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ON THE ARMELLINI-TONELLI-SANSONE THEOREM

Abstract. Sufficient conditions are given guaranteeing that all solutions of the equation

$$x'' + a(t)f(x) = 0 \quad (xf(x) > 0)$$

tend to zero as t goes to infinity. The conditions contain integrals instead of maxima and minima in earlier results. Finally, a probabilistic generalization of Armellini-Tonelli-Sansone theorem is formulated.

რეზიუმე. მოყვანილია საკმარისი პირობები, რომლებიც უზრუნველყოფენ

$$x'' + a(t)f(x) = 0 \quad (xf(x) > 0)$$

განტოლების ყველა ამონახსნის ნულისკენ მისწრაფებას, როცა t მიისწრაფვის უსასრულობისაკენ. ეს პირობები ინტეგრალურია განსხვავებით ადრინდელი შედეგებისაგან, რომლებიც შეიცავენ მაქსიმუმებს და მინიმუმებს. დასასრულ, მოყვანილია არმელინი-ტონელი-სანსონეს თეორემის ალბათური განზოგადება.

The equation

$$x'' + a(t)x = 0 \tag{1}$$

describes the oscillation of a material point of unit mass under the action of the restoring force $-a(t)x$; the function $a : [0, \infty) \to (0, \infty)$ denotes the varying elasticity coefficient.

Definition 1 (P. Hartman [6]). A function $t \mapsto x_0(t)$ existing and satisfying the equation (1) on the interval $[0, \infty)$ is called a *small solution* of (1.1) if

$$\lim_{t \to \infty} x_0(t) = 0 \tag{2}$$

holds. The zero solution is called the *trivial small solution* of (1).

Let us consider the case where the elasticity coefficient a is nondecreasing. Then the total mechanical energy

$$E(t, x, x') := \frac{(x')^2}{2} + a(t)\frac{x^2}{2} \tag{3}$$

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is nonincreasing along the motions. By using this fact it can be seen that every solution of (1) is oscillatory and the successive amplitudes of the oscillation (i.e. maxima of $|x|$ for any solution x which occur at the points where $x' = 0$) are monotone. M. Biernacki [2] raised the question of the existence of a (nontrivial) solution whose amplitudes tend to zero. H. Milloux answered this question proving the following

Theorem A (H. Milloux [9]). *If $a : [0, \infty) \to (0, \infty)$ is differentiable, nondecreasing, and satisfies*

$$\lim_{t \to \infty} a(t) = \infty, \tag{4}$$

then the equation (1) has a non-trivial small solution.

Milloux also provided an example to show that (4) cannot imply that *all solutions are small.* The problem to find conditions guaranteeing this essentially stronger property of the equation (1) is very old, it goes back at least to a paper by A. Wiman [12] in 1917. For 80 years a great number of papers have been devoted to the problem both for linear and nonlinear equations (see the history in [3, 5, 8]). Different conditions guaranteeing that all the solutions of (1) with $a(t) \to \infty$ as $t \to \infty$ are small, have a common character: they have to control the way of growth of a in some sense. The reason is that the effect of the increase of a depends on the distribution of this increase. To illuminate this phenomenon, let us consider a nontrivial solution x of (1). Let $\{t_{2n-1}\}_{n=1}^{\infty}$, and $\{t_{2n}\}_{n=1}^{\infty}$, denote all the zeros of $x(t)$, and $x'(t)$, respectively. Then $t_1 < t_2 < \cdots < t_{2n-1} < t_{2n} < \ \cdots$. Define the modified energy F by

$$F(t, x, x') := \frac{1}{2a(t)}(x')^2 + \frac{x^2}{2}. \tag{5}$$

Taking into account the equation (1), for the derivative of F with respect to (1) we obtain

$$F'(t, x, x') = -\frac{a'(t)}{2a^2(t)}(x')^2 \le 0, \tag{6}$$

i.e., $F(t, x(t), x'(t))$ is nonincreasing. To guarantee x to be small, it is enough to show $\lim\limits_{t \to \infty} F(t, x(t), x'(t)) = 0$. By the definition of $\{t_k\}$, from (6) we get

$$F'(t_{2n-1}, x(t_{2n-1}), x'(t_{2n-1})) = -\frac{1}{2}\frac{a'(t_{2n-1})}{a(t_{2n-1})}F(t_{2n-1}, x(t_{2n-1}), x'(t_{2n-1}));$$
$$F'(t_{2n}, x(t_{2n}), x'(t_{2n})) = 0.$$

We want to drive $F(t)$ to zero. The last formulae show that the increase of the function $A(t) := \ln a(t)$ makes $F(t, x(t), x'(t))$ decrease, and the increase of A is effective if "it is located at the set $\{t_{2n-1}\}$" and the increase of A is ineffective if "it is located at the set $\{t_{2n}\}$." So, if we want to have only small solutions, we have to exclude that $A(t)$ increases only very near to the points of set $\{t_{2n}\}$. G. Armellini called this type of behaviour "regular growth".

Definition 2 (G. Armellini [1]). a) Let $\{[\alpha_n,\beta_n]\}_{n=1}^{\infty}$ be a family of intervals such that $\alpha_1<\beta_1<\alpha_2<\cdots<\alpha_n<\beta_n<\alpha_{n+1}<\cdots$. Then the *density* of the set $E=\cup_{n=1}^{\infty}[\alpha_n,\beta_n]$ is defined by

$$\delta(E)=\limsup_{n\to\infty}\frac{1}{\beta_n}\sum_{k=1}^{n}(\beta_k-\alpha_k).$$

b) A continuous, nondecreasing function $A:[0,\infty)\to[0,\infty)$ with $\lim_{t\to\infty}A(t)=\infty$ is of *irregular growth* if for each $\varepsilon>0$ there is a family $\{[\alpha_n,\beta_n]\}_{n=1}^{\infty}$ of intervals such that $\delta(\cup_{n=1}^{\infty}[\alpha_n,\beta_n])<\varepsilon$ and

$$\sum_{n=1}^{\infty}(A(\alpha_{n+1})-A(\beta_n))<\infty.$$

Otherwise we say that A is of *regular growth.*

Theorem B (G. Armellini [1]–L. Tonelli [11]–G. Sansone [10]). *If $a:[0,\infty)\to(0,\infty)$ is differentiable, nondecreasing with $\lim_{t\to\infty}a(t)=\infty$, and $t\mapsto\ln a(t)$ is of regular growth, then all solutions of* (1) *are small.*

P. Hartman [6] sharpened this theorem weakening the assumption of the regular growth of $\ln a(t)$. Generalizing Hartman's result, T.A. Chanturia considered the nonlinear equation

$$x''+a(t)f(x)=0, \tag{7}$$

where $f:(-\infty,\infty)\to(-\infty,\infty)$ is continuous,

$$xf(x)>0 \quad \text{for all} \quad x\neq 0, \qquad \lim_{|x|\to\infty}\int_0^x f(r)\,dr=\infty,$$

and proved the following

Theorem C (T. A. Chanturia [4]). *Suppose that $a:[0,\infty)\to(0,\infty)$ is nondecreasing and $a(t)\to\infty$ as $t\to\infty$ regularly in the following sense: there is an $\varepsilon_0>0$ such that*

$$\sum_{n=1}^{\infty}[\ln a(\alpha_{n+1})-\ln a(\beta_n)]=\infty \tag{8}$$

for every family of intervals $\{[\alpha_n,\beta_n]\}_{n=1}^{\infty}$ satisfying the following conditions (i)–(iv)*:*

(i) $\alpha_n<\beta_n<\alpha_{n+1},\quad n=1,2,\ldots,\qquad \lim_{n\to\infty}\alpha_n=\infty;$

(ii) $\liminf_{n\to\infty}\sqrt{a(\beta_n)}(\beta_n-\alpha_n)>0;$

(iii) $\limsup_{n\to\infty} \sqrt{a(\alpha_n)}(\beta_n - \alpha_n) < \varepsilon_0$;

(iv) $0 < \liminf_{n\to\infty} \int_{\beta_n}^{\alpha_{n+1}} \sqrt{a(t)}dt \le \limsup_{n\to\infty} \int_{\beta_n}^{\alpha_{n+1}} \sqrt{a(t)}\,dt < \infty$.

Then all solutions of (7) *are small.*

Now we formulate an Armellini-Tonelli-Sansone-type theorem, in which the growth condition appears in a simple integral form.

Theorem 3. *Assume that* $a : [0, \infty) \to (0, \infty)$ *is continuous, nondecreasing, and* $\lim_{t\to\infty} a(t) = \infty$. *Suppose that for every* $\gamma > 0$ *and for every strictly increasing sequence* $\{t_n\}_{n=1}^{\infty}$ *with* $\lim_{n\to\infty} t_n = \infty$, *the inequality*

$$\liminf_{n\to\infty} \int_{t_n}^{t_{n+1}} \sqrt{a(t)}\,dt \ge \gamma \tag{9}$$

implies

$$I = \sum_{n=1}^{\infty} \int_{t_n}^{t_{n+1}} \left[\min\left\{ \frac{1}{\sqrt{a(t)}} \int_{t_n}^{t} a(s)\,ds;\ \int_{t}^{t_{n+1}} \sqrt{a(s)}ds \right\} \right]^2 d(\ln a(t)) = \\ = \infty. \tag{10}$$

Then all solutions of (7) *are small.*

To illuminate the relationship between Theorem C and Theorem 3, we formulate a corollary of Theorem 3.

Corollary 4. *Assume that* $a : [0, \infty) \to (0, \infty)$ *is continuous, nondecreasing,* $\lim_{t\to\infty} a(t) = \infty$, *and there are* $\kappa > 0$ *and* $\mu > 0$ *such that* $0 < h$ *and* $\int_{t-h}^{t} \sqrt{a(s)}\,ds < \mu$ *imply*

$$\int_{t-h}^{t} a(s)\,ds \ge \kappa \sqrt{a(t)} \int_{t-h}^{t} \sqrt{a(s)}\,ds \tag{11}$$

for all t *large enough.*

Suppose that for every $\gamma > 0$ *there is an* ε $(0 < \varepsilon < \gamma)$ *such that (8) holds for every family of intervals* $\{[\alpha_n, \beta_n]\}_{n=1}^{\infty}$ *satisfying the following conditions* (i)–(iii)*:*

(i) $\alpha_n < \beta_n < \alpha_{n+1}$, $n = 1, 2, \ldots,$ $\lim_{n\to\infty} \alpha_n = \infty$*;*

(ii) $\int_{\alpha_n}^{\beta_n} \sqrt{a(t)}\,dt = \varepsilon$, $n = 1, 2, \ldots$*;*

(iii) $0 < \gamma - \varepsilon \le \liminf_{n\to\infty} \int_{\beta_n}^{\alpha_{n+1}} \sqrt{a(t)}\,dt \le \limsup_{n\to\infty} \int_{\beta_n}^{\alpha_{n+1}} \sqrt{a(t)}\,dt < \infty$.

Then all solutions of (7) *are small.*

Proof. If a sequence $\{t_n\}$ satisfies (9), then define $\{\alpha_n, \beta_n\}$ such that

$$\alpha_n < t_n < \beta_n, \quad \int_{t_n}^{\alpha_n} \sqrt{a(t)}\,dt = \int_{\beta_n}^{t_{n+1}} \sqrt{a(t)}\,dt = \frac{\varepsilon}{2} \quad n = 1, 2, \dots$$

hold. Then the conditions (i)-(iii) are satisfied, which implies (8). We show that (10) is satisfied, too. In fact,

$$\begin{aligned} I &\geq \sum_{n=1}^{\infty} \int_{t_n}^{t_{n+1}} \left[\min\{\kappa \int_{t_n}^{t} \sqrt{a(s)}\,ds; \quad \int_{t}^{t_{n+1}} \sqrt{a(s)}\,ds\} \right]^2 d(\ln a(t)) \geq \\ &\geq \kappa^2 \sum_{n=1}^{\infty} \int_{\beta_n}^{\alpha_{n+1}} \varepsilon^2 d(\ln a(t)) = \\ &= \kappa^2 \varepsilon^2 \sum_{n=1}^{\infty} [\ln a(\alpha_{n+1}) - \ln a(\beta_{n+1})] = \\ &= \infty \end{aligned}$$

because of (8). ■

If we compare Corollary 4 with Theorem C, we can see that Corollary 4 requires (8) of less sequences $\{\alpha_n, \beta_n\}$, than Theorem C does. Among others, this is true because the condition (ii) in Corollary 4 uses integrals, while (i) and (ii) in Theorem C contain rough estimates for the same integral $\int_{\alpha_n}^{\beta_n} \sqrt{a(t)}\,dt$.

Finally, we would like to sketch a new approach to the problem. As is known, (4) alone is not sufficient for the property that *all solutions* of (1) tend to zero as t goes to infinity. On the other hand, the assumption of regular growth is too restrictive in certain cases; e.g., step function coefficients are never of regular growth. So it is natural to ask: how often does it happen that, under the only assumption (4), all solutions of (1) tend to zero Let us formulate exactly this problem for the case of (1) with step function coefficient.

Let the sequence $\{a_n\}_{n=1}^{\infty}$ be given such that

$$0 \leq a_1 < a_2 < \cdots < a_n < a_{n+1} < \cdots, \quad n = 1, 2, \dots, \quad \lim_{n\to\infty} a_n = \infty,$$

and let us choose a sequence $\{t_n\}_{n=1}^{\infty}$ at random such that

$$0 = t_0 < t_1 < \cdots < t_n < t_{n+1} < \cdots, \quad n = 1, 2, \dots, \quad \lim_{n\to\infty} t_n = \infty.$$

What is the probability that for given x_0 and x_0', the solution x with $x(0) = x_0$, $x'(0) = x_0'$ satisfies $\lim_{t\to\infty} x(t) = 0$? The following theorem can be proved.

Theorem 5 (L. Hatvani–L. Stachó). *Let $t_{n+1}-t_n$, $n=1,2,\dots$, be a uniformly distributed random variable on $[0,1]$. Then for every solution x of the equation*

$$x''+a_kx=0,\quad t_{k-1}\le t<t_k,\quad k=1,2,\dots,$$

it holds almost surely that $\lim_{t\to\infty} x(t)=0$.

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