{NCN}.$$

Hence $v(\alpha_1, \mathcal{J}) \leq v(\alpha, \mathcal{J}) \cdot |x|_{NCN}$ and $\alpha \in BV(\mathcal{J}, X)$.

Let now $\{x_n \subset NCN(\mathcal{J}, X_2)\}$ be a sequence of step-functions which converges uniformly to the function x and let $\{\varepsilon_m\}$ be a sequence of positive numbers, $\varepsilon_m \to 0$ as $m \to \infty$. Next, let $t \in [0, 1)$ be some point. Then we can easily see that the equality

$$\lim_{m \to \infty} \int_{t}^{t+\varepsilon_m} * \, d\alpha(s) \cdot x_n(s) = \alpha_\nu(t) \cdot x_n(t+)$$

is valid for every n, $n = 1, 2, \ldots$. By definition of the integral,

$$\lim_{n \to \infty} \int_{t}^{t+\varepsilon_m} * \, d\alpha(s) \cdot x_n(s) = \int_{t}^{t+\varepsilon_m} * \, d\alpha(s) \cdot x(s). \tag{3.1}$$

We have

$$\left| \int_{t}^{t+\varepsilon_m} * \, d\alpha(s) \cdot x_n(s) - \int_{t}^{t+\varepsilon_m} * \, d\alpha(s) \cdot x(s) \right| = \left| \int_{t}^{t+\varepsilon_m} * \, d\alpha(s) \cdot [x_n(s) - \right.$$

$$\left. -x(s)] \right| \leq v(\alpha, [t, t+\varepsilon_m]) \cdot |x_n - x|_{NCN} \leq v(\alpha, \mathcal{J}) \cdot |x_n - x|_{NCN}.$$

Consequently, the limit (3.1) does exist uniformly with respect to m. Then, using the theorem on transposition of passages to limit, we obtain that

$$\alpha_\nu(t+)\cdot x(t+) = \lim_{n\to\infty} \alpha_\nu(t)\cdot x_n(t+) =$$

$$= \lim_{n\to\infty}\lim_{m\to\infty} \int\limits_t^{t+\varepsilon_m} * \, d\alpha(s)\cdot x_n(s) =$$

$$= \lim_{m\to\infty}\lim_{n\to\infty} \int\limits_t^{t+\varepsilon_m} * \, d\alpha(s)\cdot x_n(s) =$$

$$= \lim_{m\to\infty} \int\limits_t^{t+\varepsilon_m} * \, d\alpha(s)\cdot x(s) = \lim_{m\to\infty}[\alpha_1(t+\varepsilon_m)-\alpha_1(t)] = \alpha_{1\nu}(t).$$

The fact that $\alpha_{1\nu}^-(t) = \alpha_\nu^-(t)x(t-)$ for every $t\in(0,1]$ is proved analogously. ∎

Theorem 3.5. *Let X_1, X_2, X, Y be B-spaces, X_1, X_2, Y being a bilinear triple. Next, let $\alpha \in BV(\mathcal{J}, X_1)$, $\mathcal{A}\in NCN(\mathcal{J}, B(X, X_2))$ and let α_1 be a function defined by*

$$\alpha_1(t)\xi = \alpha_1(t_0)\xi + \int\limits_{t_0}^{t} * \, d\alpha(s)\mathcal{A}(s)\xi,$$
$$\textit{for all}\quad t\in\mathcal{J},\quad \xi\in X, \tag{3.2}$$

where t_0 is a point from $\mathcal{J}$. Then $\alpha_1 \in BV(\mathcal{J}, B(X,\mathcal{Y}))$, and for the equality

$$\int\limits_0^1 * \, d\alpha(t)\mathcal{A}(t)x(t) = \int\limits_0^1 * \, d\alpha_1(t)x(t)$$
$$\textit{for all}\quad x\in NCN(\mathcal{J}, X) \tag{3.3}$$

to be fulfilled, it is necessary and sufficient that the function α_1 be of the form (3.2).

Proof. From Lemma 3.4 it immediately follows that $\alpha_1\in BV(\mathcal{J}, B(X,Y))$. Let (3.3) be fulfilled. Then, substituting in it the functions of the type $s\to\chi_{[t,t_0]}(s)\xi$, $s\in[0,t_0)$, $\xi\in X$ and $s\to\chi_{[t,t_0]}(s)\xi$, $s\in(t_0,t]$, $\xi\in X$, we can easily see that the function α_1 is expressed in terms of the equality (3.2). Prove the sufficiency. Let $T: NCN(\mathcal{J}, X)\to Y$ be an operator defined by

$$Tx = \int\limits_0^1 * \, d\alpha(t)\mathcal{A}(t)x(t) - \int\limits_0^1 * \, d\alpha_1(t)x(t), \quad \text{for all} \quad x\in NCN(Y,X).$$

Obviously, T is a linear operator, and

$$|Tx| \leq [v(\alpha,\mathcal{J}) \cdot |\mathcal{A}|_{NCN} + v(\alpha,\mathcal{J})] \cdot |x|, \quad x \in NCN(\mathcal{J},X).$$

It can be easily verified that the value of the operator T on step-functions equals zero. Since step-functions are dense everywhere in $NCN(\mathcal{J},X)$, a linear bounded operator vanishing on step-functions will identically be equal to zero on the whole space $NCN(\mathcal{J},X)$. ■

Theorem 3.6. *Let X_1, X_2, X, Y be B-spaces, (X_1, X_2, Y) being a bilinear triple. Next, let $\varphi \in BV(\mathcal{J}, X_1)$, $\beta \in BV(\mathcal{J}, B(X, X_2))$ and let the function φ_1 be defined by*

$$\varphi_1(t)\xi = \begin{cases} \varphi_1(0)\xi + \int_0^t * \, d\varphi(s)[\beta(s) - \beta(t)]\xi, \\ \qquad 0 \leq t \leq t_0, \quad \xi \in X, \\ \varphi_1(1)\xi + \int_1^t * \, d\varphi(s)[\beta(s) - \beta(t)]\xi, \\ \qquad t_0 < t \leq 1, \quad \xi \in X, \end{cases} \tag{3.4}$$

where t_0 is a point from $\mathcal{J}$. Then the following assertions are valid:
(a) $\varphi_1 \in BV(\mathcal{J}, B(X,Y))$, $v(\varphi_1,\mathcal{J}) \leq 2 \cdot v(\varphi,\mathcal{J}) \cdot v(\beta,\mathcal{J})$;
(b) for the equality

$$\int_0^1 * \, d\varphi(t) \int_{t_0}^t * \, d\beta(s)x(s) = \int_0^1 * \, d\varphi_1(t)x(t) \qquad \text{for} \quad x \in NCN(\mathcal{J},X) \tag{3.5}$$

to be fulfilled, it is necessary and sufficient that the function φ_1 be of the form (3.4) *and satisfy the relation*

$$[\varphi_1(0) - \varphi_1(1)]\xi = -\int_0^1 * \, d\varphi(s)[\beta(s) - \beta(t_0)]\xi, \qquad \text{for all} \quad \xi \in X. \tag{3.6}$$

Proof. Prove the assertion (a). Let $0 \leq \tau_0 < \tau_1 < \cdots < \tau_k < \tau_{k=1} < \tau_{k+1} < \cdots < \tau_n \leq 1$ be a partition of the segment $\mathcal{J}$, and let $\xi, \ldots, \xi_n$ be vectors from X, $|\xi_i| \leq 1$, $i = 1, \ldots, n$. Then for every $i \in \{1, \ldots, k\}$ we have

$$[\varphi_1(\tau_i) - \varphi_1(\tau_{i-1})]\xi =$$

$$= \int_0^{\tau_i} * \, d\varphi(s)[\beta(s) - \beta(\tau_i)]\xi_i - \int_0^{\tau_{i-1}} * \, d\varphi(s)[\beta(s) - \beta(\tau_{i-1})\xi_i =$$

$$= -\int_0^{\tau_{i-1}} * \, d\varphi(s)[\beta(\tau_i) - \beta(\tau_{i-1})]\xi_i + \int_{\tau_{i-1}}^{\tau_i} * \, d\varphi(s)[\beta(s) - \beta(\tau_i)]\xi$$

which implies that

$$\begin{aligned}
|[\varphi_1(\tau_0) - \varphi_1(\tau_{i-1})]\xi_i| \le |\varphi(\tau_{i-1}) - \varphi(0)| \cdot |[\beta(\tau_i) - \beta(\tau_{i-1})]\xi_i| + \\
+v(\varphi, [\tau_{i-1}, \tau_i]) \sup_{s\in[\tau_{i-1},\tau_i]} |[\beta(s) - \beta(\tau_i)]\xi_i| \le \\
\le v(\varphi, [0, t_0]) \cdot |\beta(\tau_i) - \beta(\tau_{i-1})| + v(\varphi, [0, t_0]) \times \\
\times v(\beta, [\tau_{i-1}, \tau_i]) \le 2 \cdot (\varphi, [0, t_0]) \cdot v(\beta, [\tau_{i-1}, \tau_i].
\end{aligned}$$

In a similar way, we can prove that for every $i \in \{k+1, \dots, n\}$ the following inequality is valid:

$$|[\varphi_1(\tau_i) - \varphi_1(\tau_{i-1})]\xi_i| \le 2v(\varphi, [t_0, 1]) \cdot (\beta, [\tau_{i-1}, \tau_i]).$$

We have

$$\begin{aligned}
\sum_{i=1}^{n} |[\varphi_1(\tau_i) - \varphi_1(\tau_{i-1})]\xi_i| \le 2 \cdot \sum_{i=1}^{k} v(\varphi, [0, t_0]) v(\beta, [\tau_{i-1}, \tau_i]) + \\
+2 \cdot \sum_{i=k+1}^{n} v(\varphi, [t_0, 1]) v(\beta, [\tau_{i-1}, \tau_i]) \le \\
\le 2v(\varphi, \mathcal{J}) \cdot \sum_{i=1}^{n} v(\beta, [\tau_{i-1}, \tau_i]) = 2v(\varphi, \mathcal{J}) \cdot v(\beta, \mathcal{J}).
\end{aligned}$$

As is easily seen, for every $\varepsilon > 0$ there exist vectors $\xi_1, \dots, \xi_n$ from X, $|\xi_i| \le 1$, $i = 1, \dots, n$, such that

$$\sum_{i=1}^{n} |\varphi_1(\tau_i) - \varphi_1(\tau_{i-1})| \le \sum_{i=1}^{n} |[\varphi_1(\tau_i) - \varphi_1(\tau_{i-1})]\xi_i| + \varepsilon.$$

Hence we obtain that

$$\sum_{i=1}^{n} |\varphi_1(\tau_i) - \varphi_1(\tau_{i-1})| \le 2v(\varphi, \mathcal{J}) \cdot v(\beta, \mathcal{J}) + \varepsilon$$

for any partition of the segment $\mathcal{J}$ and for any $\varepsilon > 0$. Thus $v(\varphi, \mathcal{J}) \le 2v(\varphi, \mathcal{J}) \cdot v(\beta, \mathcal{J})$.

Prove now the assertion (b).

The necessity. Let (3.5) be fulfilled. Consider the function $\overline{x}(t) = \chi_{[\tau,1]}(t)\xi$, $t \in \mathcal{J}$, $\xi \in X$ where τ is some point from $\mathcal{J}$. Suppose that $\tau \le t_0$. We have

$$\int_{t_0}^{t} *d\beta(s)\overline{x}(s) = \begin{cases} [\beta(\tau) - \beta(t_0)]\xi, & 0 \le t \le \tau, \\ [\beta(t) - \beta(t_0)]\xi, & \tau < t < 1. \end{cases} \tag{3.7}$$

The case where $\tau > t_0$ yields

$$\int_{t_0}^{t} *d\beta(s)\overline{x}(s) = \begin{cases} 0, & 0 \le t \le \tau, \\ [\beta(t) - \beta(\tau)]\xi, & \tau < t \le 1. \end{cases} \tag{3.8}$$

Taking into account the above obtained expressions and substituting the function x_1 in (3.5), we easily get that

$$[\varphi_1(1) - \varphi_1(\tau)]\xi =$$
$$= \begin{cases} \int_0^\tau *d\varphi(t)[\beta(t) - \beta(t_0)]\xi + \int_\tau^1 *d\varphi(t)[\beta(\tau) - \beta(t_0)]\xi, & 0 \le \tau \le t_0, \\ \int_\tau^1 *d\varphi(t)[\beta(t) - \beta(\tau)]\xi, \quad t_0 < \tau \le 1, \quad \xi \in X \end{cases} \tag{3.9}$$

which for $\tau = 0$ implies

$$[\varphi_1(1) - \varphi_1(0)]\xi = \int_0^1 *d\varphi(t)[\beta(t) - \beta(t_0)]\xi, \quad \xi \in X. \tag{3.10}$$

Multiplying this equality by -1, we obtain (3.6). The equality (3.9) results in

$$\varphi_1(t)\xi = \begin{cases} \varphi_1(1)\xi - \int_0^t *d\varphi(s)[\beta(t) - \beta(t_0)]\xi - \int_t^1 *d\varphi(s)[\beta(s) - \beta(t_0)]\xi, \\ \qquad 0 \le t \le t_0, \\ \varphi_1(1)\xi + \int_1^t *d\varphi(s)[\beta(s) - \beta(t)]\xi, \\ \qquad t_0 < t \le 1, \quad \xi \in X. \end{cases}$$

Calculating $\varphi_1(1)\xi$ from (3.10) and substituting in the upper right-hand side, we easily get (3.4).

The sufficiency. Let (3.4) and (3.6) be fulfilled. It is not difficult t_0 verify that the operator defined by the equality

$$Tx = \int_0^1 *d\varphi(t) \int_{t_0}^t *d\beta(s)x(s) - \int_0^1 *d\varphi_1(t)x(t), \quad \text{for all} \quad x \in NCN(\mathcal{J}, X)$$

is a linear bounded operator, $T : NCN(\mathcal{J}, X) \longmapsto \mathcal{Y}$. Let us calculate the value of the operator T on the functions of the kind $\overline{\overline{x}}(t) = \chi_{[0,\tau]}(t)\xi$, $t \in \mathcal{J}$, $\xi \in X$ and $\overline{x}(t) = \chi_{[\tau,1]}(t)\xi$, $t \in \mathcal{J}$, $\xi \in X$ where τ is a point from $\mathcal{J}$. First calculate $T\overline{\overline{x}}$. To this end, we will need the value of the integral $\int_{t_0}^t *d\beta(s)\overline{\overline{x}}(s)$. For $0 \le \tau \le t_0$, we have

$$\int_{t_0}^{t} *d\beta(s)\overline{\overline{x}}(s) = \begin{cases} [\beta(\tau) - \beta(t)]\xi, & 0 \le t \le \tau, \\ 0, & \tau < t \le 1, \quad \xi \in X. \end{cases}$$

The case where $\tau > t_0$ yields

$$\int\limits_{t_0}^{t} *d\beta(s)\overline{\overline{x}}(s) = \begin{cases} [\beta(t) - \beta(t_0)]\xi, & 0 \le t \le \tau, \\ [\beta(\tau) - \beta(t_0)]\xi, & \tau < t \le 1, \quad \xi \in X. \end{cases}$$

Taking into consideration the above obtained expression, for $0 \le \tau \le t_0$ we get

$$T\overline{\overline{x}} = \int\limits_0^1 *d\varphi(t)\chi_{[0,\tau]}(t)[\beta(t) - \beta(\tau)]\xi - [\varphi_1(\tau) - \varphi_1(0)]\xi = 0.$$

For $t_0 < \tau \le 1$ we have

$$T\overline{\overline{x}} = \int\limits_0^1 *d\varphi(t)[\chi_{[0,\tau]}(t)[\beta(t) - \beta(t_0)]\xi + \chi_{[\tau,1]}(t)[\beta(\tau) - \beta(t_0)]\xi] -$$

$$-[\varphi_1(\tau) - \varphi_1(0)]\xi = \int\limits_0^\tau *d\varphi(t)[\beta(t) - \beta(t_0)]\xi + \int\limits_\tau^1 *d\varphi(t)[\beta(\tau) - \beta(t_0)]\xi +$$

$$+\varphi_1(0)\xi - \varphi_1(1)\xi + \int\limits_\tau^1 *d\varphi(t)[\beta(t) - \beta(\tau)]\xi =$$

$$= \int\limits_0^1 *d\varphi(t)[\beta(t) - \beta(t_0)]\xi + \varphi_1(0)\xi - \varphi_1(1)\xi = 0.$$

Using (3.7) and (3.8) and acting analogously as when calculating $T\overline{\overline{x}}$, we easily get that $T\overline{x} = 0$.

Consequently, since any step-function is represented as a linear combination of the above considered functions $\overline{x}$, $\overline{\overline{x}}$ and also since step-functions are dense everywhere in $NCN(\mathcal{J}, X)$, from the linearity and boundedness of the operator T it follows that it is identically equal to zero. ■

From Theorems 3.5 and 3.6 we have the following

Corollary 3.7. *Let X, X_1, X_2, $\mathcal{Y}$, $\mathcal{Y}_1$, $\mathcal{Y}_2$ be B-spaces, where $(X_1, X_2, \mathcal{Y}_2)$ and $(\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y})$ are bilinear triples. Further, let $\varphi \in BV(\mathcal{J}, \mathcal{Y}_1)$, $\beta \in BV(\mathcal{J}, X_1)$, $\mathcal{A} \in NCN(\mathcal{J}, B(X, X_2))$ and let the function φ_1 be defined by the equality*

$$\varphi_1(t)\xi = \begin{cases} \varphi_1(0)\xi + \int_0^t *d\varphi(s) \int_t^s *d\beta(\tau)\mathcal{A}(\tau)\xi, \\ \qquad 0 \le t \le t_0, \\ \varphi_1(1)\xi + \int_1^t *d\varphi(s) \int_t^s *d\beta(\tau)\mathcal{A}(\tau)\xi, \\ \qquad t_0 < t \le 1, \quad \xi \in X, \end{cases} \tag{3.11}$$

where t_0 is a point from $\mathcal{J}$. Then the following statements are valid:

(a) $\varphi_1 \in BV(\mathcal{J}, B(X,Y))$, $v(\varphi_1, \mathcal{J}) \leq 2 \cdot v(\varphi, \mathcal{J}) \cdot v(\beta, \mathcal{J}) \cdot |\mathcal{A}|_{NCN}$;
(b) for the equality

$$\int_0^1 *d\varphi(t) \int_{t_0}^t *d\beta(s)\mathcal{A}(s)x(s) = \int_0^1 *d\varphi_1(t)x(t), \quad \textit{for all} \quad x \in NCN(\mathcal{J}, X),$$

to be fulfilled, it is necessary and sufficient that the function φ_1 *be of the form* (3.11) *and satisfy the relation*

$$[\varphi_1(0) - \varphi_1(1)]\xi = -\int_0^1 *d\varphi(t) \int_{t_0}^t *d\beta(s)(s)\xi, \quad \textit{for all} \quad \xi \in X.$$

The theorem below deals with the formula of integration by parts.

Theorem 3.8. *Let* X, Y, Z *be a bilinear triple,* $\sigma \in BV(\mathcal{J}, X)$ *and* $\rho \in BV(\mathcal{J}, Y)$. *Then the following formula of integration by parts is valid:*

$$\int_0^1 *\sigma(t) \cdot d\rho(t) + \int_0^1 *d\sigma(t) \cdot \rho(t) = \sigma(1) \cdot \rho(1) - \sigma(0) \cdot \rho(0) + \\ + \sum_{0 \leq \tau \leq 1} [\sigma_\nu(\tau) \cdot \rho_\nu(\tau) - \sigma_\nu^-(\tau)\rho_\nu^-(\tau)]. \tag{3.12}$$

Proof. Consider the operator

$$T : BV(\mathcal{J}, X) \to Z$$

defined by

$$T\sigma = \int_0^1 *\sigma(t) \cdot d\rho(t) + \int_0^1 *d\sigma(t) \cdot \rho(t) - \sigma(0) \cdot \rho(0) + \sigma(1) \cdot \rho(1) - \\ - \sum_{0 \leq \tau \leq 1} [\sigma_\nu(\tau) \cdot \rho_\nu(\tau) - \sigma_\nu^-(\tau) \cdot \rho_\nu^-(\tau)], \quad \text{for all} \quad \sigma \in BV(\mathcal{J}, X).$$

Obviously, T is a linear operator. Let us show that this operator is bounded. We have

$$|T\sigma| \leq |\sigma|_{NCN} \cdot v(\rho, \mathcal{J}) + v(\sigma, \mathcal{J}) \cdot |\rho|_{NCN} + |\sigma(0)| \cdot |\rho(0)| + \\ + |\sigma(1)| \cdot |\rho(1)| + \sum_{0 \leq \tau \leq 1} [|\sigma_\nu(\tau)| \cdot |\rho|_\nu(\tau)| + |\sigma_\nu^-| \cdot |\rho_\nu^-(\tau)| \leq \\ \leq (|\sigma(0)| + v(\sigma, \mathcal{J})) \cdot v(\rho, \mathcal{J}) + (|\sigma(0)| + v(\sigma, \mathcal{J})) \cdot |\rho|_{NCN} + \\ + (\sigma(0) + v(\sigma, \mathcal{J}))|\rho(0)| + (|\sigma(0)| + v(\sigma, \mathcal{J}) \cdot |\rho(1)| + v(\sigma, \mathcal{J}) \times \\ \times \sum_{0 \leq \tau \leq 1} [|\sigma_\nu(\tau)| + |\rho_\nu^-(\tau)] \leq |\sigma|_{BV} \cdot (|\rho(0)| + |\rho(1)| + 2|\rho|_{BV} + |\rho|_{NCN}).$$

Consequently, T is a bounded operator.

Let

$$\sigma_1(t) = \chi_{[0,\tau]}(t)\xi, \quad \text{for} \quad t \in \mathcal{J}, \quad \xi \in X,$$
$$\sigma_2(t) = \chi_{[\tau,1]}(t)\xi, \quad \text{for} \quad t \in \mathcal{J}, \quad \xi \in X,$$

where τ is a point from $\mathcal{J}$. We can easily see that $T\sigma_1 = 0$ and $T\sigma_2 = 0$. Since any step-function is a linear combination of functions of the above indicated types, the operator T vanishes on these functions. Let us choose the sequence $\{\sigma_n\}$ of step-functions from $BV(\mathcal{J}, X)$ such that $|\sigma_n - \sigma|_{NCN} \to 0$ as $n \to \infty$ and $v(\sigma_n, \mathcal{J}) \leq v(\sigma, \mathcal{J})$, for all $n = 1, 2, \ldots$. Then, applying Helly's theorem on the passage to limit under the integral sign [4] and taking into consideration the definition of the integral, we readily get that $T\sigma_n \to T\sigma$ as $n \to \infty$. Hence $T\sigma = 0$ for all $\sigma \in BV(\mathcal{J}, X)$. ■

Corollary 3.9. *Let (X, Y, Z) be a bilinear triple, $\sigma \in BV(\mathcal{J}, X)$ and $\rho \in BV(\mathcal{J}, Y)$. Then for any a and b, $0 \leq a < b \leq 1$, the formula*

$$\int_0^1 *\sigma(t) \cdot d\rho(t) + \int_a^b *d\sigma(t) \cdot \rho(t) = \sigma(b) \cdot \rho(b) - \sigma(a) \cdot \rho(a) + \sigma_\nu(a)\rho_\nu(a) -$$
$$-\sigma_\nu(b)\rho_\nu(b) + \sum_{0 \leq \tau \leq b} [\sigma_\nu(\tau) \cdot \rho_\nu(\tau) - \sigma_\nu^-(\tau) \cdot \rho_\nu^-(\tau)] \ \text{for all} \ \sigma \in BV(\mathcal{J}, X)$$

is valid.

Consider some results from [4] we will need in the sequel.

Theorem 3.10. *Every bounded linear functional $x^* \in CN^*(\mathcal{J}, X)$ is representable uniquely as*

$$x^* x = \int_0^1 * \, d\sigma(t) \cdot x(t) \quad \text{for all} \quad x \in CN(\mathcal{J}, X), \tag{3.13}$$

where $\sigma \in BV_0(\mathcal{J}, \mathcal{X}^*)$. Formula (3.13) specifies an isometric isomorphism between the spaces $CN^*(\mathcal{J}, X)$ and $BV_0(\mathcal{J}, \mathcal{X}^*)$, $|x^*| = |\sigma|_{BV}$.

Let X be a BK-space with the cone of positive elements X_+. Then we can see that $CN(\mathcal{J}, X)$ is also a BK-space with the cone of positive elements $CN_+(\mathcal{J}, X) = CN(\mathcal{J}, X_+)$.

Definition 3.5. The function $\sigma : \mathcal{J} \to X$ is said to be nonincreasing (nondecreasing) if for all t and s, $0 \leq t < s \leq 1$, $\sigma(t)$ and $\sigma(s)$ are congruent and the inequality $\sigma(t) \geq \sigma(s)$ (respectively, $\sigma(t) \leq \sigma(s)$) holds.

Let $x \in CN(\mathcal{J}, X)$, $x \geq 0$. Then by Corollary 3.3, the set $\{t \in \mathcal{J} | x(t) \gg 0 \wedge x(t+) \gg 0\}$ is open in $\mathcal{J}$.

Lemma 3.11. *Let $x \in CN(\mathcal{J}, X)$, $x \geq 0$, and let $\sigma \in BV(\mathcal{J}, \mathcal{X}^*)$ be a nonincreasing function. Then in order that*

$$\int_0^1 *d\sigma(t)x(t) = 0,$$

it is necessary that the following conditions be fulfilled:

(a) for any $t \in \mathcal{J}$ the equalities

$$\sigma_\nu(t)x(t+) = 0, \quad \sigma_\nu^-(t)x(t) = 0$$

hold;

(b) if $(a, b) \subseteq \mathcal{J}$ is a segment such that for all $t \in (a, b)$ $x(t)$ and $x(t+)$ belong to the interior of the cone X_+, then σ is a constant on (a, b).

If $X = \mathbb{R}^1$, then conditions (a) and (b) are sufficient as well.

§ 4. Integral Equations with Measure.

Let $\mathcal{J} = [0, 1]$ and let X, X_1 be B-spaces, where X_1 is a Banach algebra with unity id; let (X_1, X, X) be a bilinear triple, $\beta \in CN(\mathcal{J}, X_1)$ be a function of bounded variation and let $w \in CN(\mathcal{J}, X)$ be some function. Next, let $[a, b] \subseteq \mathcal{J}$ and $t_0 \in [a, b]$. Consider the equation

$$x(t) - \int_{t_0}^{t} d\beta(s)x(s) = w(t), \quad t \in [a, b], \tag{4.1}$$

where x is an unknown function. If there exists a function $x : [a, b] \to X$ satisfying the equation (4.1), then we call it a solution of this equation. The equation with measure (4.1) will be called the Volterra-Stieltjes equation.

Suppose that the solution of (4.1) does exist. Then by Lemma 3.4, the equality

$$x_\nu(t) - \beta_\nu(t)x(t+) = w_\nu(t) \quad \text{for} \quad t \in [a, b) \tag{4.2}$$

is valid. It follows from (4.1) that $x(t_0) = w(t_0)$, and from (4.2) we have

$$\begin{cases} [\mathrm{id} - \beta_\nu(t)]x(t+) = x(t) + w_\nu(t), \\ x(t) = [\mathrm{id} - \beta_\nu(t)]x(t+) - w_\nu(t) \quad \text{for} \quad t \in [a, b). \end{cases} \tag{4.3}$$

Assume $\mathrm{id} - \beta_\nu(t_0)$ to be an invertible element of the algebra X_1. Then the first equality of the system (4.3) implies that

$$x(t_0+) = [\mathrm{id} - \beta_\nu(t_0)]^{-1} w(t_0+). \tag{4.4}$$

Introduce the following functions:

$$x_2(t) = \begin{cases} x(t_0+) & \text{for} \quad t = t_0, \\ x(t) & \text{for} \quad t \in (t_0, b], \end{cases} \tag{4.5}$$

$$\beta_2(t) = \begin{cases} \beta(t_0+) & \text{for} \quad t = t_0, \\ \beta(t) & \text{for} \quad t \in (t_0, b], \end{cases} \tag{4.6}$$

$\widetilde{\beta}(t) = \beta(t) - \beta_2(t)$, $\widetilde{x}(t) = x(t) - x_2(t)$ for $t \in [t_0, b]$, and consider the equation (4.1) on the interval $[t_0, b]$. We have $x_2(t) + \widetilde{t} - \int_{t_0}^{t} *d[\beta_2(s) + \widetilde{\beta}(s)](x_2(t) + \widetilde{x}(t)) = w(t)$, for $t \in [t_0, b]$, which immediately yields

$$x_2(t) - \int\limits_{t_0}^{t} *d\beta_2(s)x_2(s) = w_2(t) \quad \text{for} \quad t \in [t_0, b], \tag{4.7}$$

where

$$w_2(t) = \begin{cases} w(t_0) + x_\nu(t_0) & \text{for} \quad t = t_0, \\ w(t) + \beta_\nu(t_0)x(t_0+) & \text{for} \quad t \in (t_0, b]. \end{cases}$$

Transforming the right-hand side of this equality and denoting the function w_1 by w, we obtain

$$\begin{gathered} w_2(t_0) = w(t_0) + x_\nu(t_0) = w(t_0) + x(t_0) - \\ -x(t_0) = [\mathrm{id} - \beta_\nu(t_0)]^{-1} w(t_0+). \\ w_2(t) = w(t) + \beta_\nu(t_0)x(t_0+) = w(t) + x_\nu(t_0) - w_\nu(t_0) = \\ = w(t) + x(t_0+) - x(t_0) - w(t_0+) + w(t_0) = \\ = w(t) - w(t_0+) + [\mathrm{id} - \beta_\nu(t_0)]^{-1} w(t_0+) = \\ = w(t) - [\mathrm{id} - [\mathrm{id} - \beta_\nu(t_0)]^{-1}] w(t_0+) \quad \text{for} \quad t \in (t_0, b] \end{gathered}$$

which implies that

$$w_2(t_0+) = w(t_0+) - [\mathrm{id} - [\mathrm{id} - \beta_\nu(t_0)]^{-1}] w(t_0+) = w_2(t_0).$$

Consequently, the function w_2 is continuous from the right at the point t_0 and expressed by the equality

$$w_2(t) = w(t) - [\mathrm{id} - [\mathrm{id} - \beta_\nu(t_0)]^{-1}] w(t_0+) \quad \text{for} \quad t \in [t_0, b]. \tag{4.8}$$

Let T_2 and L_2 be operators defined on the space $CN([t_0, b], X)$ by the equalities

$$(T_2x_2)(t) = x_2(t) - \int_{t_0}^{t} *d\beta_2(s)x_2(s) \quad \text{for all}$$
$$t \in [t_0, b], \quad x_2 \in CN([t_0, b], X),$$

$$(L_2x_2)(t) = \int_{t_0}^{t} *d\beta_2(s)x_2(s) \quad \text{for all} \quad t \in [t_0, b], \quad x_2 \in CN([t_0, b], X).$$

Since $\beta_2 \in CN(\mathcal{J}, X_1)$, by Lemma 3.4 the operators T_2 and L_2 take the values from the space $CN([t_0, b], X)$. Obviously, T_2 and L_2 are linear operators, and

$$|L_2x_2|_{CN} \leq v(\beta_2, [t_0, b])|x_2|_{CN},$$
$$|T_2| \leq |\operatorname{id} - L_2|.$$

Suppose $v(\beta_2, [t_0, b]) < 1$. Then the operator $T_2 = \operatorname{id} - L_2$ is invertible [15] and hence for any $w_2 \in CN([t_0, b], X)$, the equation (4.7) has the unique solution $x_2 = T_2^{-1}w_2$.

Reasoning analogously, we get that if $v(\beta_1, [a, t_0]) < 1$, where

$$\beta_1(t) = \begin{cases} \beta(a+) & \text{for} \quad t = a, \\ \beta(t) & \text{for} \quad t \in (a, t_0], \end{cases} \tag{4.9}$$

then the operator T_1 defined by the equality

$$(T_1x_1)(t) = x_1(t) - \int_{t_0}^{t} *d\beta_1(s)x_1(s) \quad \text{for all} \quad t \in [a, t_0]$$

maps the space $CN([a, t_0], X)$ into itself and is invertible. Thus, if w_1 is the function defined by the equality

$$w_1(t) = \begin{cases} w(a+) & \text{for} \quad t = a, \\ w(t) & \text{for} \quad t \in (a, t_0], \end{cases} \tag{4.10}$$

then the equation

$$x_1(t) - \int_{t_0}^{t} *d\beta_1(s)x_1(s) = w_1(t) \quad \text{for} \quad t \in [a, t_0], \tag{4.11}$$

has the unique solution $x_1 = T_1^{-1}w_1$.

As is seen, if x_1 and x_2 are solutions of the equations (4.11) and (4.7), respectively, then the function x defined by

$$x(t)=\begin{cases}[\mathrm{id}-\beta_\nu(a)]x_1(a)-w_\nu(a) & \text{for} \quad t=a,\\ x_1(t) \quad \text{for} \quad t\in(a,t_0],\\ x_2(t) \quad \text{for} \quad t\in(t_0,b],\end{cases}$$

is a the solution of the equation (4.1).

Formulate the obtained result in the form of

Lemma 4.1. *Let $\mathcal{J}=[0,1]$, X be a B-space, X_1 be a Banach algebra, let (X_1,X,X) be a bilinear triple and β be a function of bounded variation from the space $CN(\mathcal{J},X_1)$.*

Next, let $[a,b]\subseteq\mathcal{J}$ and $t_0\in[a,b]$. Then the following statements are valid:

(a) if $v(\beta,[a,t_0])<1+|\sigma_\nu(a)|$, then the equation (4.11), where β_1 is defined by the equality (4.9), has a unique solution $x_1\in CN([a,t_0],X)$ for any $w_1\in CN([a,t_0],X)$;

(b) if $v(\beta,[t_0,b])<1+\sigma_\nu(t_0)$, then the equation (4.7), where β_2 is defined by the equality (4.6), has the unique solution $x_2\in CN([t_0,b],X)$ for any $w_2\in CN([t_0,b],X)$;

(c) if the conditions of statements (a) and (b) are fulfilled and if $\mathrm{id}-\beta_\nu(t_0)$ is an invertible element of the algebra X, then the equation (4.1) has the unique, continuous from the left, solution $x:[a,b]\longmapsto X$ for any $w\in CN(\mathcal{J},X)$ and it is expressed by the solutions of the equations (4.7) and (4.11), respectively, as follows

$$x(t)=\begin{cases}[\mathrm{id}-\beta_\nu(a)]x_1(a)-w_\nu(a) & \textit{for} \quad t=a,\\ x_1(t) \quad \textit{for} \quad t\in(a,t_0],\\ x_2(t) \quad \textit{for} \quad t\in(t_0,b],\end{cases}$$

where w_1, w_2 are defined by the equalities (4.10), (4.8).

Let us cite the following auxiliary lemma which can be proved without any difficulties.

Lemma 4.2. *Let $\sigma\in BV(\mathcal{J},X)$. Then for any $\varepsilon>0$ there exists a partition of the segment $\mathcal{J}:0=t_0<t_1<\cdots<t_n=1$ such that the inequalities*

$$v(\sigma,[t_{i-1},t_i])<\varepsilon+|\sigma_\nu(t_{i-1})|+|\sigma_\nu^-(t_i)|, \quad i=1,2,\dots,n,$$

hold.

Theorem 4.3. *Let $\mathcal{J}=[0,1]$, X be a B-space, X_1 be a Banach algebra and let (X_1,X,X) be a bilinear triple. Next, let $t_0\in\mathcal{J}$ be some point and β be a function of bounded variation from the space $CN(\mathcal{J},X_1)$ such that for any $t\in[t_0,1]$ $\mathrm{id}-\beta_\nu(t)$ is an invertible element of the algebra X_1.*

Then for any $w \in CN(\mathcal{J}, X)$, *there exists a unique function* $x \in CN(\mathcal{J}, X)$ *satisfying the Volterra-Stieltjes equation*

$$x(t) - \int\limits_{t_0}^{t} *d\beta(s)x(s) = w(t) \quad \textit{for all} \quad t \in \mathcal{J}. \tag{4.12}$$

Proof. By Lemma 4.2, there exists a partition of the segment $\mathcal{J} : 0 = \tau'_m < \cdots < \tau'_1 < t_0 < \tau_1 < \cdots < t_n = 1$ such that the following inequalities hold:

$$\begin{aligned} &v(\beta, [\tau_{i-1}, \tau_i]) < 1 + |\beta_\nu(\tau_{i-1})| \quad \text{for} \quad i = 1, 2, \ldots, n, \\ &v(\beta, [\tau'_j, \tau'_{j-1}]) < 1 + |\beta_\nu(\tau'_j)| \quad \text{for} \quad j = 1, 2, \ldots, m, \end{aligned}$$

where $\tau_0 = \tau'_0 = t_0$. By Lemma 4.1, there exists a solution of the equation (4.12) on the interval $[\tau'_1, \tau_1]$. Denote this solution on the interval $[\tau_0, \tau_1]$ by x_1, while on the interval $[\tau'_1, \tau_0]$ by x'_1. Construct subsequently the solution of (4.12). First we construct the solution on the interval $[t_0, 1]$. Let $t \in [\tau_1, \tau_2]$. We have

$$\begin{aligned} x(t) - \int\limits_{t_0}^{\tau_1} *d\beta(s)x(s) - \int\limits_{\tau_1}^{t} *d\beta(s)x(s) = w(t), \quad \text{for} \quad t \in [\tau_1, \tau_2], \\ \int\limits_{t_0}^{\tau_1} *d\beta(s)x(s) = \int\limits_{t_0}^{\tau_1} *d\beta(s)x_1(s) = x_1(\tau_1) - w(\tau_1). \end{aligned}$$

Consequently, the equation (4.12) on the interval $[\tau_1, \tau_2]$ takes the form

$$x(t) - \int\limits_{\tau_1}^{t} *d\beta(s)x(s) = w(t) - w(\tau_1) + x_1(\tau_1) \quad \text{for} \quad t \in [\tau_1, \tau_2].$$

By Lemma 4.1, there exists a solution of this equation. Denote it by x_2. Continuing in such a manner, we obtain that on the interval $[\tau_{n-1}, \tau_n]$ there exists a solution x_n of the equation

$$x(t) - \int\limits_{\tau_{n-1}}^{t} *d\beta(s)x(s) = w(t) - w(\tau_{n-1}) + x_{n-1}(\tau_{n-1}) \quad \text{for} \quad t \in [\tau_{n-1}, \tau_n].$$

Thus the function $\overline{x}(t) = \sum_{i=1}^{n} \chi_{(\tau_{i-1}, \tau_i]}(t) x_i(t)$, $t \in [t_0, 1]$ is a solution of the equation (4.12) on the interval $[t_0, 1]$.

Analogously we can construct a solution of the equation (4.12) on the interval $[0, t_0]$. If x'_i, $i = 2, 3, \ldots, m$, are solutions of the equation

$$x(t) - \int\limits_{\tau'_{i-1}}^{t} *d\beta(s)x(s) = w(t) - w(\tau'_i) - x_{i-1}(\tau'_{i-1}) \quad \text{for} \quad t \in [\tau'_i, \tau'_{i-1}],$$

then the function $\overline{\overline{x}}(t) = \sum_{i=1}^{m} \chi_{(\tau_i,\tau_{i-1}]}(t)x_i'(t)$ for $t \in (0, t_0]$, $\overline{\overline{x}}(0) = x_m(0)$ is a solution of the equation (4.12) on the interval $[0, t_0]$. Consequently, the function $x(t) = \chi_{[0,t_0]}(t)\overline{\overline{x}}(t) + \chi_{(t_0,1]}(t)\overline{x}(t)$ for $t \in \mathcal{J}$ is a solution of the equation (4.12) on the whole interval $\mathcal{J}$. ■

Corollary 4.4. *Let $\mathcal{J} = [0,1]$, let $t_0 \in \mathcal{J}$ be a point and (X_1, X_2, X) be a bilinear triple. Next, let $\mathcal{A} \in \mathcal{CN}(\mathcal{J}, \mathcal{B}(\mathcal{X}, \mathcal{X}_\in))$ and let β be a function of bounded variation from the space $CN(\mathcal{J}, X_1)$ such that for any $t \in [t_0, 1)$ the operator* $\mathrm{id} - \beta_\nu(t)\mathcal{A}(t+)$ *is invertible. Then for any $w \in CN(\mathcal{J}, X)$ there exists a unique function $x \in CN(\mathcal{J}, X)$ satisfying the Volterra-Stieltjes equation*

$$x(t) - \int_{t_0}^{t} *d\beta(s)\mathcal{A}(s)x(s) = w(t) \quad \textit{for all} \quad t \in \mathcal{J}.$$

Now we pass to the consideration of the nonlinear equation. Let $J = [a, b]$ be a segment, $t_0 \in [a, b]$, and let β be a function of bounded variation from the space $CN(J, X_1)$. Denote by $V(\beta, t_0; \cdot)$ the function

$$v(\beta, t_0; t) = \begin{cases} v(\beta, [t_0, t_0]) & \text{for} \quad a \le t \le t_0, \\ v(\beta, [t_0, t]) & \text{for} \quad t_0 < t \le b. \end{cases}$$

Next, let $D \subset X$ be a closed set and $g : J \times D \longmapsto X_2$ be a mapping satisfying the following conditions:

(a) for every $\xi \in D$, the function $t \to g(t, \xi)$, $t \in J$, belongs to the space $CN(J, X_2)$;

(b) there exist nonnegative functions k and p from the space $CN(J, \mathbb{R}^1)$ such that the following inequalities are fulfilled:

$$|g(t, \xi)| \le k(t) \quad \text{for all} \quad t \in J, \quad \xi \in D,$$
$$|g(t, \xi_1) - g(t, \xi_2)| \le p(t)|\xi_1 - \xi_2| \quad \text{for all} \quad t \in J, \quad \xi_1, \xi_2 \in D.$$

Then, as is seen, for every $x \in CN(J, D)$ the function $t \to g(t, x(t))$, $t \in J$, belongs to the set $CN(J, X_2)$.

Consider the equation

$$x(t) - \int_{t_0}^{t} *d\beta(s)g(s, x(s)) = w(t) \quad \text{for} \quad t \in J, \tag{4.13}$$

where w is a function from the set $CN(J, D)$. By Lemma 3.4, we have

$$x(t_0+) - x(t_0) - \beta_\nu(t_0)g(t_0+, x(t_0+)) = w(t_0+) - w(t_0),$$

whence

$$x(t_0+) - \beta_\nu(t_0)g(t_0+, x(t_0+)) = w(t_0+). \tag{4.14}$$

This implies that for a solution of the equation (4.13) to exist on the interval $[t_0, b]$, it is necessary that the equation

$$\xi - \beta_\nu(t_0)g(t_0+,\xi) = w(t_0+)$$

have a solution $\xi_0 \in D$.

Theorem 4.5. *Let $J = [a,b]$, let $t_0 \in J$ be a point, (X_1, X_2, X) be a bilinear triple and let w and β be functions of bounded variation from the spaces $CN(J,X)$ and $CN(J,X_1)$, respectively. Next, let $D = \overline{B}(w(t_0), d_1) \cup \overline{B}(w(t_0+), d_2)$, let k, p be nonnegative functions from the space $CN(J,\mathbb{R}^1)$ and let $g : J \times D \longmapsto X_2$ be a mapping satisfying the conditions (a) and (b). Then for a unique solution $x \in CN(J,\mathcal{D})$ of equation (4.13) to exist, it is sufficient that the following conditions be fulfilled:*

$$(c) \qquad \sup_{t\in[a,t_0]} |w(t) - w(t_0)| + \int\limits_{t_0}^{a} *dv(\beta,t_0;t)k(t) \le d_1,$$

$$\sup_{t\in[t_0,b]} |w(t) - w(t_0+)| + \int\limits_{t_0}^{b} *dv(\beta,t_0;t)k(t) \le d_2;$$

$$(d) \qquad \int\limits_{t_0}^{a} *dV(\beta,t_0;t)p(t) < 1, \quad \int\limits_{t_0}^{b} *dv(\beta,t_0;t)p(t) < 1.$$

Proof. Consider the operator $L : B(w(t_0+), d_2) \longmapsto X$ defined by the equality

$$L(\xi) = w(t_0+) - \beta_\nu(t_0)g(t_0+,\xi), \quad \xi \in B(w(t_0+), d_2).$$

From the second inequality of condition (c) and also from the condition (a) it follows that $|L(\xi) - w(t_0+)| \le d_2$. Hence the operator L maps a closed sphere $\overline{B}(w(t_0+), d_2)$ into itself. As is seen from the condition (b),

$$|L(\xi_1) - L(\xi_2)| \le |\beta_\nu(t_0)|p(t_0+)|\xi_1 - \xi_2| \quad \text{for all} \quad \xi_1, \xi_2 \in \overline{B}(w(t_0+), d_2),$$

while from the second inequality of the condition (d) it follows $|\beta_\nu(t_0)|p(t_0+) < 1$. Consequently, B is a contraction operator [15]. Then it has a unique fixed point $\xi_0 \in \overline{B}(w(t_0+), d_2)$, i.e.,

$$\xi_0 = w(t_0+) + \beta_\nu(t_0)g(t_0+,\xi_0). \tag{4.15}$$

Since the solution of the equation (4.13) satisfies the equality (4.14), we have $x(t_0+) = \xi_0$..

From the first inequality of the condition (c) and from the condition (a), it follows that the operator T_1 defined by

$$(T_1x_1)(t) = w(t) + \int\limits_{t_0}^{t} *d\beta(s)g(s,x(s)), \quad t \in [a,t_0],$$

maps the closed set $CN([a,t_0],\overline{B}(w(t_0),d_1))$ into itself.

The first inequality of the condition (d) and the condition (b) imply that T_1 is a contraction operator. Then it has a unique fixed point $x_1 \in CN([a,t_0],\ \overline{B}(w(t_0),d_1))$.

Let $\overline{w}$ and $\overline{\beta}$ be functions defined by the relations

$$\overline{\beta}(t)=\begin{cases}\beta(t_0+) & \text{for}\quad t=t_0,\\ \beta(t) & \text{for}\quad t_0<t\le b,\end{cases}$$

$$\overline{w}(t)=\begin{cases}\xi_0 \quad \text{for}\quad t=t_0,\\ w(t)+\beta_\nu(t_0)g(t_0+,\xi_0) \quad \text{for}\quad t_0<t\le b.\end{cases}$$

It follows from (4.15) that $\overline{w}$ is right-continuous at the point t_0. Consider the operator T_2 defined by

$$(T_2x_2)(t)=\overline{w}(t)+\int\limits_{t_0}^{t} *d\overline{\beta}(s)g(s,x_2(s)),\quad t\in[t_0,b].$$

We have

$$|(T_2x_2)(t_0)-w(t_0+)|\le|\beta_\nu(t_0)|k(t_0+)\le d_2$$
$$|(T_2x_2)(t)-w(t_0+)|\le|w(t)-w(t_0+)+$$
$$+\beta_\nu(t_0)g(t_0+,\xi_0)+\int\limits_{t_0}^{t} *d\overline{\beta}(s)g(s,x_2(s))|\le$$
$$\le\sup_{t\in(t_0,b]}|w(t)-w(t_0+)|+\int\limits_{t_0}^{t} *dv(\beta,t_0;s)k(s)\le d_2.$$

Thus the operator T_2 maps the closed set $CN([t_0,b],\overline{B}(w(t_0+),d_2))$ into itself. From the second inequality of the condition (d) and from the condition (b) it follows that T_2 is a contraction operator. Then it has a unique fixed point x_2 in the set $CN([t_0,b],\overline{B}(w(t_0+),d_2))$.

It can be easily verified that the function x defined by the equality

$$x(t)=\begin{cases}x_1(t) & \text{for}\quad t\in[a,t_0],\\ x_2(t) & \text{for}\quad t\in(t_0,b],\end{cases}$$

is a solution of the equation (4.13) on the whole interval J. ■

§ 5. Optimal Problem with One-Sided Mixed Restrictions.

Suppose we are given the following objects: $\mathcal{J}=[0,1]$, $t_0\in\mathcal{J}$ is a point; X, X_1, X_2, U, Q, $\mathcal{K}_1$, $\mathcal{K}_2$ are Banach spaces; $\mathcal{K}$ and Y are $B\mathcal{K}$-spaces, (X_1,X_2,X) and $(\mathcal{K}_1,\mathcal{K}_2,\mathcal{K})$ are bilinear triples; Ω and U are convex open sets in the spaces X and U, respectively; f, g, h, q are functions from the spaces $C^1(\Omega \times U, \mathcal{K}_2)$, $C^1(\Omega \times U, X_2)$, $C^1(\Omega \times U, Y)$, $C^1(\Omega \times \Omega, Q)$ respectively; $\alpha \in BV(\mathcal{J}, \mathcal{K}_1)$; β, r are functions of bounded variation from the spaces $CN(\mathcal{J}, X_1)$, $CN(\mathcal{J}, X)$ respectively, and $r(t_0) = 0$. In what follows, the space X is said to be the phase space and v is said to be the space of controlling parameters. Functions from the set $CN(\mathcal{J}, U)$ are called admissible controls.

The problem is formulated as follows: among all functions $x \in CN(\mathcal{J}, \Omega)$, initial values $x^0 \in \Omega$ and admissible controls $u \in CN(\mathcal{J}, U)$ satisfying the restrictions

$$x(t) = x^0 + r(t) + \int\limits_{t_0}^{t} *d\beta(s) g(x(s), u(s)), \tag{5.1}$$

$$h(x(t), u(t)) \leq 0, \quad t \in \mathcal{J}, \tag{5.2}$$

$$q(x^0, x(0), x(1)) = 0, \tag{5.3}$$

find a triple (u, x, x^0) which minimizes the integral

$$\int\limits_0^1 *d\alpha(s) f(x(s), u(s)) \longmapsto \inf. \tag{5.4}$$

If such a triple does exist, we will call it an optimal process.

To obtain necessary conditions of optimality for the problem (5.1)–(5.4), we will use the method of joint covering one of modifications of which has been mentioned in §2. Following this scheme, we assume that

$$\begin{gathered} J = CN(\mathcal{J}, U) \times CN(\mathcal{J}, X) \times X \times \mathcal{K} \times CN(\mathcal{J}, Y), \\ M = CN(\mathcal{J}, U) \times CN(\mathcal{J}, \Omega) \times \Omega \times \mathcal{K}_+ \times CN(\mathcal{J}, Y_+), \\ W = \mathcal{K} \times CN(\mathcal{J}, X) \times CN(\mathcal{J}, Y) \times Q, \end{gathered} \tag{5.5}$$

and define the mapping $p : M \to W$ in a way indicated below: if $z = (u, x, x^0, k, y)$, $w = (w_1, w_2, w_3, w_4)$, $p = (p_1, p_2, p_3, p_4)$, then let $w_1 = p_1(z)$, $w_2 = p_2(z)$, $w_3 = p_3(z)$, $w_4 = p_4(z)$, where

$$\begin{gathered} p_1(z) = \int\limits_0^1 *d\alpha(t) \cdot f(x(t), u(t)) + k, \\ p_2(z, t) = x(t) - x^0 - r(t) - \int\limits_{t_0}^{t} d\beta(s) g(x(s), u(s)), \quad t \in \mathcal{J}, \\ p_3(z, t) = h(x(t), u(t)) + y(t), \quad t \in \mathcal{J}, \\ p_4(z) = q(x^0, x(0), x(1)). \end{gathered}$$

According to Lemma 2.1, if (u, x, x^0) is a minimum point of the problem (5.1)–(5.4), then $z_0 = (u, x, x^0, 0, y) \in M$, where $y(t) = -h(x(t), u(t))$, $t \in \mathcal{J}$, is a critical point of the mapping p.

For the differential $T = Dp(z_0) = (T_1, T_2, T_3, T_4)$ on the set M, it is not difficult to obtain the following expressions

$$
\begin{gathered}
T_1 \delta z = \int\limits_0^1 * d\alpha(t)\big[f_x(x(t), u(t))\delta x(t) + f_u(x(t), u(t))\delta u(t)\big] + \delta_k, \\
T_2(\delta z, t) = \delta x(t) - \delta x^0 - \int\limits_{t_0}^{t} * d\beta(s)\big[g_x(x(s), u(s))\delta x(s) + \\
+ g_u(x(s), u(s))\delta u(s)\big], \ \ t \in \mathcal{J}, \\
T_3(\delta z, t) = h_x(x(t), u(t))\delta x(t) + h_u(x(t), u(t))\delta u(t) + \delta y(t), \ \ t \in \mathcal{J}, \\
T_4 \delta z = D_1 q(x^0, x(0), x(1))\delta x^0 + D_2 q(x^0, x(0), x(1))\delta x(0) + \\
D_3 q(x^0, x(0), x(1))\delta x(1).
\end{gathered}
\tag{5.6}
$$

Determine now the conditions of nondegeneracy for the problem (5.1)–(5.4), i.e., we have to clarify under which conditions the cone $T(K)$, where $\mathcal{K} = \operatorname{cone}(M - z_0)$, contains inner points. Since the set M contains inner points, the linear manifold $\operatorname{sp}(M - z_0)$ spanned onto $M - z_0$ coincides with the whole space J. Further, if the cone $T(\mathcal{K})$ contains inner points, then evidently, $TJ = W$, and vice versa. Hence we have to determine under which conditions the image under the mapping T of the space J coincides with the whole space W.

Let $\delta w = (\delta w_1, \delta w_2, \delta w_3, \delta w_4) \in W$ be an arbitrary point. Then it is evident that for any functions $\delta u \in CN(\mathcal{J}, U)$, $\delta x \in CN(\mathcal{J}, X)$ and $\delta x^0 \in X$ there exist $\delta k \in \mathcal{K}$ and $\delta y \in CN(\mathcal{J}, Y)$ such that

$$
\begin{gathered}
T_1(\delta u, \delta x, \delta x^0, \delta k, \delta y) = \delta w_1, \\
T_3((\delta u, \delta x, \delta x^0, \delta k, \delta y), t) = \delta w_3(t), \quad t \in \mathcal{J}.
\end{gathered}
$$

It follows from the above-said that in order that $TJ = W$, it is necessary and sufficient that the following conditions be fulfilled:

(a) for any function $\delta w_2 \in CN(\mathcal{J}, X)$ and for any vector $w_4 \in Q$ there exists $\delta z \in J$ such that

$$
\begin{cases}
T_2(\delta z, t) = w_2(t), \quad t \in \mathcal{J}, \\
T_4 \delta z = w_4.
\end{cases}
\tag{5.7}
$$

Consider the equation

$$
\delta x(t) = \delta w_2(t) + \int\limits_{t_0}^{t} d\beta(s) g_x(x(s), u(s))\delta x(s), \quad t \in \mathcal{J}.
$$

Owing to Corollary 4.4, for the solution of this equation to exist for any δw_2, it is sufficient that the following condition be fulfilled:

(b) for any t, $t_0 \leq t < 1$ the operator $id - \beta_\nu(t) g_x(x(t+), u(t+))$ is invertible.

Note that if β is continuous on the interval $[t_0, 1]$, then the condition (b) is fulfilled automatically. If g_x is bounded, i.e., there exists $M > 0$ such that $|g_x(x,u)| < M$, then in order that condition (b) to be fulfilled, it is sufficient that $|\beta_\nu(t)| < 1/M$, $t_0 \leq t < 1$.

From the above arguments it follows that the question on the existence of a solution of the first equation of the system (5.7) does not depend on the choice of the vector $\delta x^0 \in X$ and of the function $\delta u \in CN(\mathcal{J}, U)$. Denote by $N(T_2)$ the set of all $\delta z \in J$ for which $T_2(\delta z, t) = 0$, $t \in \mathcal{J}$. Then it is easy to see that for the condition (a) to be fulfilled, it is sufficient that the condition

(c) $T_4(N(T_2)) = Q$

be also fulfilled.

Consequently, for the cone $T(\mathcal{K})$ to contain inner points, it is sufficient that the conditions (b) and (c) be fulfilled. The conditions (b) and (c) are those of nondegeneracy of the mapping T.

Assume $z_0 \in M$ to be a critical point of the mapping p and the condition of nondegeneracy to be fulfilled. Then by Theorem 2.9, there exists a nonzero, linear, continuous functional $w^* \in W^*$ such that for all $\delta z \in J$ for which $z_0 + \delta z \in M$, the inequality

$$w^* T \delta z \leq 0. \tag{5.8}$$

is fulfilled.

The equalities (5.5) imply $W^* = \mathcal{K}^* \times CN^*(\mathcal{J}, X) \times CN^*(\mathcal{J}, Y) \times Q^*$. Hence we have to deal with the spaces $CN^*(\mathcal{J}, X)$ and $CN^*(\mathcal{J}, Y)$. By Theorem 3.12, these spaces are isometrically isomorphic to the spaces $BV_0(\mathcal{J}, X^*)$ and $BV_0(\mathcal{J}, y^*)$, respectively.

Let us now pass to the deduction of corollaries from the relations (5.8). Let

$$w^* = (\chi, \varphi, \sigma, \psi) \in W^* = \mathcal{K} \times BV_0(\mathcal{J}, X^*) \times BV_0(\mathcal{J}, Y^*) \times Q^*.$$

Using (5.6), we obtain that for all $\delta z \in J$ for which $z_0 + \delta z \in M$, the following inequality is valid:

$$\chi \int_0^1 * d\alpha(t) f_x \delta x(t) + \chi \int_0^1 * d\alpha(t) f_u \delta u(t) + \chi \delta k +$$
$$+ \int_0^1 * d\varphi(t) \delta x(t) + \varphi(0) \delta x^0 - \varphi(1) \delta x^0 -$$

$$
-\int_{0}^{1} *d\varphi(t)\int_{t_0}^{t} *d\beta(s)g_x\delta x(s) - \int_{0}^{1} *d\varphi(t)\int_{t_0}^{t} *d\beta(s)g_u\delta u(s) +
$$
$$
+\int_{0}^{1} *d\sigma(t)h_x\delta x(t) + \int_{0}^{1} *d\sigma(t)h_u\delta u(t) + \int_{0}^{1} *d\sigma(t)\delta y(t) +
$$
$$
+\psi D_1 q\delta x^0 + \psi D_2 q\delta x(0) + \psi D_3 q\delta x(1) \le 0. \tag{5.9}
$$

Taking into account the fact that increments δu, δx, δx^0, δk, δy are independent and assuming that one of them is different from zero, while the remaining ones are equal to zero, we obtain from (5.9) five independent inequalities:

$$
\chi\delta k \le 0 \quad \text{for all} \quad \delta k \in \mathcal{K}, \quad \delta k \ge 0; \tag{5.10}
$$

$$
\begin{cases} \chi\int_0^1 *d\alpha(t)f_x\delta x(t) + \int_0^1 *d\varphi(t)\delta x(t) - \\ -\int_0^1 *d\varphi(t)\int_{t_0}^t *d\beta(s)g_x\delta x(s) + \int_0^1 *d\sigma(t)h_x\delta x(t) + \\ +\psi D_2 q\delta x(0) + \psi D_3 q\delta x(1) \le 0 \\ \text{for all } \ \delta x \in CN(\mathcal{J}, X) \ \text{ for which } \ x+\delta x \in CN(\mathcal{J}, \Omega); \end{cases} \tag{5.11}
$$

$$
\begin{cases} \varphi(0)\delta x^0 - \varphi(1)\delta x^0 + \psi D_1 q\delta x^0 \le 0 \\ \text{for all } \ \delta x^0 \in X \ \text{ for which } \ x^0 + \delta x^0 \in \Omega; \end{cases} \tag{5.12}
$$

$$
\begin{cases} \chi\int_0^1 *d\alpha(t)f_u\delta u(t) - \int_0^1 *d\varphi(t)\int_{t_0}^t *d\beta(s)g_u\delta u(s) + \\ +\int_0^1 *d\sigma(t)h_u\delta u(t) \le 0 \ \text{ for any } \ \delta u \in CN(\mathcal{J}, U) \\ \text{for which } \quad u+\delta u \in CN(\mathcal{J}, U); \end{cases} \tag{5.13}
$$

$$
\int_0^1 *d\sigma(t)\delta y(t) \le 0 \ \text{ for all } \ \delta y \in CN(\mathcal{J}, Y) \ \text{ for which}
$$
$$
-h(x(t), u(t)) + \delta y(t) \ge 0 \ \text{ for all } \ t \in \mathcal{J}. \tag{5.14}
$$

The omitted arguments of the functions in the above obtained inequalities are the same as in (5.6). From the inequality (5.10) it immediately follows that

$$
\chi \le 0. \tag{5.15}
$$

Transform the inequality (5.11). Consider the functions α_1, φ_1, σ_1 and γ which are defined by the following relations:

$$
\alpha_1(t)\xi = \alpha_1(\tau_0)\xi + \chi\int_{\tau_0}^{t} *d\alpha(s)f_x\xi, \quad t \in \mathcal{J}, \quad \xi \in X;
$$

$$\sigma_1(t)\xi = \sigma_1(\tau_0)\xi + \int_{\tau_0}^{t} *d\sigma(s)h_x\xi, \quad t \in \mathcal{J}, \xi \in X;$$

$$\gamma(t) = \begin{cases} -D_3q^*\psi & \text{for} \quad t = 0, \\ D_2q^*\psi - D_3q^*\psi & \text{for} \quad t \in (0,1), \\ D_2q^*\psi & \text{for} \quad t = 1; \end{cases}$$

$$\varphi_1(t)\xi = \begin{cases} \varphi_1(0)\xi + \int_0^t *d\varphi(s) \int_t^s *d\beta(\tau)g_x\xi & \text{for} \quad t \in [0,t_0], \quad \xi \in X, \\ \varphi_1(1)\xi + \int_1^t *d\varphi(s) \int_t^s *d\beta(\tau)g_x\xi & \text{for} \quad t \in [t_0,1], \quad \xi \in X, \end{cases}$$

$$[\varphi_1(0) - \varphi_1(1)]\xi = -\int_0^1 *d\varphi(s) \int_{t_0}^{s} *d\beta(\tau)g_x\xi, \quad \xi \in X.$$

Then, applying Theorems 3.5 and 3.6, we get from inequality (5.11) that $\int_0^1 *d\Phi(t)\delta x(t) \le 0$ for all δx from arbitrarily small neighborhood of zero of the space $CN(\mathcal{J}, X)$, where $\Phi(t) = \alpha_1(t) + \varphi(t) - \varphi_1(t) + \sigma_1(t) + \gamma(t)$ for all $t \in \mathcal{J}$. But then $\int_0^1 *d\Phi(t)\delta x(t) = 0$ for all $\delta x \in CN(\mathcal{J}, X)$. This implies that $\Phi(t) = \text{const}$. Consequently, $\Phi(t) - \Phi(s) = 0$ for all t and s from the segment $\mathcal{J}$. We have

$$\begin{gathered} \chi \int_0^t *d\alpha(s)f_x\xi + \varphi(t)\xi - \varphi(0)\xi - \int_0^t *d\varphi(s) \int_t^s *d\beta(\tau)g_x\xi + \int_0^t *d\sigma(s)h_x\xi + \\ + D_2q^*\psi\xi = 0 \quad \text{for all} \quad t \in (0,t_0], \quad \xi \in X; \\ \chi \int_t^1 *d\alpha(s)f_x\xi + \varphi(1)\xi - \varphi(t)\xi - \int_t^1 *d\varphi(s) \int_t^s *d\beta(\tau)g_x\xi + \int_t^1 *d\sigma(s)h_x\xi + \\ + D_3q^*\psi\xi = 0 \quad \text{for all} \quad t \in (t_0,1), \quad \xi \in X; \\ \chi \int_0^1 *d\alpha(s)f_x\xi + \varphi(1)\xi - \varphi(0)\xi - \int_0^1 *d\varphi(s) \int_{t_0}^s *d\beta(\tau)g_x\xi + \\ D_2q^*\psi\xi + D_3q^*\psi\xi = 0, \quad \xi \in X. \end{gathered} \tag{5.16}$$

To give the system the desired form, we have to transform some terms from this system. Namely, applying Corollary 3.11, we obtain

$$\int_0^t *d\varphi(s) \int_t^s *d\beta(\tau)g_x\xi = -\int_0^t *[\varphi(s) - \varphi(0)]d\beta(s)g_x\xi + \varphi_l(t)\xi$$

for $t \in \mathcal{J}$, $\xi \in X$, where

$$\begin{cases} \varphi_l(t)\xi = \sum_{0 \le \tau < t} \Phi_\nu(\tau)\beta_\nu(\tau) g_x(x(\tau+), u(\tau+))\xi \\ \qquad\qquad \text{for} \quad t \in (0,1], \quad \xi \in X, \\ \varphi_l(0) = 0. \end{cases} \tag{5.17}$$

$$\int_t^1 *d\varphi(s) \int_t^s *d\beta(\tau) g_x \xi = -\int_t^1 *[\varphi(s) - \varphi(1)] d\beta(s) g_x \xi + \varphi_r(t)\xi$$

for $t \in \mathcal{J}$, $\xi \in X$, where

$$\begin{cases} \varphi_r(t)\xi = \sum_{t \le \tau < 1} \varphi_\nu(\tau)\beta_\nu(\tau) g_x(x(\tau+), u(\tau+))\xi \\ \qquad\qquad \text{for} \quad t \in [0,1), \quad \xi \in X, \\ \varphi_r(1) = 0. \end{cases} \tag{5.18}$$

Using the obtained formulas, the system (5.16) takes the form

$$\begin{gathered} \varphi(t)\xi = -D_2 q^* \psi \xi + \varphi(0)\xi - \chi \int_0^t *d\alpha(s) f_x \xi - \\ - \int_0^t *[\varphi(s) - \varphi(0)] d\beta(s) g_x \xi - \int_0^t *d\sigma(s) h_x \xi + \varphi_l(t)\xi \\ \text{for} \quad 0 < t \le t_0, \quad \xi \in X, \\ \varphi(t)\xi = D_3 q^* \psi \xi + \varphi(1)\xi + \chi \int_t^1 *d\alpha(s) f_x \xi + \\ + \int_t^1 *[\varphi(s) - \varphi(1)] d\beta(s) g_x \xi + \int_t^1 *d\sigma(s) h_x \xi - \varphi_r(t) \\ \text{for} \quad t_0 < t < 1, \quad \xi \in X, \\ [\varphi(0) - \varphi(1)]\xi = D_2 q^* \psi \xi + D_3 q^* \psi \xi + \chi \int_0^1 *d\alpha(s) f_x \xi + \int_0^1 *[\varphi(s) - \varphi(0) - \\ -\varphi(1)] d\beta(s) g_x \xi + \int_0^1 *d\sigma(s) h_x \xi - [\varphi_l(t_0) + \varphi_r(t_0)]\xi, \quad \xi \in X. \end{gathered} \tag{5.19}$$

We call the above-obtained system the conjugate equation of the problem (5.1)–(5.4).

The inequality (5.12) easily results in $(\varphi(0) - \varphi(1) + D_1 q^* \psi)\delta x^0 \le 0$ for all δx^0 from arbitrarily small neighborhood of zero of the space X. This implies that $\varphi(0)-\varphi(1)+D_1q^*\psi=0$. Hence we have

$$\varphi(0)-\varphi(1)=-D_1q^*\psi. \tag{5.20}$$

The obtained equality is the condition of transversality for the conjugate equation (5.19).

Transform now the inequality (5.13). Acting in the same way as in transforming the inequality (5.11), we obtain the following system:

$$\begin{aligned}
&\chi\int_0^t *d\alpha(s)f_u\eta+\int_0^t *[\varphi(s)-\varphi(0)]d\beta(s)g_u\eta+\int_0^t *d\sigma(s)h_u\eta-\\
&\qquad -\overline{\varphi}_l(t)\eta=0 \quad \text{for all} \quad t\in[0,t_0],\quad \eta\in U;\\
&\chi\int_t^1 *d\alpha(s)f_u\eta+\int_t^1 *[\varphi(s)-\varphi(1)]d\beta(s)g_u\eta+\int_t^1 *d\sigma(s)h_u\eta-\\
&\qquad -\overline{\varphi}_r(t)\eta=0 \quad \text{for all} \quad t\in[t_0,1],\quad \eta\in U;\\
&\chi\int_0^1 *d\alpha(s)f_u\eta+\int_0^1 *[\varphi(s)-\varphi(0)-\varphi(1)]d\beta(s)g_u\eta+\int_0^1 *d\sigma(s)h_u\eta-\\
&\qquad -[\overline{\varphi}_l(t_0)+\overline{\varphi}_r(t_0)]\eta=0,\quad \eta\in U,
\end{aligned} \tag{5.21}$$

where the functions $\overline{\varphi}_l$ and $\overline{\varphi}_r$ are defined by the equality

$$\begin{cases}\overline{\varphi}_l(t)\eta=\sum_{0\le\tau<t}\varphi_\nu(\tau)\beta_\nu(\tau)g_u(x(\tau+),u(\tau+))\eta,\\ \qquad\quad \text{for} \quad t\in(0,1],\quad \eta\in U,\\ \overline{\varphi}_l(0)=0;\end{cases} \tag{5.22}$$

$$\begin{cases}\overline{\varphi}_r(t)\eta=\sum_{0\le\tau<t}\varphi_\nu(\tau)\beta_\nu(\tau)g_u(x(\tau+),u(\tau+))\eta\\ \qquad\quad \text{for} \quad t\in[0,1),\quad \eta\in U,\\ \overline{\varphi}_r(1)=0.\end{cases} \tag{5.23}$$

We call the system (5.21) the condition of maximum for the problem (5.1)–(5.4).

Transform now the inequality (5.14). Let $\delta y_1(t)=-\frac{1}{2}h(x(t)),u(t)$ for all $t\in\mathcal{J}$ and $\delta y_2(t)=\frac{1}{2}h(x(t),u(t))$ for all $t\in\mathcal{J}$. Obviously, $-h(x(t),u(t))+\delta y_i(t)\ge 0$ for all $t\in\mathcal{J}$, $i=1,2$. Then from (5.14) we have

$$\int_0^1 *d\sigma(t)h(x(t),u(t))=0. \tag{5.24}$$

Let $0\le t_1<t_2\le 1$ and $\delta y(t)=\chi_{[t_1,t_2]}(t)\eta$, where $\eta\in Y_+$. Obviously, $-h(x(t),u(t))+\delta y(t)\ge 0$. Then from (5.14) we have

$$[\sigma(t_2)-\sigma(t_1)]\eta\le 0 \quad \text{for all} \quad \eta\in Y_+.$$

Thus

$$\sigma(t_1) \geq \sigma(t_2) \quad \text{for all} \quad t_1, t_2, \quad 0 \leq t_1 < t_2 \leq 1. \tag{5.25}$$

If $\sigma(\tau_0) = 0$, where τ_0 is a point from $\mathcal{J}$, then (5.25) takes the form

$$\begin{cases} \sigma\_\text{is nonincreasing;} \\ \sigma(t) \geq 0 \quad \text{for all} \quad t \in [0, \tau_0]; \\ \sigma(t) \leq 0 \quad \text{for all} \quad t \in (\tau_0, 1]. \end{cases} \tag{5.26}$$

Now we formulate the obtained result in the form of

Theorem 5.1. *Let for problem* (5.1)–(5.4) *the conditions of nondegeneracy* (b) *and* (c) *be fulfilled, let* (u, x, x^0) *be an optimal process and let* $\tau_0 \in \mathcal{J}$ *be some point. Then there exist a nonincreasing function of bounded variation* $\sigma : \mathcal{J} \longmapsto Y^*$, $\sigma(\tau_0) = 0$, *constants* $\chi \in \mathcal{K}^*$, $\chi \leq 0$, $\psi \in Q^*$ *and a function of bounded variation* $\varphi : \mathcal{J} \longmapsto X^*$, $\varphi(\tau_0) = 0$, *such that the following conditions are fulfilled: the conjugate equation* (5.19)*; the transversality condition* (5.20)*; the maximum condition* (5.21)*; the condition of complementing nonrigidity.*

To elucidate the condition of complementing nonrigidity, we can apply Lemma 3.13.

If in the initial problem the function β is continuous, then in the conjugate equation the functions φ_l and φ_r are identically equal to zero, and in the condition of maximum the functions $\overline{\varphi}_l$ and $\overline{\varphi}_r$ are also identically equal to zero.

Let in the initial problem $\mathcal{K}_1 = X_1 = \mathbb{R}^1$, $\mathcal{K}_2 = \mathcal{K}$, $X_2 = X$ and let bilinear mappings corresponding to bilinear triples be ordinary multiplication by a number. Under these assumptions, the conjugate equation and the conditions of maximum take a more convenient form. Namely,

$$\begin{aligned} \varphi(t) = &- D_2 q^* \psi + \varphi(0) - \int_0^t * d\alpha(s) f_x^* \chi - \int_0^t * d\beta(s) g_x^* [\varphi(s) - \\ &- \varphi(0)] - \int_0^t * h_x^* d\sigma(s) + \varphi_l(t) \xi \quad \text{for} \quad 0 < t \leq t_0, \\ \varphi(t) = &D_3 q^* \psi + \varphi(1) + \int_t^1 * d\alpha(s) f_x^* \chi + \int_t^1 * d\beta(s) g_x^* [\varphi(s) - \varphi(1)] + \\ &+ \int_t^1 * h_x^* d\sigma(s) - \varphi_r(t) \quad \text{for} \quad t_0 < t \leq 1, \end{aligned}$$

$$\varphi(0)-\varphi(1)=D_2q^*\psi+D_3q^*\psi+\int_0^1 *d\alpha(s)f_x^*\chi+$$
$$+\int_0^1 *d\beta(s)g_x^*[\varphi(s)-\varphi(0)-\varphi(1)]+$$
$$+\int_0^1 *h_x^*d\sigma(s)-\varphi_l(t_0)-\varphi_r(t_0).$$
$$\int_0^t *d\alpha(s)f_u^*\chi+\int_0^t *d\beta(s)g_u^*[\varphi(s)-\varphi(0)]+\int_0^t *h_u^*d\sigma(s)-\overline{\varphi}_l(t)=0$$
$$\text{for all}\quad t\in[0,t_0],$$
$$\int_t^1 *d\alpha(s)f_u^*\chi+\int_t^1 *d\beta(s)g_u^*[\varphi(s)-\varphi(1)]+\int_t^1 *h_u^*d\sigma(s)-\overline{\varphi}_r(t)=0$$
$$\text{for all}\quad t\in(t_0,1],$$
$$\int_0^1 *d\alpha(s)f_u^*\chi+\int_0^1 *d\beta(s)g_u^*[\varphi(s)-\varphi(0)-\varphi(1)]+$$
$$\int_0^1 *h_u^*d\sigma(s)-\overline{\varphi}_l(t_0)-\overline{\varphi}_r(t_0)=0.$$

If to the above-said assumptions we add $\alpha(t)=\beta(t)=t$ for $t\in\mathcal{J}$, $t_0=0$, $\tau_0=1$ and q independent of $\delta x(0)$, then from Theorem 5.1 we directly obtain the necessary optimal conditions given in [4].

Finally it should be noted that if the space Q is finite-dimensional, then Theorem 5.1 remains valid without additional condition of nondegeneracy.

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