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SOME PROBLEMS OF THE SPECTRAL THEORY OF OPERATOR PENCILS

Abstract. For polynomial operator pencils, various properties of the spectrum, generalized eigenvalues and generalized eigen and adjoint vectors are investigated.

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Consider the operator pencil

$$P(\lambda) = \lambda^n E + \lambda^{n+1} A_1 + \cdots + \lambda A_{n-1} + A_n - A^n \tag{1}$$

in a space H and assume that the following conditions are satisfied:

(a) the spectrum $\sigma(A)$ of the operator A is continuous and coincides with $[\alpha, \beta]$;

(b) the operator function (o.f.) $\lambda^j C^{-1} A_{n-j} (\lambda^n E - A^n)^{-1} C^{-1} = R_j(\lambda)$, $j = 0, 1, \dots, n-1$ in H has an analytic continuation through the continuous spectrum of the pencil $\lambda^n E - A^n$ which, obviously, consists of n segments $[w_k\alpha, w_k\beta]$, $w_k = \sqrt[n]{1}$, $k = \overline{1,n}$. Then the following theorem is valid.

Theorem 1. *Let linear operators $A_1, A_2, \dots, A_n$ be such that $A_j(A^n + iE)^{-1} \in \sigma_\infty(H)$, $j = \overline{1,n}$. Then the spectrum of the pencil* (1) *consists at most of a countable number of points with possible limit points in $w_k\alpha$ and $w_k\beta$ and of the segments $[w_k\alpha, w_k\beta]$, $k = \overline{1,n}$.*

Denote $\Omega_k^+ = \{\lambda \in C | \arg w_k < \arg \lambda < \arg w_{k+1}\}$, $k = \overline{1,n}$, where $|w_{n+1}| = |w_1|$; $\arg w_{n+1} = \arg w_1 + 2\pi$ and assume that (o.f.) $R_j(\lambda)$ acting in H_1, has a bounded analytic continuation $R_j^+(\lambda)$ from the domain Ω_k^+ into a wider Ω_k so that $\Omega_k^+ \subset \Omega_k$ and $(w_k\alpha, w_k\beta) \subset \Omega_k$, $k = \overline{1,n}$;

(c) at any $\lambda \in \Omega_k$ there is $R_j^+(\lambda) \in \sigma_\infty(H)$;

(d) o.f. $\widetilde{C}^{-1}(\lambda^n E - A^n)^{-1} C^{-1} = R_\lambda^+(A)$ is holomorphic in the domain Ω_k, $k = \overline{1,n}$ and belongs to $\sigma_\infty(H)$.

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Theorem 2. *Let $D(A_n) \cap H_n$ be everywhere dense in H and let the vectors $f_0, \dots, f_{n-1}$ belong to $D(A_n) \cap mH_1$. Further, let the conditions $(a) - -(d)$ be satisfied, and $\lim_{\lambda \to \infty} \|R_j(\lambda)\| = 0$. Then the following expanssion holds:*

$$\widetilde{C}^{-1} f_j = \frac{1}{2\pi i} \int_{\Gamma} \lambda^j \widetilde{C}^{-1} P \tag{2}$$

where Γ is a contour overlapping all the spectrum of the pencil $P(\lambda)$, and the convergence of the integral is understood in the sense of the metrics of H.

Consider now a polynomial operator pencil of the form

$$L(\lambda) \equiv A_0 + \lambda A_1 + \dots + \lambda^n A_n + B_0 + \lambda B_1 + \dots + \lambda^{n-1} B_{n-1}, \tag{3}$$

where A_0 is a linear closed operator with $\overline{D(A_0)} = H$, and $A_j, B_{j-1} \in L(H)$, $j = \overline{1,n}$. Components of the spectrum of the pencil $A(\lambda) \equiv A_0 + \lambda A_1 + \cdot + \lambda^n A_n$ are assumed to consist of the following sets:

$$\sigma(A(\lambda)) = \{\lambda_1, \lambda_2, \dots, \lambda_m\} \bigcup_{j=1}^{k} [w_j \alpha_j, w_j \beta_j],$$

$$\sigma_p(A(\lambda)) = \{\lambda_1, \lambda_2, \dots, \lambda_m\} \bigcup_{j=1}^{k} \{\mu_1^{(j)}, \mu_2^{(j)}, \dots, \mu_j^{(j)}\},$$

$$\sigma_c(A(\lambda)) = \sigma(A(\lambda)) \backslash \sigma_p(A(\lambda)),$$

where $\mu^{(j)} \in [w_i \alpha_j, w_j P_j]$, $(j = 1, 2, \dots, s, j = 1, 2, \dots, k)$. Then the following theorem holds.

Theorem 3. *Let $w_j \alpha_j \in \Omega_j^+$, $w_j \beta_j \in \Omega_j^+$ or $w_j \alpha_j \in \Omega_j^+$, $\beta_j = \infty$ $(j = \overline{1,k})$. Then the pencil $L(\lambda)$ has a finite number of generalized eigenvalues from $(w_j \alpha_j, w_j \beta_j) \cup (w_{j+1} \alpha_{j+1, w_{j+1}} \beta_{j+1})$ $(j = \overline{1,k})$ (Spectral singularities).*

Consider the polynomial operator pencil of the form

$$L(\lambda) \equiv A_0 + \lambda A_1 + \cdot + \lambda^n A_n + B_0 + \lambda B_1 + \cdot + \lambda^{n-1} B_{n-1},$$

where A_0 is a linear closed operator with the everywhere dense domain $D(A_0)$ in a Hilbert space H, and

$$A_j \in L(H), \quad B_{j-1} \in L(H), \quad j = 1, 2, \dots, n.$$

Assume that the components of the spectrum of the pencil $A(\lambda) \equiv A_0 + \lambda A_1 + \dots + \lambda^n A_n$ consist of the following sets:

$$\sigma(A(\lambda)) = \{\lambda_1, \lambda_2, \dots, \lambda_m\} \bigcup_{j=1}^{k} [w_j \alpha_j, w_j \beta_j],$$

$$\sigma_p(A(\lambda)) = \{\lambda_1, \lambda_2, \dots, \lambda_m\} \bigcup_{j=1}^{k} \{\mu_1^{(j)}, \mu_2^{(j)}, \dots, \mu_j^{(j)}\},$$

$$\sigma_c(A(\lambda)) = \sigma(A(\lambda)) \backslash \sigma_p(A(\lambda)),$$

where $\mu^{(j)} \in [w_i\alpha_j, w_jP_j]$, $(j = 1,2,\dots,s_j, j = 1,2,\dots,k)$, $[w_j\alpha_j, w_j\beta_j]$ are segments in the complex plane C such that $0 \le \alpha_j < \beta_j \le +\infty$ $(j = 1,2,\dots,k)$, $\arg w_1 < \arg w_2 < \dots < \arg w_k$.

The use is made of the following notation:

$$\Omega_j = \{\lambda \in C | \arg w_j < \arg\lambda < \arg w_{j+1}\}, \quad (j = 1,2,\dots,K),$$
$$|w_{k+1}| = |w_1|, \quad \arg w_{k+1} = \arg w_1 + 2\pi.$$

Let the resolvent $R(A(\lambda))$ of the pencil $A(\lambda)$ be finite meromorphic in the domain $C \setminus \bigcup_{j=1}^{k} [w_j\alpha_j, w_j\beta_j]$ and let the following condition be satisfied:

(a) for every $j = \{1,2,\dots,k\}$ there exists at least one point $\lambda_0^{(j)} \in \Omega_j$ which is a common point of regularity of the pencils $A(\lambda)$ and $L(\lambda)$ such that

$$R(A(\lambda_0^{(j)})) \quad B_s \in \sigma_\infty(H) \quad (j = 1,2,\dots,K; S = 0,1,\dots,n-1 \quad).$$

Then the following theorem holds.

Theorem 4. *In the set* $C \setminus \bigcup_{j=1}^{k} [w_j\alpha_j, w_j\beta_j]$ *there exists only the point spectrum of the pencil consisting at most of a countable number of isolated eigenvalues with finite algebraic multiplicity and with possible limit points in* $\bigcup_{j=1}^{k} [w_j\alpha_j, w_j\beta_j]$ *and at* ∞. *The set* $\bigcup_{j=1}^{k} [w_j\alpha_j, w_j\beta_j]$ *belongs to the spectrum of the pencil* $L(\lambda)$ *and the sets* $[w_j\alpha_j, w_j\beta_j] \setminus (S_j \cup G_j)$ $(j = 1,2,\dots,k)$ *belong to the continuous spectrum of the pencil* $L(\lambda)$, *where*

$$S_j \equiv \{\lambda \in [w_j\alpha_j, w_j\beta_j] \ker L(\lambda) \ne \{0\}\},$$
$$G_j \equiv \{\lambda \in [w_j\alpha_j, w_j\beta_j] \setminus S_j | \ker L^*(\overline{\lambda}) \ne \{0\}\} \quad (j = 1,2,\dots,k).$$

The resolvent $R(L(\lambda))$ of the pencil $L(\lambda)$ is finite meromorphic o.f. in the domain $C \setminus \bigcup_{j=1}^{k} [w_j\alpha_j, w_j\beta_j]$.

(b) the operators B_j admit extensions $\overline{B}_j$ to all the space H such that $\overline{B}_j \in L(H_-, H_+)$ $(j = 0,1,\dots,n-1)$;

(c) the o.f. $P_\lambda \equiv C^{-1}R(A(\lambda))C^{-1}$ admits a finite meromorphic continuation of $R_\lambda^j \equiv \overline{C}^{-1}R^{(j)}(A(\lambda))C^{-1}$ from the domain Ω_j into the domain Ω_j^+ such that $\Omega_j \subset \Omega_j^+$, $(w_j\alpha_j, w_j\beta_j) \subset \Omega_j^+$ and $(w_{j+1}\alpha_{j+1}, w_{j+1}\beta_{j+1}) \in \Omega_j^+$ $(j = \overline{1,k})$.

Denote by D_j the set of all poles of o.f. $P_\lambda^{(j)}$ from the domain Ω_j^+. Assume $T_s \equiv C\overline{B}_s\overline{C}$ $(s = 0,1,\dots,n-1)$ and let the following condition be satisfied:

(d) $P_\lambda^{(j)} T_s \in \sigma_\infty(H)$ for all $\lambda \in \Omega_j^+ \setminus D_j$ $(j=1,2,\dots,k; s=0,1,\dots,n-1)$.

Further we assume that the conditions (a)–(d) are satisfied.

Theorem 5. *The o.f.* $T_\lambda \equiv \overline{C}^{-1} R(L(\lambda))C^{-1}$ *admits a finite meromorphic continuation* $T_\lambda^{(j)} \equiv \overline{C}^{-1} R^{(j)}(L(\lambda))C^{-1}$ *from the domain* Ω_j *into the domain* Ω_j^+ $(j = 1, 2, \dots, k)$.

If

(e) $\lim_{|\lambda|\to\infty} \|\lambda^{n-1}P^{(j)}(\lambda)\| = 0$ $(j = \overline{1,k})$,
then the following theorem holds.

Theorem 6. *Let* $w_j\alpha_j \in \Omega_j^+$, $w_j\beta_j \in \Omega_j^+$ *or* $w_j\alpha_j \in \Omega_j^+$, $\beta_j = \infty$ $(j = 1, 2, \dots, k)$. *Then the pencil* $L(\lambda)$ *has a finite number of eigenvalues and also a finite number of g.e.v. from* $(w_j\alpha_j, w_j\beta_j) \cup (w_{j+1}\alpha_{j+1}, w_{j+1}\beta_{j+1})$ $(j = 1, 2, \dots, k)$ (*spectral singularities*).

Let the conditions of Theorem 5 and the conditions (a)–(e) be satisfied. Then the pencil $L(\lambda)$ has a finite number of ganeralized eigenvalues and spectral singularities. Denote by $\lambda_1, \lambda_2, \dots, \lambda_s$ the eigenvalues of the pencil which do not lie in the set $\bigcup_{j=1}^{k} [w_j\alpha_j, w_j\beta_j]$ and by $\mu_{j\nu}^{(\nu)}$ $(j_\nu = 1, 2, \dots, s_\nu, \nu = 1, 2, \dots, k)$ the generalized ligenvalues of the pencil from $(w_\nu\alpha\nu, w_\nu\beta_\nu)$ and $(w_{\nu+1}\alpha_{\nu+1}, w_{nu} + 1\beta_{\nu+1})$. Here the enumeration of the numbers is such that $\mu_{t_\nu}^{(\nu)} = \mu_{t_{\nu+1}}^{(\nu+1)}$, $t_\nu \le S_\nu$, $t_\nu \le S_{\nu+1}$ $(\nu = 1, 2, \dots, k)$, $\mu_{t_{k+1}}^{(k+1)} \equiv \mu_{t_1}^{(1)}$, $S_{k+1} \equiv S_1$.

Denote by Γ_0 the contour formed by a subsegment of the segment $[w_{\nu+1}\alpha_{\nu+1}, w_{\nu+1}\beta_{\nu+1}]$ and by semicircles of sufficiently small radii with centers at the points $\mu_1^{(\nu)} \dots, \mu_{s_\nu}^{(\nu)}$ in the domain Ω_ν and at the points $\mu_{t\nu+1}^{(\nu+1)}, \dots, \mu_{s\nu}$ in the domain $\Omega_{\nu+1}$ $(\nu = 1, 2, \dots)$, where we assume that $\Omega_{k+1} \equiv \Omega$.

The following notation is used:

$$\phi_1(\lambda) \equiv -(A_0 + B_0),\ \phi_j(\lambda) \equiv -(A_0 + B_0 + \sum_{s=1}^{j-1} \lambda^s (A_s + B_s)),\ j = 2, 3, \dots, n;$$

$$P_j(\lambda) \equiv \frac{1}{\lambda^j}(L(\lambda) + \phi_j(\lambda)) = \sum_{s=j}^{n-1} \lambda^{s-j}(A_s + B_s) + \lambda^{n-j} A_n,\ \ j = 1, 2, \dots, n.$$

Under these assumptions and notation, we have the following

Theorem 7. *Let* $f_j \in D(A_0) \cap H_+$, $A_i f_j \in H_+$ $(i = 0, 1, \dots, n)$ $j = 0, 1, \dots, n-1)$. *Then there exists an* n*-fold expansion of generalized eigen-*

and adjoint vectors of the pencil $L(\lambda)$:

$$\overline{C}^{-1} f_j = \frac{1}{2\pi i} \sum_{\nu=1}^{k} \int_{\Gamma_\gamma} \lambda^j (T_\lambda^{(\nu+1)} - T_\nu^{(\nu)}) C \sum_{m=1}^{n} P_m(\lambda) f_{m-1} d\lambda +$$

$$+ \sum_{k=1}^{k} \Re s \left[\lambda^j T_\lambda C \sum_{m=1}^{n} P_m(\lambda) f_{m-1} \right]_{\lambda=\lambda_1} +$$

$$+ \sum_{\nu=1}^{k} \sum_{i=1}^{t_0} \Re s \left[\lambda^j T_\lambda^{(\lambda)} C \sum_{m=1}^{n} P_m(\lambda) f_{m-1} \right]_{\lambda=\mu^{(\nu)}},$$

$j = 0, 1, \ldots, n-1$, where $T_\lambda^{(k+1)} \equiv T_\lambda^{(1)}$. The integrals in the right-hand sides converge in the metrics of H.

Further, the spectral analysis of a rational operator pencil of nonself-adjoint operators with the continuous–point spectrum is carried out.

In a Hilbert space H we consider the rational operator pencil

$$L(\gamma) \equiv A - \lambda I + B_0 + \frac{B_1}{(\lambda - C)^m} + \frac{B_2}{(\lambda - C)^{m-1}} + \cdots + \frac{B_m}{\lambda - C},$$

where A is a self-adjoint operator in H having only the continuous spectrum coinciding with the segment

$$[\alpha, \beta], \quad -\infty \le \alpha \le +\infty \quad B_i \in L(H), \quad (i = 0, 1, \ldots, m).$$

Let the following conditions be satisfied:

(a) there exists at least one point, $\lambda_0^{\pm} \in \mathbb{C}^{\pm} = \{\lambda \in \mathbb{C} | \operatorname{Im} \neq 0\}$, $\lambda_0^1 \neq C$ is any fixed number from $\mathbb{C}$ such that

$$R_{\lambda_0^{\pm}}(A) B_j \in \sigma_\infty(H), \quad j = 0, 1, \ldots, m.$$

Then following theorem is valid.

Theorem 8. *In the set $\mathbb{C} \backslash ([\alpha, \beta] \cup \{C\})$ there is only the discrete spectrum of the pencil consisting at most of a countable number of isolated eigenvalues with finite algebraic multiplicities and with possible limit points in $[\alpha, \beta] \cup \{C\}$ and at ∞. The set $[\alpha, \beta] \cup \{C\}$ belongs to the spectrum and the set $[\alpha, \beta] \cup \{C\}$ belongs to the continuous spectrum of the pencil $L(\lambda)$, where*

$$M_1 = \{\lambda \in [\alpha, \beta] | \ker L(\lambda) \neq \{0\}\},$$
$$M_2 = \{\lambda \in [\alpha, \beta] \backslash | \ker L^*(\lambda) \neq \{0\}\}.$$

The resolvent $R(L(\lambda))$ of the pencil $L(\lambda)$ is a finite meromorphic o.f. in the domain $\mathbb{C} \backslash ([\alpha, \beta] \cup \{C\})$.

Let also the following conditions be satisfied:

(b) the operators B_i admit extensions $\overline{B}_i$ to the all space H such that $\overline{B}_i \in L(H_-, H_+)$ $(i = 0, 1, \ldots, m)$;

(c) the o.f. $P_\lambda \equiv \overline{C}^{-1} R_\lambda(A) C^{-1}$ admits an analytic continuation $P_\lambda^\pm \equiv \overline{C}^{-1} R_\lambda^+(A) C^{-1}$ from the domain $C^\pm$ into the domain $\Omega^\pm$ such that $(\alpha, \beta) \subset \Omega^\pm$;

(d) $P_\lambda^\pm K_i \in \sigma_\infty(H)$, $\lambda \in \Omega^\pm$, where it is assumed that $K_i - C\overline{B}_i\overline{C}$ $(i = 0, 1, \ldots, m)$.

Under the conditions (a)–(d) the following theorem is valid.

Theorem 9. *The o.f.* $T_\lambda = \overline{C}^{-1} R(L(\lambda)) C^{-1}$ *admits a finite meromorphic continuation* $T_\lambda^\pm = \overline{C}^{-1} R^\pm(L(\lambda)) C^{-1}$ *from the domain* $C^\pm \backslash \{c\}$ *into the domain* $\Omega^\pm \backslash \{c\}$.

Let also the following conditions be satisfied:

(e) $\lim_{\substack{|\lambda| \to \infty \\ \lambda \in \Omega^\pm}} \|P_{\lambda^\pm}\| = 0$;

(f) $K_i \equiv CB_iC$ $(i = 1, 2, \ldots, m)$ are finite-dimensional operators in H.

Then we have the following

Theorem 10. *Let one of the conditions* $\alpha, \beta \in \Omega^\pm$ *or* $\alpha \in \Omega^\pm$, $\beta = +\infty$ *be satisfied. Then the pencil* $L(\lambda)$ *has a finite number of generalized eigenvalues from* (α, β) *(spectral singularities).*

Under the conditions (a)–(d) the following theorem holds.

Theorem 11. *If* $\lambda_0 \in \Omega^\pm$ $(\lambda_0 \neq c)$ *is a generalized eigen–and adjoint vectors of the pencil* $L(\lambda)$*, then there exists a canonic system* $\varphi_0^{(j)}, \varphi_1^{(j)}, \ldots, \varphi_{r-1}^{(j)}$ $(j = 1, 2, \ldots, n)$ *of generalized eigenvalue of the pencil* $L(\lambda)$ *coresponding to the generalized eigenvalue* λ_0 *and a canonic* $\psi_0^{(j)}, \psi_1^{(j)}, \ldots, \psi_{r-1}^{(j)}$ $(j = 1, 2, \ldots, n)$ *system of eigen-and adjoint vectors of o.f.* $I + (R_\lambda^\pm(A)\overline{B}(\gamma))^*$ *coresponding to the eigenvalue* λ_0 *such that we have the expansion*

$$T_\lambda^\pm f = \sum_{j=1}^{n} \sum_{k=1}^{r_j} (\lambda - \lambda_0)^{-k} \sum_{i=1}^{k} \sum_{s=0}^{r_j - i} \frac{1}{(k-i)} \times$$
$$\times \left(\frac{d^{k-i}}{d\lambda_0^{k-i}} f, \overline{C}^{-1} \psi_s^{(j)} \overline{C}^{-1} \psi_s^{(j)} \right) \overline{C}^{-1} \varphi_{r_i - i - s}^{(j)}, \quad f \in H. \tag{4}$$

If λ_0 is a generalized eigenvalue of the o.f. $L(\lambda)$, then under the condition (f) the expansion (4) holds.

For the family of differential operators generated by the differential expression

$$l'_\lambda(y) = iy^{(2n-1)} + P_2(, \lambda) y^{(2n-3)} + P_3(x, \lambda) y^{(2n-4)} = \cdots + (P_{2n-1}(x, \lambda) + \lambda^{2n-1}) y,$$

where $P_k(x,\lambda) = \lambda^{k-1}P_{k1}(x) + \lambda^{k-2}P_{k2}(x) + \cdots + P_{kk}(x)$, $k = \overline{2, 2n-1}$ and $P_{k_j}(x)$, $j \le k$ are complex-valued functions summable on $[0,\infty)$, in the space $L_2(0,\infty)$ the boundary conditions of the form

$$\begin{aligned}
U_\nu(y) &= \sum_{j=0}^{2n-2} \alpha_{\nu j}(\lambda) y^{(j)}(o,\lambda) = 0,\ \nu = 1,2,\ldots,n-1, \quad \text{if} \quad \lambda \in S'_{mH} \\
U_\nu(y) &= 0, \quad \nu = 1,2,\ldots,n \quad \text{if} \quad \lambda \in S'_m \\
U_\nu(y) &= 0, \quad \nu = 1,2,\ldots,n \quad \text{if} \quad \lambda \in S''_m \\
U_\nu(y) &= 0, \quad \nu = 1,2,\ldots,n-1 \quad \text{if} \quad \lambda \in S''_{mB}
\end{aligned}$$

are given Here $\alpha_{\nu k}(\lambda)$ are meromorphic functions of a complex variable in the sectors S'_m and S''_m and such that for $|\lambda| \to \infty$

$$\alpha_{\nu k}(\lambda) = \alpha_p^{\nu k} \lambda^{p(k)}[1 + O(1/|\lambda|)],$$

where $P(2n-2) \ge P(2n-3) \ge \cdots \ge P(0)$, $P(k)$ are natural numbers or zero.

The sectors $S'_m(S''_m)$ are defined by the inequalities

$$\frac{(m-1/2)\pi}{2n-1} < \arg\lambda < \frac{(m+1/2)\pi}{2n-1}, \quad m = 1,2,\ldots,4n-2,$$

for odd (even) n. Here $S'_m = S'_{mH} \cup S'_{mb}$ and $S''_m = S_{mH} \cup S_{mb''}$.

It is easy to show that under the condition

$$|P_{kj}(x)| \le Ce^{\varepsilon x} j \le k, \quad C = \text{const}, \quad \varepsilon > 0, \tag{5}$$

the equation $l_\lambda(y) = 0$ has $2n-1$ linearly independent solutions $y_i(x,\lambda)$ $(i = 1,\ldots,2n-1)$ which, together with their derivatives, are functions continuous in (x,λ) and holomorphic in λ for each fixed $x \in [0,\infty)$. Then, if $\lambda \in S'_{mH}$ or $\lambda \in S''_{mb}$, then the equation

$$A_1(\lambda) \equiv \begin{vmatrix} U_1(y_1) & \cdots\cdots & U_1(y_{n-1}) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ U_{n-1}(y_1) & \cdots\cdots & U_{n-1}(y_{n-1}) \end{vmatrix} = 0$$

defines the eigenvalues of the family $L(\lambda)$. If $\lambda \in S''_{mb}$ or $\lambda \in S'_{mH}$, then the eigenvalues of the family $L(\lambda)$ are defined by the equation

$$A_2(\lambda) \equiv \begin{vmatrix} U_1(y_1) & \cdots\cdots & U_1(y_n) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ U_n(y_1) & \cdots\cdots & U_n(y_n) \end{vmatrix} = 0.$$

We call all zeros of the functions $A_1(\lambda)$ and $A_2(\lambda)$ we call singular numbers of the family of operators $L(\lambda)$; moreover, it is shown that $A_1(\lambda)$ and $A_2(\lambda)$ are not identically equal to zero.

Thus we have the following

Theorem 12. *Let* (5) *hold. Then the set of singular numbers of the family of operators* $L(\lambda)$ *is finite, and the continuous spectrum consists of the rays* L'_m *and* L''_m.

To obtain the expansion formula, for any sufficiently large by modulus λ an estimation of the resolvent kernel was found in each bounded square $0 \leq x, \zeta \leq a$, $a > 0$.

Theorem 13. *Let* (5) *hold and let singular numbers of the operator* $L(\lambda)$ *be simple. Then for arbitrary test functions* f_j, $j = 0, \overline{n-2}$, *differentiable* $2n - j - 2$ *times for* $j \geq 3$, *there exists a* $2n-1$*-fold expansion for even (odd)* n *of the form*

$$f_j = \sum_{s=1}^{k} \lambda_s^j a_s y_s(x) + \frac{1}{2\pi i} \sum_{m=1}^{2n-1} \sum_{i=1}^{n} \int\limits_{L'_m (L''_m)} \lambda^j [B\Gamma_1(x,\lambda)] d\lambda$$

$$\Gamma_i(x,\lambda) = \sigma^{i+1}(x, \lambda\sigma)(F_i)(\lambda\sigma) - y_i(x,\lambda)F_i(\lambda), \quad \sigma = e^{i\pi/(2n-1)}$$

Here

$$[D\phi] = \phi(\lambda) - \sum_{k=p+1}^{1} B_k(\lambda)\phi(\lambda_k), \quad \{B_k(\lambda)\}_{\lambda=\lambda_k} = \begin{cases} 1, & k = k', \\ 0, & k \neq k,' \end{cases}$$

k *is the number of eigenvalues, the spectral singularities* $F_i(\lambda)$ *are known functions and the integrals converge uniformly and absolutely for all* $x \in [0, \infty)$.

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