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NONLINEAR TRANSMISSION PROBLEMS

Abstract. An existence theorem for solutions of non-linear transmission problems is proved using certain special estimates of inverses of Toeplitz operators. A framework for construction of analytic discs attached to singular targets is described, and a mechanism of creating such discs is presented.

რეზიუმე. ტეპლიცის ოპერატორების შეუღლებულების სპეციალური შეფასებების გამოყენებით არაწრფივი გადატანის ამოცანებისათვის დამტკიცებულია ამონახსნის არსებობის თეორემა. აღწერილია ზოგადი ჩარჩოები სინგულარულ სამიზნეებთან დაკავშირებული ანალიზური დისკოების ასაგებად და მოყვანილია ასეთი დისკოების შექმნის მექანიზმი.

1. Introduction

In course of a long historical development, the famous Riemann-Hilbert problem became an "organizing center" for a number of important topics of complex analysis, differential equations, topology and operator theory. Most of these topics are developing quite actively and continue suggesting new interesting problems and interrelations.

The problem itself has many faces and goes with several names, such as boundary value problem of Riemann-Hilbert (or Hilbert) type or as transmission problem; in operator theory it is intimately related with Toeplitz operators and singular integral equations, and, more recently, in a geometric setting it appears in connection with analytic discs. These names do not only depend on context and history but they also stress different aspects of the topic.

In this paper, we prefer to think of the problem as of a transmission problem, where two functions Φ_- and Φ_+ which are holomorphic in an interior and an exterior domain, respectively, are to be determined from a nonlinear coupling condition on the common boundary of their domains. As a generalization of the linear transmission problem

$$\Phi_+(t) = G(t) \cdot \Phi_-(t) + g(t),$$

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which was investigated in seminal papers by N.I. Muskhelishvili, I.N. Vekua, and F.D. Gakhov, we also admit nonlinear conditions

$$\Phi_+(t) = G(t, \Phi_-(t)). \tag{1}$$

From the operator theoretic point of view, the linear problem is related to Toeplitz operators, i.e., to the interaction of *multiplication* with the Riesz projection, while the nonlinear problem concerns interaction of the Riesz projection with *superposition*. In particular, Toeplitz operators are one of the main tools for studying (2).

In the next section, we introduce a special class of nonlinear transmission problems and obtain a rather complete description of their solutions. At the end we relate nonlinear transmission problems with the existence problem for so-called analytic discs and some non-traditional aspects of singularity theory. In particular, we describe an appropriate setting for nonlinear transmission problems with singular target manifolds with isolated plane curve singularities [2].

2. Transmission Problems and Toeplitz Operators

For a given continuously differentiable function $G\colon\ \mathbb{T}\times\mathbb{C}\to\mathbb{C}$, we study the nonlinear transmission problem

$$\Phi_+(t) = G(t, \Phi_-(t)), \qquad \forall t \in \mathbb{T}. \tag{2}$$

It is supposed that the unknown functions Φ_+ and Φ_- extend holomorphically from the complex unit circle $\mathbb{T}$ into its interior $\mathbb{D}$ and its exterior $\mathbb{E}$, respectively, and that Φ_- vanishes at infinity.

If G is linear in z and $\overline{z}$, $G(.,z) = g_0 + g_1 z + g_2\overline{z}$, we get the linear transmission problem with conjugation ([6], [7]).

The 'holomorphic case' $\overline{\partial}_z G \equiv 0$ was studied in Wolfersdorf's paper [13]. More generally, the nonlinear problem (2) is said to be *elliptic* if

$$|\overline{\partial}_z G(t,z)| < |\partial_z G(t,z)|, \qquad \forall (t,z) \in \mathbb{T}\times\mathbb{C}. \tag{3}$$

Another case of particular interest corresponds to real-valued G, pertaining to the 'parabolic case', since then $|\partial_z G| = |\overline{\partial}_z G|$. In this situation Φ_+ must be holomorphic in $\mathbb{D}$ and real-valued on $\mathbb{T}$ and hence (2) is equivalent to the scalar Riemann-Hilbert problem

$$G(t, \Phi_-(t)) = const. \tag{4}$$

(see [5], [9], [10], for instance). In contrast to the general nonlinear transmission problem (2), there is a rather complete geometric theory of Riemann-Hilbert problems (4) (see [10]).

We say that (2) has a solution in W^1_r, if the functions Φ_- and Φ_+ have boundary functions in the Sobolev space $W^1_r(\mathbb{T})$. The following existence theorem does also cover the linear elliptic case with continuously differentiable coefficients and index zero.

Theorem 1. *Let $G\colon\ \mathbb{T}\times\mathbb{C}\to\mathbb{C}$ be continuously differentiable with uniformly bounded first derivatives.*

(i) *If there exist a positive constant δ and a smooth unimodular function $g\colon\ \mathbb{T}\to\mathbb{T}$ with winding number zero, $\operatorname{wind} g=0$, such that*

$$\big|\partial_z G(t,z)\big| - |\overline{\partial}_z G(t,z)| \geq \delta > 0 \quad \forall (t,z)\in\mathbb{T}\times\mathbb{C}, \tag{5}$$

$$\operatorname{Re}\,\big(g(t)\,\partial_z G(t,z)\big) \geq \delta > 0 \quad \forall (t,z)\in\mathbb{T}\times\mathbb{C}, \tag{6}$$

then the transmission problem (2) has a solution in W_r^1 for each $r\in(1,\infty)$.

(ii) *The solution is unique if, in addition to the above assumptions,*

$$\operatorname{Re}\big(g(t)\,\partial_z G(t,z)\big) - |\overline{\partial}_z G(t,z)| \geq \delta > 0 \quad \forall (t,z)\in\mathbb{T}\times\mathbb{C}. \tag{7}$$

The proof is prepared by several observations. First of all, we remark that the function g in the condition (6) admits a factorization $g=g_H/g_R$, where g_R and g_H are smooth functions on $\mathbb{T}$, g_R is real and strictly positive and g_H extends to a holomorphic function in $\mathbb{D}$ without zeros. This allows to rewrite the boundary relation as

$$\widetilde{\Phi}_+ := g_H\cdot\Phi_+ = g_R\cdot g\cdot G(.,\Phi_-) =: \widetilde{G}(.,\Phi_-).$$

If G satisfies (5), (6) (and (7)), then $\widetilde{G}$ satisfies the same conditions with $g\equiv 1$. Consequently we can assume that $g\equiv 1$.

The following constructions serve to transform the transmission problem (2) into a fixed-point equation for a compact operator K. The idea is to differentiate the boundary relation along $\mathbb{T}$ (Wolfersdorf [12]), which gives rise to a quasi-linear transmission problem with conjugation. The main ingredient of the operator K is a primitive of an appropriate solution of this auxiliary problem.

Fix $s\in(1,\infty)$. For a given scalar complex valued function $\varphi\in W_s^1(\mathbb{T})$, we define

$$a(t) := \partial_z G\big(t,\varphi(t)\big),\quad b(t) := \overline{\partial}_z G\big(t,\varphi(t)\big),\quad c(t) := i\,t\,\partial_t G\big(t,\varphi(t)\big), \tag{8}$$

where $i\,t\,\partial_t\equiv\partial_\tau$ denotes the derivative with respect to the polar angle τ of $t\equiv e^{i\tau}\in\mathbb{T}$. Note that a, b, and c are continuous functions.

We denote by H_+^r (resp. H_-^r) the Hardy spaces of functions φ which extend holomorphically into $\mathbb{D}$ (resp. in $\mathbb{E}$ with $\varphi(\infty)=0$), and let $H_\pm^r := H_+^r\times H_-^r$.

Lemma 1. *Let G be subject to the assumptions of Theorem 1 with $g\equiv 1$, fix $r,s\in(1,\infty)$, let $\varphi\in W_s^1$, and let a, b, and c be given by (8).*

(i) *For each $\varphi\in W_s^1(\mathbb{T})$, the linear transmission problem*

$$\widetilde{\Phi}_+ = a\,\widetilde{\Phi}_- + b\,\overline{\widetilde{\Phi}_-} + c \tag{9}$$

has a unique solution $\widetilde{\Phi} := (\widetilde{\Phi}_+,\widetilde{\Phi}_-)\in H_\pm^r$.

(ii) *For each value of the constant δ in Theorem* 1, *there exists an $r > 1$ such that the $H^r_\pm$-norm of the solution $\widetilde{\Phi} \equiv (\widetilde{\Phi}_+, \widetilde{\Phi}_-)$ to* (9) *is bounded by a constant not depending on the choice of φ.*

Proof. 1. Existence and uniqueness of the solution follow from [7], Section 9.3 (see also [6], Theorem 17.1).

2. In order to prove (ii), we derive a representation of the solutions which involves the inverses of a certain Toeplitz operator.

The function w defined on $\mathbb{T}$ by $w(t) := \big(\overline{\Phi_-(t)}/t, \Phi_+(t)\big)$ extends holomorphically into $\mathbb{D}$. With the definitions $f := -(\operatorname{Re} c, \operatorname{Im} c)$, and

$$A := \begin{bmatrix} \overline{a}+b & -1 \\ i(\overline{a}-b) & i \end{bmatrix} \cdot \begin{bmatrix} t & 0 \\ 0 & 1, \end{bmatrix} \tag{10}$$

the problem (9) is equivalent to

$$R\,w := Re\,A\,w = f. \tag{11}$$

Let $P\colon\ L^r \to H^r_+$ denote the Riesz projection of $L^r(\mathbb{T})$ onto the Hardy space H^r_+ along H^r_-. We introduce the 'adjoint Riemann-Hilbert operator'

$$S\colon\ L^r \to H^r_+,\ x \mapsto P\,\overline{t}\,\overline{A}^{-1}\operatorname{Re} x. \tag{12}$$

A straightforward verification shows that SR is a Toeplitz operator, $2\,SR = T := P\widetilde{B}P$. The symbol $\widetilde{B} := \overline{t}\,\overline{A}^{-1}A$ of T has the representation $\widetilde{B} = \frac{1}{a}\,J\,B$, where

$$J := \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \qquad B := \begin{bmatrix} |a|^2-|b|^2 & \overline{bt} \\ -bt & 1 \end{bmatrix}. \tag{13}$$

Since $|a| > |b|$ and $\operatorname{wind} a = 0$, the Toeplitz operator T is invertible, which implies that the solution of (11) admits the representation

$$w = 2\,T^{-1}\,S\,f. \tag{14}$$

3. Remember that S and T depend on the choice of φ in (8). It is obvious that the norm of S is bounded by a constant not depending on the choice of φ in (8). In what follows, we prove that the norms of the inverse $T^{-1} \in \mathcal{L}(H^{1/(r-1)}, H^r)$ are also uniformly bounded with respect to φ, provided that $r > 1$ is sufficiently small. Since J is constant, we can replace T by $\widehat{T} := P(\frac{1}{a})BP$.

4. Because $\operatorname{Re}(Bz, z) = \big(|a|^2-|b|^2\big)\,|z_1|^2 + |z_2|^2 \geq m\,\|z\|^2$, for some positive number $m = m(\delta)$, Lemma 1 of [11] shows that the inverses of the Toeplitz operators $T_B := PBP\colon\ H^2_+ \to H^2_+$ are uniformly bounded, $\|T_B^{-1}\| \leq 1/m$. The invertibility of T_B implies that the (continuous) symbol B admits a generalized Wiener-Hopf factorization (canonical factorization) $B = B_-B_+$, where B_-, B_+, B_-^{-1}, $B_+^{-1} \in L^p$ for each $p < \infty$ ([1], Section 5.5,

see also [3], [7]). Since $T_B^{-1} = B_+^{-1} P B_-^{-1} P$ and $B_+^{-1} P B_-^{-1} = B_+^{-1} P B_-^{-1} P$ on L^2, the multiplication operator

$$f \mapsto B_+^{-1} P B_-^{-1} f \tag{15}$$

is bounded on L^2 (uniformly with respect to $\varphi \in W_s^1$).

5. The function a is continuous and its range lies in a compact subset of the right-half complex plane (independent of φ). Consequently a admits a Wiener-Hopf factorization $a = a_+ \cdot a_-$, where $a_+ = \exp(P \log a)$ and $a_- = \exp((I-P)\log a)$. Since $|\mathrm{Im}\, \log a| \le \gamma(\delta) < \pi/2$, Zygmund's lemma applies to estimate the norms of the factors a_+ and a_- in $L^{2+\varepsilon}$ (recall that $P = \frac{1}{2}(-iH + I + P_0)$, where H denotes the conjugation operator). The result is

$$\|a_+\|_{2+\varepsilon} \le C(\delta), \qquad \|a_-\|_{2+\varepsilon} \le C(\delta), \tag{16}$$

for some sufficiently small positive $\varepsilon = \varepsilon(\delta)$.

6. So far we have the factorization $(1/a)\, B = a_-^{-1} B_- B_+ a_+^{-1}$ almost everywhere on $\mathbb{T}$. Using (15) and (16), we get that the operator

$$H_+^{1/(r-1)} \to H_+^r \colon\ w \mapsto a_+ B_+^{-1} P B_-^{-1} a_- w \tag{17}$$

is bounded (uniformly with respect to φ).

In order to prove that (17) is the inverse of $\widehat{T}$, we remark that $\widehat{T} w \equiv P a_-^{-1} B_- B_+ a_+^{-1} w = f$ is equivalent to $B_+ a_+^{-1} w = P B_-^{-1} a_- w$ (note that $B_+ a_+^{-1} w \in H_+^{2-\varepsilon}$ with $\varepsilon > 0$), which implies that $w = a_+ B_+^{-1} P B_-^{-1} a_- f$ almost everywhere on $\mathbb{T}$. ■

We continue the construction of the fixed point equation. For any scalar complex valued function $\varphi \in W_s^1$, we denote by $\widetilde{\Phi}_+, \widetilde{\Phi}_-$ the solution of the associated transmission problem

$$\widetilde{\Phi}_+ = a\, \widetilde{\Phi}_- + b\, \overline{\widetilde{\Phi}_-} + c, \tag{18}$$

with a, b, and c from (8). With

$$\widehat{\Phi}_-(e^{i\tau}) := \int_0^\tau \widetilde{\Phi}_-(e^{i\sigma})\, \mathrm{d}\sigma, \qquad P_0 \widehat{\Phi}_- := \frac{1}{2\pi} \int_0^{2\pi} \widehat{\Phi}_-(e^{i\sigma})\, \mathrm{d}\sigma,$$

the operator $K\colon\ W_s^1 \to W_r^1$ is given by $K\varphi := \widehat{\Phi}_- - P_0 \widehat{\Phi}_-$. The definition of $\widehat{\Phi}_-$ makes sense since $P_0 \widetilde{\Phi}_- = 0$.

Lemma 2.

(i) *The operator $K\colon\ W_s^1 \to W_r^1$ is compact for any $r, s \in (1, \infty)$.*

(ii) *The image of $K\colon\ W_r^1 \to W_r^1$ is bounded if $r > 1$ is sufficiently small.*

(iii) *The pair $(\Phi_+, \Phi_-) \in W_r^1$ is a solution of the transmission problem (2) if and only if $K\Phi_- = \Phi_-$ and $\Phi_+ = G(., \Phi_-)$.*

Proof. 1. The embedding $W_s^1(\mathbb{T}) \to C(\mathbb{T})$ is compact, and hence (i) follows once it is shown that $K\colon\ C(\mathbb{T}) \to W_r^1(\mathbb{T})$ is continuous. The superposition operators $\varphi \mapsto a := \partial_z G(.,\varphi)$, $\varphi \mapsto b := \overline{\partial}_z G(.,\varphi)$, $\varphi \mapsto f := i\,t \cdot \partial_t G(.,\varphi)$ are continuous in $C(\mathbb{T})$, and thus the associated Toeplitz operators $T := P\widetilde{B}P$ with $\widetilde{B} := \frac{1}{a}JB$ with J and B from (13), and the 'adjoint Riemann-Hilbert operators' S from (12) depend continuously on φ. Since all these operators are invertible, the solutions in $H_\pm^r$ to the transmission problems (9) also depend continuously on φ (cf. (14)). Integrating these solutions along $\mathbb{T}$ proves the continuity of $K\colon\ C(\mathbb{T}) \to W_r^1(\mathbb{T})$.

2. If $r > 1$ is sufficiently small, then according to Lemma 1, the solutions $\widetilde{\Phi}_\pm$ of (9) are bounded in $H_\pm^r$ uniformly with respect to the choice of $\varphi \in W_r^1$, and hence the Kw are uniformly bounded in W_r^1.

3. Let $\Phi = (\Phi_+, \Phi_-) \in W_r^1$ be a solution of $\Phi_+ = G(.,\Phi_-)$. Differentiating this boundary relation with respect to the polar angle τ, we obtain that $\widetilde{\Phi} := \partial_\tau \Phi \equiv it\partial_t \Phi$ is a (unique) solution of the auxiliary transmission problem (9). Consequently,

$$K\Phi_-(e^{i\tau}) = const + \int_0^\tau \widetilde{\Phi}_-(e^{i\sigma})\ \mathrm{d}\sigma =$$

$$= const + \int_0^\tau \partial_\tau \Phi_-(e^{i\sigma})\,\mathrm{d}\sigma = const + \Phi_-(e^{i\tau}).$$

The constant on the right-hand side vanishes, since $P_0\,K\Phi_- = 0$ and $P_0\Phi_- = 0$.

4. Conversely, let $\Phi_- \in W_r^1$, $K\Phi_- = \Phi_-$, and $\Phi_+ := G(.,\Phi_-)$. We prove that Φ_+ and Φ_- are holomorphic in $\mathbb{D}$ and $\mathbb{E}$, respectively, and $P_0\Phi_- = 0$.

First of all, $\partial_\tau\Phi_- = \widetilde{\Phi}_-$. Since $\widetilde{\Phi}_-$ is holomorphic in $\mathbb{E}$, so is Φ_-. Further, $P_0\Phi_- = P_0\,K\Phi_- = 0$. Inserting $\widetilde{\Phi}_- = \partial_\tau\Phi_-$ into (9) shows that

$$\widetilde{\Phi}_+ = a\,\widetilde{\Phi}_- + b\,\overline{\widetilde{\Phi}_-} + c = \frac{\mathrm{d}}{\mathrm{d}\tau}\,G(.,\Phi_-) = \partial_\tau\Phi_+.$$

Consequently, Φ_+ is holomorphic in $\mathbb{D}$ and $\Phi := (\Phi_+, \Phi_-) \in W_r^1$ solves (2). ■

By virtue of Lemma 2, the existence result (i) of Theorem 1 is a consequence of Schauder's fixed-point principle.

It remains to prove that the solution of (2) is unique under the assumption (7). Let $\Phi^{(1)}, \Phi^{(2)} \in H_\pm^\infty \cap W_1^r$ be two solutions of (2). The difference $\Delta\Phi \equiv (\Delta\Phi_+, \Delta\Phi_-) := \Phi^{(2)} - \Phi^{(1)}$ solves the homogeneous linear transmission problem

$$\Delta\Phi_+ = a \cdot \Delta\Phi_- + b \cdot \overline{\Delta\Phi_-}, \tag{19}$$

where

$$a := \int_0^1 \partial_z G\big(.,\lambda\Phi_-^{(1)} + (1-\lambda)\Phi_-^{(2)}\big)\mathrm{d}\lambda, \quad b := \int_0^1 \overline{\partial}_z G\big(.,\lambda\Phi_-^{(1)} + (1-\lambda)\Phi_-^{(2)}\big)\mathrm{d}\lambda,$$

and $\Delta\Phi_-(\infty) = 0$. The assumption (7) on $\partial_z G$ and $\overline{\partial}_z G$ (with $g \equiv 0$) ensures that

$$\operatorname{Re} a - |b| \geq \delta > 0 \quad \text{on} \quad \mathbb{T}, \tag{20}$$

and hence (19) has only the trivial solution (cf. [7], Section 9.3).

We conjecture that the solution is unique even without the strengthened assumption (7).

3. Singular Targets

Now we will outline a framework for constructing analytic discs with boundaries in targets possessing isolated singular points. For simplicity, we consider only the case of two variables $n = 2$.

A germ X of a plane curve singularity [2] will be called almost real if all tangent planes outside the singular point are represented by totally real subspaces of $\mathbb{C}^2$. Examples of such situations arise quite naturally in symplectic geometry and singularity theory, suffice it to recall the so-called open Whitney umbrella [2]: $\mathbb{R}^2 \to \mathbb{R}^4$, $(x, y) \mapsto (x^2, y, xy, 2x^3/3)$.

An extensive list of examples is provided by suitable non-holomorphic perturbations of certain singularities.

Proposition 1. Any quasihomogeneous singularity in two variables may be embedded in $\mathbb{C}^2$ as an almost real plane curve singularity.

The proof runs as follows. One perturbs the equation of singularity by small scalar multiples of non-holomorphic monomials of sufficiently high degree (greater than the Milnor number) and observes that this does not change the singularity type [2]. Then using a Jacobian criterion of total reality from [11], it is easy to check that for a generic choice of deformation parameters, the corresponding Jacobian vanishes only at the origin, which means that it is an almost real germ.

It turns out that if a target manifold has such almost real isolated singularities, then one may often construct analytic discs attached to nearby nonsingular curves, in other words solve the corresponding nonlinear transmission problem. Recall that for a nearby smooth perturbation $\tilde{X}$ of a plane curve singularity X with the Milnor number μ (see [8]), its first (co)homology group $H^1(X, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^\mu$. Cycles forming a basis of this homology group are called vanishing cycles of X and are known to be representable by smoothly embedded circles.

Proposition 2. If X is an almost real embedding of an elementary catastrophe in the sense of R. Thom [2], then there always exist analytic discs spanning vanishing cycles of a sufficiently close smooth perturbation of X.

This result is proved by a direct verification but there is good evidence that similar results hold in much greater generality. In particular, we conjecture that nearby analytic discs exist for all quasihomogeneous plane curve singularities.

In conclusion, we remark that another way of obtaining nearby analytic discs is to guarantee existence of elliptic complex points in nearby smooth deformations, which may be done in terms of Maslov indices of vanishing cycles.

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