Mem. Differential Equations Math. Phys. 21(2000), 134–137

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ON LIAPUNOV STABILITY OF LINEAR SYSTEMS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

(Reported on March 6, 2000)

Consider the linear system of generalized ordinary differential equations

$$dx(t) = dA(t) \cdot p(t) \cdot x(t) + df(t), \tag{1}$$

where $A : [0, +\infty[\to \mathbb{R}^{n\times n}$ and $f : [0, +\infty[\to \mathbb{R}^n$ are, respectively, real matrix- and vector-functions with locally bounded variation components, and $p : [0, +\infty[\to \mathbb{R}^{n\times n}$ is a matrix-function, locally integrable with respect to A.

In this paper we give some sufficient conditions guaranteeing the stability with respect to, small perturbation in the Liapunov sense, of the system (1).

Before passing to the statement of the basic results, we give some notation and definitions.

$\mathbb{R} =]-\infty, +\infty[$ is the set of all real numbers, $[a, b]$ and $]a, b[$ are, respectively, closed and open intervals, $\mathbb{R}_+ = [0, +\infty[$.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$-matrices $x = (x_{ij})_{ij=1}^{n,m}$ with the norm

$$\|x\| = \max_{i=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|.$$

$\mathbb{R}_+^{n\times m} = \{(x_{ij})^{n,m} : x_{ij} \geq 0 \quad (i = 1,\dots,n;\ j = 1,\dots,m)\}$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$.

If $x \in \mathbb{R}^{n\times n}$, then x^{-1} and $\det(x)$ are, respectively, the inverse to x matrix and the determinant of x; I_n is the identity $n \times n$ matrix.

$\overset{d}{\underset{c}{V}} = \sup\{\overset{b}{\underset{a}{V}}(x) : c < a < b < d\}$, where $\overset{b}{\underset{a}{V}}(x)$ is the sum of total variations on the closed interval $[a, b]$ of the components x_{ij} $(i = 1,\dots,n;\ j = 1,\dots,m)$ of the matrix-function $x :]c, d[\to \mathbb{R}^{n\times m}$, $v(x)(t) = (v(x_{ij})(t))_{i,j=1}^n$, where $v(x_{ij})(t) = (\overset{t}{\underset{-\infty}{V}} x_{ij})$ for $t \in]c, d[$ $(i = 1,\dots,n;\ j = 1,\dots,m)$[1].

$x(t-)$ and $x(t+)$ are the left and the right limits of the matrix-function $x :]c, d[\to \mathbb{R}^{n\times m}$ at the point $t \in]c, d[$, $d_1x(t) = x(t) - x(t-)$, $d_2x(t) = x(t+) - x(t)$.

$BV_{\text{loc}}([0, +\infty[, \mathbb{R}^{n\times m})$ is the set of all real matrix-functions $x : [0, +\infty[\to \mathbb{R}^{n\times m}$ of bounded variation on every closed interval from $[0, +\infty[$.

If $g : [0, +\infty[\to \mathbb{R}$ is a nondecreasing function, $x : [0, +\infty[\to \mathbb{R}$ and $0 \leq s < t < +\infty$, then

$$\int_s^t x(\tau)\, dg(\tau) = \int_{]s,t[} x(\tau)\, dg_1(\tau) - \int_{]s,t[} x(\tau)\, dg_2(\tau) + \\ + \sum_{s<\tau\leq t} x(\tau)\, d_1g(\tau) - \sum_{s\leq\tau<t} x(\tau)\, d_2g(\tau),$$

2000 *Mathematics Subject Classification.* 34B05.

Key words and phrases. Stability in the Liapunov sense, linear systems of generalized ordinary differential equations.

[1] x_{ij} as a constant outside $[a, b]$ is assumed to be continuous.

where $g_j : [0,+\infty[\to \mathbb{R}$ $(j=1,2)$ are continuous nondecreasing functions such that the function $g_1 - g_2$ is identically equal to the continuous part of g, and $\int_{]s,t[} dg_j(\tau)$ is the Lebesgue-Stieltjes integral over the open interval $]s,t[$ with respect to the measure corresponding to the function $g_j (j=1,2)$, (if $s=t$, then $\int_s^t x(\tau)dg(\tau)=0$);

$L_{\rm loc}([0,+\infty[,\ \ell; g)$ is the set of all real functions $x : [0,+\infty[\to \mathbb{R}$ $\mu(g)$-measurable (i.e., measurable with respect to measures $\mu(g_1)$ and $\mu(g_2)$) and integrable on the closed interval $[0,b]$ for every $b\in[0,+\infty[$.

A matrix-function is said to be nondecreasing if each of its components is such.

If $G = (g_{ik})_{i,k}^{\ell,m} : [0,+\infty[\to \mathbb{R}^{\ell\times n}$ is a nondecreasing matrix-function, then $L([0,+\infty[,\mathbb{R}^{n\times n})$ is the set of all matrix-functions $x=(x_{kj})_{k,j}^{n,m} : [0,+\infty[\to \mathbb{R}^{n\times m}$ such that $x_{kj}\in L([0,+\infty[,,\mathbb{R}; g_{ik})$ $(i=1,\dots,\ell,; k=1,\dots,n; j=1,\dots,m)$

$$\int_s^t dG(\tau)\cdot x(\tau) = \Big(\sum_{k=1}^n \int_s^t x_{kj}(\tau)\, dg_{ik}(\tau)\Big)_{i,j=1}^{\ell,m} \quad \text{for } 0\le s\le t<+\infty.$$

If $G_j : [0,+\infty[\to \mathbb{R}^{\ell\times n}$ $(j=1,2)$ are nondecreasing matrix-functions, $G\equiv G_1-G_2$ and $x : [0,+\infty[\to \mathbb{R}^{n\times m}$, then

$$\int_s^t dG(\tau)\cdot x(\tau) = \int_s^t dG_1(\tau)\cdot x(\tau) - \int_s^t dG_2(\tau)\cdot x(\tau) \quad \text{for } 0\le s\le t<+\infty;$$

$$L([0,+\infty[,\ \mathbb{R}^{n\times m}; G) = \bigcap_{j=1}^{2} L([0,+\infty[,\ \mathbb{R}^{n\times m}; G_j).$$

$r(H)$ is the spectral radius of the matrix $H\times\mathbb{R}^{n\times n}$.

Under a solution of the system (1) is understood a vector-function $x\in BV_{\rm loc}([0,+\infty[,\mathbb{R}^n)$ such that

$$x(t)-x(s) = \int_s^t dA(\tau)\cdot p(\tau)\cdot x(\tau) + f(t) - f(s) \quad \text{for } 0\le s\le t<+\infty.$$

We will assume that $f\in BV_{\rm loc}([0,+\infty[;\mathbb{R}^n)$; $A\in BV_{\rm loc}([0,+\infty[,\mathbb{R}^{n\times n}$ and $p\in L_{\rm loc}([0,+\infty[,\mathbb{R}^{n\times n}, A)$ are such that

$$\det(I_n + (-1)^j\, d_jA(t)\cdot p(t)) \ne 0 \quad \text{for } t\in\mathbb{R}_+ \ (j=1,2). \tag{2}$$

Let $x_0\in BV_{\rm loc}([0,,+\infty[,\mathbb{R}^n)$ be a solution of the system (1).

Definition 1. Let $\xi : \mathbb{R}_+\to\mathbb{R}_+$ be a nondecreasing function such that

$$\lim_{t\to+\infty}\xi(t) = +\infty.$$

The solution x_0 of the system (1) is called *ξ-exponentially asymptotically stable* if there exists a positive number η such that for every $\varepsilon>0$ there exists a positive number $\delta=\delta(\varepsilon)$ such that an arbitrary solution x of the system (1) satisfying the inequality

$$\|x(t_0)-x_0(t_0)\| < \delta$$

for some $t_0\in\mathbb{R}_+$ admits the estimate

$$\|x(t)-x_0(t_0)\| < \varepsilon\exp(-\eta(\xi(t)-\xi(t_0)) \quad \text{for } t\ge t_0.$$

Stability, uniform stability and asymptotic stability of the solution x_0 are defined just in the same way as for systems of ordinary differential equations (see, e.g., [1] or [2]), i.e., in the case, where $A(t)$ is the diagonal matrix-function with diagonal elements equal to t). Note that exponential asymptotic stability ([1], [2]) is a particular case of ξ-exponential asymptotic stability ($\xi(t)\equiv t$).

Definition 2. The system (1) is called *stable (uniformly stable, asymptotically stable, ξ-exponentially asymptotically stable)* if every solution of this system is stable (uniformly stable, asymptotically stable, ξ-exponentially asymptotically stable).

Alongside with the system (1) we consider the corresponding homogeneous system

$$dx(t) = dA(t) \cdot p(t) \cdot x(t). \tag{1_0}$$

Proposition 1. *The system* (1) *is stable* (*uniformly stable, asymptotically stable, ξ-exponentially asymptotically stable*) *if and only if the zero solution of the system* (1_0) *is stable* (*uniformly stable, asymptotically stable, ξ-exponentially asymptotically stable*).

Proposition 2. *The system* (1) *is stable* (*uniformly stable, asymptotically stable, ξ-exponentially asymptotically stable*) *if and only if some solution of that system is stable* (*uniformly stable, asymptotically stable, ξ-exponentially asymptotically stable*).

Therefore the stability (in all senses) of the system (1) is the property of the matrix-functions A and p.

Definition 3. A pair (A, p) of matrix-functions $A \in BV_{\text{loc}}([0, +\infty[, \mathbb{R}^{n\times n})$ and $p \in ([0, +\infty[, \mathbb{R}^{n\times n}\ A)$ satisfying the condition (2) is called *stable (uniformly stable, asymptotically stable, ξ-exponentially asymptotically stable)* if the system (1) is stable, (uniformly stable, asymptotically stable, ξ-exponentially asymptotically stable).

Now we formulate the basic lemma which will be applied in proving theorems below.

Lemma 1. *Let the condition* (2) *hold. Moreover, let the matrix-functions $A_0 \in BV_{\text{loc}}([0, +\infty[, \mathbb{R}^{n+\times n})$ and $p_0 \in L_{\text{loc}}([0, +\infty[, \mathbb{R}^{n+\times n}, A_0)$ be such that the following conditions are valid:*

$$(a)\, \det(I_n + (-1)^j\, dj A_0(t) \cdot p(t)) \neq 0 \ \ for \ \ t \in \mathbb{R}_+ \ \ (j = 1, 2); \tag{3}$$

(*b*) *for some $t_0 \in \mathbb{R}_+$, the Cauchy matrix u_0 of the system*

$$dx(t) = dA_0(t) \cdot p_0(t) \cdot x(t)$$

satisfies the inequality

$$|u(t, t_0)| \leq \Omega e^{-\xi(t) + \xi(t_0)} \ \ for \ \ t \geq t_0,$$

where $\Omega \in \mathbb{R}_+^{n\times n}$, and $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function satisfying (3);

(*c*) *there exists a matrix $H \in \mathbb{R}_+^{n\times n}$ such that $r(H) < 1$ and*

$$\int_{t_0}^{t} e^{\xi(t) - \xi(\tau)} |u(t, \tau)| dV(B)(\tau) < H \ \ for \ \ t \geq t_0,$$

where

$$B(A, p, A_0, p_0)(t) \equiv \int_0^t dA(\tau) \cdot p(\tau) - \int_0^t dA_0(\tau) \cdot p_0(t) +$$
$$+ \sum_{0<\tau\leq t} d_1 A_0(\tau) \cdot p_0(\tau) \big(I_n - d_1 A_0(\tau) \cdot p_0)\big)^{-1} \big(d_1 A(\tau) \cdot p(\tau) - d_1 A_0(\tau) \cdot p_0(\tau)\big) -$$
$$- \sum_{0\leq\tau<t} d_2 A_0(\tau) \cdot p_0(\tau) \big(I_n + d_2 A_0(\tau) \cdot p_0(\tau)\big)^{-1} \big(d_2 A(\tau) \cdot p(\tau) - d_2 A_0(\tau) \cdot p_0(\tau)\big).$$

Then an arbitrary solution x of the system (1_0) *admits the estimate*

$$|x(t)| \leq Q |x(t_0)| e^{-\xi(t) + \xi(t_0)} \ \ for \ \ t \geq t_0,$$

where $Q(I_n - H)^{-1}\Omega$.

Theorem 1. *Let the conditions* (2) *and* (4) *hold, where* $A,\ A_0 \in BV_{\rm loc}([0,+\infty[,\mathbb{R}^{n\times n})$, $p \in L_{\rm loc}([0,+\infty[,\mathbb{R}^{n\times n};A)$ *and* $p_0 \in L_{\rm loc}([0,+\infty[,\mathbb{R}^{n\times n};A_0)$. *Moreover, let the pair* (A_0,p_0) *be uniformly stable and*

$$\mathop{V}_{0}^{+\infty}(B) < +\infty, \tag{4}$$

where the matrix-function $B(A,p;\,A_0,p_0):\mathbb{R}_+\to\mathbb{R}^{n\times n}$ *is defined by* (5). *Then the pair* (A,p) *is uniformly stable as well.*

Theorem 2. *Let the conditions* (2) *and* (4) *hold, where* $A,A_0 \in BV_{\rm loc}([0,+\infty[,\mathbb{R}^{n\times n})$, $p \in L_{\rm loc}([0,+\infty[,\mathbb{R}^{n\times n};A)$ *and* $p_0 \in L_{\rm loc}([0,+\infty[,\mathbb{R}^{n\times n};A_0)$. *Moreover, let the pair* (A_0,p_0) *be* ξ*-exponentially asymptotically stable and the condition*

$$\lim_{t\to+\infty}\mathop{V}_{t}^{\nu(\xi)(t)} B = 0$$

hold, where $\xi:\mathbb{R}_+\to\mathbb{R}_+$ *is a nondecreasing function satisfying* (3),

$$\nu(\xi)(t) = \sup\{\tau \ge t : \xi(\tau) \le \xi(t)+1\},$$

and the matrix-function $B(A,p;A_0,p_0):\mathbb{R}_+\to\mathbb{R}^{n\times n}$ *is defined by* (5). *Then the pair* (A,p) *is* ξ*-exponentially asymptotically stable as well.*

Remark 1. If the pair (A,p) is ξ-exponentially asymptotically stable and the conditions (2) and

$$\lim_{t\to+\infty}\mathop{V}_{t}^{\nu(\xi)(t)} \widetilde{B} = 0,$$

hold, where

$$\widetilde{B}(A,p,f) \equiv f(t) + \sum_{0<\tau\le t} d_1A(\tau)\cdot p(\tau)(I_n - d_1A(\tau)\cdot p(\tau))^{-1}\cdot d_1f(\tau) -$$
$$- \sum_{0\le\tau<t} d_2A(\tau)\cdot p(\tau)(I_n + d_2A(\tau)\cdot p(\tau))^{-1}\cdot d_2f(\tau)$$

and the function $\nu(\xi):\mathbb{R}_+\to\mathbb{R}_+$ is defined as in Theorem 2, then an arbitrary solution x of the system (1) satisfies the condition

$$\lim_{t\to+\infty}\|x(t)\| = 0.$$

Analogous results were obtained in [2] for linear systems of ordinary differential equations.

References

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