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# NECESSARY CONDITIONS OF OPTIMALITY FOR AN OPTIMAL PROBLEM WITH VARIABLE DELAYS AND WITH A CONTINUOUS CONDITION

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Necessary conditions of optimality in the form of an integral maximum principle are obtained for an initial function and control. The condition at the optimal initial moment, unlike the early known condition [1], contains new additional terms. The continuity of a condition means that at the initial moment of time the value of the initial function always coincides with the value of the trajectory.

Let $J = [a\ ,b]$ be a finite interval, $O \subset R^n$, $G \subset R^r$, be open sets and let the function $f : J \times O^s \times G^\nu \to R^n$ satisfy the following conditions:

1. For a fixed $t \in J$ the function $f(t,\ x_1, \dots, x_s,\ u_1, \dots, u_\nu)$ is continuous with respect to $(x_1, \dots, x_s,\ u_1, \dots, u_\nu) \in O^s \times G^\nu$ and continuously differentiable with respect to $(x_1, \dots, x_s) \in O^s$;

2. For a fixed $(x_1, \dots, x_s,\ u_1, \dots, u_\nu) \in O^s \times G^\nu$ the functions $f,\ f_{x_i},\ i = 1, \dots, s$, are measurable with respect to $t$. For arbitary compacts $K \subset O$, $V \subset G$ there exists a function $m_{K,V}(\cdot) \in L(J,\ R_0^+)$, $R_0^+ = [0,\ \infty)$, such that

$$| f(t,\ x_1, \dots, x_s,\ u_1, \dots, u_\nu) | + \sum_{i=1}^{s} | f_{x_i}(\cdot) | \leq m_{K,V}(t)$$
$$\forall (t,\ x_1, \dots, x_s,\ u_1, \dots, u_\nu) \in J \times K^s \times V^\nu.$$

Let now $\tau_i(t)$, $i = 1, \dots, s$, $t \in J$, be absolutely continuous functions satisfying the conditions: $\tau_i(t) \leq t$, $\dot{\tau}_i(t) > 0$; $\Phi$ be a set of continuous functions $\varphi :\ \ J_1 = [\tau,\ b] \to M$, $\tau = \min(\tau_1(a), \dots, \tau_s(a))$, with $M \subset O$ convex bounded. Let the functions $\theta_i(t)$, $i = 1, \dots, \nu$, $t \in R$ satisfy commeasurability condition, i. e., let there exist absolutely continuous function $\theta(t) < t$, $\dot{\theta}(t) > 0$, such that $\theta_i(t) = \theta^{k_i}(t)$, where $k_\nu > \dots > k_1 \geq 0$ are natural numbers, $\theta^i(t) = \theta(\theta^{i-1}(t))$, $\theta^0(t) = t$; $\Omega$ be a set of measurable functions $u : J_2 = [\theta,\ b] \to U$, $\theta = \theta_\nu(a)$, such that $\mathrm{cl}\{u(t) :\ \ t \in J_2\}$ is compact in $G$; let $U \subset G$ be an arbitrary set; $q^i(t_0,\ t_1,\ x_0,\ x_1)$, $i = 0, \dots, l$, $(t_0,\ t_1,\ x_0,\ x_1) \in J^2 \times O^2$ be continuously differentiable scalar functions.

We consider the differential equation in $R^n$

$$\dot{x}(t) = f(t,\ x(\tau_1(t)), \dots, x(\tau_s(t)),\ u(\theta_1(t)), \dots, u(\theta_\nu(t)),\ \ t \in [t_0,\ t_1] \subset J, \tag{1}$$

with the continuous condition

$$x(t) = \varphi(t), \quad t \in [\tau,\ t_0]. \tag{2}$$

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**Definition 1.** The function $x(t) = x(t,\ \sigma) \in O$, $\sigma = (t_0,\ t_1,\ \varphi(\cdot)\ u(\cdot)) \in A = J_2 \times \Phi \times \Omega$, $t_0 < t_1$, defined on the interval $[\tau\ , t_1]$, is said to be a solution corresponding to the element $\sigma \in A$ if on the interval $[\tau\ , t_0]$ it satisfies the condition (2), while on the interval $[t_0\ , t_1]$ the trajectory $x(t)$ is absolutely continuous and almost everywhere satisfies the equation (1).

**Definition 2.** The element $\sigma \in A$ is said to be admissible if the corresponding soluton $x(t)$ satisfies the conditions

$$q^i(t_0,\ t_1,\ x(t_0),\ x(t_1)) = 0, \quad i = 1, \ldots, l.$$

The set of admissible elements will be denoted by $A_0$.

**Definition 3.** The elemant $\tilde{\sigma} = (\tilde{t_0},\ \tilde{t_1},\ \tilde{\varphi}(\cdot),\ \tilde{u}(\cdot)) \in A_0$ is said to be optimal if for an arbitrary element $\sigma \in A_0$ the inequality

$$q^0(\tilde{t_0},\ \tilde{t_1},\ \tilde{x}(\tilde{t_0}),\ \tilde{x}(\tilde{t_1})) \le q^0(t_0,\ t_1,\ x(t_0),\ x(t_1)), \quad \tilde{x}(t) = x(t,\ \tilde{\sigma}),$$

holds.

The problem of optimal control consists in finding an optimal element.

**Theorem 1.** *Let $\tilde{\sigma} \in A_0$ be an optimal element, $\tilde{t_0} \in (a\ , b)$, $\tilde{t_1} \in (a\ , b]$ and the following conditions hold:*

a) *the function $\tilde{\varphi}(t)$ is absolutely continuous in some left semi-neighborhood of the point $\tilde{t_0}$;*

b) *there exist the finite limits: $\dot{\varphi}^- = \dot{\tilde{\varphi}}(\tilde{t}_0-)$,*

$$\begin{cases} \lim\limits_{\omega \to \omega_0} \tilde{f}(\omega) = f_0^-, \quad \omega \in R_{\tilde{t_0}}^- \times O^s, \quad \omega_0 = (\tilde{t_0},\ \tilde{x}(\tau_1(\tilde{t_0})), \ldots, \tilde{x}(\tau_s(\tilde{t_0})), \\ \tilde{f}(\omega) = \tilde{f}(t,\ x_1, \ldots, x_s) = f(t,\ x_1, \ldots, x_s, \tilde{u}(\theta_1(t), \ldots, \tilde{u}(\theta_\nu(t)) \end{cases}.$$

c) *there exist the finite limit*

$$\lim_{\omega \to \omega_0} \tilde{f}(\omega) = f_1^-, \quad \omega \in R_{\tilde{t_0}}^- \times O^s, \quad \omega_1 = (\tilde{t_1},\ \tilde{x}(\tau_1(\tilde{t_1})), \ldots, \tilde{x}(\tau_s(\tilde{t_1})).$$

*Then there exists a non-zero vector $\pi = (\pi_0, \ldots, \pi_s)$, $\pi_0 \le 0$, and a solution $\psi(t)$, $t \in [\tilde{t_0},\ \gamma]$, $\gamma = \max(\gamma_1(b), \ldots, \gamma_s(b))$, $\gamma_i(t) = \gamma_i^{-1}(t)$, of the equation*

$$\dot{\psi}(t) = -\sum_{i=1}^{s} \psi(\gamma_i(t)) \tilde{f}_{x_i}[\gamma_i(t)] \dot{\gamma}_i(t), \quad t \in [\tilde{t}_0,\ \tilde{t}_1], \tag{3}$$

$$\psi(\tilde{t}_1) = \pi \frac{\delta \tilde{Q}}{\delta x_1}, \quad \psi(t) = 0, \quad t \in (\tilde{t}_1,\ \gamma],$$

*such that the following conditions are fulfelled:*

$$\sum_{i=1}^{s} \int\limits_{\tau_i(\tilde{t}_0)}^{\tilde{t}_0} \psi(t) \tilde{f}_{x_i}[\gamma_i(t)] \dot{\gamma}_i(t) \tilde{\varphi}(t) dt \ge \sum_{i=1}^{s} \int\limits_{\tau_i(\tilde{t}_0)}^{\tilde{t}_0} \psi(t) \tilde{f}_{x_i}[\gamma_i(t)] \dot{\gamma}_i(t) \varphi(t) dt;$$

$$\forall \varphi(\cdot) \in \Delta; \tag{4}$$

$$(\pi \tilde{Q}_{x_0} + \psi(\tilde{t}_0)) \tilde{\varphi}(\tilde{t}_0) \ge (\pi \tilde{Q}_{x_0} + \psi(\tilde{t}_0)) \varphi, \quad \forall \varphi \in M; \tag{5}$$

$$\int\limits_{\tilde{t}_0}^{\tilde{t}_1} \psi(t) \tilde{f}[t] dt \ge \int\limits_{\tilde{t}_0}^{\tilde{t}_1} \psi(t) f(t,\ \tilde{x}(\tau_1(t)), \ldots, \tilde{x}(\tau_s(t)), u(\theta_1(t)), \ldots, u(\theta_\nu(t)) dt,$$

$$\forall u(\cdot) \in \Omega; \tag{6}$$

$$\pi\tilde{Q}_{t_1} \geq -\psi(\tilde{t}_1)f_1^-; \tag{7}$$

$$\pi(\tilde{Q}_{t_0} + \tilde{Q}_{x_0}\dot{\varphi}^-) \geq \psi(\tilde{t}_0)(f_0^- - \dot{\varphi}^-). \tag{8}$$

*Here*

$$\tilde{f}[t] = \tilde{f}(t,\ \tilde{x}(\tau_1(t)), \dots, \tilde{x}(\tau_s(t))), \quad \tilde{f}_{x_i}[t] = \tilde{f}_{x_i}(t,\ \tilde{x}(\tau_1(t)), \dots, \tilde{x}(\tau_s(t)))$$

*and the tilde over* $Q = (q^0, \dots, q^l)^T$ *means that the corresponding gradient is calculated at the point* $(\tilde{t}_0,\ \tilde{t}_1,\ \tilde{x}(\tilde{t}_0),\ \tilde{x}(\tilde{t}_1))$.

**Theorem 2.** *Let* $\tilde{\sigma} \in A_0$ *be an optimal element,* $\tilde{t}_0 \in [a,\ b)$, $\tilde{t}_1 \in (a,\ b)$ *and the following condition hold:*

d) *the function* $\tilde{\varphi}(t)$ *is absolutely continuous in some right semi-neighborhood of the point* $t_0$*;*

e) *there exist the finite limits:* $\dot{\varphi}^+ = \dot{\tilde{\varphi}}(\tilde{t}_0+)$,

$$\lim_{\omega \to \omega_0} \tilde{f}(\omega) = f_0^+, \quad \omega \in R^+_{\tilde{t}_0} \times O^s;$$

f) *there exist the finite limit*

$$\lim_{\omega \to \omega_1} \tilde{f}(\omega) = f_1^+, \quad \omega \in R^+_{\tilde{t}_1} \times O^s.$$

*Then there exists a non-zero vector* $\pi = (\pi_0, \dots, \pi_s)$, $\pi_0 \leq 0$, *and a solution* $\psi(t)$ *of the equation* (3) *such that* (4)–(6) *are fulfilled. Moreover*

$$\pi\tilde{Q}_{t_1} \leq -\psi(\tilde{t}_1)f_1^+, \tag{9}$$

$$\pi(\tilde{Q}_{t_0} + \tilde{Q}_{x_0}\dot{\varphi}^+ \leq \psi(\tilde{t}_0)(f_0{}^+ - \dot{\varphi}^+). \tag{10}$$

**Theorem 3.** *Let* $\tilde{\sigma} \in A_0$ *be an optimal element,* $\tilde{t}_0,\ \tilde{t}_1 \in (a,\ b)$ *and the assumptions of Theorems* 1, 2 *hold. Let, besides*

g) $\dot{\varphi}^- = \dot{\varphi}^+ = \dot{\varphi}$, $f^-{}_0 = f^+{}_0 = f_0$;

h) $f^-{}_1 = f^+{}_1 = f_1$.

*Then there exsit a non-zero vector* $\pi = (\pi_0, \dots, \pi_l)$, $\pi_0 \leq 0$, *and a solution* $\psi(t)$ *of the equation* (3) *such that* (5)–(6) *are fulfilled. Moreover*

$$\pi\tilde{Q}_{t_1} = -\psi(\tilde{t}_1)f_1, \tag{11}$$

$$\pi(\tilde{Q}_{t_0} + \tilde{Q}_{x_0}\dot{\varphi}) = \psi(\tilde{t}_0)(f_0 - \dot{\varphi}). \tag{12}$$

If

$$\operatorname{rank}(\tilde{Q}_{t_0} + \tilde{Q}_{x_0}\dot{\varphi},\ \tilde{Q}_{t_1},\ \tilde{Q}_{x_1}) = 1 + l,$$

then in Theorem 3 $\psi(t) \not\equiv 0$.

If $\tilde{\varphi}(\tilde{t}_0) \in \operatorname{int} M$, then from (5), (12) it follows

$$\begin{cases} \pi\tilde{Q}_{x_0} = -\psi(\tilde{t}_0), \\ \pi\tilde{Q}_{t_0} = \psi(\tilde{t}_0)f_0. \end{cases} \tag{13}$$

The condition (13) in the case $\tau_i(t) = t - \tau_i$, $i = 1, \dots, s$, $\tau_s > \dots > \tau_1 = 0$, $\theta_i(t) = t - m_i h$, $i = 1, \dots, \nu$, where $m_1 = 0$, $h > 0$, while $m_\nu > \dots > m_2$ are natural numbers, is obtaind in [1].

**Theorem 4.** *Let* $\tilde{\sigma} \in A_0$ *be an optimal element,* $\tilde{t}_0,\ \tilde{t}_1 \in (a\ ,b)$ *and the conditions* a), b), f) *hold. Then there exists* $\pi = (\pi_0, \dots, \pi_l) \neq 0$, $\pi_0 \leq 0$, *and a solution* $\psi(t)$ *of the equation* (3) *such that* (4)–(6), (8), (10) *are fulfilled.*

**Theorem 5.** *Let $\tilde{\sigma} \in A_0$ be an optimal element, $\tilde{t}_0$, $\tilde{t}_1 \in (a\ ,b)$ and the conditions* a), b), h) *hold. Then there exists $\pi = (\pi_0, \dots, \pi_l) \neq 0$, $\pi_0 \leq 0$, and a solution $\psi(t)$ of the equation* (3) *such that* (4)–(6), (8), (11) *are fulfilled.*

**Theorem 6.** *Let $\tilde{\sigma} \in A_0$ be an optimal element, $\tilde{t}_0 \in [a,\ b)$, $\tilde{t}_1 \in (a\ ,b]$ and the conditions* d), e), c) *hold. Then there exists $\pi = (\pi_0, \dots, \pi_l) \neq 0$, $\pi_0 \leq 0$, and a solution of the equation* (3) *such that* (4)–(6), (7), (10) *are fulfilled.*

**Theorem 7.** *Let $\tilde{\sigma} \in A_0$ be an optimal element, $\tilde{t}_0 \in [a,\ b)$, $\tilde{t}_1 \in (a\ ,b)$ and the conditions* d), e), h) *hold. Then there exists $\pi = (\pi_0, \dots, \pi_l) \neq 0$, $\pi_0 \leq 0$, and a solution $\psi(t)$ of the equation* (3) *such that* (4)–(6), (10), (11) *are fulfilled.*

**Theorem 8.** *Let $\tilde{\sigma} \in A_0$ be an optimal element, $\tilde{t}_0 \in (a,\ b)$, $\tilde{t}_1 \in (a\ ,b]$ and the conditions* a), b) c), d), e), g) *hold. Then there exists $\pi = (\pi_0, \dots, \pi_l) \neq 0$, $\pi_0 \leq 0$, and a solution $\psi(t)$ of the equation* (3) *such that* (4)–(6), (7), (12) *are fulfilled.*

**Theorem 9.** *Let $\tilde{\sigma} \in A_0$ be an optimal element, $\tilde{t}_0$, $\tilde{t}_1 \in (a\ ,b)$ and the conditions* a), b) d), e), f), g) *hold. Then there exists $\pi = (\pi_0, \dots, \pi_l) \neq 0$, $\pi_0 \leq 0$, and a solution $\psi(t)$ of the equation* (3) *such that* (4)–(6), (9), (12) *ARE fulfilled.*

Finally we note that optimal problems with a discontinuous initial condition, i. e., when, generally speakng, $\varphi(t_0) \neq x(t_0)$ are considered in [2 ,3].

## References


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