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ON CONSTANT SIGN SOLUTIONS OF A PERIODIC TYPE BOUNDARY VALUE PROBLEMS FOR FIRST ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. In this paper the question on the existence and uniqueness of a constant sign solution of a periodic type boundary value problem is studied. More precisely, the nonimprovable effective sufficient conditions for a linear bounded operator $\ell : C([a,b];R) \to L([a,b];R)$ are established guaranteeing that the problem

$$u'(t) = \ell(u)(t) + q(t), \qquad u(a) - \lambda u(b) = c,$$

where $q \in L([a,b];R_+)$, $\lambda \in R_+$, has a unique solution, and this solution does not change its sign.

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Introduction

The following notation is used throughout.

N is the set of all natural numbers.

R is the set of all real numbers, $R_+ = [0, +\infty[$.

$C([a,b];R)$ is the Banach space of continuous functions $u : [a,b] \to R$ with the norm $\|u\|_C = \max\{|u(t)| : a \le t \le b\}$.

$C([a,b];R_+) = \{u \in C([a,b];R) : u(t) \ge 0 \text{ for } t \in [a,b]\}$.

$\widetilde{C}([a,b];D)$, where $D \subseteq R$, is the set of absolutely continuous functions $u : [a,b] \to D$.

$L([a,b];R)$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to R$ with the norm $\|p\|_L = \int\limits_a^b |p(s)|ds$.

$L([a,b];R_+) = \{p \in L([a,b];R) : p(t) \ge 0 \text{ for almost all } t \in [a,b]\}$.

$\mathcal{M}_{ab}$ is the set of measurable functions $\tau : [a,b] \to [a,b]$.

$\mathcal{L}_{ab}$ is the set of linear bounded operators $\ell : C([a,b];R) \to L([a,b];R)$.

$\mathcal{P}_{ab}$ is the set of linear operators $\ell \in \mathcal{L}_{ab}$ transforming the set $C([a,b];R_+)$ into the set $L([a,b];R_+)$.

We will say that $\ell \in \mathcal{L}_{ab}$ is a t_0-Volterra operator, where $t_0 \in [a,b]$, if for arbitrary $a_1 \in [a,t_0]$, $b_1 \in [t_0,b]$, $a_1 \ne b_1$, and $v \in C([a,b];R)$ satisfying the condition

$$v(t) = 0 \quad \text{for} \quad t \in [a_1,b_1],$$

we have

$$\ell(v)(t) = 0 \quad \text{for} \quad t \in [a_1,b_1].$$

$[x]_+ = \frac{1}{2}(|x| + x)$, $[x]_- = \frac{1}{2}(|x| - x)$.

By a solution of the equation

$$u'(t) = \ell(u)(t) + q(t), \tag{0.1}$$

where $\ell \in \mathcal{L}_{ab}$ and $q \in L([a,b];R)$, we understand a function $u \in \widetilde{C}([a,b];R)$ satisfying the equation (0.1) almost everywhere in $[a,b]$. The special case of the equation (0.1) is the differential equation with deviating arguments

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)) + q(t), \tag{0.2}$$

where $p, g \in L([a,b];R_+)$, $q \in L([a,b];R)$, $\tau, \mu \in \mathcal{M}_{ab}$.

We will consider the problem on the existence and uniqueness of a nonnegative, resp. nonpositive solution u of (0.1) with $q \in L([a,b];R_+)$ satisfying the condition

$$u(a) - \lambda u(b) = c, \tag{0.3}$$

where $\lambda, c \in R_+$.

Along with the equation (0.1), resp. (0.2), and the condition (0.3) we will consider the corresponding homogeneous equation

$$u'(t) = \ell(u)(t), \tag{0.1_0}$$

resp.

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)), \tag{0.2_0}$$

and the corresponding homogeneous condition

$$u(a) - \lambda u(b) = 0. \tag{0.3_0}$$

The following result is well–known from the general theory of boundary value problems for functional differential equations (see, e.g., [1, 2, 13]).

Theorem 0.1. *The problem* (0.1), (0.3) *is uniquely solvable iff the corresponding homogeneous problem* (0.1_0), (0.3_0) *has only the trivial solution.*

Introduce the definition

Definition 0.1. We will say that an operator $\ell \in \mathcal{L}_{ab}$ belongs to the set $V^+(\lambda)$ (resp. $V^-(\lambda)$), if the homogeneous problem (0.1_0), (0.3_0) has only the trivial solution, and for arbitrary $q \in L([a,b];R_+)$ and $c \in R_+$, the solution of the problem (0.1), (0.3) is nonnegative (resp. nonpositive).

Remark 0.1. From Definition 0.1 it immediately follows that $\ell \in V^+(\lambda)$ (resp. $\ell \in V^-(\lambda)$), iff a certain theorem on differential inequalities holds for the equation (0.1), i.e., whenever $u, v \in \widetilde{C}([a,b];R)$ satisfy the inequalities

$$u'(t) \le \ell(u)(t) + q(t), \qquad v'(t) \ge \ell(v)(t) + q(t) \quad \text{for} \quad t \in [a,b],$$
$$u(a) - \lambda u(b) \le v(a) - \lambda v(b),$$

then $u(t) \le v(t)$ (resp. $u(t) \ge v(t)$) for $t \in [a,b]$.

Therefore theorems formulated below are, in fact, theorems on differential inequalities. In paper [8] there are established conditions guaranteeing the inclusion $\ell \in V^+(\lambda)$ in the case $\lambda \in [0,1]$. In this paper, there are established conditions guaranteeing the inclusion $\ell \in V^-(\lambda)$ in the case $\lambda \in R_+$ and the conditions guaranteeing the inclusion $\ell \in V^+(\lambda)$ in the case $\lambda \ge 1$.

Another types of theorems on differential inequalities for the systems of functional differential equations can be found, e.g., in [3-12].

Remark 0.2. Note also that if $\ell \in \mathcal{P}_{ab}$ and $\ell \in V^+(\lambda)$, then $\lambda < 1$, and if $-\ell \in \mathcal{P}_{ab}$ and $\ell \in V^-(\lambda)$, then $\lambda > 1$.

In what follows if $\lambda = 1$, then the operator $\ell \in \mathcal{L}_{ab}$ is supposed to be nontrivial.

1. On the Set $V^-(\lambda)$

1.1. Main Results.

Theorem 1.1. *Let* $\lambda \in]0,1]$,

$$\ell \in \mathcal{P}_{ab}, \tag{1.1}$$

and

$$\frac{1-\lambda}{\lambda} < \int_a^b \ell(1)(s)ds \leq 1. \tag{1.2}$$

Then $\ell \in V^-(\lambda)$.

Remark 1.1. Theorem 1.1 is nonimprovable in the sense that the strict inequality in (1.2) cannot be replaced by the nonstrict one and the nonstrict inequality in (1.2) cannot be replaced by the inequality

$$\int_a^b \ell(1)(s)ds \leq 1+\varepsilon, \tag{1.3}$$

no matter how small $\varepsilon > 0$ would be (see Examples 4.1 and 4.2).

Theorem 1.2. *Let* $\lambda \in]0,1]$, $\ell \in \mathcal{P}_{ab}$, *let there exist* $\delta \in]0,\lambda]$ *such that* $\ell \notin V^+(\lambda-\delta)$ *and*

$$\int_a^b \ell(1)(s)ds \leq 1. \tag{1.4}$$

Then $\ell \in V^-(\lambda)$.

Remark 1.2. Theorem 1.2 is nonimprovable. More precisely, the condition (1.4) cannot be replaced by the condition (1.3), no matter how small $\varepsilon > 0$ would be (see Example 4.2).

Note also that if $\lambda = 1$ and $\ell \in \mathcal{P}_{ab}$, then there exist $\delta \in]0,\lambda]$ such that $\ell \notin V^+(\lambda-\delta)$. Indeed, the function

$$u(t) = 1 + \int_a^t \ell(1)(s)ds \quad \text{for} \quad t \in [a,b] \tag{1.5}$$

is a nontrivial nonnegative solution of the problem

$$u'(t) \leq \ell(u)(t), \qquad u(a) - (1-\delta)u(b) = 0, \tag{1.6}$$

where $\delta = \frac{\|\ell(1)\|_L}{1+\|\ell(1)\|_L}$. Since we suppose that in this case ℓ is a nontrivial operator, $\delta \in]0,1]$, and thus, by Proposition 1.1 in [8], $\ell \notin V^+(1-\delta)$.

Nevertheless, if $\lambda \in]0,1[$, then the assumption $\delta \in]0,\lambda]$ in Theorem 1.2 cannot be replaced by the assumption $\delta \in [0,\lambda]$ (see Example 4.3).

Corollary 1.1. *Let $\lambda \in]0,1]$, $\ell \in \mathcal{P}_{ab}$, and let there exist $\gamma \in \widetilde{C}([a,b];R)$ satisfying*

$$\gamma'(t) \le \ell(\gamma)(t) \quad for \quad t \in [a,b], \tag{1.7}$$

$$\gamma(a) < \lambda\gamma(b), \tag{1.8}$$

and

$$\gamma(b) > 0. \tag{1.9}$$

If, moreover, the inequality (1.4) *holds, then $\ell \in V^-(\lambda)$.*

Remark 1.3. Corollary 1.1 is nonimprovable. More precisely, the condition (1.4) cannot be replaced by the condition (1.3), no matter how small $\varepsilon > 0$ would be (see Example 4.2). Furthermore, the strict inequality (1.9) cannot be weakened (see On Remark 1.3).

Note also that if $\lambda = 1$, then there exist a function $\gamma \in \widetilde{C}([a,b];R)$ satisfying the conditions (1.7), (1.8), and (1.9). Indeed, in this case the operator ℓ is supposed to be nontrivial and thus, the function

$$\gamma(t) = 1 + \int_a^t \ell(1)(s)ds \quad \text{for} \quad t \in [a,b]$$

satisfies the inequalities (1.7), (1.8), and (1.9).

Nevertheless, if $\lambda \in]0,1[$, then the strict inequality (1.8) cannot be replaced by the inequality

$$\gamma(a) \le \lambda\gamma(b) \tag{1.10}$$

(see Example 4.3).

Theorem 1.3. *Let $\lambda \in]0,1]$, $\ell \in \mathcal{P}_{ab}$ be a $b-$Volterra operator, and let there exist $\delta \in]0,\lambda]$ such that $\ell \notin V^+(\lambda-\delta)$. If, moreover, there exist a function $\beta \in \widetilde{C}([a,b];R_+)$ satisfying*

$$\beta(t) > 0 \quad for \quad t \in]a,b], \tag{1.11}$$

$$\beta'(t) \ge \ell(\beta)(t) \quad for \quad t \in [a,b], \tag{1.12}$$

then $\ell \in V^-(\lambda)$.

Remark 1.4. Theorem 1.3 is nonimprovable in a certain sense. More precisely, the assumption (1.11) cannot be replaced by the assumption

$$\beta(t) > 0 \quad \text{for} \quad t \in]a_1, b], \tag{1.13}$$

where $a_1 \in]a, b[$ is an arbitrary fixed point (see Example 4.4).

Note also that if $\lambda = 1$, then there exist $\delta \in]0, \lambda]$ such that $\ell \notin V^+(\lambda - \delta)$ (see Remark 1.2).

Nevertheless, if $\lambda \in]0, 1[$ then the assumption $\delta \in]0, \lambda]$ cannot be replaced by the assumption $\delta \in [0, \lambda]$ (see Example 4.3).

Corollary 1.2. *Let $\lambda \in]0, 1]$, $\ell \in \mathcal{P}_{ab}$, ℓ be a $b-$Volterra operator, and let there exist a function $\gamma \in \widetilde{C}([a, b]; R)$ satisfying the inequalities* (1.7), (1.8), *and* (1.9). *If, moreover, there exists a function $\beta \in \widetilde{C}([a, b]; R_+)$ satisfying the inequalities* (1.11) *and* (1.12), *then $\ell \in V^-(\lambda)$.*

Remark 1.5. Corollary 1.2 is nonimprovable in the sense that the strict inequality (1.9) cannot be weakened (see On Remark 1.5). Furthermore, the assumption (1.11) cannot be replaced by the assumption (1.13), where $a_1 \in]a, b[$ is an arbitrary fixed point (see Example 4.4).

Note also that if $\lambda = 1$, then there exist a function $\gamma \in \widetilde{C}([a, b]; R)$ satisfying the conditions (1.7), (1.8), and (1.9) (see Remark 1.3).

Nevertheless, if $\lambda \in]0, 1[$, then the strict inequality (1.8) cannot be replaced by the inequality (1.10) (see Example 4.3).

Proposition 1.1. *Let $\lambda \in]1, +\infty[$ and $-\ell \in \mathcal{P}_{ab}$. Then $\ell \in V^-(\lambda)$ iff the problem*

$$u'(t) \geq \ell(u)(t), \qquad u(a) - \lambda u(b) = 0$$

has no nontrivial nonnegative solution.

Theorem 1.4. *Let $\lambda \in]1, +\infty[$ and $-\ell \in \mathcal{P}_{ab}$. Then $\ell \in V^-(\lambda)$ iff there exists $\gamma \in \widetilde{C}([a, b];]0, +\infty[)$ satisfying the inequalities* (1.7) *and* (1.8).

Corollary 1.3. *Let $\lambda \in]1, +\infty[$, $-\ell \in \mathcal{P}_{ab}$, and at least one of the following items be fulfilled:*

a) *ℓ is a $b-$Volterra operator and*

$$\int_a^b |\ell(1)(s)| ds < \ln \lambda ; \tag{1.14}$$

b) *there exist natural numbers m, k and a constant $\alpha \in]0, 1[$ such that $m > k$ and*

$$\rho_m(t) \leq \alpha \rho_k(t) \quad \textit{for} \quad t \in [a, b],$$

where $\rho_1 \equiv 1$ *and*

$$\rho_{i+1}(t) \stackrel{def}{=} -\frac{1}{\lambda-1}\int_a^b \ell(\rho_i)(s)ds - \int_t^b \ell(\rho_i)(s)ds$$
$$\text{for } t \in [a,b], \quad i \in N;$$

c) *there exists* $\overline{\ell} \in \mathcal{P}_{ab}$ *such that*

$$\frac{1}{\lambda}\exp\left(\int_a^b |\ell(1)(s)|ds\right) + \\ + \int_a^b \overline{\ell}(1)(s)\exp\left(\int_a^s |\ell(1)(\xi)|d\xi\right)ds < 1 \tag{1.15}$$

and on the set $\{v \in C([a,b];R_+) : v(a) = \lambda v(b)\}$ *the inequality*

$$\ell(1)(t)\vartheta(v)(t) - \ell(\vartheta(v))(t) \le \overline{\ell}(v)(t) \quad \text{for} \quad t \in [a,b]$$

holds, where

$$\vartheta(v)(t) \stackrel{def}{=} -\frac{1}{\lambda-1}\int_a^b \ell(v)(s)ds - \int_t^b \ell(v)(s)ds \quad \text{for} \quad t \in [a,b].$$

Then $\ell \in V^-(\lambda)$.

Remark 1.6. The conditions in Corollary 1.3 are nonimprovable. More precisely, the assumption $\alpha \in\,]0,1[$ cannot be replaced by the assumption $\alpha \in\,]0,1]$, and the strict inequalities (1.14) and (1.15) cannot be replaced by the nonstrict ones (see On Remark 1.6).

Theorem 1.5. *Let* $\lambda \in [1,+\infty[$, $\ell \in \mathcal{P}_{ab}$, ℓ *be a* $b-$*Volterra operator, and let there exist a function* $\gamma \in \widetilde{C}([a,b];R_+)$ *such that*

$$\begin{aligned} \gamma(t) &> 0, \quad \text{for} \quad t \in\,]a,b], \\ \gamma'(t) &\ge \ell(\gamma)(t) \quad \text{for} \quad t \in [a,b]. \end{aligned} \tag{1.16}$$

Then $\ell \in V^-(\lambda)$.

Remark 1.7. Theorem 1.5 is nonimprovable. More precisely, the condition (1.16) cannot be replaced by the condition

$$\gamma(t) > 0, \quad \text{for} \quad t \in\,]a_1,b],$$

where $a_1 \in\,]a,b[$ is an arbitrary fixed point (see On Remark 1.6).

Theorem 1.6. *Let* $\lambda \in [1,+\infty[$, $\ell \in \mathcal{P}_{ab}$, *and at least one of the following items be fulfilled:*

a) $$\int_a^b \ell(1)(s)ds \le \frac{1}{\lambda}; \tag{1.17}$$

b) *ℓ is a $b-$Volterra operator and the inequality* (1.4) *holds.*
Then $\ell \in V^-(\lambda)$.

Remark 1.8. Theorem 1.6 is nonimprovable in the sense that the inequalities (1.17) and (1.4) cannot be replaced by the inequalities

$$\int_a^b \ell(1)(s)ds \le \frac{1}{\lambda} + \varepsilon$$

and (1.3), no matter how small $\varepsilon > 0$ would be (see On Remark 1.6).

Corollary 1.4. *Let $\lambda \in [1, +\infty[$, $\ell \in \mathcal{P}_{ab}$, ℓ be a $b-$Volterra operator, and let*

$$\int_a^b \widetilde{\ell}(1)(s) \exp\left(\int_s^b \ell(1)(\xi)d\xi\right) ds \le 1, \tag{1.18}$$

where

$$\widetilde{\ell}(v)(t) \stackrel{def}{=} \ell(\widetilde{\theta}(v))(t) - \ell(1)(t)\widetilde{\theta}(v)(t) \quad for \quad t \in [a,b],$$

$$\widetilde{\theta}(v)(t) \stackrel{def}{=} -\int_t^b \ell(\widetilde{v})(s)ds \quad for \quad t \in [a,b],$$

$$\widetilde{v}(t) \stackrel{def}{=} v(t)\exp\left(-\int_t^b \ell(1)(s)ds\right) \quad for \quad t \in [a,b].$$

Then $\ell \in V^-(\lambda)$.

Remark 1.9. Corollary 1.4 is nonimprovable in the sense that the inequality (1.18) cannot be replaced by the inequality

$$\int_a^b \widetilde{\ell}(1)(s) \exp\left(\int_s^b \ell(1)(\xi)d\xi\right) ds \le 1 + \varepsilon,$$

no matter how small $\varepsilon > 0$ would be (see On Remark 1.6).

Theorem 1.7. *Let $\lambda \in]1, +\infty[$ and the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$. If, moreover, $\ell_0 \in V^-(\lambda)$ and $-\ell_1 \in V^-(\lambda)$, then $\ell \in V^-(\lambda)$.*

Remark 1.10. Theorem 1.7 is nonimprovable in the sense that the assumption

$$\ell_0 \in V^-(\lambda), \qquad -\ell_1 \in V^-(\lambda)$$

can be replaced neither by the assumption

$$(1-\varepsilon)\ell_0 \in V^-(\lambda), \qquad -\ell_1 \in V^-(\lambda),$$

nor by the assumption

$$\ell_0 \in V^-(\lambda), \qquad -(1-\varepsilon)\ell_1 \in V^-(\lambda),$$

no matter how small $\varepsilon > 0$ would be (see On Remark 1.6).

1.2. Equations With Deviating Arguments. The theorems established in Section 1.1 imply the following results for the differential equations with deviating arguments.

Theorem 1.8. *Let* $\lambda \in \,]0,1]$, $p \in L([a,b];R_+)$, $\tau \in \mathcal{M}_{ab}$, *and*

$$\frac{1-\lambda}{\lambda} < \int_a^b p(s)ds \le 1. \tag{1.19}$$

Then the operator ℓ *defined by*

$$\ell(v)(t) \stackrel{def}{=} p(t)v(\tau(t)) \tag{1.20}$$

belongs to the set $V^-(\lambda)$.

Remark 1.11. Theorem 1.8 is nonimprovable. More precisely, the strict inequality in the condition (1.19) cannot be replaced by the nonstrict one and the nonstrict inequality in (1.19) cannot be replaced by the inequality

$$\int_a^b p(s)ds \le 1 + \varepsilon, \tag{1.21}$$

no matter how small $\varepsilon > 0$ would be (see Examples 4.1 and 4.2).

Theorem 1.9. *Let* $\lambda \in \,]0,1[$, $p \in L([a,b];R_+)$, $p \not\equiv 0$, $\tau \in \mathcal{M}_{ab}$, *and let there exist* $x \in \left[\ln\frac{1}{\lambda}, +\infty\right[$ *such that*

$$\operatorname{ess\,inf}\left\{\int_t^{\tau(t)} p(s)ds : t \in [a,b]\right\} > \frac{\|p\|_L}{x}\left(x + \ln\frac{(1-\lambda)x}{\|p\|_L\,(e^x-1)}\right), \tag{1.22}$$

$$\|p\|_L \le 1. \tag{1.23}$$

Then the operator ℓ *defined by* (1.20) *belongs to the set* $V^-(\lambda)$.

Remark 1.12. Theorem 1.9 is nonimprovable in the sense that the inequality (1.23) cannot be replaced by (1.21), no matter how small $\varepsilon > 0$ would be (see Example 4.2).

Corollary 1.5. *Let $\lambda \in \,]0,1[$, $p \in L([a,b];R_+)$, $p \not\equiv 0$, $\tau \in \mathcal{M}_{ab}$,*

$$\text{ess inf}\left\{\int_t^{\tau(t)} p(s)ds : t \in [a,b]\right\} > \frac{\|p\|_L}{|\ln\lambda|}\ln\frac{|\ln\lambda|}{\|p\|_L},$$

and the condition (1.23) *hold. Then the operator ℓ defined by* (1.20) *belongs to the set $V^-(\lambda)$.*

Theorem 1.10. *Let $\lambda \in \,]0,1[$, $p \in L([a,b];R_+)$, $\tau \in \mathcal{M}_{ab}$, $\tau(t) \geq t$ for $t \in [a,b]$, and let*

$$\text{ess sup}\left\{\int_t^{\tau(t)} p(s)ds : t \in [a,b]\right\} < \vartheta^*, \tag{1.24}$$

where

$$\vartheta^* = \sup\left\{\frac{1}{x}\ln\left(x + \frac{x}{\exp\left(x\int_a^b p(s)ds\right) - 1}\right) : x > 0\right\}. \tag{1.25}$$

If, moreover,

$$\int_a^b p(s)ds > \ln\frac{1}{\lambda}, \tag{1.26}$$

then the operator ℓ defined by (1.20) *belongs to the set $V^-(\lambda)$.*

Corollary 1.6. *Let $\lambda \in \,]0,1[$, $p \in L([a,b];R_+)$, $\tau \in \mathcal{M}_{ab}$, $\tau(t) \geq t$ for $t \in [a,b]$, the condition* (1.26) *hold, and*

$$\int_t^{\tau(t)} p(s)ds \leq \frac{1}{e} \quad \text{for} \quad t \in [a,b]. \tag{1.27}$$

Then the operator ℓ defined by (1.20) *belongs to the set $V^-(\lambda)$.*

Remark 1.13. It is clear that for the ordinary differential equations, i.e., if ℓ is defined by

$$\ell(v)(t) \stackrel{def}{=} p(t)v(t) \quad \text{for} \quad t \in [a,b], \tag{1.28}$$

where $p \in L([a,b];R_+)$, the conditions (1.24) and (1.27) are fulfilled, and the condition (1.26) is sufficient and necessary condition for the operator ℓ given by (1.28) to belong to the set $V^-(\lambda)$ with $\lambda \in\,]0,1]$. Thus, the condition (1.26) in Theorem 1.10 and Corollary 1.6 cannot be weakened.

Theorem 1.11. *Let $\lambda \in\,]1,+\infty[$, $g \in L([a,b];R_+)$, $\mu \in \mathcal{M}_{ab}$, and at least one of the following items be fulfilled:*

a) *$\mu(t) \geq t$ for $t \in [a,b]$ and*

$$\int_a^b g(s)ds < \ln\lambda; \tag{1.29}$$

b) *there exists $\alpha \in\,]0,1[$ such that*

$$\frac{1}{\lambda-1}\int_a^b g(s)\left(\int_{\mu(s)}^b g(\xi)d\xi\right)ds + \int_t^b g(s)\left(\int_{\mu(s)}^b g(\xi)d\xi\right)ds \leq$$

$$\leq \left(\alpha - \frac{1}{\lambda-1}\int_a^b g(s)ds\right)\left(\frac{1}{\lambda-1}\int_a^b g(s)ds + \int_t^b g(s)ds\right) \quad \textit{for} \quad t \in [a,b];$$

c) $$\frac{1}{\lambda}\exp\left(\int_a^b g(s)ds\right) +$$

$$+\int_a^b g(s)\sigma(s)\left(\int_{\mu(s)}^s g(\xi)d\xi\right)\exp\left(\int_a^s g(\eta)d\eta\right)ds < 1, \tag{1.30}$$

where $\sigma(t) = \frac{1}{2}(1+\operatorname{sgn}(t-\mu(t)))$ for $t \in [a,b]$.

Then the operator ℓ defined by

$$\ell(v)(t) \stackrel{def}{=} -g(t)v(\mu(t)) \tag{1.31}$$

belongs to the set $V^-(\lambda)$.

Remark 1.14. Theorem 1.11 is nonimprovable in the sense that the assumption $\alpha \in\,]0,1[$ cannot be replaced by the assumption $\alpha \in\,]0,1]$ and the strict inequalities (1.29) and (1.30) cannot by replaced by the nonstrict ones (see On Remark 1.6).

Theorem 1.12. *Let $\lambda \in]1,+\infty[$, $g \in L([a,b];R_+)$, $g \not\equiv 0$, $\mu \in \mathcal{M}_{ab}$, and let there exist $x \in]0, \ln\lambda]$ such that*

$$\operatorname{ess\,sup}\left\{\int\limits_{\mu(t)}^{t} g(s)ds : t \in [a,b]\right\} < \frac{\|g\|_L}{x}\left(x + \ln\frac{(\lambda-1)x}{\lambda\|g\|_L\,(e^x-1)}\right).$$

Then the operator ℓ defined by (1.31) *belongs to the set $V^-(\lambda)$.*

Corollary 1.7. *Let $\lambda \in]1,+\infty[$, $g \in L([a,b];R_+)$, $g \not\equiv 0$, $\mu \in \mathcal{M}_{ab}$, and let*

$$\operatorname{ess\,sup}\left\{\int\limits_{\mu(t)}^{t} g(s)ds : t \in [a,b]\right\} < \frac{\|g\|_L}{\ln\lambda}\ln\frac{\ln\lambda}{\|g\|_L}.$$

Then the operator ℓ defined by (1.31) *belongs to the set $V^-(\lambda)$.*

Theorem 1.13. *Let $\lambda \in [1,+\infty[$, $p \in L([a,b];R_+)$, $\tau \in \mathcal{M}_{ab}$, and at least one of the following items be fulfilled:*

a)
$$\int\limits_a^b p(s)ds \le \frac{1}{\lambda}; \tag{1.32}$$

b) *$\tau(t) \ge t$ for $t \in [a,b]$ and*

$$\int\limits_a^b p(s)ds \le 1; \tag{1.33}$$

c) *$\tau(t) \ge t$ for $t \in [a,b]$, $p \not\equiv 0$ and*

$$\int\limits_a^b p(s)\int\limits_s^{\tau(s)} p(\xi)\exp\left(\int\limits_s^{\tau(\xi)} p(\eta)d\eta\right)d\xi ds \le 1; \tag{1.34}$$

d) *$\tau(t) \ge t$ for $t \in [a,b]$, $p \not\equiv 0$, and the inequality* (1.24) *is fulfilled, where ϑ^* is defined by* (1.25).

Then the operator ℓ defined by (1.20) *belongs to the set $V^-(\lambda)$.*

Remark 1.15. The assumptions a), b), and c) in Theorem 1.13 are nonimprovable. More precisely, the inequalities (1.32), (1.33), and (1.34) cannot be replaced by the inequalities

$$\int\limits_a^b p(s)ds \le \frac{1}{\lambda} + \varepsilon, \qquad \int\limits_a^b p(s)ds \le 1 + \varepsilon,$$

and

$$\int_a^b p(s) \int_s^{\tau(s)} p(\xi) \exp\left(\int_s^{\tau(\xi)} p(\eta)d\eta \right) d\xi ds \le 1 + \varepsilon,$$

no matter how small $\varepsilon > 0$ would be (see On Remark 1.6).

Theorem 1.14. *Let $\lambda \in\,]1, +\infty[$, $p, g \in L([a,b]; R_+)$, $\tau, \mu \in \mathcal{M}_{ab}$, and let the functions p, τ satisfy at least one of the conditions* a), b), c) *or* d) *in Theorem* 1.13, *while the functions g, μ satisfy at least one of the conditions* a), b) *or* c) *in Theorem* 1.11 *or the assumptions of Theorem* 1.12. *Then the operator ℓ defined by*

$$\ell(v)(t) \stackrel{def}{=} p(t)v(\tau(t)) - g(t)v(\mu(t)) \tag{1.35}$$

belongs to the set $V^-(\lambda)$.

Remark 1.16. Theorem 1.14 is nonimprovable in a certain sense (see On Remark 1.6).

2. On the Set $V^+(\lambda)$

2.1. Main Results.

Theorem 2.1. *Let $\lambda \in [1, +\infty[$, $-\ell \in \mathcal{P}_{ab}$, and*

$$\lambda - 1 < \int_a^b |\ell(1)(s)|ds \le 1.$$

Then $\ell \in V^+(\lambda)$.

Theorem 2.2. *Let $\lambda \in [1, +\infty[$, $-\ell \in \mathcal{P}_{ab}$, there exist $\delta > 0$ such that $\ell \notin V^-(\lambda + \delta)$, and*

$$\int_a^b |\ell(1)(s)|ds \le 1. \tag{2.1}$$

Then $\ell \in V^+(\lambda)$.

Corollary 2.1. *Let $\lambda \in [1, +\infty[$, $-\ell \in \mathcal{P}_{ab}$, and let there exist $\gamma \in \widetilde{C}([a,b]; R)$ satisfying the inequalities*

$$\gamma'(t) \ge \ell(\gamma)(t) \quad \text{for} \quad t \in [a,b], \tag{2.2}$$

$$\gamma(a) > \lambda\gamma(b), \tag{2.3}$$

$$\gamma(a) > 0. \tag{2.4}$$

If, moreover, the inequality (2.1) *holds, then $\ell \in V^+(\lambda)$.*

Theorem 2.3. *Let* $\lambda \in [1, +\infty[$, $-\ell \in \mathcal{P}_{ab}$, ℓ *be an* $a-$ *Volterra operator, and let there exist* $\delta > 0$ *such that* $\ell \notin V^-(\lambda + \delta)$. *If, moreover, there exists a function* $\beta \in \widetilde{C}([a,b]; R_+)$ *satisfying*

$$\beta(t) > 0 \quad \textit{for} \quad t \in [a, b[\,, \tag{2.5}$$

$$\beta'(t) \le \ell(\beta)(t) \quad \textit{for} \quad t \in [a, b], \tag{2.6}$$

then $\ell \in V^+(\lambda)$.

Corollary 2.2. *Let* $\lambda \in [1, +\infty[$, $-\ell \in \mathcal{P}_{ab}$, ℓ *be an* $a-$ *Volterra operator, and let there exist a function* $\gamma \in \widetilde{C}([a,b]; R)$ *satisfying the inequalities* (2.2), (2.3), *and* (2.4). *If, moreover, there exists a function* $\beta \in \widetilde{C}([a,b]; R_+)$ *satisfying the inequalities* (2.5) *and* (2.6), *then* $\ell \in V^+(\lambda)$.

Remark 2.1. Theorems 2.1–2.3 and Corollaries 2.1 and 2.2 are nonimprovable in a certain sense (see On Remark 2.1).

2.2. Equations with Deviating Arguments.

Theorem 2.4. *Let* $\lambda \in [1, +\infty[$, $g \in L([a,b]; R_+)$, $\mu \in \mathcal{M}_{ab}$, *and*

$$\lambda - 1 < \int_a^b g(s)ds \le 1.$$

Then the operator ℓ *defined by* (1.31) *belongs to the set* $V^+(\lambda)$.

Theorem 2.5. *Let* $\lambda \in]1, +\infty[$, $g \in L([a,b]; R_+)$, $g \not\equiv 0$, $\mu \in \mathcal{M}_{ab}$, *and let there exist* $x \in [\ln \lambda, +\infty[$ *such that*

$$\operatorname{ess\,inf}\left\{ \int_{\mu(t)}^{t} g(s)ds : t \in [a,b] \right\} > \frac{\|g\|_L}{x}\left(x + \ln \frac{(\lambda - 1)x}{\lambda \|g\|_L \left(e^x - 1\right)} \right), \tag{2.7}$$

$$\|g\|_L \le 1. \tag{2.8}$$

Then the operator ℓ *defined by* (1.31) *belongs to the set* $V^+(\lambda)$.

Corollary 2.3. *Let* $\lambda \in]1, +\infty[$, $g \in L([a,b]; R_+)$, $g \not\equiv 0$, $\mu \in \mathcal{M}_{ab}$,

$$\operatorname{ess\,inf}\left\{ \int_{\mu(t)}^{t} g(s)ds : t \in [a,b] \right\} > \frac{\|g\|_L}{\ln \lambda} \ln \frac{\ln \lambda}{\|g\|_L},$$

and let the condition (2.8) *hold. Then the operator* ℓ *defined by* (1.31) *belongs to the set* $V^+(\lambda)$.

Theorem 2.6. *Let $\lambda \in]1,+\infty[$, $g \in L([a,b];R_+)$, $\mu \in \mathcal{M}_{ab}$, $\mu(t) \leq t$ for $t \in [a,b]$, and*

$$\text{ess sup}\left\{\int_{\mu(t)}^{t} g(s)ds : t \in [a,b]\right\} < \eta^*, \tag{2.9}$$

where

$$\eta^* = \sup\left\{\frac{1}{x}\ln\left(x + \frac{x}{\exp\left(x\int_a^b g(s)ds\right) - 1}\right) : x > 0\right\}. \tag{2.10}$$

If, moreover,

$$\int_a^b g(s)ds > \ln\lambda, \tag{2.11}$$

then the operator ℓ defined by (1.31) *belongs to the set $V^+(\lambda)$.*

Corollary 2.4. *Let $\lambda \in]1,+\infty[$, $g \in L([a,b];R_+)$, $\mu \in \mathcal{M}_{ab}$, $\mu(t) \leq t$ for $t \in [a,b]$, the condition* (2.11) *hold, and*

$$\int_{\mu(t)}^{t} g(s)ds \leq \frac{1}{e} \quad \textit{for} \quad t \in [a,b].$$

Then the operator ℓ defined by (1.31) *belongs to the set $V^+(\lambda)$.*

Remark 2.2. Theorems 2.4–2.6 and Corollaries 2.3 and 2.4 are nonimprovable in a certain sense (see On Remark 2.2).

3. Proofs

Proof of Theorem 1.1. Let u be a nontrivial solution of the problem (0.1), (0.3) with $q \in L([a,b];R_+)$ and $c \in R_+$. We will show that

$$u(t) \leq 0 \quad \text{for} \quad t \in [a,b]. \tag{3.1}$$

Assume that u changes its sign. Put

$$M = \max\{u(t) : t \in [a,b]\}, \qquad m = -\min\{u(t) : t \in [a,b]\}, \tag{3.2}$$

and choose $t_M, t_m \in [a,b]$ such that

$$u(t_M) = M, \qquad u(t_m) = -m. \tag{3.3}$$

Obviously $M > 0$, $m > 0$, and either

$$t_M < t_m \tag{3.4}$$

or

$$t_M > t_m. \tag{3.5}$$

First suppose that (3.4) is fulfilled. The integration of (0.1) from t_M to t_m, in view of (1.1), (3.2), (3.3), and the assumption $q \in L([a,b]; R_+)$, results in

$$M + m = -\int_{t_M}^{t_m} \ell(u)(s)ds - \int_{t_M}^{t_m} q(s)ds \le m \int_{t_M}^{t_m} \ell(1)(s)ds \le m \int_a^b \ell(1)(s)ds.$$

According to (1.2), we obtain $M + m \le m$, a contradiction.

Now suppose that (3.5) holds. The integration of (0.1) from a to t_m and from t_M to b, on account of (1.1), (3.2), (3.3), and the assumption $q \in L([a,b]; R_+)$, yields

$$u(a) + m = -\int_a^{t_m} \ell(u)(s)ds - \int_a^{t_m} q(s)ds \le m \int_a^{t_m} \ell(1)(s)ds, \tag{3.6}$$

$$M - u(b) = -\int_{t_M}^{b} \ell(u)(s)ds - \int_{t_M}^{b} q(s)ds \le m \int_{t_M}^{b} \ell(1)(s)ds. \tag{3.7}$$

Multiplying both sides of (3.7) by λ and taking into account (1.1) and the assumptions $\lambda \in]0,1]$ and $m > 0$, we get

$$\lambda M - \lambda u(b) \le m \int_{t_M}^{b} \ell(1)(s)ds.$$

Summing the last inequality and (3.6), on account of (0.3), (1.1), and the assumptions $c \in R_+$ and $m > 0$, results in

$$\lambda M + m \le c + \lambda M + m \le m \int_a^b \ell(1)(s)ds.$$

The last inequality, on account of (1.2), yields the contradiction $\lambda M + m \le m$. Therefore, u does not change its sign.

Now assume the that (3.1) is not valid. Due to above–proved we have

$$u(t) \ge 0 \quad \text{for} \quad t \in [a,b], \qquad u \not\equiv 0. \tag{3.8}$$

From this inequality, (0.1), (1.1), and the assumption $q \in L([a,b]; R_+)$, it follows that

$$u'(t) \ge 0 \quad \text{for} \quad t \in [a,b]. \tag{3.9}$$

The integration of (0.1) from a to b, in view of the assumption $q \in L([a,b];R_+)$ implies

$$u(a) - u(b) = -\int_a^b \ell(u)(s)ds - \int_a^b q(s)ds \le -\int_a^b \ell(u)(s)ds.$$

Hence, by (0.3), (1.1), (3.9), and the assumption $c \in R_+$, we obtain

$$u(b)(\lambda - 1) \le c + \lambda u(b) - u(b) \le$$
$$\le -u(a)\int_a^b \ell(1)(s)ds \le -\lambda u(b)\int_a^b \ell(1)(s)ds. \tag{3.10}$$

Obviously, (3.8) and (3.9) imply $u(b) > 0$, and therefore, from (3.10) we get

$$\frac{1-\lambda}{\lambda} \ge \int_a^b \ell(1)(s)ds,$$

which contradicts the first inequality in (1.2).

We have proved that if u is a nontrivial solution of the problem (0.1), (0.3) with $q \in L([a,b];R_+)$ and $c \in R_+$, then the inequality (3.1) is satisfied. Now suppose that the homogeneous problem (0.1_0), (0.3_0) has a nontrivial solution u_0. Obviously, $-u_0$ is a nontrivial solution of the problem (0.1_0), (0.3_0), as well, and, according to above–proved, we have

$$u_0(t) \le 0, \qquad -u_0(t) \le 0 \quad \text{for} \quad t \in [a,b],$$

i.e., $u_0 \equiv 0$, a contradiction. □

Proof of Theorem 1.2. Let u be a solution of the problem (0.1), (0.3) with $q \in L([a,b];R_+)$ and $c \in R_+$. We will show that (3.1) holds.

Assume that (3.1) is not valid. Analogously to the proof of Theorem 1.1, in view of (1.4), we obtain that u does not change its sign. Therefore (3.8) holds. From (0.1), (1.1), (3.8), and the assumption $q \in L([a,b];R_+)$ it follows that

$$u'(t) \ge \ell(u)(t) \ge 0 \quad \text{for} \quad t \in [a,b]. \tag{3.11}$$

Obviously, (3.8) and (3.11) imply $u(b) > 0$, and therefore, by virtue of (0.3) and the assumptions $c \in R_+$ and $\delta \in\,]0,\lambda]$, we have

$$u(a) = \lambda u(b) + c \ge \lambda u(b) > (\lambda - \delta)u(b) \ge 0. \tag{3.12}$$

According to (3.11) and (3.12), we obtain

$$u(t) > 0 \quad \text{for} \quad t \in [a,b].$$

From this inequality, (3.11), (3.12), and Theorem 1.1 in [8] it follows that $\ell \in V^+(\lambda - \delta)$, which contradicts the assumption of the theorem.

We have proved that if u is a solution of the problem (0.1), (0.3) with $q \in L([a,b];R_+)$ and $c \in R_+$, then the inequality (3.1) is satisfied. Now we will show that the homogeneous problem (0.1_0), (0.3_0) has only the trivial solution. Indeed, let u_0 be a solution of this problem. Obviously, $-u_0$ is a solution of this problem, as well, and, according to above–proved, we have

$$u_0(t) \le 0, \qquad -u_0(t) \le 0 \quad \text{for} \quad t \in [a,b].$$

Therefore $u_0 \equiv 0$. □

Proof of Corollary 1.1. We will show that the conditions (1.7), (1.8), and (1.9) imply $\ell \notin V^+(\lambda - \delta)$ for a suitable $\delta \in\,]0,\lambda]$. Indeed, according to (1.7) and (1.1), we obtain

$$[\gamma(t)]'_+ \le \ell([\gamma]_+)(t) \quad \text{for} \quad t \in [a,b].$$

Moreover, (1.8) and (1.9) yield

$$[\gamma(a)]_+ < \lambda[\gamma(b)]_+.$$

According to the last inequality, there exists $\delta \in\,]0,\lambda]$ such that

$$[\gamma(a)]_+ = (\lambda - \delta)[\gamma(b)]_+.$$

Since $\gamma(b) > 0$, the function $[\gamma]_+$ is a nontrivial nonnegative solution of the problem

$$u'(t) \le \ell(u)(t), \qquad u(a) - (\lambda - \delta)u(b) = 0.$$

Thus, by Proposition 1.1 in [8], we have $\ell \notin V^+(\lambda - \delta)$.

Therefore the assumptions of Theorem 1.2 are fulfilled. □

Proof of Theorem 1.3. Let u be a solution of the problem (0.1), (0.3) with $q \in L([a,b];R_+)$ and $c \in R_+$. We will show that (3.1) holds.

Suppose that $u(a) > 0$. Then there exists $t_0 \in\,]a,b[$ such that

$$u(t) > 0 \quad \text{for} \quad t \in [a,t_0]. \tag{3.13}$$

Since ℓ is a $b-$Volterra operator, the restriction of u to the interval $[t_0,b]$ is a solution of the equation (0.1) with the condition $u(t_0) > 0$. Moreover, the restriction of β to the interval $[t_0,b]$ is a positive absolutely continuous function satisfying the inequality

$$\beta'(t) \ge \ell(\beta)(t) \quad \text{for} \quad t \in [t_0,b]. \tag{3.14}$$

According to Theorem 1.1 in [8] (for $\lambda = 0$, $a = t_0$) and the assumptions $\ell \in \mathcal{P}_{ab}$, $u(t_0) > 0$, and $q \in L([a,b];R_+)$, we get

$$u(t) > 0 \quad \text{for} \quad t \in [t_0,b]. \tag{3.15}$$

The inequality (3.15), together with (3.13), yields

$$u(t) > 0 \quad \text{for} \quad t \in [a,b]. \tag{3.16}$$

In view of (0.1), (0.3), (3.16), and the assumptions $q \in L([a,b];R_+)$, $c \in R_+$, and $\delta \in \,]0,\lambda]$, we have

$$u'(t) \geq \ell(u)(t) \quad \text{for} \quad t \in [a,b], \tag{3.17}$$

$$u(a) \geq \lambda u(b) > (\lambda - \delta)u(b). \tag{3.18}$$

However, (3.16), (3.17), and (3.18) guarantee that $\ell \in V^+(\lambda - \delta)$ (see Theorem 1.1 in [8]), which contradicts the assumption of the theorem.

Therefore $u(a) \leq 0$. In view of (0.3) and the assumptions $\lambda > 0$ and $c \in R_+$, the inequality

$$u(b) \leq 0 \tag{3.19}$$

holds. Since ℓ is a $b-$Volterra operator, according to (1.11), (1.12), (3.19), and Theorem 1.6 in [7], the inequality (3.1) is fulfilled.

We have proved that if u is a solution of the problem (0.1), (0.3) with $q \in L([a,b];R_+)$ and $c \in R_+$, then the inequality (3.1) is satisfied. Now we will show that the homogeneous problem (0.1_0), (0.3_0) has only the trivial solution. Indeed, let u_0 be a solution of this problem. Obviously, $-u_0$ is a solution of this problem, as well, and, according to above–proved, we have

$$u_0(t) \leq 0, \qquad -u_0(t) \leq 0 \quad \text{for} \quad t \in [a,b].$$

Therefore $u_0 \equiv 0$. □

Proof of Corollary 1.2. By the same arguments as in the proof of Corollary 1.1, in view of (1.7), (1.8), and (1.9), we get $\ell \notin V^+(\lambda-\delta)$ for a suitable $\delta \in \,]0,\lambda]$. Therefore the assumptions of Theorem 1.3 are fulfilled. □

Introduce the following notation. Let $\lambda \in [1,+\infty[$, $\ell \in \mathcal{L}_{ab}$, and $q \in L([a,b];R)$. Let the operator $\psi : L([a,b];R) \to L([a,b];R)$ be defined by

$$\psi(w)(t) \stackrel{def}{=} w(a+b-t) \quad \text{for} \quad t \in [a,b]$$

and let φ be a restriction of ψ to the space $C([a,b];R)$. Put $\vartheta = \frac{1}{\lambda}$, $\widehat{c} = -\frac{c}{\lambda}$,

$$\widehat{\ell}(w)(t) \stackrel{def}{=} -\psi(\ell(\varphi(w)))(t), \qquad \widehat{q}(t) = -\psi(q)(t) \quad \text{for} \quad t \in [a,b].$$

It is clear that if $u \in \widetilde{C}([a,b];R)$ is a solution of the problem

$$u'(t) = \ell(u)(t) + q(t), \qquad u(a) - \lambda u(b) = c, \tag{3.20}$$

then the function $v \stackrel{def}{=} \varphi(u)$ is a solution of the problem

$$v'(t) = \widehat{\ell}(v)(t) + \widehat{q}(t), \qquad v(a) - \vartheta v(b) = \widehat{c}, \tag{3.21}$$

and vice versa, if $v \in \widetilde{C}([a,b];R)$ is a solution of the problem (3.21), then the function $u \stackrel{def}{=} \varphi(v)$ is a solution of the problem (3.20). Therefore the following proposition is valid.

Proposition 3.1. $\ell \in V^+(\lambda)$ *(resp.* $\ell \in V^-(\lambda)$*) iff* $\widehat{\ell} \in V^-(\vartheta)$ *(resp.* $\widehat{\ell} \in V^+(\vartheta)$*).*

Proofs of Proposition 1.1, *Theorems* 1.4–1.7 *and Corollaries* 1.3 and 1.4. The conditions guaranteeing the inclusion $\ell \in V^+(\lambda)$ with $\lambda \in [0,1]$ have been established in [8]. According to Proposition 3.1, Proposition 1.1, Theorems 1.4–1.7 and Corollaries 1.3 and 1.4 can be immediately derived from Proposition 1.1, Theorems 1.1–1.4 and Corollaries 1.1 and 1.2 from [8]. □

Proof of Theorem 1.8. It immediately follows from Theorem 1.1. □

Proof of Theorem 1.9. According to (1.22), there exists $\varepsilon \in\,]1,+\infty[$ such that

$$\int_t^{\tau(t)} p(s)ds \geq \frac{\|p\|_L}{\varepsilon x} \ln \frac{\varepsilon x e^{\varepsilon x}}{\|p\|_L \left(e^{\varepsilon x} + \frac{\lambda e^x - 1}{1-\lambda}\right)} \quad \text{for} \quad t \in [a,b]. \tag{3.22}$$

Put

$$x_0 = \frac{\varepsilon x}{\|p\|_L}. \tag{3.23}$$

Obviously, $x_0 > 0$. By (3.22), (3.23), and the assumption $x \in \left[\ln \frac{1}{\lambda}, +\infty\right[$, we obtain

$$\int_t^{\tau(t)} p(s)ds \geq \frac{1}{x_0} \ln \frac{x_0 e^{x_0 \int_a^{\tau(t)} p(s)ds}}{e^{x_0 \int_a^{\tau(t)} p(s)ds} + \frac{\lambda e^x - 1}{1-\lambda}} \quad \text{for} \quad t \in [a,b], \tag{3.24}$$

$$x_0 \|p\|_L > x. \tag{3.25}$$

Define the function $\gamma \in \widetilde{C}([a,b];R)$ by

$$\gamma(t) = e^{x_0 \int_a^t p(s)ds} + \frac{\lambda e^x - 1}{1-\lambda} \quad \text{for} \quad t \in [a,b]. \tag{3.26}$$

Obviously, if ℓ is defined by (1.20), then (1.23) implies (1.4), and, by virtue of (3.24), (3.25), and (3.26), the function γ satisfies the inequalities (1.7), (1.8), and (1.9). Therefore, according to Corollary 1.1, $\ell \in V^-(\lambda)$. □

Proof of Corollary 1.5. It immediately follows from Theorem 1.9 for $x = \ln \frac{1}{\lambda}$. □

Proof of Theorem 1.10. According to (1.24), there exists $\varepsilon > 0$ such that

$$\int_t^{\tau(t)} p(s)ds < \vartheta^* - \varepsilon \quad \text{for} \quad t \in [a,b]. \tag{3.27}$$

In view of (1.25) we can choose $x_1 > 0$ and $\delta \in]0,1[$ such that

$$\frac{1}{x_1} \ln \left(x_1 + \frac{x_1(1-\delta)}{\exp\left(x_1 \int_a^b p(s)ds\right) - (1-\delta)} \right) > \vartheta^* - \varepsilon. \tag{3.28}$$

Put

$$\beta(t) = \exp\left(x_1 \int_a^t p(s)ds \right) - (1-\delta) \quad \text{for} \quad t \in [a,b].$$

It is not difficult to verify that $\beta(t) > 0$ for $t \in [a,b]$ and, according to (3.27) and (3.28), the inequalities (1.11) and (1.12) are fulfilled.

It can be easily shown that according to (1.26) and the assumption $\tau(t) \geq t$ for $t \in [a,b]$, the inequality (1.22) is fulfilled with $x = \ln\frac{1}{\lambda}$. Therefore, analogously to the proof of Theorem 1.9, there exists a function $\gamma \in \widetilde{C}([a,b];R)$ satisfying the inequalities (1.7), (1.8), and (1.9).

Therefore, the assumptions of Corollary 1.2 are fulfilled. □

Proof of Corollary 1.6. It immediately follows from Theorem 1.10. □

Proofs of Theorems 1.11–1.14 *and Corollary* 1.7. The conditions guaranteeing the inclusion $\ell \in V^+(\lambda)$ with $\lambda \in [0,1]$ have been established in [8]. According to Proposition 3.1, Theorems 1.11–1.14 and Corollary 1.7 can be immediately derived from Theorems 2.1–2.4 and Corollary 2.1 from [8]. □

Proofs of Theorems 2.1–2.6 *and Corollaries* 2.1–2.4. According to Proposition 3.1, Theorems 2.1–2.6 and Corollaries 2.1–2.4 can be immediately derived from Theorems 1.1–1.3 and 1.8–1.10 and Corollaries 1.1, 1.2, 1.5, and 1.6. □

4. Comments and Examples

Example 4.1. Let $p \in L([a,b];R_+)$ and ℓ be defined by (1.20). If

$$\int_a^b p(s)ds < 1 - \lambda,$$

then, according to Remark 1.2 in [8], $\ell \in V^+(\lambda)$. Consequently, $\ell \notin V^-(\lambda)$. Therefore, suppose that

$$1-\lambda \le \int_a^b p(s)ds \le \frac{1-\lambda}{\lambda}. \tag{4.1}$$

Define

$$u(t) = \frac{1}{1-\lambda}\int_a^t p(s)ds + \frac{\lambda}{1-\lambda}\int_t^b p(s)ds \quad \text{for} \quad t \in [a,b].$$

Obviously, in view of (4.1),

$$u(a) = \frac{\lambda}{1-\lambda}\int_a^b p(s)ds \le 1, \qquad u(b) = \frac{1}{1-\lambda}\int_a^b p(s)ds \ge 1.$$

Consequently, there exists $t_0 \in [a,b]$ such that $u(t_0) = 1$. Put $\tau(t) = t_0$ for $t \in [a,b]$. Then u is a nontrivial solution of the problem

$$u'(t) = p(t)u(\tau(t)), \qquad u(a) - \lambda u(b) = 0. \tag{4.2}$$

Therefore $\ell \notin V^-(\lambda)$.

Example 4.1 shows that the strict inequality in (1.2) in Theorem 1.1 cannot be weakened.

This example also shows that the strict inequality in (1.19) in Theorem 1.8 cannot be weakened.

Example 4.2. Let $\lambda \in \,]0,1]$, $\varepsilon > 0$, $\tau \equiv b$, and let $p \in L([a,b];R_+)$ be such that

$$\int_a^b p(s)ds = 1+\varepsilon.$$

It is clear that the operator ℓ defined by (1.20) satisfies

$$\int_a^b \ell(1)(s)ds = 1+\varepsilon.$$

Put

$$u(t) = \varepsilon - \int_a^t p(s)ds \quad \text{for} \quad t \in [a,b].$$

Let $\delta \in \,]0,\lambda]$. Then, evidently, u is a solution of the problem

$$u'(t) = \ell(u)(t), \qquad u(a) - (\lambda-\delta)u(b) = \varepsilon + \lambda - \delta.$$

However, $u(b) = -1 < 0$, therefore, $\ell \notin V^+(\lambda - \delta)$. Moreover, $\gamma \equiv -u$ satisfies the inequalities (1.7), (1.8), and (1.9), and if $\lambda > \frac{1}{e}$, then there exists $x \in \left[\ln \frac{1}{\lambda}, +\infty\right[$ such that the inequality (1.22) is fulfilled.

On the other hand, u is a solution of the problem

$$u'(t) = \ell(u)(t), \qquad u(a) - \lambda u(b) = \varepsilon + \lambda.$$

Since $u(a) = \varepsilon > 0$, we have $\ell \notin V^-(\lambda)$.

Example 4.2 shows that the nonstrict inequality in (1.2) in Theorem 1.1 and the inequality (1.4) in Theorem 1.2 and Corollary 1.1 cannot be replaced by the inequality (1.3), no matter how small $\varepsilon > 0$ would be.

This example also shows that the nonstrict inequality in (1.19) in Theorem 1.8 and the inequality (1.23) in Theorem 1.9 and Corollary 1.5 cannot be replaced by the inequality (1.21), no matter how small $\varepsilon > 0$ would be.

Example 4.3. Let $\lambda \in]0, 1[$, $\tau \equiv b$, and let $p \in L([a, b]; R_+)$ be such that

$$\int_a^b p(s)ds = 1 - \lambda. \tag{4.3}$$

It is clear that the condition (1.4) is fulfilled, where ℓ is defined by (1.20). Moreover, the function

$$\beta(t) = \lambda + \int_a^t p(s)ds \quad \text{for} \quad t \in [a, b]$$

satisfies the inequalities (1.11) and (1.12), and the function $\gamma \equiv \beta$ satisfies the conditions (1.7), (1.9), and (1.10).

On the other hand, β is a nontrivial solution of the problem (0.1_0), (0.3_0). Therefore $\ell \notin V^+(\lambda)$. However, $\ell \notin V^-(\lambda)$, as well.

Example 4.3 shows that the assumption $\delta \in]0, \lambda]$ in Theorems 1.2 and 1.3 cannot be replaced by the assumption $\delta \in [0, \lambda]$. This example also shows that the strict inequality (1.8) in Corollaries 1.1 and 1.2 cannot be replaced by the nonstrict one.

On Remark 1.3. We will show that the inequality (1.9) in Corollary 1.1 cannot be weakened. Indeed, let (1.4) hold and let there exist a function $\gamma \in \widetilde{C}([a, b]; R)$ satisfying (1.7), (1.8), and $\gamma(b) \leq 0$. By virtue of (1.8), we have

$$\gamma(a) < 0. \tag{4.4}$$

Put $M = \max\{\gamma(t) : t \in [a, b]\}$ and choose $t_0 \in [a, b]$ such that $u(t_0) = M$.

Suppose that

$$M \geq 0. \tag{4.5}$$

Then, on account of (4.4), we find $t_0 \neq a$ and the integration of (1.7) from a to t_0, on account of (1.1), (1.4), and (4.5), yields

$$M - \gamma(a) \le \int_a^{t_0} \ell(\gamma)(s)ds \le M \int_a^b \ell(1)(s)ds \le M,$$

which contradicts (4.4).

Therefore $M < 0$ and the function $-\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ satisfies the inequalities

$$(-\gamma(t))' \ge \ell(-\gamma)(t) \quad \text{for} \quad t \in [a,b], \tag{4.6}$$

$$-\gamma(a) > \lambda(-\gamma(b)). \tag{4.7}$$

Thus, according to Theorem 1.1 in [8], we have $\ell \in V^+(\lambda)$. Consequently $\ell \notin V^-(\lambda)$.

Example 4.4. Let $\lambda \in]0,1]$, $\varepsilon > 0$, and let $a_1 \in]a,b[$ be an arbitrary fixed point. Choose $p, q \in L([a,b];R_+)$ and $c \in R_+$ such that $p \not\equiv 0$ in $]a_1, a_1+\delta_0]$ for some $\delta_0 > 0$ and

$$\int_a^{a_1} p(s)ds > 1, \qquad \int_{a_1}^b p(s)ds = 1,$$

$$q(t) = 0 \quad \text{for} \quad t \in [a,a_1], \qquad \int_{a_1}^b q(s)ds = \frac{\varepsilon}{\int_a^{a_1} p(s)ds - 1}.$$

Put

$$\beta(t) = \begin{cases} 0 & \text{for } t \in [a,a_1[\\ \int_{a_1}^t p(s)ds & \text{for } t \in [a_1,b] \end{cases}, \qquad \gamma(t) = \begin{cases} \int_a^t p(s)ds & \text{for } t \in [a,a_1[\\ \int_a^{a_1} p(s)ds & \text{for } t \in [a_1,b] \end{cases},$$

$$\tau(t) = \begin{cases} a_1 & \text{for } t \in [a,a_1[\\ b & \text{for } t \in [a_1,b] \end{cases}.$$

Let ℓ be defined by (1.20). Obviously, the function γ satisfies the inequalities (1.7), (1.8), and (1.9), which also implies that there exists $\delta \in]0,\lambda]$ such that $\ell \notin V^+(\lambda - \delta)$ (see the proof of Corollary 1.1). It is also evident that the operator ℓ is a $b-$Volterra operator and the function β satisfies the inequality (1.12) and the condition (1.13).

On the other hand, the function

$$u(t) = \begin{cases} \varepsilon - \frac{\varepsilon}{\int_a^{a_1} p(s)ds - 1} \int_a^t p(s)ds & \text{for } t \in [a, a_1[\\ -\frac{\varepsilon}{\int_a^{a_1} p(s)ds - 1} + \frac{\varepsilon - c}{\lambda} \int_{a_1}^t p(s)ds + \int_{a_1}^t q(s)ds & \text{for } t \in [a_1, b] \end{cases}$$

is a solution of the problem (0.1), (0.3) with $u(a) = \varepsilon > 0$. Consequently, $\ell \notin V^-(\lambda)$.

Example 4.4 shows that the condition (1.11) in Theorem 1.3 and Corollary 1.2 cannot be replaced by the condition (1.13), where $a_1 \in]a, b[$ is an arbitrary fixed point.

On Remark 1.5. We will show that the inequality (1.9) in Corollary 1.2 cannot be weakened. Indeed, let there exist a function $\beta \in \widetilde{C}([a, b]; R_+)$ satisfying the inequalities (1.11) and (1.12), and let there exist a function $\gamma \in \widetilde{C}([a, b]; R)$ satisfying (1.7), (1.8), and $\gamma(b) \le 0$. Then the condition (1.8) implies (4.4). Obviously, γ is a solution of the equation (0.1) with

$$q(t) = \gamma'(t) - \ell(\gamma)(t) \le 0 \quad \text{for} \quad t \in [a, b]. \tag{4.8}$$

It is also evident that there exists $t_0 \in]a, b[$ such that

$$\gamma(t) < 0 \quad \text{for} \quad [a, t_0]. \tag{4.9}$$

Since ℓ is a $b-$Volterra operator, the restriction of γ to the interval $[t_0, b]$ is a solution of the equation (0.1) with the condition $\gamma(t_0) < 0$. Moreover, the restriction of β to the interval $[t_0, b]$ is a positive absolutely continuous function satisfying the inequality (3.14). According to (1.1), (4.8), $\gamma(t_0) < 0$, and Theorem 1.1 in [8] (for $\lambda = 0$, $a = t_0$), we get

$$\gamma(t) < 0 \quad \text{for} \quad t \in [t_0, b].$$

This inequality, together with (4.9), yields $\gamma(t) < 0$ for $t \in [a, b]$ and so the function $-\gamma \in \widetilde{C}([a, b];]0, +\infty[)$ satisfies the inequalities (4.6) and (4.7). Thus, according to Theorem 1.1 in [8], we have $\ell \in V^+(\lambda)$. Consequently, $\ell \notin V^-(\lambda)$.

On Remark 1.6. According to Proposition 3.1, the optimality of Theorems 1.5–1.7 and Corollaries 1.3 and 1.4 follows from the optimality of Theorems 1.2–1.4 and Corollaries 1.1 and 1.2 from [8], where the examples showing the optimality of the obtained results can be found.

On Remark 2.1. According to Proposition 3.1, the optimality of Theorems 2.1–2.3 and Corollaries 2.1 and 2.2 follows from the optimality of Theorems 1.1–1.3 and Corollaries 1.1 and 1.2.

On Remark 2.2. According to Proposition 3.1, the optimality of Theorems 2.4–2.6 and Corollaries 2.3 and 2.4 follows from the optimality of Theorems 1.8–1.10 and Corollaries 1.5 and 1.6.

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