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ON CONVERGENCE OF SEQUENCES OF INTERNAL SUPERPOSITION OPERATORS IN IDEAL SPACES

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Dedicated to Lina Fazylovna Rakhmatullina and
Nikolaĭ Viktorovich Azbelev
on the occasion of their jubilees

1. Introduction

Theory of functional differential equations is based on studies of the properties of an internal superposition operator in different function spaces [1]. In particular, the question of correct solvability of boundary value problems for functional differential equations requires the establishment of conditions for convergence of sequences of such operators [2], [3], [4].

Recall the following definition.

Definition 1.1. Let $A_\nu : X \to Y$, $A : X \to Y$, $\nu \in \mathbf{N}$, are mappings between two Banach spaces X and Y. One says that the sequence A_ν converges to A *strongly* (pointwise), if $A_\nu x \to Ax$ in Y for all $x \in X$.

Denote by $\mathbf{R}^n$ the space of n-dimensional real vectors $\alpha = (\alpha_1, \dots \alpha_n)$ with the norm $||\alpha|| = \max_{1\le i\le n} |\alpha_i|$. The same symbol $||\cdot||$ will be used for the norm of $n\times n$ - matrix coordinated with the norm in $\mathbf{R}^n$. The triple $(\mathbf{E}, \mathbf{\Sigma}, \mathbf{m})$, consisting of a set $\mathbf{E} \subset \mathbf{R}^n$, some σ-algebra $\mathbf{\Sigma}$ of subsets of $\mathbf{E}$ and a measure $\mathbf{m}$ defined on $\mathbf{\Sigma}$ will be called the space with measure. The measure $\mathbf{m}$ is assumed to be complete positive σ-finite and non-atomic.

In this paper, given the sequence of measurable functions $g^\nu : \mathbf{E} \to \mathbf{R}^n$ and $g : \mathbf{E} \to \mathbf{R}^n$, we are interested in the convergence of a sequence of respective linear inner superposition operators $S_{g^\nu} : \mathbf{X}(\mathbf{E}, \mathbf{\Sigma}, \mathbf{m}; \mathbf{R}^n) \to \mathbf{Y}(\mathbf{E}, \mathbf{\Sigma}, \mathbf{m}; \mathbf{R}^n)$ given by the formula

$$(S_{g^\nu} x)(t) = \begin{cases} x(g^\nu(t)), & g^\nu(t) \in \mathbf{E}, \\ 0, & g^\nu(t) \notin \mathbf{E} \end{cases} \tag{1}$$

to another inner superposition $S_g : \mathbf{X}(\mathbf{E}, \mathbf{\Sigma}, \mathbf{m}; \mathbf{R}^n) \to \mathbf{Y}(\mathbf{E}, \mathbf{\Sigma}, \mathbf{m}; \mathbf{R}^n)$ given by

$$(S_g x)(t) = \begin{cases} x(g(t)), & g(t) \in \mathbf{E}, \\ 0, & g(t) \notin \mathbf{E}. \end{cases} \tag{2}$$

For the above operators to be well defined on the classes of measurable functions, we are to impose the following conditions on the functions g^ν and g (denoting from now on for the sake of brevity $g^0 = g$) :

$$e \subset \mathbf{E}, \quad meas \;\; e = 0 \Rightarrow meas \;\; (g^\nu)^{-1}(e) = 0, \quad \nu \in \mathbf{N} \cup \{0\}. \tag{3}$$

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2. Basic Notions, Definitions and Facts

We give here some notions and definitions from [6], [10] that we are going to use in what follows, as well as the necessary material of the measure theory [7], [8].

Denote by $\mathcal{M}(\mathbf{E},\mathbf{\Sigma},\mathbf{m};\mathbf{R}^n)$ (shortly $\mathcal{M}^n(\mathbf{E})$ or $\mathcal{M}^n$) the space of measurable functions $x:\mathbf{E}\to\mathbf{R}^n$ with convergence topology in the sense of the measure $\mathbf{m}$ on each set $H\in\mathbf{\Sigma},\quad \mathbf{m}(H)<\infty$.

Definition 2.1. 1) A linear subset $\mathbf{X}\subset\mathcal{M}^n$ is called an ideal space (IS) on $(\mathbf{E},\mathbf{\Sigma},\mathbf{m})$ if $x\in\mathbf{X},\quad y\in\mathcal{M}^n,\quad \|y(t)\|\leq\|x(t)\|,\quad t\in\mathbf{E}$, imply that $y\in\mathbf{X}$.

Let us use the notation $\mathbf{X}$ for the ideal space $\mathbf{X}(\mathbf{E},\mathbf{\Sigma},\mathbf{m};\mathbf{R}^n)$.

2) An ideal space $\mathbf{X}$ with $\operatorname{supp}\mathbf{X}=\mathbf{E}$ ($\operatorname{supp}\mathbf{X}$ stays for the support of the space $\mathbf{X}$) is called the fundamental space (FS) on $(\mathbf{E},\mathbf{\Sigma},\mathbf{m})$.

3) A (IS) supplied with a norm is called a normed ideal space (NIS) on $(\mathbf{E},\mathbf{\Sigma},\mathbf{m})$.

4)A complete normed fundamental space is called a Banach fundamental space (BFS).

Definition 2.2. We say that condition **(A)** is satisfied in (NIS) $\mathbf{X}$ if $x_k\downarrow 0$ implies $\|x_k\|_{\mathbf{X}}\to 0$.

Definition 2.3. We say that condition **(C)** is satisfied in (NIS) $\mathbf{X}$ if $0\leq x_k\uparrow x$, $x\in\mathbf{X}$, imply $\|x_k\|_{\mathbf{X}}\to\|x\|_{\mathbf{X}}$.

Definition 2.4. We say that condition **(B)** is satisfied in (NIS) $\mathbf{X}$ if $0\leq x_k\uparrow$, $x_k\in\mathbf{X}$, $k\in\mathbf{N}$, $\sup\|x_k\|<\infty$ imply the existence of $x\in\mathbf{X}$ such that $x_k\uparrow x$.

Let us define by $\mathbf{X}'$ the set of all $x'\in\mathcal{M}^n$ such that $\mathbf{m}(\operatorname{supp}\quad x'\setminus\operatorname{supp}\mathbf{X})=0$ and for each $x\in\mathbf{X}$

$$\int\limits_{\mathbf{E}}\|x(s)\|\|x'(s)\|d\mathbf{m}(s)<\infty.$$

$\mathbf{X}'$ is called dual to the ideal space $\mathbf{X}$. Let us define a norm on $\mathbf{X}$ as follows:

$$\|x\|_{\mathbf{X}}=sup_{\|x'\|_{\mathbf{X}'}\leq 1}\int\limits_{\mathbf{E}}\|x(s)\|\|x'(s)\|d\mathbf{m}(s).$$

In what follows we will consider the following concrete ideal spaces.

1. $\mathbf{L}_p(\mathbf{E},\mathbf{\Sigma},\mathbf{m};\mathbf{R}^n)$ (shortly $\mathbf{L}_p^n(\mathbf{E})$ or $\mathbf{L}_p^n$),$1\leq p<\infty$, is the space of the functions $x\in\mathcal{M}(\mathbf{E},\mathbf{\Sigma},\mathbf{m};\mathbf{R}^n)$ with the norm

$$\|x\|_{\mathbf{L}_p^n}=\Big[\int\limits_{\mathbf{E}}\|x(s)\|^p d\mathbf{m}(s)\Big]^{\frac{1}{p}}.$$

For $1\leq p<\infty\quad \mathbf{L}_p^n$ is a (BFS) with conditions **(A)** and **(B)**.

2. $\mathbf{L}_\infty(\mathbf{E},\mathbf{\Sigma},\mathbf{m};\mathbf{R}^n)$ (shortly $\mathbf{L}_\infty^n(\mathbf{E})$ or $\mathbf{L}_\infty^n$) is the space of essentially bounded on $\mathbf{E}$ functions $x\in\mathcal{M}(\mathbf{E},\mathbf{\Sigma},\mathbf{m};\mathbf{R}^n)$ with the norm

$$\|x\|_{\mathbf{L}_\infty^n}=\operatorname{ess\,sup}_{s\in\mathbf{E}}\|x(s)\|.$$

$\mathbf{L}_\infty^n$ is a (BFS) with conditions **(B)** and **(C)**.

An even, convex, positive for $u\neq 0$ continuous function M defined on $(-\infty,\infty)$ is called the N-function if

$$\lim_{u\to 0}\frac{M(u)}{u}=0,\quad \lim_{u\to\infty}\frac{M(u)}{u}=\infty.$$

For each N-function M the function $M^*(u)=\sup_{-\infty<v<\infty}(uv-M(v))$ is a complementary N-function. Let us fix some N-function M.

A collection of functions $x \in \mathcal{M}(\mathbf{E}, \mathbf{\Sigma}, \mathbf{m}; \mathbf{R}^n)$ such that

$$\int\limits_{\mathbf{E}} M(\alpha\|x(s)\|)d\mathbf{m}(s) < \infty$$

is called the Orlich class $\mathbf{L}_M^\alpha(\mathbf{E}, \mathbf{\Sigma}, \mathbf{m}; \mathbf{R}^n), \quad \alpha > 0$.

Clearly, if $\mathbf{m}(\mathbf{E}) < \infty$, then $\mathbf{L}_\infty \subset \mathbf{L}_M^\alpha$.

The Orlich space $\mathbf{L}_M(\mathbf{E}, \mathbf{\Sigma}, \mathbf{m}; \mathbf{R}^n)$ (shortly $\mathbf{L}_M^n(\mathbf{E})$ or $\mathbf{L}_M^n$) is the union of the functions $x \in \mathcal{M}(\mathbf{E}, \mathbf{\Sigma}, \mathbf{m}; \mathbf{R}^n)$ such that there exists an integer $\lambda = \lambda(x)$ for which

$$\int\limits_{\mathbf{E}} M\Big(\frac{\|x(s)\|}{\lambda}\Big)d\mathbf{m}(s) < \infty.$$

It is evident that $\mathbf{L}_M^\alpha \subset \mathbf{L}_M^n$.

On the Orlich space $\mathbf{L}_M^n$ we will consider the following norm

$$\|x\|_{\mathbf{L}_M^n} = \sup\Big\{ \int\limits_{\mathbf{E}} \|x(s)\|\|y(s)\|d\mathbf{m}(s) : \int\limits_{\mathbf{E}} M^*(\|y(s)\|)d\mathbf{m}(s) \le 1\Big\}.$$

The Orlich space $\mathbf{L}_M^n$ with the above defined norm is a (BFS) with conditions **(B)** and **(C)**.

Define by $\mathcal{E}_M^n$ the closure of $\mathbf{L}_\infty^n$ in the Orlich space $\mathbf{L}_M^n$ (here $\mathbf{m}(\mathbf{E}) < \infty$). We consider on $\mathcal{E}_M^n$ the same norm $\|\cdot\|_{\mathbf{L}_M^n}$.

The space $\mathcal{E}_M^n$ is a (BFS) with condition **(A)**.

3. Strong Convergence of a Sequence of Internal Superposition Operators in Ideal Spaces

Let a measurable function $z : \mathbf{E} \to [0, \infty)$ be defined in space $(\mathbf{E}, \mathbf{\Sigma}, \mathbf{m})$, and let $H \in \mathbf{\Sigma}$. Define on $\mathbf{\Sigma}$ the function $\mu_H(z, g, \mathbf{m})$ as follows:

$$\mu_H(z, g, \mathbf{m})(e) = \int\limits_{\{t \in H : g(t) \in e\}} z(s)d\mathbf{m}(s), \quad e \in \mathbf{\Sigma}.$$

It was proved in [5] that there exists a measurable function $\frac{d\mu_H(z,g,\mathbf{m})}{d\mathbf{m}} : \mathbf{E} \to [0, \infty)$ ($\frac{d\mu(z,g,\mathbf{m})}{d\mathbf{m}}$ if $H = \mathbf{E}$), which connects the measures $\mu_H(z, g, \mathbf{m})$ and $\mathbf{m}$ by the equality

$$\mu_H(z, g, \mathbf{m})(e) = \int\limits_{e} \frac{d\mu_H(z, g, \mathbf{m})}{d\mathbf{m}}(s)d\mathbf{m}(s).$$

Note, that if $\mathbf{m}(\mathbf{E}) < \infty$, then the last statement follows from the Radon–Nikodym theorem.

In what follows we will essentially exploit the following

Lemma 3.1 ([5], **Lemma 2**). *For convergence* $(\forall x \in \mathcal{M}^n)$ *of a sequence of functions* $\{S_{g_k}x\}_{k=1}^\infty$ *to the function* $S_g x$ *in the space* $\mathcal{M}^n$ *it is necessary and sufficient, that on every set* $H \in \mathbf{\Sigma}, \quad \mathbf{m}(H) < \infty$, *the following conditions hold:*

1) $\lim_{k\to\infty} \mathbf{m}(\{t \in H : g_k(t) \in \mathbf{E}\}\Delta\{t \in H : g(t) \in \mathbf{E}\}) = 0$ (*here* Δ *denotes the notion of the symmetric difference*);

2) *for any* $\sigma > 0$

$$\lim_{k\to\infty} \mathbf{m}(\{t \in H \cap g^{-1}(\mathbf{E}) : \|g_k(t) - g(t)\| \ge \sigma\}) = 0;$$

3) *the sequence* $\{\frac{d\mu_H}{d\mathbf{m}}(1, g_k, \mathbf{m})\}_{k=1}^{\infty}$ *has equipotentially absolutely continuous integrals.*

In the following statements of this section we will assume that $\mathbf{m}(\mathbf{E}) < \infty$.

Theorem 3.1. *Let* $\mathbf{X}_1(\mathbf{E}, \mathbf{\Sigma}, \mathbf{m}; \mathbf{R}^n)$ *and* $\mathbf{X}_2(\mathbf{E}, \mathbf{\Sigma}, \mathbf{m}; \mathbf{R}^n)$ *be (BFS) with condition* **(A)**. *Moreover, let* $\mathbf{X}_2$ *be a symmetric space. A sequence* $\{S_{g_k} : \mathbf{X}_1 \to \mathbf{X}_2\}_{k=1}^{\infty}$ *strongly converges to a continuous operator* $S_g : \mathbf{X}_1 \to \mathbf{X}_2$ *iff the following conditions are satisfied:*

1) *the sequence* $\{S_{g_k}x\}_{k=1}^{\infty}$ *converges to* $S_g x$ *for any* $x \in \mathbf{X}_1$ *in the sense of measure;*
2) $\sup_{k\geq 1}\{||S_{g_k}||_{\mathbf{X}_1 \to \mathbf{X}_2}\} \leq c < \infty$.

Proof. The necessity of the condition **1)** follows from Theorem 1 in ([6], p. 139), the necessity of the condition **2)** – from the Banach–Steinhaus theorem (see, for example, [6], p. 271). Let us prove the sufficiency. The set of continuous functions is dense everywhere in $\mathbf{X}_1$. Indeed, in virtue of the Fréchet theorem (see, for example, p.63) for each measurable almost everywhere finite function x it is possible to find a sequence of continuous functions $\{x_k\}_{k=1}^{\infty}$ converging to x almost everywhere. Then the condition **(A)** implies the convergence of $\{x_k\}_{k=1}^{\infty}$ to x in the norm of $\mathbf{X}_1$.

Let $x : \mathbf{E} \to \mathbf{R}^n$ be a uniformly continuous function. Let us show that

$$\lim_{k\to\infty} ||(S_{g_k} - S_g)x||_{\mathbf{X}_2} = 0.$$

Indeed, the following inequality is true:

$$||(S_{g_k}x - S_g x)||_{\mathbf{X}_2} \leq ||\chi_{g_k^{-1}(\mathbf{E})\cap g^{-1}(\mathbf{E})}[(S_{g_k}x - S_g x]||_{\mathbf{X}_2} + \\ +||\chi_{g_k^{-1}(\mathbf{E})\Delta g^{-1}(\mathbf{E})}[(S_{g_k}x - S_g x]||_{\mathbf{X}_2}.$$

Here and below χ_e stands for the characteristic function of the set e.

Now let us estimate every summand in the right hand side of the last inequality.

Since in the space $\mathbf{X}_2$ the condition **(A)** is satisfied, then the condition **1)** of the theorem implies that the difference

$$||\chi_{g_k^{-1}(\mathbf{E})\cap g^{-1}(\mathbf{E})}[(S_{g_k}x - S_g x]||_{\mathbf{X}_2}$$

tends to zero when $k \to \infty$. Then,

$$||\chi_{g_k^{-1}(\mathbf{E})\Delta g^{-1}(\mathbf{E})}[(S_{g_k}x - S_g x]||_{\mathbf{X}_2} \leq (||(S_{g_k}||_{\mathbf{X}_1\to\mathbf{X}_2} + \\ +||(S_g||_{\mathbf{X}_1\to\mathbf{X}_2})||x||_{\mathbf{X}_1}||\chi_{g_k^{-1}(\mathbf{E})\Delta g^{-1}(\mathbf{E})}||_{\mathbf{X}_2} \leq 2c||x||_{\mathbf{X}_1}||\chi_{g_k^{-1}(\mathbf{E})\Delta g^{-1}(\mathbf{E})}||_{\mathbf{X}_2}.$$

By virtue of the symmetricity of $\mathbf{X}_2$ the value $||\chi_{g_k^{-1}(\mathbf{E})\Delta g^{-1}(\mathbf{E})}||_{\mathbf{X}_2}$ depends on $m(g_k^{-1}(\mathbf{E})\Delta g^{-1}(\mathbf{E}))$ only (see [9], p.22) and therefore, by virtue of **Lemma 3.1** it tends to zero for $k \to \infty$.

Thus, the sequence $\{S_{g_k} : \mathbf{X}_1 \to \mathbf{X}_2\}$ tends to the operator $S_g : \mathbf{X}_1 \to \mathbf{X}_2$ on everywhere dense in $\mathbf{X}_1$ set of continuous functions. Reference to the Banach–Steinhaus theorem completes the proof. □

The essence of our assumption that the condition **(A)** is satisfied in the spaces $\mathbf{X}_1$ and $\mathbf{X}_2$ follows from the example of $\mathbf{L}_{\infty}^n$ studied in [5]. Note finally that the spaces $\mathbf{L}_p^n$, $\mathbf{L}_M^n$ are symmetric and in $\mathcal{E}_M^n$, as it has been mentioned in the section 2, the condition **(A)** is fulfilled.

Definition 3.1 (see [10]**).** A convex function Q is called principal part of some N - function M if $Q(u) = M(u)$ for the large values of the argument u.

Let M and M_1 be N - functions and let the superposition $M[M_1^{-1}]$ be a principal part of some N - function Q.

Corollary 3.1. *The sequence* $\{S_{g_k} : \mathcal{E}_M^n \to \mathcal{E}_{M_1}^n\}_{k=1}^{\infty}$ *strongly converges to the operator* $S_g : \mathcal{E}_M^n \to \mathcal{E}_{M_1}^n$ *if the conditions*

1. $\lim_{k\to\infty} \mathbf{m}(g_k^{-1}(\mathbf{E})\Delta g^{-1}(\mathbf{E})) = 0$;

2. *for any* $\sigma > 0$

$$\lim_{k\to\infty} \mathbf{m}(\{t \in g^{-1}(\mathbf{E}) : ||g_k(t) - g(t)|| \geq \sigma\}) = 0;$$

3. $\sup_{k\geq 1} \left\{ \int_{g_k^{-1}(\mathbf{E})} Q^*(\frac{d\mu(1,g_k,\mathbf{m})}{d\mathbf{m}}(s))d\mathbf{m}(s) \right\} < \infty$

are satisfied.

Proof. The boundedness on the average of the sequence $\{(\frac{d\mu(1,g_k,\mathbf{m})}{d\mathbf{m}}\}_{k=1}^{\infty}$ in $\mathbf{L}^1_{Q^*}$ implies, in virtue of the Vallée-Poussin theorem, the equipotential absolute continuity of the integrals of this sequence. Thus, all the conditions of **Lemma 3.1** are fulfilled, which means that $\{S_{g_k}x\}_{k=1}^{\infty}$ converges in measure to $S_g x$ for any $x \in \mathcal{E}_M^n$. The condition **3)** implies ([10], p. 89) that $\sup_{k\geq 1}\{||\frac{d\mu(1,g_k,\mathbf{m})}{d\mathbf{m}}||_{\mathbf{L}^1_{Q^*}}\} < \infty$. Then, in virtue of the following estimate (see [4])

$$||S_g||_{\mathbf{L}^n_M \to \mathbf{L}^n_{M_1}} \leq 2\left\|\frac{d\mu(1,g,\mathbf{m})}{d\mathbf{m}}\right\|_{\mathbf{L}^1_{\{M[M_1^{-1}]\}^*}} + 1$$

we have $\sup_{k\geq 1} ||S_{g_k}||_{\mathcal{E}^n_M \to \mathcal{E}^n_{M_1}} < \infty$. Thus, all the conditions of **Theorem 3.1** are fulfilled and the reference to this theorem completes the proof. □

The following two statements can be proved analogously.

Corollary 3.2. *The sequence of operators* $\{S_{g_k} : \mathcal{E}_M^n \to \mathcal{E}_M^n\}_{k=1}^{\infty}$ *strongly converges to the operator* $S_g : \mathcal{E}_M^n \to \mathcal{E}_M^n$ *if the conditions* **1** *and* **2** *of* ***Corollary* 3.1** *and the estimate*

$$\sup_{k\geq 1}\left\{\left\|\frac{d\mu(1,g_k,\mathbf{m})}{d\mathbf{m}}\right\|_{\mathbf{L}^1_\infty}\right\} < \infty$$

are satisfied.

Definition 3.2 (see [10]**).** N - functions M_1 and M_2 are called equivalent ($M_1 \sim M_2$), if there exist positive constants k_1, k_2, and u_0 such that

$$M_1(k_1 u) \leq M_2(u) \leq M_1(k_2 u), \quad u > u_0.$$

Definition 3.3 (see [10]**).** N - function satisfies the $\mathbf{\Delta}_\Phi$ - condition, where Φ is some fixed N – function, if $\Phi[M] \sim M$.

Corollary 3.3. *Let* N *- function* M *satisfy the* $\mathbf{\Delta}_\Phi$ *- condition. The sequence* $\{S_{g_k} : \mathcal{E}_M^n \to \mathcal{E}_M^n\}_{k=1}^{\infty}$ *strongly converges to the operator* $S_g : \mathcal{E}_M^n \to \mathcal{E}_M^n$ *if the conditions* **1** *and* **2** *of* ***Corollary* 3.1** *and the estimate*

$$\sup_{k\geq 1}\left\{ \int\limits_{g_k^{-1}(\mathbf{E}} \Phi^*\left(\frac{d\mu(1,g_k,\mathbf{m})}{d\mathbf{m}}(s)\right)d\mathbf{m}(s)\right\} < \infty$$

are satisfied.

For the completeness, let us quote a statement on the strong convergence of a sequence of internal superposition operators in the Lebesgue spaces $\mathbf{L}_p(\mathbf{E},\mathbf{\Sigma},\mathbf{m};\mathbf{R}^n)$, $1 \leq p < \infty$, from [5].

Consider a sequence of operators $\{S_k\}_{k=1}^{\infty}$, defined as follows

$$(S_k x)(t) = B_k(t)(S_{g_k}x)(t), \quad t \in \mathbf{E}, \quad k = 1,2,\dots, \tag{4}$$

where B_k, $k = 1, 2, \dots$, are $n \times n$ - matrices of measurable almost everywhere finite functions from $\mathbf{E}$ into $\mathbf{R}$.

The next corollary establishes conditions for the strong convergence of $\{S_k : \mathbf{L}_p^n \to \mathbf{L}_r^n\}_{k=1}^{\infty}$, $1 \le r \le p < \infty$, to $S : \mathbf{L}_p^n \to \mathbf{L}_r^n$, defined by the following equality

$$(Sx)(t) = B(t)(S_g x)(t). \tag{5}$$

Corollary 3.4 (see [5]). *Let* $\forall H \in \mathbf{\Sigma}, \mathbf{m}(H) < \infty)$ *the following conditions be valid:*
1) $\lim_{k\to\infty} \mathbf{m}(\{t \in H : b(t) \ne 0, \quad g_k(t) \in \mathbf{E}\}\Delta\{t \in H : b(t) \ne 0, \quad g(t) \in \mathbf{E}\}) = 0;$
2) *for any* $\sigma > 0$

$$\lim_{k\to\infty} \mathbf{m}(\{t \in H \cap g^{-1}(\mathbf{E}) : b(t) \ne 0, \quad ||g_k(t) - g(t)|| \ge \sigma\}) = 0;$$

3) *the sequence* $\frac{d\mu_H}{d\mathbf{m}}(1, g_k, \mathbf{m})_{k=1}^{\infty}$ *has equipotentially absolutely continuous integrals;*
4) *for any* $\sigma > 0$

$$\lim_{k\to\infty} \mathbf{m}(\{t \in H \cap g^{-1}(\mathbf{E}) : ||B_k(t) - B(t)|| \ge \sigma\}) = 0;$$

5) $\sup_{k\ge1} \left\{ \left\| \frac{d\mu}{d\mathbf{m}}(b^r, g_k, \mathbf{m}) \right\|_{\mathbf{L}^1_{\frac{p}{p-r}}} \right\} < \infty.$

Then the sequence of operators

$$\{S_k : \mathbf{L}_p^n \to \mathbf{L}_r^n\}_{k=1}^{\infty}, \quad 1 \le r \le p < \infty,$$

strongly converges to the operator $S : \mathbf{L}_p^n \to \mathbf{L}_r^n$.
Here $b_k(t) \equiv ||B_k(t)||, \quad k = 1, 2, \dots, \quad b(t) \equiv ||B(t)||.$

Let us point out that in the last corollary the only assumption on the measure is the σ - finiteness, i.e., the equality $\mathbf{m}(\mathbf{E}) = \infty$ is permitted.

In conclusion let us mention that some other types of convergence and connections between them for a sequence of internal superposition operators are studied in [11], [12].

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