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ON PARTIALLY IRREGULAR ALMOST PERIODIC SOLUTIONS OF DIFFERENTIAL SYSTEMS WITH DIAGONAL RIGHT-HAND SIDE

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Let D be a compact subset of $\mathbb{R}^n$. Consider the system

$$\dot{x} = f(t, x), \quad t \in \mathbb{R}, \quad x \in D, \tag{1}$$

where the vector valued function $f(t, x)$ is continuous on $\mathbb{R} \times D$ and almost periodic in t uniformly for $x \in D$. By $\mathrm{mod}(f)$ we denote a frequency module of $f(t, x)$, i.e. $\mathrm{mod}(f)$ is the smallest additive group of real numbers that contains all Fourier exponents of $f(t, x)$.

The existence problem of almost periodic solutions to (1) is a significant problem of qualitative theory of ordinary differential equations. Many authors have investigated this problem. Most of them considered only the regular solutions $x(t)$, i.e the solutions with $\mathrm{mod}(x) \subset \mathrm{mod}(f)$ (see e.g. [1 – 7]). However, there can be various relations between $\mathrm{mod}(x)$ and $\mathrm{mod}(f)$. In [8] J.Kurzweil and O.Veivoda have shown that there exists a system (1) having an almost periodic solution $x(t)$ such that $\mathrm{mod}(x) \cap \mathrm{mod}(f) = \{0\}$. We say that such solutions are irregular. In [9, 10] we have obtained necessary and sufficient conditions for existence of irregular almost periodic solutions to (1). In [11] we have shown that some classes of quasiperiodic systems admit quasiperiodic solutions that have some of right part frequencies. It is interesting to investigate similar phenomena for almost periodic systems.

Definition. Let $\mathrm{mod}(f)$ be the frequency module of the right part of system (1) and $\mathrm{mod}(f) = L_1 \oplus L_2$. An almost periodic solution $x(t)$ of the system (1) is called irregular with respect to L_2 (or partially irregular) if $(\mathrm{mod}(x) + L_1) \cap L_2 = \{0\}$.

In [12] regular almost periodic solutions of the system (1) with $f(t, x) = X(t, x) + Y(t, x)$ are considered. In [13, 14] we have obtained necessary and sufficient conditions for existence of almost periodic irregular with respect to $\mathrm{mod}(Y)$ solutions of such systems with $\mathrm{mod}(X) \cap \mathrm{mod}(Y) = \{0\}$.

Let $F(t_1, t_2, x)$ be a continuous on $\mathbb{R}^2 \times D$ vector valued function. We assume that $F(t_1, t_2, x)$ is almost periodic in t_j $(j = 1, 2)$ uniformly for the rest of the arguments and L_j is the module of $F(t_1, t_2, x)$ with respect to t_j $(j = 1, 2)$. In the sequel we will suppose that

$$f(t, x) \equiv F(t, t, x), \quad \mathrm{mod}(f) = L_1 \oplus L_2. \tag{2}$$

Note that similar systems are studied in [15, 16].

The aim of this paper is to establish the existence conditions for partially irregular almost periodic solutions of the system (1), where $f(t, x)$ is represented in the form (2).

Following [15], we define the mean value of $f(t, x)$ with respect to the module L_2 by

$$\hat{f}_{L_2}(t, x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T F(t, \tau, x) d\tau.$$

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Now let us consider the system

$$\dot{x} = \hat{f}_{L_2}(t,x), \quad f(t,x) - \hat{f}_{L_2}(t,x) = 0. \tag{3}$$

Theorem. *Suppose that* (2) *holds and a function* $x(t)$ *is an almost periodic solution of* (1). *The solutions* $x(t)$ *is irregular with respect to* L_2 *iff* $x(t)$ *is a solution of* (3).

Proof. Suppose that $x(t)$ is an almost periodic solution to (1) and $(\text{mod}(x)+L_1)\cap L_2 = \{0\}$. Let $N^{(2)} = \{\nu_1^{(2)}, \nu_2^{(2)}, \dots\}$ be the frequency set of $F(t_1,t_2,x)$ with respect to t_2. By (2), the frequency set of $f(t,x)$ contains $N^{(2)}$ and the module L_2 is generated by $N^{(2)}$. Let

$$f(t,x) - \hat{f}_{L_2}(t,x) \sim \sum_{k,\ \nu_k^{(2)}\neq 0} a_k(t,x)\exp(i\nu_k^{(2)}t) \tag{4}$$

be the Fourier-series expansion of $f(t,x)$ with respect to module L_2. Then

$$a_k(t,x) = \lim_{T\to\infty}\frac{1}{T}\int_0^T F(t,\tau,x)\exp(-i\nu_k^{(2)}\tau)d\tau \quad (k=1,2,\dots;\ \nu_k^{(2)}\neq 0).$$

It follows from [1, p. 30] that $\hat{f}_{L_2}(t,x)$ and $a_k(t,x)$ $(k=1,2,\dots)$ are almost periodic in t uniformly for $x\in D$. By [1, p. 27], the functions $f_{L_2}^x = \hat{f}_{L_2}(t,x(t))$, and $a_k^x = a_k(t,x(t))$ $(k=1,2,\dots)$ are almost periodic and $\text{mod}(\hat{f}_{L_2}^x)\subset(L_1+\text{mod}(x))$, $\text{mod}(a_k^x)\subset(L_1+\text{mod}(x))$ $(k=1,2,\dots)$. Let $\{\mu_1,\mu_2,\dots\}$ be a frequency set of $a_k(t,x(t))$ $(k=1,2,\dots)$. Then we have

$$a_k(t,x(t)) \sim \sum_m a_{km}\exp(i\mu_m t), \tag{5}$$

where

$$a_{km} = \lim_{T\to\infty}\frac{1}{T}\int_0^T a_k(x(\tau))\exp(-i\mu_m\tau)d\tau, \quad (\mu_m\in L_1;\ \ k,m=1,2,\dots).$$

It follows from (4) and (5) that

$$f(t,x(t)) - \hat{f}_{L_2}(t,x(t)) \sim \sum_{k,\ \nu_k^{(2)}\neq 0}\sum_m a_{km}\exp(i(\nu_k^{(2)}+\mu_m)t).$$

Put $-\dot{x}(t) + \hat{f}_{L_2}(t,x(t)) \equiv a_0(t)$. It is clear that $a_0(t)$ is almost periodic and $\text{mod}(a_0)\subset(\text{mod}(x)+L_1)$. Let $\{\tilde{\mu}_1,\tilde{\mu}_2,\dots\}$ be the frequency set of $a_0(t)$. Then we can write

$$a_0(t)\sim\sum_s a_{0s}\exp(i\tilde{\mu}_s t)dt, \quad a_{0s} = \lim_{T\to\infty}\frac{1}{T}\int_0^T f_0(t)\exp(-i\tilde{\mu}_s t)dt.$$

Since $x(t)$ is a solution to (2), we have

$$0\equiv a_0(t) + \hat{f}_{L_2}(t,x(t)) + f(t,x(t)) \sim$$
$$\sim\sum_s a_{0s}\exp(i\tilde{\mu}_s t) + \sum_{k,\ \nu_k^{(2)}\neq 0}\sum_m a_{km}\exp((i(\nu_k^{(2)}+\mu_m)t). \tag{6}$$

Since $\text{mod}(a_r)\cap L_2 = \{0\}$ $(r=0,1,\dots)$, we have $\tilde{\mu}_s\neq\nu_k^{(2)}+\mu_m$ $(\nu_k^{(2)}\neq\neq 0;\ s,k,m=1,2,\dots)$. Hence, all the Fourier coefficients in (6) are equal to zero. By the uniqueness theorem for almost periodic functions, we obtain $a_0(t)\equiv 0$, $f(t,x(t)) - \hat{f}_{L_2}(t,x(t))\equiv 0$. This implies that $x(t)$ satisfies (3).

Conversely, let $x(t)$ be an almost periodic irregular with respect to L_2 solution of the system (3). Then $f(t,x(t)) - \hat{f}(t,x(t))\equiv 0$. Hence, $x(t)$ satisfies (1). This completes the proof. □

Corollary 1. *Thes* (1) *has an irregular with respect to* L_2 *almost periodic solution* $x(t)$ *iff* $x(t)$ *satisfies the system*

$$\dot{x} = F(t, \tau, x)$$

for each $\tau \in \mathbb{R}$.

Corollary 2. *A function* $x(t)$ *is an irregular with respect to* L_2 *almost periodic solution of system* (1) *iff* $x(t)$ *satisfies the conditions*

$$\dot{x} = F(t, t_0, x), \quad f(t, x) - F(t, t_0, x) = 0$$

for some $t_0 \in \mathbb{R}$.

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