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THE CONSTRUCTIVE STUDY OF DIFFERENTIAL EQUATIONS WITH DEVIATED ARGUMENT

Abstract. A constructive technique for the study of linear differential equations with arbitrary deviations is considered. This technique allows to recognize the property of the correct solvability of the so called Principal Boundary Value Problem for a class of differential systems and n-th order differential equations, as well as to construct approximate solutions of such problems with guaranteed error bounds.

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1. Notation

The following notation is used.

R^n denotes the linear space of all real columns $\alpha = \mathrm{col}\{\alpha_1, \dots, \alpha_n\}$ with the norm $\|\alpha\|_n = \max_{1\le i\le n} |\alpha_i|$; $|\alpha| \overset{\mathrm{def}}{=} \mathrm{col}\{|\alpha_1|, \dots, |\alpha_n|\}$; for $\alpha \in R^n$, $\beta \in R^n$, the inequality $\alpha \le \beta$ means $\alpha_i \le \beta_i$, $i = 1, \dots, n$;

$R^{n\times n}$ denotes the linear space of all real $n \times n$-matrices $A = \{a_{ij}\}_{i,j=1}^n$ with the norm $\|A\|_{R^{n\times n}} = \max_{1\le i\le n} \sum_{j=1}^n |a_{ij}|$; $|A| = \{|a_{ij}|\}$;

$L_p^n[0,T]$, $1 \le p < \infty$, is the Banach space of measurable functions $z : [0,T] \to R^n$, $z(\cdot) = \mathrm{col}\{z_1(\cdot), \dots, z_n(\cdot)\}$, such that

$$\|z\|_{L_p^n[0,T]} = \max_{1\le i\le n} \left(\int_0^T |z_i(s)|^p \, ds \right)^{\frac{1}{p}} < \infty;$$

$L_\infty^n[0,T]$ denotes the Banach space of essentially bounded measurable functions $z : [0,T] \to R^n$, $z(\cdot) = \mathrm{col}\{z_1(\cdot), \dots, z_n(\cdot)\}$; $\|z\|_{L_\infty^n[0,T]} = \max_{1\le i\le n} \operatorname{vrai\,sup}_{0\le s\le T} |z_i(s)|$;

$L_p^{n\times n}[0,T]$, $1 \le p < \infty$ is the Banach space of measurable functions $Z : [0,T] \to R^{n\times n}$, $Z(t) = \{z_{ij}(t)\}_{i,j=1}^n$, such that $\|Z\|_{L_p^{n\times n}[0,T]} = \max_{1\le i\le n} \sum_{j=1}^n \|z_{ij}\|_{L_p^1[0,T]}$;

$D_p^n[0,T]$ denotes the Banach space of absolutely continuous functions x: $[0,T] \to R^n$ such that $\dot{x} = \frac{dx}{dt} \in L_p^n[0,T]$; $\|x\|_{D_p^n[0,T]} = \|x(0)\|_n + \|\dot{x}\|_{L_p^n[0,T]}$.

Let us fix a collection of points $t_1, \dots, t_m$; $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$. Denote $B_q = [t_{q-1}, t_q)$, $q = 1, \dots, m$; $B_{m+1} = [t_m, T]$; $\chi_q(t) = \begin{cases} 1, & t \in B_q, \\ 0, & t \notin B_q \end{cases}$ is the characteristic function of B_q.

$DS_p^n[0,T](m) = DS_p^n[0, t_1, \dots, t_m, T]$ is the Banach space of all functions $y : [0,T] \to R^n$ with $\dot{y} \in L_p^n[0,T]$ and the representation

$$y(t) = y(0) + \int_0^t \dot{y}(s)ds + \sum_{q-1}^m \chi_{[t_q,T]}(t)\Delta y(t_q),$$

where

$$\Delta y(t_q) = y(t_q) - y(t_q - 0); \chi_{[t_q,T]}(t) = \begin{cases} 1, & t \in [t_q, T], \\ 0, & t \notin [t_q, T] \end{cases}$$

is the characteristic function of $[t_q, T]$;

$\Delta y \overset{\text{def}}{=} \operatorname{col}\{y(0), \Delta y(t_1), \ldots, \Delta y(t_m)\}$;

$\|y\|_{DS_p^n[0,T]}(m) = \|\dot{y}\|_{L_p^n[0,T]} + \|\Delta y\|_{mn+n}$;

$WS_p^n{}_{[0,T]}(m)$ denotes the Banach space of function $y : [0,T] \to R^1$, $y^{(n)} \in L_p^1[0,T]$ with $y^{(i)} \in DS_p^1[0,T](m)$, $i = 0, \ldots, n-1$, and the representation

$$y(t) = \int_0^t \frac{(t-s)^{(n-1)}}{(n-1)!} y^{(n)}(s)ds \; + \; \sum_{i=0}^{n-1} \frac{t^i}{i!} y^{(i)}(0) \; + \\ + \sum_{i=0}^{n-1} \sum_{q=1}^{m} \frac{(t-t_q)^i}{i!} \chi_{[t_q,T]}(t) \Delta y^{(i)}(t_q),$$

where

$$\begin{aligned} \Delta y^{(i)}(t_q) \; &= \; y^{(i)}(t_q) - y^{(i)}(t_q - 0), \\ \Delta^n y \; &= \; \operatorname{col}\Big\{ y(0), y^{(1)}(0), \ldots, y^{(n-1)}(0), \Delta y(t_1), \\ &\qquad \Delta y^{(1)}(t_1), \ldots, \Delta y^{(n-1)}(t_1), \ldots, \Delta y^{(1)}(t_m), \\ &\qquad \Delta y^{(2)}(t_m), \ldots, \Delta y^{(n-1)}(t_m) \Big\}; \end{aligned}$$

$\|y\|_{WS_p^n[0,T]}(m) = \|\dot{y}\|_{L_p^1[0,T]} + \|\Delta^n y\|_{mn+n}$;

p' denotes the exponent adjoint to p: $p' = \begin{cases} \dfrac{p}{p-1}, & p > 1, \\ \infty, & p = 1; \end{cases}$

I is the identity operator;
E_n is the identity $n \times n$-matrix;
O_n is the zero $n \times n$-matrix;

Let Λ be a linear operator acting from B_1^n, a space n-vector functions, into B_2^n, a space of n-vector functions, and Z be an $n \times n$-matrix with columns from B_1^n. In this case ΛZ means the $n \times n$-matrix such that each its column is a result of applying the operator Λ to the corresponding column of the matrix Z.

Let B be a Banach space, $x_0, x_k \in B$, $k = 1, 2, \ldots, \infty$. We write $x_k \to x_0$ to denote the convergence of $\{x_k\}$ to x_0 in B:

$$\lim_{k \to \infty} \|x_0 - x_k\|_B = 0.$$

2. Preliminaries

Let us cite some facts from the theory of functional differential equations [1, 2, 3] which are necessary for the main presentation. Consider the equation

$$(\mathcal{L}y)(t) = f(t), \quad t \in [0,T], \tag{1}$$

where $\mathcal{L}$: $DS_p^n[0,T](m) \to L_p^n[0,T]$ is a linear bounded operator; $f \in L_p^n[0,T]$. An element $y \in DS_p^n[0,T](m)$ is said to be a solution to (1) iff the equality (1) is fulfilled almost everywhere (a.e.) on $[0,T]$. The representation

$$y(t) = \int_0^t \dot{y}(s)ds + y(0) + \sum_{q=1}^{m} \Delta y(t_q)\chi_{[t_q,T]}(t), \quad t \in [0,T],$$

implies

$$(\mathcal{L}y)(t) = (Q\dot{y})(t) + A_0(t)y(0) + \sum_{q=1}^{m} A_q(t)\Delta y(t_q), \tag{2}$$

$t \in [0,T]$, where $Q : L_p^n[0,T] \to L_p^n[0,T]$ is a linear bounded operator, the so called principal part of $\mathcal{L}$; $Qz = \mathcal{L}\Big(\int_0^{(\cdot)} z(s)ds\Big)$; $A_q \in L_p^{n\times n}[0,T]$, $q = 0,\dots,m$; $A_q(t) = (\mathcal{L}E_n)(t)\chi_{[t_q,T]}(t)$, $q = 1,\dots,m$, $A_0(t) = (\mathcal{L}E_n)(t)$. For a wide class of functional differential equations, the operator Q is a Fredholm one and has the form $Q = I - K$, where $K : L_p^n[0,T] \to L_p^n[0,T]$ is an integral completely continuous operator,

$$(Kz)(t) = \int_0^T K(t,s)z(s)ds.$$

The following equations are included in the above class:

Example 1. Ordinary differential system

$$\begin{gathered}(\mathcal{L}y)(t) \equiv \dot{y}(t) + P(t)y(t) = f(t), \quad t \in [0,T];\\ P \in L_p^{n\times n}[0,T];\ f \in L_p^n[0,T].\end{gathered} \tag{3}$$

In this case $(Kz)(t) = -\int_0^T P(t)\chi(t,s)z(s)\,ds$, where $\chi(\cdot,\cdot)$ is the characteristic function of the set

$$\{(t,s) \in [0,T] \times [0,T] :\ 0 \le s \le t\}.$$

Example 2. Differential system with deviated arguments

$$(\mathcal{L}y)^i(t) \equiv \dot{y}_i(t) \;+\; \sum_{j=1}^{n}\sum_{k=1}^{n_{ij}} p_{ij}^k(t) y_j[h_{ij}^k(t)] \;=\; f_i(t);$$
$$y_j(\xi) = \phi_j(\xi),\ \xi \notin [0,T]; \quad i,j = 1,\dots,n; \quad t \in [0,T]; \tag{4}$$

$p_{ij}^k \in L_p^1[0,T]$; $f_i \in L_p^1[0,T]$; $h_{ij}^k : [0,T] \to R^1$ is a given measurable function, $i,j = 1,\dots,n$, $k = 1,\dots,n_{ij}$. The equation (4) admits the form

$$(\mathcal{L}y)^i(t) \equiv \dot{y}_i(t) \;+\; \sum_{j=1}^{n}\sum_{k=1}^{n_{ij}} p_{ij}^k(t)(S_{h_{ij}^k}y)(t) \;=$$
$$=\; f_i(t) - \sum_{j=1}^{n}\sum_{k=1}^{n_{ij}} p_{ij}^k(t)\phi^{h_{ij}^k}(t), \quad t \in [0,T],$$

where, for some $r : [0,T] \to R^1$,

$$(S_r y)(t) \;\stackrel{\text{def}}{=}\; \begin{cases} y[r(t)], & r(t) \in [0,T]; \\ 0, & r(t) \notin [0,T], \end{cases}$$

$$v^r(t) \;\stackrel{\text{def}}{=}\; \begin{cases} 0, & r(t) \in [0,T]; \\ v[r(t)], & r(t) \notin [0,T]. \end{cases}$$

Let us write $\mathcal{L}y = \left\{(\mathcal{L}y)^i\right\}_{i=1}^n$. In this case $K = \{K_{ij}\}_{i,j=1}^n$, K_{ij}: $L_p^1[0,T] \to L_p^1[0,T]$,

$$(K_{ij}z)(t) \;=\; -\int_0^T \sum_{k=1}^{n_{ij}} p_{ij}^k(t)\chi_{ij}^k(t,s)z(s)\,ds,$$

where $\chi_{ij}^k(\cdot,\cdot)$ is the characteristic function of the set

$$\left\{(t,s) \in [0,T]\times[0,T] :\ 0 \le s \le h_{ij}^k(t) \le T\right\}.$$

Note that the operator Q in Example 1 is invertible, but the operator Q in Example 2 is not invertible in general. The invertibility of the operator Q takes place if $h_{ij}^k(t) \le t$, $t \in [0,T]$, $i,j = 1,\dots,n$, $k = 1,\dots,n_{ij}$ (the case of delayed argument).

Let us cite some results of the general theory of functional differential equations $[1,2,3]$ concerning the equation (1) under the assumption that Q has the bounded inverse $Q^{-1} : L_p^n[0,T] \to L_p^n[0,T]$. The solutions space of the corresponding homogeneous equation (i.e., the null-space of the operator $\mathcal{L}$)

$$(\mathcal{L}y)(t) \;=\; 0, \quad t \in [0,T], \tag{5}$$

is finite-dimensional with the dimension $mn+n$. The basis $\{y_1, \dots, y_{mn+n}\}$ of the null-space of $\mathcal{L}$ constitutes the fundamental system of solutions to (5). The matrix $Y \in DS_p^{n\times(mn+n)}$ defined by

$$Y(\cdot) = \{y_1(\cdot), \dots, y_{mn+n}(\cdot)\}$$

is said to be the fundamental matrix of the equation (5). In what follows we put $\Delta Y = E_{mn+n}$. The principal boundary value problem (PBVP)

$$\begin{aligned} (\mathcal{L}y)(t) &= f(t), \quad t \in [0,T], \\ \Delta y &= \alpha, \end{aligned} \tag{6}$$

is uniquely solvable for any $f \in L_p^n[0,T]$ and $\alpha \in R^{mn+n}$. The solution of the semihomogeneous problem

$$\begin{aligned} (\mathcal{L}y)(t) &= f(t), \quad t \in [0,T], \\ \Delta y &= 0 \end{aligned} \tag{7}$$

admits the representation

$$y(t) = \int_0^t \left(Q^{-1}f\right)(s)ds \stackrel{\text{def}}{=} (Gf)(t) \stackrel{\text{def}}{=} \int_0^T G(t,s)f(s)\,ds,$$

$t \in [0,T]$, where $G : L_p^n[0,T] \to D_p^n[0,T]$ is the so called Green operator, $G(t,s)$ is the Green matrix. In case the operator Q^{-1} is Volterra, this representation gets the form

$$y(t) = \int_0^t C(t,s)f(s)ds, \quad t \in [0,T]. \tag{8}$$

The kernel $C(t,s)$ of the integral representation (8) is called the Cauchy matrix.

3. A Class of Functions and Operators

The constructive techniques and algorithms for the study of systems with deviated arguments described below are based on a specific approximation of original problems within the class of computable functions and operators [5].

Remark. Everywhere in what follows we assume that the spaces $DS_p^n[0,T](m)$ and $WS_p^n[0,T](m)$ are constructed by means of the partition

$$0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T, \tag{9}$$

where t_q, $q = 1, \dots, m+1$, are rational numbers. The sets $B_q = [t_{q-1}, t_q)$, $q = 1, \dots, m$; $B_{m+1} = [t_m, T]$ are defined with respect to the same partition.

Definition 1. We say that a function $y \in DS_p^n[0,T](m)$ possesses the property $\mathcal{C}$ (is computable) if its components as well as the components of the functions $\dot{y}$ and Vy take rational values at rational values of argument.

Let $y \in DS_p^n[0,T](m)$. The property $\mathcal{C}$ is fulfilled for the function of the form

$$y(t) = \sum_{q=1}^{m} \chi_q(t) p_q(t), \qquad t \in [0,T], \tag{10}$$

where the components of $p_q : [0,T] \to R^n$, $q = 1,\dots,m$, are polynomials with rational coefficients. We denote by $\mathcal{P}_m^n$ the set of all $y \in DS_p^n[0,T](m)$ of the form (10).

Definition 2. We say that a function $y \in WS_p^n[0,T](m)$ possesses the property $\mathcal{C}$ (is computable) if this function as well as the functions $y^{(i)}$, $i = 1,\dots,n$ and Vy take rational values at the rational values of argument.

Definition 3. We say that a function $h : [0,T] \to R^1$, is computable over partition (9) if h possesses the property $\mathcal{C}$ and for every $j \in \{1,\dots,m\}$ there exists an integer q_j, $0 \le q_j \le j$, such that $h(t) \in B_{q_j}$ as $t \in B_j$.

An example of a function being computable over (9) is given by the function $h : [0,T] \to R^1$ of the form

$$h(t) = \sum_{q=1}^{m+1} \chi_q(t)\, h_q, \quad h_q \le t_q, \quad t \in [0,T],$$

where h_q, $q = 1,\dots,m+1$, are rational constants.

Definition 4. We say that a function $h : [0,T] \to R^1$ is computable over (9) in the generalized sense, if h possesses the property $\mathcal{C}$ and for every $j \in \{1,\dots,m\}$, there exists an integer $q_j = 0 \le q_j \le m+1$, such that $h(t) \in B_{q_j}$, as $t \in B_j$.

A function $h : [0,T] \to R^1$ of the form

$$h(t) = \sum_{q=1}^{m+1} \chi_q(t)\, h_q, \quad t \in [0,T],$$

with rational constants h_q, $q = 1,\dots,m+1$, gives an example of a function which is computable over (9) in the generalized sense.

Definition 5. A linear bounded operator $\mathcal{L}$: $DS_p^n[0,T](m) \to L_p^n[0,T]$ possesses the property $\mathcal{C}$ (is computable) if it maps the set $\mathcal{P}_m^n$ into itself.

Some examples of computable operators are as follows:

- the operator $\mathcal{L}y \equiv \dot{y} + P(\cdot)y$ such that the columns of $P(\cdot)$ are elements of $\mathcal{P}_m^n$;

– the operator

$$(\mathcal{L}y)(t) \equiv \dot{y}(t) + P(t)y[h(t)], \quad t \in [0,T],$$

$$y[h(t)] = \begin{cases} y[h(t)], & h(t) \in [0,T], \\ 0, & h(t) \notin [0,T], \end{cases}$$

is computable when the columns of $P(\cdot)$ are elements of $\mathcal{P}_m^n$ and the function h is computable in the generalized sense.

4. Key Ideas and Constructions Illustrated by the Case of the Scalar Differential Equation with Deviated Argument

The two main points of our approach are as follows:

- the constructive study of an approximate problem (AP) with computable parameters for the correct solvability (this AP is constructed by means of the original PBVP);
- the construction of an approximate solution to AP (in the case of the correct solvability) in the way which allows to obtain a guaranteed error bound (of a quite high accuracy).

It is principal for our aims that algorithms of constructing these error bound provide (theoretically) the possibility of attaining as high accuracy as we wish.

To outline the key ideas and constructions of our techniques we consider in this section the case of scalar differential equation with deviated argument in the space $D_p^1[0,T]$. The general case will be considered in Section 5.

Consider the Cauchy problem

$$(\mathcal{L}x)(t) \equiv \dot{x}(t) + b(t)x[h(t)] = f(t), \quad t \in [0,T],$$
$$x(\xi) = \begin{cases} \varphi_0(\xi), & \xi < 0, \\ \varphi_T(\xi), & \xi > T, \end{cases} \quad x(0) = \alpha, \tag{11}$$

where $b, f \in L_p^1[0,T]$, $1 \le p < \infty$; α is a rational number; a given function $h : [0,T] \to R^1$ is strictly increasing and continuous.

Let us take h_a^0, h_a^T as rational approximations of $h(0)$ and $h(T)$, respectively, as well as a rational h_v such that

$$h_a^0 \le h(0) \le h_a^0 + h_v; \quad h_a^T \le h(T) \le h_a^T + h_v.$$

Fix a constant m and construct the collection h_ν^a as follows:

$$h_\nu^a = h_a^0 + \nu \frac{h_a^T + h_0^a}{m}, \quad \nu = 0, \ldots, m,$$

and

$$h_{\nu-1}^a + h_v < h_\nu^a, \; \nu = 1, \ldots, m.$$

For $\nu = 1, \ldots, m-1$, define $\{{}^1t_\nu^a, {}^1t_\nu^v\}$ and $\{{}^2t_\nu^a, {}^2t_\nu^v\}$ such that

$$h^{-1}(h_\nu^a) \in \left[{}^1t_\nu^a, {}^1t_\nu^a + {}^1t_\nu^v\right], \quad h^{-1}(h_\nu^a + h_v) \in \left[{}^2t_\nu^a, {}^2t_\nu^a + {}^2t_\nu^v\right],$$

and

$$ {}^1t_\nu^a + {}^1t_\nu^v < {}^2t_\nu^a \quad 0 < {}^1t_1^a, \quad {}^2t_{m-1}^a + {}^2t_{m-1}^v < T. $$

Define the points $t_0, t_1, \dots, t_m$ by the equalities

$$ t_\nu = \frac{{}^1t_\nu^a + {}^1t_\nu^v + {}^2t_\nu^a}{2}, \quad \nu = 1, \dots, m-1, \quad t_0 = 0, \quad t_m = T. $$

Note that $0 = t_0 < t_1 < \dots < t_{m-1} < t_m = T$ and

$$ h_\nu^a \le h(t_\nu) \le h_\nu^a + h_v, \quad \nu = 0, \dots, m. $$

Denote $B_\nu = [t_{\nu-1}, t_\nu)$, $\nu = 1, \dots, m-1$, $B_m = [t_{m-1}, T]$ and $\chi_\nu(\cdot) = \chi_{B_\nu}(\cdot)$. Define the function $h_a : [0,T] \to R^1$ by

$$ h_a(t) = \sum_{\nu=1}^{m} \chi_\nu(t) h_{\nu-1}^a, \quad t \in [0,T]. $$

Obviously h_a is computable in generalized sense over $\{t_\nu\}_{\nu=0}^m$. We assume bellow that

$$ \varphi_0 \in D_p^1[h_0^*, 0], \quad \varphi_T \in D_p^1[T, h_T^*], $$

where h_0^*, h_T^* are defined by

$$ h_0^* = h_0^a; \qquad h_T^* = h_m^a + h_v. $$

Next we approximate the functions b, f, φ_0, φ_T by polynomials b_a, f_a, φ_0^a, φ_T^a with rational coefficients. Let b_v, f_v, ${}^0\varphi_0^v$, ${}^1\varphi_0^v$, ${}^0\varphi_T^v$, ${}^1\varphi_T^v$, be corresponding error bounds of the approximation

$$ \begin{aligned} & b_v \ge \|b - b_a\|_{L_p^1[0,T]}, \quad f_v \ge \|f - f_a\|_{L_p^1[0,T]}, \\ & {}^0\varphi_0^v \ge \left|\varphi_0(h_*) - \varphi_0^a(h_*)\right|, \quad {}^1\varphi_0^v \ge \left\|\dot\varphi - \dot\varphi_0^a\right\|_{L_p^1[h_0^*,0]}, \\ & {}^0\varphi_T^v \ge \left|\varphi_T(h_*) - \varphi_T^a(h_*)\right|, \quad {}^1\varphi_T^v \ge \left\|\dot\varphi - \dot\varphi_T^a\right\|_{L_p^1[T,h_T^*]}. \end{aligned} $$

We call the Cauchy problem

$$ \begin{aligned} & \mathcal{L}_a \equiv \dot x(t) + b_a(t) x[h_a(t)] = f_a(t), \quad t \in [0,T], \\ & x(\xi) = \begin{cases} \varphi_0^a(\xi), & \xi < 0, \\ \varphi_T^a(\xi), & \xi > T, \end{cases} \quad x(0) = \alpha, \end{aligned} \tag{12} $$

to be the approximating Cauchy problem to the problem (11). Note that $\mathcal{L}_a$: $D_p^1[0,T] \to L_p^1[0,T]$ is a computable operator.

As is known (see, for instance, [1, 2, 3]), the problem (11) (and (12) as well) is in general not uniquely solvable. Hence we have at first to study (11) for the property of the unique solvability.

With this aim we define the integral operator $K : L_p^1[0,T] \to L_p^1[0,T]$ by

$$\begin{aligned}(Kz)(t) &= \int_0^T \mathcal{K}(t,s)z(s)\,ds, \quad t \in [0,T], \\ \mathcal{K}(t,s) &= -b(t)\chi(t,s), \quad (t,s) \in [0,T] \times [0,T],\end{aligned} \tag{13}$$

where $\chi(t,s)$ is the characteristic function of

$$\{(t,s) \in [0,T \times [0,T] : 0 \le s \le h(t) \le T\},$$

and rewrite (11) in the integral form:

$$[(I-K)\,z]\,(t) \;=\; g(t), \quad t \in [0,T], \tag{14}$$

where

$$g(t) \;=\; f(t) - b(t)\chi(t,0)\alpha - b(t)\,[1-\chi(t,0)]\,\varphi[h(t)];$$

$$\varphi[h(t)] = \begin{cases} \varphi_0[h(t)], & h(t) < 0, \\ 0, & 0 \le h(t) \le T, \\ \varphi_T[h(t)], & h(t) > T. \end{cases}$$

Next we define the integral operator $K_a : L_p^1[0,T] \to L_p^1[0,T]$ by the equality

$$\begin{aligned}(K_a z)(t) &= \int_0^T \mathcal{K}_a(t,s)z(s)\,ds, \quad t \in [0,T], \\ \mathcal{K}_a(t,s) &= -b_a(t)\chi_a(t,s), \quad (t,s) \in [0,T] \times [0,T],\end{aligned} \tag{15}$$

where $\chi_a(t,s)$ is the characteristic function of

$$\{(t,s) \in [0,T \times [0,T] : 0 \le s \le h_a(t) \le T\},$$

and rewrite (12) in the integral form

$$[(I-K_a)\,z]\,(t) \;=\; g_a(t), \quad t \in [0,T], \tag{16}$$

where

$$g_a(t) \;=\; f_a(t) - b_a(t)\chi_a(t,0)\alpha - b_a(t)\,[1-\chi_a(t,0)]\,\varphi_a[h_a(t)],$$

$$\varphi_a[h_a(t)] = \begin{cases} \varphi_0^a[h_a(t)], & h_a(t) < 0, \\ 0, & 0 \le h_a(t) \le T, \\ \varphi_T^a[h_a(t)], & h_a(t) > T. \end{cases}$$

The representation

$$\chi_a(t,s) \;=\; \sum_{\nu=1}^{m} \chi_\nu(t)\chi_{[0,d_\nu^a]}(s), \quad (t,s) \in [0,T] \times [0,T],$$

where $d_\nu^a = \begin{cases} h_{\nu-1}^a, & h_{\nu-1}^a \in [0,T], \\ 0, & h_{\nu-1}^a \notin [0,T], \end{cases}$ implies that $\mathcal{K}_a(\cdot,\cdot)$ is a degenerate kernel of the form

$$\mathcal{K}_a(t,s) = \sum_{\nu=1}^{m} u_\nu(t) v_\nu(s), \quad (t,s) \in [0,T]\times[0,T], \tag{17}$$

where

$$u_\nu(t) = -b_a(t)\chi_\nu(t), \qquad v_\nu(s) = \chi_{[0,d_\nu^a]}(s).$$

Following the known way of solving integral equations with degenerate kernel (see e.g. [4]), we will try to find a solution z_a to (16) in the form

$$z_a(t) \ = \ \sum_{\nu=1}^{m} u_\nu(t) c_\nu \ + \ g_a(t). \tag{18}$$

Define the matrix Λ by the equality

$$\Lambda \ = \ \{\lambda_{\nu j}\}_{\nu,j=1}^m \,, \quad \lambda_{\nu j} \ = \ \delta_{\nu j} - \int_0^T u_j(s) v_\nu(s)\, ds, \tag{19}$$

where $\delta_{\nu j}$ is the Kronecker symbol. In the case where Λ is invertible, the vector $c = \mathrm{col}\,\{c_1, \ldots, c_m\}$ can be calculated:

$$c \ = \ \Lambda^{-1}\beta, \tag{20}$$

where $\beta = \{\beta_\nu\}_{\nu=1}^m$, $\beta_\nu = \int_0^T v_\nu(s) g_a(s)\, ds$, $\nu = 1, \ldots, m$. Let $M = \Lambda^{-1}$, $M = \{m_{\nu j}\}_{\nu,j=1}^m$. In this case the equation (16) is uniquely solvable, the operator $(I - K_a)$ is invertible and there exists the resolvent operator $R_a : L_p^1[0,T] \to L_p^1[0,T]$ such that $(I + R_a) = (I - K_a)^{-1}$ and

$$\begin{aligned} (R_a g_a)(t) \ &= \ \int_0^T r_a(t,s) g_a(s)\, ds, \\ r_a(t,s) \ &= \ \sum_{\nu=1}^{m}\sum_{j=1}^{m} u_\nu(t) v_j(s) m_{\nu j}, \\ z_a(t) \ &= \ [(I + R_a)\, g_a](t), \quad t \in [0,T]. \end{aligned}$$

There takes place the estimate

$$\|R_a\|_{L_p^1[0,T]\to L_p^1[0,T]} \ \le \ \rho_0, \tag{21}$$

where

$$\rho_0 \ = \ \sum_{\nu=1}^{m}\sum_{j=1}^{m} |m_{qj}|\, \|u_\nu\|_{L_p^1[0,T]}\, \|v_j\|_{L_{p'}^1[0,T]}\,.$$

Define the operator ΔK: $L_p^1[0,T] \to L_p^1[0,T]$, $\Delta K = K - K_a$, by

$$(\Delta K z)(t) = \int_0^T \mathcal{K}_\Delta(t,s) z(s)\, ds, \quad t \in [0,T], \tag{22}$$
$$\mathcal{K}_\Delta(t,s) = -b(t)\chi(t,s) + b_a(t)\chi_a(t,s),$$
$$(t,s) \in [0,T] \times [0,T].$$

In what follows, an estimate of $\|\Delta K\|_{L_p^1[0,T]\to L_p^1[0,T]}$ is required. To obtain such an estimate, we construct at first an estimate for $|\chi(\cdot,\cdot) - \chi_a(\cdot,\cdot)|$. There are the following cases:

1. One of the conditions $h_0^a > T$, $h_m^a + h_v < 0$, holds. In this case

$$|\chi(t,s) - \chi_a(t,s)| \equiv 0, \quad (t,s) \in [0,T] \times [0,T]; \tag{23}$$

2. The condition $0 \le h_0^a < h_m^a + h_v \le T$ is fulfilled. In this case

$$|\chi(t,s) - \chi_a(t,s)| \le \sum_{\nu=1}^{m} \chi_\nu(t)\chi_{[h_{\nu-1}^a, h_\nu^a + h_v]}(s), \quad (t,s) \in [0,T] \times [0,T]; \tag{24}$$

3. The equation $h(t) = 0$ has a solution belonging to the set $B_{\nu_0^*}$ and the equation $h(t) = 0$ has a solution belonging to the set $B_{\nu_T^*}$. In this case

$$|\chi(t,s) - \chi_a(t,s)| \le$$
$$\le \sum_{\nu=\nu_0^*+1}^{\nu_T^*-1} \chi_\nu(t)\chi_{[h_{\nu-1}^a, h_\nu^a + h_v]}(s) + \chi_{\nu_0^*}(t) + \chi_{\nu_T^*}(t), \tag{25}$$
$$(t,s) \in [0,T] \times [0,T].$$

The estimate

$$\|\Delta K\|_{L_p^1[0,T]\to L_p^1[0,T]} \le \delta_0 \tag{26}$$

holds, where a rational δ_0 is defined as follows.

In case 1:

$$\delta_0 = 0;$$

In case 2:

$$\delta_0 \overset{\text{def}}{\ge} \max_{1\le\nu\le m-1} \left(\|b_a\|_{L_p^1[t_{\nu-1},t_\nu]} + \|b_a\|_{L_p^1[t_\nu,t_{\nu+1}]} \right) \times$$
$$\times \sqrt[p']{t_{\nu+1} - t_{\nu-1}} + b_v \sqrt[p']{T}.$$

In case 3:

$$\delta_0 \overset{\text{def}}{\geq} \max_{\nu_0^*+1\leq\nu\leq\nu_T^*-1} \left(\|b_a\|_{L_p^1[t_{\nu-1},t_\nu]} + \|b_a\|_{L_p^1[t_\nu,t_{\nu+1}]}\right) \sqrt[p']{t_{\nu+1}-t_{\nu-1}} +$$
$$+ \left(\|b_a\|_{L_p^1\left[t_{\nu_0^*-1},t_{\nu_0^*}\right]} + \|b_a\|_{L_p^1\left[t_{\nu_T^*-1},t_{\nu_T^*}\right]} + b_v\right) \sqrt[p']{T}.$$

Now we can formulate

Theorem 1. *Let the matrix* Λ *defined by* (19) *be invertible and the condition*

$$\delta_0 < \frac{1}{1+\rho_0} \tag{27}$$

hold, where δ_0 *and* ρ_0 *are defined by* (26) *and* (21), *respectively. Then the Cauchy problem* (11) *is uniquely solvable for any* $f \in L_p^1[0,T]$ *and* $\alpha \in R^1$.

Proof. The condition (27) implies the estimate

$$\|\Delta K\|_{L_p^1[0,T]\to L_p^1[0,T]} < \frac{1}{\|I+R_a\|_{L_p^1[0,T]\to L_p^1[0,T]}},$$

which provides the invertibility of the operator $I-K$ due to the theorem on the invertible operator (see [6, Th.3.6.3]). □

Under the condition of Theorem 1 there exists the resolvent operator R: $L_p^1[0,T] \to L_p^1[0,T]$,

$$(I+R) = (I-K)^{-1}, \tag{28}$$

and the representation $z(t) = [(I+R)\,g](t)$, $t \in [0,T]$, of z, the solution to (14), holds. As is known [6], there takes place the estimate

$$\|R-R_a\|_{L_p^1[0,T]\to L_p^1[0,T]} \leq \rho_v, \text{ with } \rho_v = \frac{\delta_0\,(1+\rho_0)^2}{1-\delta_0\,(1+\rho_0)}. \tag{29}$$

The difference $g-g_a$ can be written in the form

$$g(t)-g_a(t) = f(t) - f_a(t) + [b(t) - b_a(t)]\,\zeta_1(t) -$$
$$- b_a(t)\,\{\alpha\zeta_2(t) + \zeta_3(t)\}, \quad t \in [0,T],$$

where

$$\zeta_1(t) = \chi(t,0)\alpha + [1 - \chi(t,0)]\,\varphi[h(t)],$$
$$\zeta_2(t) = \chi(t,0) - \chi_a(t,0),$$
$$\zeta_3(t) = [1 - \chi(t,0)]\,\varphi[h(t)] - [1 - \chi_a(t,0)]\,\varphi_a[h_a(t)].$$

Construct the estimate of $|\zeta_i(t)|$, $t \in [0,T]$, $i = 1,3$. Fix $\nu = 1,\ldots,m$, $t \in B_\nu$, and consider the following cases:

(1) The condition $h_\nu^a + h_v < 0$ holds. In this case $\chi(t,0) \equiv \chi_a(t,0) \equiv 0$ and the following estimates take place:

$$|\zeta_1(t)| \le \left|\varphi_{_0}^a(h_{_0}^*)\right| + {}^0\varphi_{_0}^v + \sqrt[p']{h_\nu^a + h_v - h_{_0}^*} \times 0 \\ \times \left(\left\|\dot\varphi_{_0}^a\right\|_{L_p^1[h_{_0}^*, h_\nu^a + h_v]} + {}^1\varphi_{_0}^v\right) \overset{\text{def}}{=} \gamma_\nu^1, \tag{30}$$

$$|\zeta_2(t)| = 0 \overset{\text{def}}{=} \gamma_\nu^2, \tag{31}$$

$$|\zeta_3(t)| \le {}^0\varphi_{_0}^v + {}^1\varphi_{_0}^v \sqrt[p']{h_\nu^a + h_v - h_{\nu-1}^a} \overset{\text{def}}{=} \gamma_\nu^3. \tag{32}$$

(2) The condition $h_{\nu-1}^a > T$ holds. Then $\chi(t,0) \equiv \chi_a(t,0) \equiv 0$ and the estimates

$$|\zeta_1(t)| \le \left|\varphi_{_T}^a(T)\right| + {}^0\varphi_{_T}^v + \sqrt[p']{h_\nu^a + h_v - T} \times \\ \times \left(\left\|\dot\varphi_{_T}^a\right\|_{L_p^1[T, h_\nu^a + h_v]} + {}^1\varphi_{_T}^v\right) \overset{\text{def}}{=} \gamma_\nu^1, \tag{33}$$

$$|\zeta_2(t)| = 0 \overset{\text{def}}{=} \gamma_\nu^2, \tag{34}$$

$$|\zeta_3(t)| \le {}^0\varphi_{_T}^v + {}^1\varphi_{_T}^v \sqrt[p']{h_\nu^a + h_v - h_{\nu-1}^a} \overset{\text{def}}{=} \gamma_\nu^3 \tag{35}$$

hold.

(3) The conditions $h_{\nu-1}^a < 0$, $h_\nu^a + h_v \ge 0$ are fulfilled (the equation $h(t) = 0$ has a solution in B_ν). In this case $\chi_a(t,0) \equiv 0$, the sign of h can not be established by calculations, and the estimates

$$|\zeta_1(t)| \le |\alpha| + \left|\varphi_{_0}^a(h_{_0}^*)\right| + {}^0\varphi_{_0}^v + \\ + \left(\left\|\dot\varphi_{_0}^a\right\|_{L_p^1[h_{_0}^*, 0]} + {}^1\varphi_{_0}^v\right) \sqrt[p']{-h_{_0}^*} \overset{\text{def}}{=} \gamma_\nu^1, \tag{36}$$

$$|\zeta_2(t)| = 1 \overset{\text{def}}{=} \gamma_\nu^2, \tag{37}$$

$$|\zeta_3(t)| \le {}^0\varphi_{_0}^v + {}^1\varphi_{_0}^v \sqrt[p']{-h_{\nu-1}^a} + \left|\varphi_{_0}^a(h_{_0}^*)\right| + \\ + \left\|\dot\varphi_{_0}^a\right\|_{L_p^1[h_{_0}^*, h_{\nu-1}^a]} \sqrt[p']{h_{\nu-1}^a - h_{_0}^*} \overset{\text{def}}{=} \gamma_\nu^3 \tag{38}$$

hold.

(4) The conditions $h_{\nu-1}^a \le T$, $h_\nu^a + h_v > T$ are fulfilled (the equation $h(t) = T$ has a solution in B_ν). In this case $\chi_a(t,0) \equiv 1$ and there

take place the estimates

$$|\zeta_1(t)| \le \left|\varphi^a_T(h^*_0)\right| + {}^0\varphi^v_T + \sqrt[p']{h^a_\nu + h_v - T} \times$$
$$\times \left(\|\dot\varphi^a_T\|_{L^1_p[T,h^a_\nu+h_v]} + {}^1\varphi^v_0\right) + |\alpha| \overset{\text{def}}{=} \gamma^1_\nu, \tag{39}$$

$$|\zeta_2(t)| = 1 \overset{\text{def}}{=} \gamma^2_\nu, \tag{40}$$

$$|\zeta_3(t)| \le \left|\varphi^a_T(T)\right| + {}^0\varphi^v_T + \sqrt[p']{h^a_\nu + h_v - T} \times$$
$$\times \left(\|\dot\varphi^a_T\|_{L^1_p[T,h^a_\nu+h_v]} + {}^1\varphi^v_T\right) \overset{\text{def}}{=} \gamma^3_\nu. \tag{41}$$

(5) The conditions $h^a_{\nu-1} \ge 0$, $h^a_\nu + h_v \le T$ hold. In this case $\chi(t,0) \equiv \chi_a(t,0) \equiv 1$ and

$$|\zeta_1(t)| = |\alpha| \overset{\text{def}}{=} \gamma^1_\nu; \tag{42}$$

$$|\zeta_2(t)| = 0 \overset{\text{def}}{=} \gamma^2_\nu; \tag{43}$$

$$|\zeta_3(t)| = 0 \overset{\text{def}}{=} \gamma^3_\nu. \tag{44}$$

Thus

$$|\zeta_1(t)| \le \sum_{\nu=1}^m \chi_\nu(t)\gamma^1_\nu \le \max_{1\le\nu\le m} \gamma^1_\nu,$$
$$|\zeta_2(t)| \le \sum_{\nu=1}^m \chi_\nu(t)\gamma^2_\nu \le \max_{1\le\nu\le m} \gamma^2_\nu,$$
$$|\zeta_3(t)| \le \sum_{\nu=1}^m \chi_\nu(t)\gamma^3_\nu \le \max_{1\le\nu\le m} \gamma^3_\nu.$$

The above estimates alow us to construct a constant g_v such that

$$\|g - g_a\|_{L^1_p[0,T]} \le g_v, \tag{45}$$
$$g_v \overset{\text{def}}{\ge} f_v + b_v \max_{1\le\nu\le m} \gamma^1_\nu +$$
$$+ \|b\|_{L^1_p[0,T]} \left(|\alpha| \max_{1\le\nu\le m} \gamma^2_\nu + \max_{1\le\nu\le m} \gamma^3_\nu\right),$$

where γ^1_ν is defined by one of (30), (33), (36), (39), (42); γ^2_ν is defined by one of (31), (34), (37), (40), (43); γ^3_ν is defined by one of (32), (35), (38), (41), (44). The representation

$$z(t) - z_a(t) = [(R - R_a)\, g]\,(t) + [(I + R_a)\,(g - g_a)]\,(t),$$

$t \in [0,T]$, where z_a is the solution of (16) defined by (18), implies the estimate

$$\|z - z_a\|_{L^1_p[0,T]} \leq z_v, \tag{46}$$
$$z_v = (1+\rho_0)\, g_v + \rho_v \left(\|g_a\|_{L^1_p[0,T]} + g_v \right),$$

where the constant ρ_0 is defined by (21), the constant ρ_v is defined by (29). To conclude, we define an approximate solution x_a to (11) as well as an error bound $x_v \geq \|\dot{x} - \dot{x}_a\|_{L^1_p[0,T]}$ (here x is the exact solution of (11)) as follows:

$$x_a(t) = \alpha + \int_0^t z_a(s)\, ds, \quad x_v = z_v. \tag{47}$$

Asymptotic accuracy of the error bound. Let the problem (11) be uniquely solvable. The questions we consider in this subsection are:

(1) Is the proposed method an effective one? In other words, is there an opportunity to recognize the property of the unique solvability in any case, at least theoretically?
(2) (very close to 1) Does the estimate (46) possess the property of the asymptotic accuracy?

Define the constants h^*_0, h^*_T by the equalities

$$h^*_0 = \min_{t\in[0,T]} \{h(t)\} - 1, \quad h^*_T = \max_{t\in[0,T]} \{h(t)\} + 1.$$

Let $k = k_0, k_0+1, \ldots$, $k_0 = 1$ and ${}^a h^0_k$, ${}^a h^T_k$ be such that

$${}^a h^0_k \leq h(0) \leq {}^a h^0_k + h^v_k; \quad {}^a h^T_k \leq h(T) \leq {}^a h^T_k + h^v_k, \quad h^v_k \stackrel{\text{def}}{=} \frac{1}{k^2}.$$

Next construct the collection of points ${}^a h^\nu_k$ as follows:

$${}^a h^\nu_k = {}^a h^0_k + \nu \frac{{}^a h^T_k + {}^a h^k_0}{k}, \quad \nu = 0, \ldots, m,$$

and assume that the conditions ${}^a h^{\nu-1}_k + h^v_k <^a h^\nu_k$, $\nu = 1, \ldots, k$, are fulfilled.

For $\nu = 1, \ldots, k-1$, define ${}^a_1 t^\nu_k$, ${}^a_2 t^\nu_k$, in such a way that

$$h^{-1}({}^a h^\nu_k) \in [{}^a_1 t^\nu_k, {}^a_1 t^\nu_k + h^v_k], \quad h^{-1}({}^a h^\nu_k + h^v_k) \in [{}^a_2 t^\nu_k, {}^a_2 t^\nu_k + h^v_k]$$

and

$${}^a_1 t^\nu_k + h^v_k < {}^a_2 t^\nu_k. \quad 0 < {}^a_1 t^1_k, \quad {}^a_2 t^{k-1}_k + h^v_k < T.$$

Next define t^ν_k by

$$t^\nu_k = \frac{{}^a_1 t^\nu_k + h^v_k + {}^a_2 t^\nu_k}{2}, \quad t^0_k = 0, \quad t^k_k = T.$$

Note that this implies

$${}^a h^\nu_k \leq h(t^\nu_k) \leq {}^a h^\nu_k + h^v_k, \quad \nu = 0, \ldots, k.$$

Introduce the sets $B_k^\nu = [t_k^{\nu-1}, t_k^\nu)$, $\nu = 1, \ldots, k-1$, and $B_k^k = [t_k^{k-1}, T]$ and their characteristic functions $\chi_k^\nu(\cdot)$, $\nu = 1, \ldots, k$. Let us define h_k by the equality

$$h_k(t) = \sum_{\nu=1}^{k} \chi_k^\nu(t)\, {}^a h_k^{\nu-1}, \quad t \in [0,T].$$

The functions b, f, φ_0,φ_T are to be approximated by polynomials b_k^a, f_k^a, ${}^a\varphi_k^0$, ${}^a\varphi_k^T$ with rational coefficients in such a way that the error bounds b_k^v, f_k^v, ${}^v_0\varphi_k^0$, ${}^v_1\varphi_k^0$, ${}^v_0\varphi_k^T$, ${}^v_1\varphi_k^T$ satisfy the inequalities

$$\begin{aligned}
&\|b - b_k^a\|_{L_p^1[0,T]} \le \frac{1}{k} \overset{\text{def}}{=} b_k^v,\\
&\|f - f_k^a\|_{L_p^1[0,T]} \le \frac{1}{k} \overset{\text{def}}{=} f_k^v,\\
&\left|\varphi_0(h_0^*) - {}^a\varphi_k^0(h_0^*)\right| \le \frac{1}{k} \overset{\text{def}}{=} {}^v_0\varphi_k^0,\\
&\|\dot\varphi_0 - {}^a\dot\varphi_k^0\|_{L^1_{p'}[h_0^*,0]} \le \frac{1}{k} \overset{\text{def}}{=} {}^v_1\varphi_k^0,\\
&|\varphi_T(T) - {}^a\varphi_k^T(T)| \le \frac{1}{k} \overset{\text{def}}{=} {}^v_1\varphi_k^T,\\
&\|\dot\varphi_T - {}^a\dot\varphi_k^T\|_{L^1_{p'}[T,h_T^*]} \le \frac{1}{k} \overset{\text{def}}{=} {}^v_1\varphi_k^T.
\end{aligned}$$

Consider the Cauchy problem approximating the problem (11)

$$(\mathcal{L}_k x)(t) \equiv \dot x(t) + b_k^a(t) x[h_k(t)] = f_k^a(t), \quad t \in [0,T], \tag{48}$$

$$x(\xi) = \begin{cases} {}^a\varphi_k^0(\xi), & \xi < 0,\\ {}^a\varphi_k^T(\xi), & \xi > T, \end{cases} \quad x(0) = \alpha.$$

Here the operator $\mathcal{L}_k : D_p^1[0,T] \to L_p^1[0,T]$ is computable.

The principal part $Q : L_p^1[0,T] \to L_p^1[0,T]$ of the operator $\mathcal{L}$ in (11) has the form $Q = I - K$, where the integral operator $K : L_p^1[0,T] \to L_p^1[0,T]$ is defined by (13). The principal part $Q_k : L_p^1[0,T] \to L_p^1[0,T]$ of $\mathcal{L}_k$ (48) has the form $Q_k = I - K_k$, where the operator $K_k : L_p^1[0,T] \to L_p^1[0,T]$ is an integral one:

$$(K_k z)(t) = \int_0^T \mathcal{K}_k(t,s) z(s)\, ds, \quad t \in [0,T], \tag{49}$$

$$\mathcal{K}_k(t,s) = -b_k^a(t)\chi_k(t,s), \quad (t,s) \in [0,T] \times [0,T],$$

with $\chi_k(\cdot,\cdot)$ being the characteristic function of the set

$$\{(t,s) \in [0,T] \times [0,T] : 0 \le s \le h_k(t) \le T\}.$$

The estimate (26) implies

$$\|Q - Q_k\|_{L_p^1[0,T] \to L_p^1[0,T]} \le \delta_k^0, \tag{50}$$

where δ_k^0 is defined as follows:

Case 1. Either the inequality ${}^a h_k^0 > T$ or the inequality ${}^a h_k^m + h_k^v < 0$ holds. In this case

$$\delta_k^0 = 0.$$

Case 2. The condition $0 \leq^a h_k^0 <^a h_k^m + h_k^v \leq T$ is fulfilled. Then

$$\delta_k^0 \overset{\text{def}}{\geq} \max_{1 \leq \nu \leq k-1} \left(\|b_k^a\|_{L_p^1[t_k^{\nu-1}, t_k^\nu]} + \|b_k^a\|_{L_p^1[t_k^\nu, t_k^{\nu+1}]} \right) \times \sqrt[p']{t_k^{\nu+1} - t_k^{\nu-1}} + b_k^v \sqrt[p']{T}.$$

Case 3. The equation $h(t) = 0$ has a solution in $B_k^{\nu_0^*}$ and the equation $h(t) = T$ has a solution in $B_k^{\nu_T^*}$. Then

$$\delta_k^0 \overset{\text{def}}{\geq} \max_{\nu_0^*+1 \leq \nu \leq \nu_T^*-1} \left(\|b_k^a\|_{L_p^1[t_k^{\nu-1}, t_k^\nu]} + \|b_k^a\|_{L_p^1[t_k^\nu, t_k^{\nu+1}]} \right) \sqrt[p']{t_k^{\nu+1} - t_k^{\nu-1}} +$$
$$+ \left(\|b_k^a\|_{L_p^1\left[t_k^{\nu_0^*-1}, t_k^{\nu_0^*}\right]} + \|b_k^a\|_{L_p^1\left[t_k^{\nu_T^*-1}, t_k^{\nu_T^*}\right]} + b_k^v \right) \sqrt[p']{T}.$$

The way of constructing $\mathcal{L}_k$ implies that, for any $\nu = 1, \ldots, k$, mes $\left[t_k^{\nu-1}, t_k^\nu\right] \to 0$ as $k \to \infty$. Due to absolute continuity of the Lebesgue integral this implies, for $\nu = 1, \ldots, k$, $\|b\|_{L_p^1[t_k^{\nu-1}, t_k^{\nu+1}]} \to 0$ as $k \to \infty$. Thus due to the estimate (50) we have $\delta_k^0 \to 0$ $(Q_k \to Q)$ as $k \to \infty$, and $\mathcal{L}_k \to \mathcal{L}$ as $k \to \infty$. Since under the above assumption Q is invertible, the established convergence of Q_k implies that there exists $N_* \geq k_0$ such that the problem (48) is uniquely solvable for any $k \geq N_*$. Below we assume $k \geq N_*$. Let x_k be a solution of (48). Rewrite the problem (48) in the form

$$\begin{aligned} [(I - K_k) z](t) &= g_k(t), \quad t \in [0, T], \qquad (51) \\ g_k(t) &= f(t) - b(t)\chi_k(t, 0)\alpha - b(t)\left[1 - \chi_k(t, 0)\right] \varphi[h_k(t)], \end{aligned}$$
$$\varphi[h_k(t)] = \begin{cases} \varphi_0[h_k(t)], & h_k(t) < 0, \\ 0, & 0 \leq h_k(t) \leq T, \\ \varphi_T[h_k(t)], & h_k(t) > T \end{cases}$$

(the operator K_k is defined by (49)). Let the equation $h(t) = 0$ have a solution in $B_k^{\nu_0^*}$, and the equation $h(t) = T$ have a solution in $B_k^{\nu_T^*}$. Then, due to (45), we have

$$\begin{aligned}\|g-g_k\|_{L_p^1[0,T]} \le & \sum_{\substack{1\le\nu\le k\\ \nu\ne\nu_0^*,\,\nu\ne\nu_T^*}}^{k} \|g-g_k\|_{L_p^1[t_k^{\nu-1},t_k^{\nu}]} + \\ & + \|g-g_k\|_{L_p^1\left[t_k^{\nu_0^*-1},t_k^{\nu_0^*}\right]} + \\ & + \|g-g_k\|_{L_p^1\left[t_k^{\nu_T^*-1},t_k^{\nu_T^*}\right]} \le g_k^v, \qquad (52)\\ g_k^v \stackrel{\text{def}}{=} & \, 3f_k^v + 3b_k^v\gamma_k^1 + \|b\|_{L_p^1[0,T]}\gamma_k^3 + \\ & + \left(\|b\|_{L_p^1\left[t_k^{\nu_0^*-1},t_k^{\nu_0^*}\right]} + \|b\|_{L_p^1\left[t_k^{\nu_T^*-1},t_k^{\nu_T^*}\right]}\right)\times \\ & \times\left(|\alpha|+\gamma_k^1+\gamma_k^3\right);\end{aligned}$$

here

$$\begin{aligned}\gamma_k^1 = & |\alpha| + \left|\varphi_0(h_0^*)\right| + \|\dot\varphi_0\|_{L_p^1[h_0^*,0]}\sqrt[p']{-h_0^*} + \\ & + |\varphi_T(T)| + \|\dot\varphi_T\|_{L_p^1[T,h_T^*]}\sqrt[p']{h_T^*-T},\\ \gamma_k^3 = & {}_0^v\varphi_k^0 + {}_1^v\varphi_k^0\sqrt[p']{-h_0^*} + {}_0^v\varphi_k^T + {}_1^v\varphi_k^T\sqrt[p']{h_T^*-T}.\end{aligned}$$

Due to the estimate (52), convergence $\gamma_k^3 \to 0$ as $k\to\infty$, and boundedness of γ_k^1, there takes place the convergence $g_k^v \to g$ as $k\to\infty$, i.e., $g_k \to 0$ as $k\to\infty$ ($\|b\|_{L_p^1[t_k^{\nu-1},t_k^{\nu}]} \to 0$ as $k\to\infty$ for absolute continuity of the Lebesgue integral for any $\nu = 1,\dots,k$). Thus there exists a nonnegative g_N such that $\|g_k\|_{l_p^1[0,T]} \le g_N$, $k = N_*, N_*+1,\dots$.

Under the above assumptions there exists the resolvent operator $R_k : L_p^1[0,T] \to L_p^1[0,T]$, $(I+R_k) = (I-K_k)^{-1}$, $k = N_*, N_*+1,\dots$, (with K_k defined by (49)); therewith the convergence $R_k \to R$ ((28)) as $k\to\infty$ takes place. Hence there exists ρ_N such that $\|R_k\|_{L_p^1[0,T]\to L_p^1[0,T]} \le \rho_N$ for any $k = N_*, N_*+1,\dots$.

Assume the condition $\delta_k^0(1+\rho_N) < 1$ $\forall k = N_*, N_*+1,\dots$, to be fulfilled (with δ_k^0 defined by (50)). These conditions can be provided by proper choice of the parameter k_0. Let us use the invertible operator theorem again to obtain the estimate

$$\|R-R_k\|_{L_p^1[0,T]\to L_p^1[0,T]} \le \frac{\delta_k^0\,(1+\rho_N)^2}{1-\delta_k^0\,(1+\rho_N)}.$$

This estimate and the representation

$$z(t) - z_k(t) = \left[(R-R_k)\,g\right](t) + \left[(I+R_k)\,(g-g_k)\right](t),$$

$t \in [0,T]$, imply

$$\|\dot{x} - \dot{x}_k\|_{L_p^1[0,T]} = \|z - z_k\|_{L_p^1[0,T]} \le x_k^v, \tag{53}$$

$$x_k^v = (1+\rho_N)\, g_k^v + \frac{\delta_k^0 (1+\rho_N)^2}{1 - \delta_k^0 (1+\rho_N)} (g_N + g_k^v),$$

which provides the convergence $x_k^v \to 0$ as $k \to \infty$.

Thus we have

Theorem 2. *Let the Cauchy problem* (11) *be uniquely solvable. Then there exists a sequence of computable operators* $\mathcal{L}_k : D_p^1[0,T] \to L_p^1[0,T]$, $\mathcal{L}_k \to \mathcal{L}$ (*with* $\mathcal{L}$ *defined by* (11)) *such that the following conditions are fulfilled:*

- *there exists* N_* *such that the Cauchy problem* (48) *is uniquely and everywhere solvable for any* $k \ge N_*$ *and the solution* x_k *to* (48) *satisfies the estimate* (52);
- *the estimate* (52) *possesses the property of asymptotic accuracy* (*i.e.*, $x_k^v \to 0$ *as* $k \to \infty$).

5. Differential System with Arbitrary Deviations

Consider in the space $DS_p^n[0,T](m)$, $1 \le p < \infty$, the PBVP

$$\begin{aligned} (\mathcal{L}y)^i(t) &\equiv \dot{y}_i(t) + \sum_{j=1}^{n} \sum_{k=1}^{n_{ij}} p_{ij}^k(t) y_j[h_{ij}^k(t)] = f_i(t), \\ y_j(\xi) &= \begin{cases} \varphi_j^0(\xi), & \xi < 0, \\ \varphi_j^T(\xi), & \xi > T, \end{cases} \quad t \in [0,T], \\ \Delta y_i &= \alpha_i, \quad \alpha_i = \operatorname{col}\{\alpha_i^1, \dots, \alpha_i^{m+1}\} \end{aligned} \tag{54}$$

with $p_{ij}^k, f_i \in L_p^1[0,T]$, $\alpha_i^j \in R^1$, and the functions $h_{ij}^k(\cdot)$, $i,j = 1,\dots,n$, $k = 1,\dots,n_{ij}$ being continuous and strictly monotone on each set B_q.

The investigation of the problem (54) includes the following main stages:

- approximating the problem under consideration within the class of computable operators;
- constructing the solution of the approximate problem (in the case where it does exist);
- the study of the original problem for the solvability with the use of the solution of the approximate problem;
- computing a guaranteed error bound of the approximate solution in the case where the unique solvability of the original problem is established.

In the sequel we consider these stages in details.

The approximation of the problem within the class of computable operators. The numbers α_i^j are to be approximated by rational numbers ${}^a\alpha_i^j$

with rational error bounds ${}^v\alpha_i^j \geq \left|\alpha_i^j - {}^a\alpha_i^j\right|$. Introduce the vectors

$$\alpha_i^a = \mathrm{col}\left\{{}^a\alpha_i^1, \dots, {}^a\alpha_i^{m+1}\right\}, \qquad \alpha_i^v = \mathrm{col}\left\{{}^v\alpha_i^1, \dots, {}^v\alpha_i^{m+1}\right\},$$

$i = 1, \dots, n$. Fixing $q = 1, \dots, m+1$, approximate the functions p_{ij}^k, f_i on the set B_q by polynomials ${}_a^q p_{ij}^k$ and ${}^a f_i^q$ with rational coefficients and define rational error bounds

$${}_v^q p_{ij}^k \ \geq \ \left\|p_{ij}^k - {}_a^q p_{ij}^k\right\|_{L_p^1[t_{q-1},t_q]}, \quad {}^v f_i^q \ \geq \ \|f_i - {}^a f_i^q\|_{L_p^1[t_{q-1},t_q]}.$$

Now define

$$\begin{aligned} {}^a p_{ij}^k(t) &= \sum_{q=1}^{m+1} \chi_q(t) {}_a^q p_{ij}^k(t), \quad t \in [0,T], \\ f_i^a(t) &= \sum_{q=1}^{m+1} \chi_q(t) {}^a f_i^q(t), \quad t \in [0,T], \end{aligned}$$

$i, j = 1, \dots, n$; $k = 1, \dots, n_{ij}$.

Fixing $i, j = 1, \dots, n$, $k = 1, \dots, n_{ij}$, calculate rational approximations ${}_a^q \bar{h}_{ij}^k$ of $h_{ij}^k(t_q)$, $q = 0, \dots, m+1$, and define a rational h_v such that

$${}_a^q \bar{h}_{ij}^k \ - \ h_v \ \leq \ h_{ij}^k(t_q) \ \leq \ {}_a^q \bar{h}_{ij}^k \ + \ h_v.$$

Let ${}_a^\nu h_{ij}^k$ be defined by

$$\begin{aligned} {}_a^\nu h_{ij}^k \ &= \ \min_{0 \leq q \leq m+1} \left\{{}_a^q \bar{h}_{ij}^k\right\} \ + \ \nu \, \frac{\max\limits_{0 \leq q \leq m+1} \left\{{}_a^q \bar{h}_{ij}^k\right\} \ - \ \min\limits_{0 \leq q \leq m+1} \left\{{}_a^q \bar{h}_{ij}^k\right\}}{m_{ij}^k}, \\ \nu &= 0, \dots, m_{ij}^k, \end{aligned}$$

where

$$m_{ij}^k \ = \ {}_\triangle^\nu h_{ij}^k \, \nu_0, \qquad {}_\triangle^\nu h_{ij}^k \ = \ \left\{{}_a^q \bar{h}_{ij}^k\right\}_{q=0}^{m+1}.$$

The parameter ν_0, having positive integer values, define the accuracy of approximation to the function h_{ij}^k. We assume therewith that the conditions ${}_a^{\nu-1} h_{ij}^k \ + \ h_v \ <_a^\nu \ h_{ij}^k - h_v$, $\nu = 1, \dots, m_{ij}^k$ are satisfied. Let ${}^q\nu_{ij}^k$, $q = 0, \dots, m+1$, be a value of ν such that ${}_a^{{}^q\nu_{ij}^k} h_{ij}^k = {}_a^q \bar{h}_{ij}^k$. For $q = 1, \dots, m+1$, denote by ${}^q m_{ij}^k + 1$ the number of values ${}_a^\nu h_{ij}^k$ which are detectable as reliably belonging to the inverse image of the function h_{ij}^k on the set B_q. Define the elements of ${}^q\mathcal{I}_{ij}^k = \left\{{}_a^{q\nu} \tilde{h}_{ij}^k\right\}_{\nu=0}^{{}^q m_{ij}^k}$ as follows:

1. ${}_a^{q\nu} \tilde{h}_{ij}^k \ = \ {}_a^{{}^{q-1}\nu_{ij}^k + \nu} h_{ij}^k$, $\nu = 0, \dots, {}^q m_{ij}^k$, if h_{ij}^k is strictly increasing on B_q;
2. ${}_a^{q\nu} \tilde{h}_{ij}^k \ = \ {}_a^{{}^q\nu_{ij}^k - \nu} h_{ij}^k$, $\nu = 0, \dots, {}^q m_{ij}^k$, if h_{ij}^k is strictly decreasing on B_q.

In the way described in Section 4, construct by means of the set ${}^{q}\mathcal{I}_{ij}^{k}$ collection of points ${}^{q\nu}t_{ij}^{k}$, $\nu = 0,\dots,{}^{q}m_{ij}^{k}$:

$$t_{q-1} = {}^{q0}t_{ij}^{k} < {}^{q1}t_{ij}^{k} < \dots < {}^{q({}^{q}m_{ij}^{k}-1)}t_{ij}^{k} < {}^{q\,{}^{q}m_{ij}^{k}}t_{ij}^{k} = t_q, \qquad (55)$$

such that

$${}^{q\nu}_{a}\tilde{h}_{ij}^{k} - h_v \leq h({}^{q\nu}t_{ij}^{k}) \leq {}^{q\nu}_{a}\tilde{h}_{ij}^{k} + h_v, \quad \nu = 0,\dots,{}^{q}m_{ij}^{k},$$

hold. Next define

$$\begin{aligned} {}^{q\nu}B_{ij}^{k} &= [{}^{q(\nu-1)}t_{ij}^{k}, {}^{q\nu}t_{ij}^{k}), \quad \nu = 1,\dots,{}^{q}m_{ij}^{k} - 1, \\ {}^{q\,{}^{q}m_{ij}^{k}}B_{ij}^{k} &= \begin{cases} [{}^{q({}^{q}m_{ij}^{k}-1)}t_{ij}^{k}, t_q), & q < m+1, \\ [{}^{q({}^{q}m_{ij}^{k}-1)}t_{ij}^{k}, T], & q = m+1, \end{cases} \end{aligned}$$

and the characteristic function ${}^{q\nu}\chi_{ij}^{k}(\cdot)$ of the set ${}^{q\nu}B_{ij}^{k}$, $\nu = 1,\dots,{}^{q}m_{ij}^{k}$. The function ${}^{a}h_{ij}^{k}(\cdot)$ is constructed by

$${}^{a}h_{ij}^{k}(t) = \sum_{q=1}^{m+1}\sum_{\nu=1}^{{}^{q}m_{ij}^{k}} {}^{q\nu}\chi_{ij}^{k}(t)\,{}^{q(\nu-1)}_{a}\tilde{h}_{ij}^{k}, \quad t \in [0,T]. \qquad (56)$$

It is easy to see that ${}^{a}h_{ij}^{k}$ is computable over (55).

We assume below that $\phi_j^0 \in D_p^1[h_0^*, 0]$, $\phi_j^T \in D_p^1[T, h_T^*]$, $j = 1,\dots,n$, the constants h_0^*, h_T^* are defined as follows:

$$\begin{aligned} h_0^* &= \min_{\substack{1\leq i,j\leq n\\ 1\leq k\leq n_{ij}}} \left\{{}^{0}b_{ij}^{k}\right\}, \\ {}^{0}b_{ij}^{k} &= \min\left\{\min_{t\in[0,T]}\left\{h_{ij}^{k}(t)\right\}, \min_{0\leq\nu\leq m_{ij}^{k}}\left\{{}^{\nu}_{a}h_{ij}^{k} - h_v\right\}\right\}, \\ h_T^* &= \max_{\substack{1\leq i,j\leq n\\ r1\leq k\leq n_{ij}}} \left\{{}^{T}b_{ij}^{k}\right\}, \\ {}^{T}b_{ij}^{k} &= \max\left\{\min_{t\in[0,T]}\left\{h_{ij}^{k}(t)\right\}, \max_{0\leq\nu\leq m_{ij}^{k}}\left\{{}^{\nu}_{a}h_{ij}^{k} + h_v\right\}\right\}. \end{aligned}$$

Let us approximate the functions ϕ_j^0, ϕ_j^T by polynomials ${}^{a}\phi_j^0$ and ${}^{a}\phi_j^T$, respectively, with rational coefficients and with rational error bounds $\{{}^{0}_{0}\phi_j^v, {}^{0}_{1}\phi_j^v\}$, $\{{}^{T}_{0}\phi_j^v, {}^{T}_{1}\phi_j^v\}$:

$$\begin{aligned} {}^{0}_{0}\phi_j^v &\geq \left|\phi_j^0(h_0^*) - {}^{a}\phi_j^0(h_0^*)\right|, \quad {}^{0}_{1}\phi_j^v \geq \left\|\dot{\phi}_j^0 - {}^{a}\dot{\phi}_j^0\right\|_{L_p^1[h_0^*,0]}, \\ {}^{T}_{0}\phi_j^v &\geq \left|\phi_j^T(T) - {}^{a}\phi_j^T(T)\right|, \quad {}^{T}_{1}\phi_j^v \geq \left\|\dot{\phi}_j^T - {}^{a}\dot{\phi}_j^T\right\|_{L_p^1[T,h_T^*]}. \end{aligned}$$

Finally we obtain the PBVP which approximates the problem (54):

$$(\mathcal{L}_a y)^i(t) \equiv \dot{y}_i(t) + \sum_{j=1}^{n}\sum_{k=1}^{n_{ij}} {}^a p_{ij}^k(t) y_j[{}^a h_{ij}^k(t)] = f_i^a(t), \tag{57}$$

$$y_j(\xi) = \begin{cases} {}^a\varphi_j^0(\xi), & \xi < 0, \\ {}^a\varphi_j^T(\xi), & \xi > T, \end{cases} \quad t \in [0,T],$$

$$\Delta y_i \ = \ \alpha_i^a, \quad i,j = 1,\ldots,n.$$

Recall that the operator $\mathcal{L}_a$: $DS_p^n[0,T](m) \to L_p^n[0,T]$ defined by $\mathcal{L}_a = \left\{(\mathcal{L}_a y)^i\right\}_{i=1}^n$ is computable.

Construction of the solution of the approximate problem. Let $\chi_{ij}^k(\cdot,\cdot)$ and ${}^a\chi_{ij}^k(\cdot,\cdot)$ be the characteristic functions of the sets

$$\begin{gathered} \left\{(t,s) \in [0,T]\times[0,T]:\ 0 \le s \le h_{ij}^k(t) \le T\right\}, \\ \left\{(t,s) \in [0,T]\times[0,T]:\ 0 \le s \le {}^a h_{ij}^k(t) \le T\right\}, \end{gathered} \tag{58}$$

respectively; $i,j = 1,\ldots,n$; $k = 1,\ldots,n_{ij}$. Define the functions $\kappa_{ij}(\cdot,\cdot)$ and $\kappa_{ij}^a(\cdot,\cdot)$ by

$$\begin{aligned} \kappa_{ij}(t,s) \ &= \ -\sum_{k=1}^{n_{ij}} p_{ij}^k(t)\chi_{ij}^k(t,s), \\ \kappa_{ij}^a(t,s) \ &= \ -\sum_{k=1}^{n_{ij}} {}^a p_{ij}^k(t)\,{}^a\chi_{ij}^k(t,s), \\ (t,s) &\in [0,T]\times[0,T], \quad i,j = 1,\ldots,n. \end{aligned} \tag{59}$$

Put

$$\mathcal{K}(\cdot,\cdot) \ = \ \{\kappa_{ij}(\cdot,\cdot)\}_{i,j=1}^n, \quad \mathcal{K}_a(\cdot,\cdot) \ = \ \left\{\kappa_{ij}^a(\cdot,\cdot)\right\}_{i,j=1}^n. \tag{60}$$

There takes place the representation

$$\begin{aligned} y_j[h_{ij}^k(t)] = &\int_0^T \chi_{ij}^k(t,s)\dot{y}_j(s)ds + y_j(0)\chi_{ij}^k(t,0) + \\ &+ \left[1 - \chi_{ij}^k(t,0)\right]\phi_j[h_{ij}^k(t)] + \\ &+ \sum_{q=1}^{m} \Delta y(t_q)\chi_{ij}^k(t,t_q), \quad t \in [0,T], \end{aligned} \tag{61}$$

$$\phi_j[h_{ij}^k(t)] \ = \begin{cases} \varphi_j^0[h_{ij}^k(t)], & h_{ij}^k(t) < 0, \\ 0, & 0 \le h_{ij}^k(t) \le T, \\ \varphi_j^T[h_{ij}^k(t)], & h_{ij}^k(t) > T. \end{cases}$$

Using the representation (61) and (59), (60), we rewrite the problem (54) in the integral form

$$\begin{aligned} \left[(I-K)\,z\right](t) &= g(t),\ K: L_p^n[0,T] \to L_p^n[0,T], \\ (Kz)(t) &= \int\limits_0^T \mathcal{K}(t,s)z(s)ds, \quad t \in [0,T], \end{aligned} \tag{62}$$

where $z(\cdot) = \dot{y}(\cdot)$. The components $g_i(\cdot)$, $i = 1, \dots, n$ of $g(\cdot)$ are defined by

$$\begin{aligned} g_i(t) = f_i(t) \;-\; &\sum_{j=1}^{n}\sum_{k=1}^{n_{ij}} p_{ij}^k(t) \sum_{q=0}^{m} \alpha_j^{q+1} \chi_{ij}^k(t,t_q) \;- \\ &- \sum_{j=1}^{n}\sum_{k=1}^{n_{ij}} p_{ij}^k(t)\left[1 \;-\; \chi_{ij}^k(t,0)\right] \phi_j[h_{ij}^k(t)]. \end{aligned} \tag{63}$$

Analogously rewrite the problem (57):

$$\begin{aligned} \left[(I-K_a)\,z\right](t) &= g_a(t),\ K_a: L_p^n[0,T] \to L_p^n[0,T], \\ (K_a z)(t) &= \int\limits_0^T \mathcal{K}_a(t,s)z(s)ds, \quad t \in [0,T], \end{aligned} \tag{64}$$

where the components $g_i^a(\cdot)$, $i = 1, \dots, n$, of $g_a(\cdot)$ are defined by

$$\begin{aligned} g_i^a(t) = f_i^a(t) - &\sum_{j=1}^{n}\sum_{k=1}^{n_{ij}} {}^a p_{ij}^k(t) \sum_{q=0}^{m} {}^a\alpha_j^{q+1}\, {}^a\chi_{ij}^k(t,t_q) - \\ &- \sum_{j=1}^{n}\sum_{k=1}^{n_{ij}} {}^a p_{ij}^k(t)\left[1 \;-\; {}^a\chi_{ij}^k(t,0)\right] \phi_j^a[{}^a h_{ij}^k(t)], \end{aligned} \tag{65}$$

$$\phi_j^a[{}^a h_{ij}^k(t)] \;=\; \begin{cases} {}^a\varphi_j^0[{}^a h_{ij}^k(t)], & {}^a h_{ij}^k(t) < 0, \\ 0, & 0 \le {}^a h_{ij}^k(t) \le T, \\ {}^a\varphi_j^T[{}^a h_{ij}^k(t)], & {}^a h_{ij}^k(t) > T. \end{cases}$$

The function ${}^a\chi_{ij}^k(\cdot,\cdot)$ admits the representation

$${}^a\chi_{ij}^k(t,s) \;=\; \sum_{q=1}^{m+1}\sum_{\nu=1}^{{}^q m_{ij}^k} {}^{q\nu}\chi_{ij}^k(t)\chi_{[0,\,{}^{q\nu}_{a}d_{ij}^k]}(s), \tag{66}$$

$(t,s) \in [0,T] \times [0,T]$, where

$${}^{q\nu}_{a}d_{ij}^k \;=\; \begin{cases} {}^{q(\nu-1)}_{a}\tilde{h}_{ij}^k, & \text{if } {}^{q(\nu-1)}_{a}\tilde{h}_{ij}^k \in [0,T], \\ 0, & \text{otherwise}, \end{cases}$$

$i, j = 1, \ldots, n$; $k = 1, \ldots, n_{ij}$. Define the functions ${}^{q\nu}u_{ij}^k(\cdot)$, ${}^{q\nu}v_{ij}^k(\cdot)$ by

$$\begin{aligned} {}^{q\nu}u_{ij}^k(t) &= {}_a^q p_{ij}^k(t)\, {}^{q\nu}\chi_{ij}^k(t), \quad t \in [0,T], \\ {}^{q\nu}v_{ij}^k(s) &= \chi_{[0,{}_a^{q\nu} d_{ij}^k]}(s), \qquad s \in [0,T], \end{aligned} \tag{67}$$

$i, j = 1, \ldots, n$; $k = 1, \ldots, n_{ij}$, $\nu = 1, \ldots, {}^q m_{ij}^k$. Due to (67), the functions κ_{ij}^a defined by (59) admit the representation

$$\kappa_{ij}^a(t,s) = \sum_{k=1}^{n_{ij}} \sum_{q=1}^{m+1} \sum_{\nu=1}^{{}^q m_{ij}^k} {}^{q\nu}u_{ij}^k(t)\, {}^{q\nu}v_{ij}^k(s), \tag{68}$$

$(t,s) \in [0,T] \times [0,T]$; $i, j = 1, \ldots, n$. Define the sets $\mathcal{J}_{ij}^k$ as follows:

$$\mathcal{J}_{ij}^k = \left\{0,\ {}^{11}t_{ij}^k, \ldots,\ {}^{1\,{}^1m_{ij}^k}t_{ij}^k,\ {}^{21}t_{ij}^k, \ldots,\ {}^{(m+1)\,({}^{m+1}m_{ij}^k-1)}t_{ij}^k, T\right\},$$

$i, j = 1, \ldots, n$, $k = 1, \ldots, n_{ij}$. Denote

$$\mathcal{J} = \bigcup_{i,j=1}^{n} \bigcup_{k=1}^{n_{ij}} \mathcal{J}_{ij}^k.$$

Let t_i, $i = 0, \ldots, n_0$, be such that

$$\mathcal{J} = \{0 = \bar{t}_0 < \bar{t}_1 < \cdots < \bar{t}_{n_0-1} < \bar{t}_{n_0} = T\}.$$

Define the sets $\bar{B}_\sigma = [\bar{t}_{\sigma-1}, \bar{t}_\sigma)$, $\sigma = 1, \ldots, n_0 - 1$, $\bar{B}_{n_0)} = [\bar{t}_{n_0-1}, T]$ and denote by $\bar{\chi}_\sigma$ the characteristic function of $\bar{B}_\sigma$, $\sigma = 1, \ldots, n_0$. Taking into account (66), we have for ${}^a\chi_{ij}^k(\cdot)$ the representation

$$ {}^a\chi_{ij}^k(t,s) = \sum_{\sigma=1}^{n_0} \bar{\chi}_\sigma(t) \chi_{[0,{}_a^\sigma \bar{d}_{ij}^k]}(s), \tag{69}$$

$(t,s) \in [0,T] \times [0,T]$, where ${}_a^\sigma \bar{d}_{ij}^k = {}_a^{q\nu} d_{ij}^k$, if $\bar{B}_\sigma \subset^{q\nu} B_{ij}^k$, $i, j = 1, \ldots, n$, $k = 1, \ldots, n_{ij}$. In the similar way we can transform the representation (68) of the function $\kappa_{ij}^a(\cdot,\cdot)$:

$$\kappa_{ij}^a(t,s) = \sum_{k=1}^{n_{ij}} \sum_{\sigma=1}^{n_0} {}^\sigma\hat{u}_{ij}^k(t)\, {}^\sigma\hat{v}_{ij}^k(s), \tag{70}$$

$(t,s) \in [0,T] \times [0,T]$, where

$$\begin{aligned} {}^\sigma\hat{u}_{ij}^k(t) &= {}_a^q p_{ij}^k(t)\, \bar{\chi}_\sigma(t), \quad t \in [0,T], \\ {}^\sigma\hat{v}_{ij}^k(s) &= \chi_{[0,{}_a^\sigma \bar{d}_{ij}^k]}(s), \qquad s \in [0,T]. \end{aligned} \tag{71}$$

Define the vector-functions $\bar{u}_{ij}^\nu(\cdot)$ and x $\bar{v}_{ij}^\nu(\cdot)$ by

$$\begin{aligned} \bar{u}_{ij}^\sigma(\cdot) &= \operatorname{row}\left[u_{ij}^{1\sigma}(\cdot), \ldots, u_{ij}^{n_{ij}\sigma}(\cdot)\right], \\ \bar{v}_{ij}^\sigma(\cdot) &= \operatorname{col}\left\{v_{ij}^{1\sigma}(\cdot), \ldots, v_{ij}^{n_{ij}\sigma}(\cdot)\right\}, \end{aligned} \tag{72}$$

$i, j = 1, \dots, n$; $\sigma = 1, \dots, n_0$, and form $n \times n^2$-matrices U_σ and $n^2 \times n$-matrices V_σ as follows:

$$U_\sigma(\cdot) = row\,[U_{1\sigma}(\cdot), U_{2\sigma}(\cdot), \dots, U_{n\sigma}(\cdot)]\,, \tag{73}$$

where

$$U_{1\sigma}(\cdot) = \begin{pmatrix} \bar{u}^\sigma_{11}(\cdot) & \bar{u}^\sigma_{12}(\cdot) & \dots & \bar{u}^\sigma_{1n} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

$$U_{2\sigma}(\cdot) = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \bar{u}^\sigma_{21}(\cdot) & \bar{u}^\sigma_{22}(\cdot) & \dots & \bar{u}^\sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

$$\vdots$$

$$U_{n\sigma}(\cdot) = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{u}^\sigma_{n1}(\cdot) & \bar{u}^\sigma_{n2}(\cdot) & \dots & \bar{u}^\sigma_{nn} \end{pmatrix},$$

$$V_\sigma(\cdot) = \begin{Bmatrix} V_{1\sigma}(\cdot) \\ V_{2\sigma}(\cdot) \\ \vdots \\ V_{n\sigma}(\cdot) \end{Bmatrix} \tag{74}$$

with

$$V_{i\sigma} = \operatorname{diag}\{\bar{v}^\sigma_{i1}, \bar{v}^\sigma_{i2}, \dots, \bar{v}^\sigma_{in}\}\,, \quad i = 1, \dots, n.$$

The representations (73), (74) allow us to define the elements of the matrices U_σ, V_σ as follows:

$$U_\sigma(t) = \left\{\tilde{u}^\sigma_{ij}(t)\right\}, i = 1, \dots, n;\ j = 1, \dots, n^2, \tag{75}$$

$$\tilde{u}^\sigma_{ij}(t) = \begin{cases} \bar{u}^\sigma_{ij}(t), & \text{if } i \le j \le n + i - 1, i = 1, \dots, n, \\ \text{row}\{\underbrace{0, \dots, 0}_{n_{ij}}\}, & \text{otherwise}, \end{cases}$$
$$t \in [0, T],$$

$$V_\sigma(s) = \left\{\tilde{v}^\sigma_{ij}(s)\right\}, i = 1, \dots, n^2;\ j = 1, \dots, n, \tag{76}$$

$$\tilde{v}_{ij}^{\sigma}(s) = \begin{cases} \bar{v}_{ij}^{\sigma}(s), & \text{if } n(k-1)+1 \le i \le nk, \\ & j = ki, \quad k = 1,\ldots,n; \\ \mathrm{col}\{\underbrace{0,\ldots,0}_{n_{ij}}\}, & \text{otherwise}, \end{cases}$$
$$s \in [0,T].$$

Put $\tilde{u}_{ij}^{\sigma}(\cdot) = \mathrm{row}\left\{{}^{\sigma}\tilde{u}_{ij}^{\theta}(\cdot)\right\}_{\theta=1}^{n_{ij}}$, $\tilde{v}_{ij}^{\sigma}(\cdot) = \mathrm{col}\left\{{}^{\sigma}\tilde{v}_{ij}^{\theta}(\cdot)\right\}_{\theta=1}^{n_{ij}}$. With (68), (72), (73) and (74), the kernel $\mathcal{K}_a(\cdot,\cdot)$ defined by (60) can be written in the form

$$\mathcal{K}_a(t,s) = \sum_{\sigma=1}^{n_0} U_\sigma(t)V_\sigma(s), \quad (t,s) \in [0,T] \times [0,T]. \tag{77}$$

Hence the equation (64) is a Fredholm equation of the second kind with a degenerate kernel. Following the known way of solving such equations, we will find a solution z_a to (64) in the form

$$z_a(t) = \sum_{\sigma=1}^{n_0} U_\sigma(t)\gamma_\sigma + g_a(t), \quad t \in [0,T]; \tag{78}$$

here the n^2-vector γ_σ has the elements γ_σ^θ:

$$\begin{aligned} \gamma_\sigma^\theta &= \mathrm{col}\left\{{}^1\gamma_\sigma^\theta,\ldots,{}^{n_{i_\theta j_\theta}}\gamma_\sigma^\theta\right\}, \quad \theta = 1,\ldots,n^2, \\ i_\theta &= [\theta/n]+1, \\ j_\theta &= \theta - (i_\theta - 1)n \end{aligned} \tag{79}$$

(here and below $[(\cdot)]$ means the greatest integer in $(\cdot)$). The vectors γ_σ, $\sigma = 1,\ldots,n_0$, may be found via solving the system

$$\Lambda\gamma = \beta. \tag{80}$$

Here $\gamma = \mathrm{col}\{\gamma_1,\ldots,\gamma_{n_0}\}$, $\beta = \mathrm{col}\{\beta_1,\ldots,\beta_{n_0}\}$,

$$\beta_\sigma = \int_0^T V_\sigma(s)g_a(s)ds, \quad \sigma = 1,\ldots,n_0.$$

The $(n^2n_0)\times(n^2n_0)$-matrix Λ has the form

$$\Lambda = \begin{pmatrix} \tilde{E}-P_{11} & -P_{12} & \ldots & -P_{1n_0} \\ -P_{21} & \tilde{E}-P_{22} & \ldots & -P_{2n_0} \\ \vdots & \vdots & \ddots & \vdots \\ -P_{n_01} & -P_{n_02} & \ldots & \tilde{E}-P_{n_0n_0} \end{pmatrix} \tag{81}$$

with $n^2 \times n^2$-matrices $P_{\sigma\theta}$ and $\tilde{E}$ defined by

$$P_{\sigma\theta} = \int_0^T V_\sigma(s)U_\theta(s)ds; \qquad \tilde{E} = \operatorname{diag}\left\{\tilde{E}_1, \dots, \tilde{E}_{n^2}\right\}.$$

Here $\tilde{E}_\tau$ is the identity matrix of $n_{i_\tau j_\tau}$-th order, $i_\tau = \left[\frac{\tau}{n}\right] + 1$; $j_\tau = \tau - (i_\tau - 1)n$. The matrices $P_{\sigma\theta}$ are of the form

$$P_{\sigma\theta} = \{\tilde{p}^{r\tau}_{\sigma\theta}\}^{n^2}_{r,\tau=1},$$

where the elements $\tilde{p}^{r\tau}_{\sigma\theta}$ are the following matrices:

$$\begin{aligned} \tilde{p}^{r\tau}_{\sigma\theta} &= \left\{{}^{ij}p^{r\tau}_{\sigma\theta}\right\}; \quad i = 1, \dots, n_{\bar{i}_i \bar{j}_i},\ j = 1, \dots, n_{\bar{i}_j \bar{j}_j}, \\ &\bar{i}_i = [i/n] + 1; \quad \bar{i}_j = [j/n] + 1; \\ &\bar{j}_i = i - (\bar{i}_i - 1)n; \qquad \bar{j}_j = j - (\bar{i}_j - 1)n. \end{aligned}$$

Define the constant n_Σ by

$$n_\Sigma = \sum_{i=1}^n \sum_{j=1}^n n_{ij}.$$

Construct the $n_\Sigma \times n_\Sigma$-matrices $\bar{P}_{\sigma\theta}$ in the natural way by means of the matrices $P_{\sigma\theta}$ having the elements ${}^{ij}p^{r\tau}_{\sigma\theta}$. Define the $(n_0 n_\Sigma) \times (n_0 n_\Sigma)$-matrix $\bar{\Lambda}$:

$$\bar{\Lambda} = \begin{pmatrix} E_{n_\Sigma} - \bar{P}_{11} & -\bar{P}_{12} & \dots & -\bar{P}_{1n_0} \\ -\bar{P}_{21} & E_{n_\Sigma} - \bar{P}_{22} & \dots & -\bar{P}_{2n_0} \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{P}_{n_0 1} & -\bar{P}_{n_0 2} & \dots & E_{n_\Sigma} - \bar{P}_{n_0 n_0} \end{pmatrix}. \tag{82}$$

Let $\bar{\Lambda}$ be invertible, i.e., PBVP (57) be uniquely solvable. Denote $\bar{B} = \bar{\Lambda}^{-1}$. Following the structure of the matrix Λ (81), we construct by means of the matrix $\bar{B}$ the $(n^2 n_0) \times (n^2 n_0)$-matrix B:

$$B = \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1n_0} \\ B_{21} & B_{22} & \dots & B_{2n_0} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n_0 1} & B_{n_0 2} & \dots & B_{n_0 n_0} \end{pmatrix}; \tag{83}$$

$$\begin{aligned} B_{\sigma\theta} &= \left\{\tilde{b}^{r\tau}_{\sigma\theta}\right\}^{n^2}_{r,\tau=1}, \\ \tilde{b}^{r\tau}_{\sigma\theta} &= \left\{{}^{ij}b^{r\tau}_{\sigma\theta}\right\}, \quad i = 1, \dots, n_{\bar{i}_i \bar{j}_i},\ j = 1, \dots, n_{\bar{i}_j \bar{j}_j}. \end{aligned}$$

Then the desired vectors γ_σ are

$$\gamma_\sigma \;=\; \sum_{\theta=1}^{n_0} B_{\sigma\theta}\beta_\theta, \quad \sigma = 1,\dots,n_0. \tag{84}$$

It remains to write the solution z_a to (64)

$$z_a(t) \;=\; \sum_{\sigma=1}^{n_0}\sum_{\theta=1}^{n_0} U_\sigma(t) B_{\sigma\theta}\beta_\theta \;+\; g_a(t), \quad t \in [0,T], \tag{85}$$

as well as the solution y_a of PBVP (57):

$$y_a(t) = \int_0^t z_a(s)ds \;+\; y_a(0) \;+ \\ + \sum_{q=1}^{m} \Delta y_a(t_q)\chi_{[t_q,T]}(t), \quad t \in [0,T]. \tag{86}$$

A norm estimate of the resolvent operator. Let the problem (57) be uniquely solvable, i.e., there exist the resolvent operator $R_a : L_p^n[0,T] \to L_p^n[0,T]$ such that $[I + R_a] = [I - K_a]^{-1}$. Now the solution z_a to (64) can be written in the form

$$z_a(t) = [(I + R_a)\, g_a]\,(t), \quad t \in [0,T], \tag{87}$$

where

$$\begin{aligned} (R_a g_a)\,(t) \;&=\; \int_0^T \mathcal{R}_a(t,s) g_a(s)ds, \; t \in [0,T], \\ \mathcal{R}_a(t,s) \;&=\; \sum_{\sigma=1}^{n_0}\sum_{\theta=1}^{n_0} U_\sigma(t) B_{\sigma\theta} V_\theta(s), \; (t,s) \in [0,T]\times[0,T]. \end{aligned} \tag{88}$$

Denote by ζ_{ij}^a the elements of $\mathcal{R}_a(\cdot,\cdot)$. To obtain an estimate of $\|R_a\|_{L_p^n[0,T]\to L_p^n[0,T]}$, use the matrix representation of the operator $R_a = \{R_{ij}^a\}_{i,j=1}^n$, $R_{ij}^a : L_p^1[0,T] \to L_p^1[0,T]$,

$$\left(R_{ij}^a z\right)(t) \;=\; \int_0^T \zeta_{ij}^a(t,s) z(s)ds, \; t \in [0,T].$$

Due to (88) we have

$$\zeta_{ij}^a(t,s) = \sum_{\sigma=1}^{n_0}\sum_{\theta=1}^{n_0}\sum_{\tau=1}^{n^2}\sum_{r=1}^{n^2}\sum_{\varpi=1}^{n_{i\tau}}\sum_{\delta=1}^{n_{rj}} {}^{\sigma}\tilde{u}_{ij}^{\varpi}(t)\, {}^{\varpi\delta}b_{\tau r}^{\sigma\theta}\, {}^{\theta}\tilde{v}_{rj}^{\delta}(s). \tag{89}$$

The desired estimate is

$$\|R_a\|_{L^n_p[0,T]\to L^n_p[0,T]} \le \varrho_0, \tag{90}$$

$$\varrho_0 = \max_{1\le i\le n}\sum_{j=1}^{n}\varrho_0^{ij}, \quad \varrho_0^{ij} \ge \Big\| \|\zeta^a_{ij}(\cdot,s)\|_{L^1_p[0,T]} \Big\|_{L^1_{p'}[0,T]}.$$

An estimate of $\|g - g_a\|_{L^n_p[0,T]}$. Making use of (63), (65), obtain the representation of the i-th component of $[g(t) - g_a(t)]$, $t \in [0,T]$:

$$g_i(t) - g_i^a(t) = \sum_{q=1}^{m+1}[f_i(t) - {}^af_i^q(t)] -$$
$$-\sum_{j=1}^{n}\sum_{k=1}^{n_{ij}}\sum_{q=0}^{m}\sum_{\nu=1}^{{}^qm^k_{ij}}\Bigg\{\left[p^k_{ij}(t) - {}^{q+1}_{\;a}p^k_{ij}(t)\right]\alpha_j^{q+1}\times$$
$$\times\,\chi_{[t_q,T]}\left[h^k_{ij}(t)\right] + \left(\chi_{[t_q,T]}\left[h^k_{ij}(t)\right] - \chi_{[t_q,T]}\left[{}^ah^k_{ij}(t)\right]\right)\times$$
$$\times\,{}^{q+1}_{\;a}p^k_{ij}(t)\,{}^a\alpha_j^{q+1} + \chi_{[t_q,T]}\left[{}^ah^k_{ij}(t)\right]\times$$
$$\times\,{}^{q+1}_{\;a}p^k_{ij}(t)\left(\alpha_j^{q+1} - {}^a\alpha_j^{q+1}\right)\Bigg\}\,{}^{(q+1)\nu}\chi^k_{ij}(t) -$$
$$-\sum_{j=1}^{n}\sum_{k=1}^{n_{ij}}\sum_{q=1}^{m+1}\sum_{\nu=1}^{{}^qm^k_{ij}}\Bigg\{\left[p^k_{ij}(t) - {}^q_ap^k_{ij}(t)\right]\left[1 - \chi^k_{ij}(t,0)\right]\times$$
$$\times\,\phi_j[h^k_{ij}(t)] + {}^q_ap^k_{ij}(t)\Bigg([1 - \chi^k_{ij}(t,0)]\phi_j[h^k_{ij}(t)] -$$
$$-\,[1 - {}^a\chi^k_{ij}(t,0)]\phi^a_j\left[{}^ah^k_{ij}(t)\right]\Bigg)\Bigg\}\,{}^{q\nu}\chi^k_{ij}(t). \tag{91}$$

Fix $i,j = 1,\dots,n$, $k = 1,\dots,n_{ij}$, $q = 1,\dots,m+1$, $\nu = 1,\dots,{}^qm^k_{ij}$, $t \in {}^{q\nu}B^k_{ij}$. Define the constants ${}^{q\nu}_{\;0}h^k_{ij}$, ${}^{q\nu}_{\;1}h^k_{ij}$,

$${}^{q\nu}_{\;0}h^k_{ij} \overset{\text{def}}{=} \min\left\{{}^{q\nu-1}_{\;\;a}\tilde h^k_{ij} - h_v,\ {}^{q\nu}_{\;a}\tilde h^k_{ij} - h_v\right\},$$
$${}^{q\nu}_{\;1}h^k_{ij} \overset{\text{def}}{=} \max\left\{{}^{q\nu-1}_{\;\;a}\tilde h^k_{ij} + h_v,\ {}^{q\nu}_{\;a}\tilde h^k_{ij} + h_v\right\}$$

such that ${}^{q\nu}_{\;0}h^k_{ij} \le h^k_{ij}(t) \le {}^{q\nu}_{\;1}h^k_{ij}$. Denote

$${}^{q\nu}_{\;1}\zeta^k_{ij}(t) = \left|\chi_{[t_{q-1},T]}\left[h^k_{ij}(t)\right]\right|;$$
$${}^{q\nu}_{\;2}\zeta^k_{ij}(t) = \left|\chi_{[t_{q-1},T]}\left[h^k_{ij}(t)\right] - \chi_{[t_{q-1},T]}\left[{}^ah^k_{ij}(t)\right]\right|;$$
$${}^{q\nu}_{\;3}\zeta^k_{ij}(t) = \left|\chi_{[t_{q-1},T]}\left[{}^ah^k_{ij}(t)\right]\right|;$$

$$\begin{aligned} {}^{q\nu}_{4}\zeta^{k}_{ij}(t) &= \big|\big[1 - \chi^{k}_{ij}(t,0)\big]\,\phi_j[h^{k}_{ij}(t)]\big|\,; \\ {}^{q\nu}_{5}\zeta^{k}_{ij}(t) &= \big|\big[1 - \chi^{k}_{ij}(t,0)\big]\,\phi_j[h^{k}_{ij}(t)] - \\ &\quad - \big[1 - \chi^{k}_{ij}(t,0)\big]\,\phi^{a}_j[{}^{a}h^{k}_{ij}(t)]\big|\,. \end{aligned}$$

Now construct the estimates for ${}^{q\nu}_{i}\zeta^{k}_{ij}(t)$, $i = 1,\dots,3$. There are the following cases.

Case 1. $t_q \le {}^{q\nu}_{0}h^{k}_{ij}$ and ${}^{q\nu}_{1}h^{k}_{ij} \le T$. In this case

$$\begin{aligned} {}^{q\nu}_{1}\zeta^{k}_{ij}(t) &= 1 \overset{\text{def}}{=} {}^{q\nu}_{1}\gamma^{k}_{ij}, \\ {}^{q\nu}_{2}\zeta^{k}_{ij}(t) &= 0 \overset{\text{def}}{=} {}^{q\nu}_{2}\gamma^{k}_{ij}, \\ {}^{q\nu}_{3}\zeta^{k}_{ij}(t) &= 1 \overset{\text{def}}{=} {}^{q\nu}_{3}\gamma^{k}_{ij}. \end{aligned} \tag{92}$$

Case 2. Either ${}^{q\nu}_{1}h^{k}_{ij} < t_q$ or ${}^{q\nu}_{0}h^{k}_{ij} > T$. In this case

$$\begin{aligned} {}^{q\nu}_{1}\zeta^{k}_{ij}(t) &= 0 \overset{\text{def}}{=} {}^{q\nu}_{1}\gamma^{k}_{ij}, \\ {}^{q\nu}_{2}\zeta^{k}_{ij}(t) &= 0 \overset{\text{def}}{=} {}^{q\nu}_{2}\gamma^{k}_{ij}, \\ {}^{q\nu}_{3}\zeta^{k}_{ij}(t) &= 0 \overset{\text{def}}{=} {}^{q\nu}_{3}\gamma^{k}_{ij}. \end{aligned} \tag{93}$$

Case 3. Either $\big({}^{q\nu}_{a}\tilde{h}^{k}_{ij} > T$ and ${}^{q\nu}_{0}h^{k}_{ij} \le T\big)$ or $\big({}^{q\nu}_{a}\tilde{h}^{k}_{ij} < t_q$ and ${}^{q\nu}_{1}h^{k}_{ij} \ge t_q\big)$. In this case

$$\begin{aligned} {}^{q\nu}_{1}\zeta^{k}_{ij}(t) &= 1 \overset{\text{def}}{=} {}^{q\nu}_{1}\gamma^{k}_{ij}, \\ {}^{q\nu}_{2}\zeta^{k}_{ij}(t) &= 1 \overset{\text{def}}{=} {}^{q\nu}_{2}\gamma^{k}_{ij}, \\ {}^{q\nu}_{3}\zeta^{k}_{ij}(t) &= 0 \overset{\text{def}}{=} {}^{q\nu}_{3}\gamma^{k}_{ij}. \end{aligned} \tag{94}$$

Case 4. Either $\big(t_q \le {}^{q\nu}_{a}\tilde{h}^{k}_{ij} \le T$ and ${}^{q\nu}_{0}h^{k}_{ij} < t_q\big)$ or $\big(t_q \le {}^{q\nu}_{a}\tilde{h}^{k}_{ij} \le T$ and ${}^{q\nu}_{1}h^{k}_{ij} > T\big)$. In this case

$$\begin{aligned} {}^{q\nu}_{1}\zeta^{k}_{ij}(t) &= 1 \overset{\text{def}}{=} {}^{q\nu}_{1}\gamma^{k}_{ij}, \\ {}^{q\nu}_{2}\zeta^{k}_{ij}(t) &= 1 \overset{\text{def}}{=} {}^{q\nu}_{2}\gamma^{k}_{ij}, \\ {}^{q\nu}_{3}\zeta^{k}_{ij}(t) &= 1 \overset{\text{def}}{=} {}^{q\nu}_{3}\gamma^{k}_{ij}. \end{aligned} \tag{95}$$

Now construct the estimates for ${}^{q\nu}_{i}\zeta^{k}_{ij}(t)$, $i = 4, 5$. There are the following cases.

Case 1. ${}^{q\nu}_{1}h^k_{ij} < 0$. In this case

$$\begin{aligned} {}^{q\nu}_{4}\zeta^k_{ij}(t) \le {} & \left|{}^a\phi^0_j(h^*_0)\right| + {}^0_0\phi^v_j + \\ & + \left(\left\|{}^a\dot{\phi}^0_j\right\|_{L^1_p[h^*_0,\, {}^{q\nu}_{1}h^k_{ij}]} + {}^0_1\phi^v_j\right) \times \\ & \times \sqrt[p']{{}^{q\nu}_{1}h^k_{ij} - h^*_0} \overset{\text{def}}{=} {}^{q\nu}_{4}\gamma^k_{ij}, \\ {}^{q\nu}_{5}\zeta^k_{ij}(t) \le {} & {}^0_0\phi^v_j + {}^0_1\phi^v_j \sqrt[p']{{}^{q\nu}_{1}h^k_{ij} - {}^{q\nu}_{0}h^k_{ij}} \overset{\text{def}}{=} {}^{q\nu}_{5}\gamma^k_{ij}. \end{aligned} \tag{96}$$

Case 2. ${}^{q\nu}_{0}h^k_{ij} > T$. In this case

$$\begin{aligned} {}^{q\nu}_{4}\zeta^k_{ij}(t) \le {} & \left|{}^a\phi^T_j(T)\right| + {}^T_0\phi^v_j + \\ & + \left(\left\|{}^a\dot{\phi}^T_j\right\|_{L^1_p[T,\, {}^{q\nu}_{1}h^k_{ij}]} + {}^T_1\phi^v_j\right) \times \\ & \times \sqrt[p']{{}^{q\nu}_{1}h^k_{ij} - T} \overset{\text{def}}{=} {}^{q\nu}_{4}\gamma^k_{ij}, \\ {}^{q\nu}_{5}\zeta^k_{ij}(t) \le {} & {}^T_0\phi^v_j + {}^T_1\phi^v_j \sqrt[p']{{}^{q\nu}_{1}h^k_{ij} - {}^{q\nu}_{0}h^k_{ij}} \overset{\text{def}}{=} {}^{q\nu}_{5}\gamma^k_{ij}. \end{aligned} \tag{97}$$

Case 3. ${}^{q\nu}_{a}\tilde{h}^k_{ij} < 0$ and ${}^{q\nu}_{1}h^k_{ij} \ge 0$. In this case

$$\begin{aligned} {}^{q\nu}_{4}\zeta^k_{ij}(t) \le {} & \left|{}^a\phi^0_j(h^*_0)\right| + {}^0_0\phi^v_j + \\ & + \left(\left\|{}^a\dot{\phi}^0_j\right\|_{L^1_p[h^*_0,\, 0]} + {}^0_1\phi^v_j\right) \times \\ & \times \sqrt[p']{-h^*_0} \overset{\text{def}}{=} {}^{q\nu}_{4}\gamma^k_{ij}, \\ {}^{q\nu}_{5}\zeta^k_{ij}(t) \le {} & {}^0_0\phi^v_j + {}^0_1\phi^v_j \sqrt[p']{-{}^{q\nu}_{0}h^k_{ij}} + \left|{}^a\phi^0_j(h^*_0)\right| + \\ & + \left\|{}^a\dot{\phi}^0_j\right\|_{L^1_p[h^*_0,\, {}^{q\nu}_{a}\tilde{h}^k_{ij}]} \times \\ & \times \sqrt[p']{{}^{q\nu}_{a}\tilde{h}^k_{ij} - h^*_0} \overset{\text{def}}{=} {}^{q\nu}_{5}\gamma^k_{ij}. \end{aligned} \tag{98}$$

Case 4. ${}^{q\nu}_{a}\tilde{h}^k_{ij} \ge 0$ and ${}^{q\nu}_{0}h^k_{ij} < 0$. In this case

$$\begin{aligned} {}^{q\nu}_{4}\zeta^k_{ij}(t) = {}^{q\nu}_{5}\zeta^k_{ij}(t) \le {} & \left|{}^a\phi^0_j(h^*_0)\right| + {}^0_0\phi^v_j + \\ & + \left(\left\|{}^a\dot{\phi}^0_j\right\|_{L^1_p[h^*_0,\, 0]} + {}^0_1\phi^v_j\right) \times \\ & \times \sqrt[p']{-h^*_0} \overset{\text{def}}{=} {}^{q\nu}_{4}\gamma^k_{ij} \overset{\text{def}}{=} {}^{q\nu}_{5}\gamma^k_{ij}. \end{aligned} \tag{99}$$

Case 5. ${}^{q\nu}_{a}\tilde{h}^{k}_{ij} > T$ and ${}^{q\nu}_{0}h^{k}_{ij} \le T$. In this case

$$
\begin{aligned}
{}^{q\nu}_{4}\zeta^{k}_{ij}(t) \le{}& \left|{}^{a}\phi^{T}_{j}(T)\right| + {}^{T}_{0}\phi^{v}_{j} + \\
&+ \left(\left\|{}^{a}\dot{\phi}^{T}_{j}\right\|_{L^{1}_{p}[T,\,{}^{q\nu}_{1}h^{k}_{ij}]} + {}^{T}_{1}\phi^{v}_{j}\right) \times \\
&\times \sqrt[p']{{}^{q\nu}_{1}h^{k}_{ij} - T} \stackrel{\text{def}}{=} {}^{q\nu}_{4}\gamma^{k}_{ij}, \\
{}^{q\nu}_{5}\zeta^{k}_{ij}(t) \le{}& {}^{T}_{0}\phi^{v}_{j} + {}^{0}_{1}\phi^{v}_{j} \times \\
&\times \sqrt[p']{{}^{q\nu}_{1}h^{k}_{ij} - T} + \left|{}^{a}\phi^{T}_{j}(T)\right| + \\
&+ \left\|{}^{a}\dot{\phi}^{T}_{j}\right\|_{L^{1}_{p}[T,\,{}^{q\nu}_{a}\tilde{h}^{k}_{ij}]} \times \\
&\times \sqrt[p']{{}^{q\nu}_{a}\tilde{h}^{k}_{ij} - T} \stackrel{\text{def}}{=} {}^{q\nu}_{5}\gamma^{k}_{ij}.
\end{aligned}
\tag{100}
$$

Case 6. ${}^{q\nu}_{a}\tilde{h}^{k}_{ij} \le T$ and ${}^{q\nu}_{1}h^{k}_{ij} > T$. In this case

$$
\begin{aligned}
{}^{q\nu}_{4}\zeta^{k}_{ij}(t) = {}^{q\nu}_{5}\zeta^{k}_{ij}(t) \le{}& \sqrt[p']{{}^{q\nu}_{1}h^{k}_{ij} - T} \times \\
&\times \left(\left\|{}^{a}\dot{\phi}^{T}_{j}\right\|_{L^{1}_{p}[T,\,{}^{q\nu}_{1}h^{k}_{ij}]} + {}^{T}_{1}\phi^{v}_{j}\right) + \\
&+ \left|{}^{a}\phi^{T}_{j}(T)\right| + {}^{T}_{0}\phi^{v}_{j} \stackrel{\text{def}}{=} {}^{q\nu}_{4}\gamma^{k}_{ij} \stackrel{\text{def}}{=} {}^{q\nu}_{5}\gamma^{k}_{ij}.
\end{aligned}
\tag{101}
$$

Case 7. ${}^{q\nu}_{0}h^{k}_{ij} \ge 0$ and ${}^{q\nu}_{1}h^{k}_{ij} \le T$. In this case

$$
{}^{q\nu}_{4}\zeta^{k}_{ij}(t) = {}^{q\nu}_{5}\zeta^{k}_{ij}(t) = 0 \stackrel{\text{def}}{=} {}^{q\nu}_{4}\gamma^{k}_{ij} \stackrel{\text{def}}{=} {}^{q\nu}_{5}\gamma^{k}_{ij}. \tag{102}
$$

The above estimates imply conclusively

$$
\|g - g_a\|_{L^{n}_{p}[0,T]} \le g_v, \quad g_v = \max_{1\le i\le n} g^{v}_{i}. \tag{103}
$$

Here the rational constants g^{v}_{i}, $i = 1, \dots, n$, are defined by the inequality

$$
\begin{aligned}
g^{v}_{i} \stackrel{\text{def}}{\ge}{}& \sum_{q=1}^{m+1} {}^{v}f^{q}_{i} + \\
&+ \sum_{j=1}^{n}\sum_{k=1}^{n_{ij}}\sum_{q=0}^{m} \Bigg\{ {}^{q+1}_{v}p^{k}_{ij}\left(|{}^{a}\alpha^{q+1}_{j}| + {}^{v}\alpha^{q+1}_{j}\right) \max_{1\le\nu\le{}^{q}m^{k}_{ij}} {}^{q\nu}_{1}\gamma^{k}_{ij} + \\
&\qquad + \left\|{}^{q+1}_{a}p^{k}_{ij}\right\|_{L^{1}_{p}[t_q,t_{q+1}]} \left|{}^{a}\alpha^{q+1}_{j}\right| \max_{1\le\nu\le{}^{q}m^{k}_{ij}} {}^{q\nu}_{2}\gamma^{k}_{ij} + \\
&\qquad + \left\|{}^{q+1}_{a}p^{k}_{ij}\right\|_{L^{1}_{p}[t_q,t_{q+1}]} {}^{v}\alpha^{q+1}_{j} \max_{1\le\nu\le{}^{q}m^{k}_{ij}} {}^{q\nu}_{3}\gamma^{k}_{ij} \Bigg\} + \\
&+ \sum_{j=1}^{n}\sum_{k=1}^{n_{ij}}\sum_{q=1}^{m+1} \Bigg\{ {}^{q}_{v}p^{k}_{ij} \max_{1\le\nu\le{}^{q}m^{k}_{ij}} {}^{q\nu}_{4}\gamma^{k}_{ij} +
\end{aligned}
$$

$$+ \left\| {}^{q}_{a}p^{k}_{ij} \right\|_{L^{1}_{p}[t_{q-1},t_{q}]} \max_{1 \le \nu \le {}^{q}m^{k}_{ij}} {}^{q\nu}_{5}\gamma^{k}_{ij} \Bigg\} . \tag{104}$$

Constructing an estimate of $\|K - K_a\|_{L^n_p[0,T] \to L^n_p[0,T]}$. Define the operator K_Δ: $L^n_p[0,T] \to L^n_p[0,T]$ as follows:

$$K_\Delta = K - K_a, \tag{105}$$

where

$$\left(K_\Delta z\right)(t) = \int_0^T \mathcal{K}_\Delta(t,s) z(s) ds, \ t \in [0,T],$$

$$\mathcal{K}_\Delta(t,s) = \mathcal{K}(t,s) - \mathcal{K}_a(t,s), \ (t,s) \in [0,t] \times [0,T],$$

and the matrices $\mathcal{K}(\cdot,\cdot)$, $\mathcal{K}_a(\cdot,\cdot)$ are defined by (60). To estimate $\|K_\Delta\|_{L^n_p[0,T] \to L^n_p[0,T]}$, use the matrix representation

$$K_\Delta = \left\{K^\Delta_{ij}\right\}^n_{i,j=1}, \quad K^\Delta_{ij} : L^1_p[0,T] \to L^1_p[0,T],$$

$$\left(K^\Delta_{ij} z\right)(t) = \int_0^T \kappa^\Delta_{ij}(t,s) ds, \quad t \in [0,T],$$

$$\kappa^\Delta_{ij}(t,s) = \kappa_{ij}(t,s) - \kappa^a_{ij}(t,s), \quad (t,s) \in [0,t] \times [0,T].$$

Here $\kappa_{ij}(\cdot,\cdot)$ and $\kappa^a_{ij}(\cdot,\cdot)$ are defined by (59). For $\kappa^\Delta_{ij}(\cdot,\cdot)$ we have

$$\begin{aligned} \kappa^\Delta_{ij}(t,s) = &- \sum_{k=1}^{n_{ij}} \sum_{q=1}^{m+1} \left[p^k_{ij}(t) - {}^{q}_{a}p^k_{ij}(t)\right] \chi^k_{ij}(t,s) - \\ &- \sum_{k=1}^{n_{ij}} \sum_{q=1}^{m+1} {}^{q}_{a}p^k_{ij}(t) \left[\chi^k_{ij}(t,s) - {}^{a}\chi^k_{ij}(t,s)\right], \\ &(t,s) \in [0,T] \times [0,T], \end{aligned} \tag{106}$$

$i,j = 1,\dots,n$. For each triple $\left({}^{\nu-1}_{a}h^k_{ij}, {}^{\nu}_{a}h^k_{ij}, {}^{\nu+1}_{a}h^k_{ij}\right)$, $\nu = 1,\dots,m^k_{ij} - 1$, such that ${}^{\nu-1}_{a}h^k_{ij} - h_v \ge 0$ and ${}^{\nu+1}_{a}h^k_{ij} + h_v \le T$, we form the set ${}^{\nu}\mathcal{G}^k_{ij}$ of the values q, $1 \le q \le m+1$, such that ${}^{\nu-1}_{a}h^k_{ij}, {}^{\nu}_{a}h^k_{ij}, {}^{\nu+1}_{a}h^k_{ij} \in^{q} \mathcal{I}^k_{ij}$ (by the definition, the set ${}^{q}\mathcal{I}^k_{ij}$ includes the inverse image of $h^k_{ij}(\cdot)$ on the set B_q). Denote by ${}^{q\sigma_{\nu-1}}t^k_{ij}$, ${}^{q\sigma_{\nu}}t^k_{ij}$, ${}^{q\sigma_{\nu+1}}t^k_{ij}$ the points of the partition of B_q, which are appropriate to fixed triple $\left({}^{\nu-1}_{a}h^k_{ij}, {}^{\nu}_{a}h^k_{ij}, {}^{\nu+1}_{a}h^k_{ij}\right)$, and define the numbers ${}^{\nu}\varpi^k_{ij}$ by

$${}^{\nu}\varpi^k_{ij} = \sum_{q \in {}^{\nu}\mathcal{G}^k_{ij}} \left\| {}^{q}_{a}p^k_{ij} \right\|_{L^1_p\left[{}^{q\sigma_{\nu-1}}t^k_{ij}, {}^{q\sigma_{\nu+1}}t^k_{ij}\right]} \sqrt[p']{{}^{q\sigma_{\nu+1}}t^k_{ij} - {}^{q\sigma_{\nu-1}}t^k_{ij}}. \tag{107}$$

Let ${}^0\nu_{ij}^k \in [1, m_{ij}^k]$ be such that the segment $\left[{}_a^{{}^0\nu_{ij}^k-1}h_{ij}^k,\ {}_a^{{}^0\nu_{ij}^k}h_{ij}^k\right]$ includes 0. Define the set ${}^0\mathcal{G}_{ij}^k$ of the values of q, $1 \le q \le m+1$, such that ${}_a^{{}^0\nu_{ij}^k-1}h_{ij}^k$, ${}_a^{{}^0\nu_{ij}^k}h_{ij}^k \in {}^q\mathcal{I}_{ij}^k$. Denote by ${}^{q\sigma_0-1}t_{ij}^k$, ${}^{q\sigma_0}t_{ij}^k$ the points of partition of B_q, which are appropriate to the fixed pair $\left({}_a^{{}^0\nu_{ij}^k-1}h_{ij}^k,\ {}_a^{{}^0\nu_{ij}^k}h_{ij}^k\right)$ and define ${}^0\varpi_{ij}^k$ by

$$ {}^0\varpi_{ij}^k \;=\; \sum_{q\in{}^0\mathcal{G}_{ij}^k} \|{}_a^q p_{ij}^k\|_{L_p^1[{}^{q\sigma_0-1}t_{ij}^k,\, {}^{q\sigma_0}t_{ij}^k]}\ \sqrt[p']{T}. \tag{108} $$

Let ${}^T\nu_{ij}^k \in [1, m_{ij}^k]$ be such that the segment $\left[{}_a^{{}^T\nu_{ij}^k-1}h_{ij}^k,\ {}_a^{{}^T\nu_{ij}^k}h_{ij}^k\right]$ includes T. Define the set ${}^T\mathcal{G}_{ij}^k$ of the values of q, $1 \le q \le m+1$, such that ${}_a^{{}^T\nu_{ij}^k-1}h_{ij}^k$, ${}_a^{{}^T\nu_{ij}^k}h_{ij}^k \in {}^q\mathcal{I}_{ij}^k$. Denote by ${}^{q\sigma_T-1}t_{ij}^k$, ${}^{q\sigma_T}t_{ij}^k$ the points of the partition of B_q, which are appropriate to the pair $\left({}_a^{{}^T\nu_{ij}^k-1}h_{ij}^k,\ {}_a^{{}^T\nu_{ij}^k}h_{ij}^k\right)$ and define ${}^T\varpi_{ij}^k$ by

$$ {}^T\varpi_{ij}^k \;=\; \sum_{q\in{}^T\mathcal{G}_{ij}^k} \|{}_a^q p_{ij}^k\|_{L_p^1[{}^{q\sigma_T-1}t_{ij}^k,\, {}^{q\sigma_T}t_{ij}^k]}\ \sqrt[p']{T}. \tag{109} $$

Using the way of constructing the estimate (26), we obtain with (106) conclusively

$$ \|K_\Delta\|_{L_p^n[0,T]\to L_p^n[0,T]} \;\le\; \delta_0, \qquad \delta_0 \ge \max_{1\le i\le n} \sum_{j=1}^{n} \sum_{k=1}^{n_{ij}} \delta_{ij}^k, \tag{110} $$

where δ_{ij}^k are defined as follows:

1.

$$ \delta_{ij}^k \stackrel{\text{def}}{=} 0, \tag{111} $$

if one of the condition ${}_0^a h_{ij}^k - h_v > T$, ${}_{m_{ij}^k}^{a} h_{ij}^k + h_v < 0$ holds.

2.

$$ \delta_{ij}^k \;\stackrel{\text{def}}{\ge}\; \max_{1\le\nu\le m_{ij}^k-1} {}^\nu\varpi_{ij}^k \;+\; \sum_{q=1}^{m+1} {}_v^q p_{ij}^k\ \sqrt[p']{T}, \tag{112} $$

if $0 \le {}_0^a h_{ij}^k - h_v < {}_a^{m_{ij}^k} h_{ij}^k + h_v \le T$.

3.

$$ \delta_{ij}^k \;\stackrel{\text{def}}{\ge}\; \max_{{}^0\nu_{ij}^k+1\le\nu\le{}^T\nu_{ij}^k-1} {}^\nu\varpi_{ij}^k \;+\; \sum_{q=1}^{m+1} {}_v^q p_{ij}^k\ \sqrt[p']{T} \;+ $$
$$ +\,{}^0\varpi_{ij}^k \;+\; {}^T\varpi_{ij}^k, \tag{113} $$

with ${}^{\nu}\varpi_{ij}^{k}$ defined by (107) and ${}^{0}\varpi_{ij}^{k}$, ${}^{T}\varpi_{ij}^{k}$ defined by (108), (109) respectively, if there exist ${}^{0}\nu_{ij}^{k}$, ${}^{T}\nu_{ij}^{k}$.

The study of the original problem for the solvability and the construction of the error bound. Let the approximating problem (57) be uniquely solvable, i.e. the resolvent operator R_a (88) exist. Assume that

$$\delta_0 \; < \; \frac{1}{1+\varrho_0}, \tag{114}$$

where ϱ_0 is defined by (90), and δ_0 is defined by (110). In this case the estimate

$$\|K_\Delta\|_{L_p^n[0,T]\to L_p^n[0,T]} \; < \; \frac{1}{\|1+R_a\|_{L_p^n[0,T]\to L_p^n[0,T]}}$$

holds, and due to the invertible operator theorem there takes place the invertibility of the operator $(I-K)$ as well as the existence of the resolvent operator R: $L_p^n[0,T] \to L_p^n[0,T]$, $(I+R) = (I-K)^{-1}$, giving the representation of the solution z to (62):

$$z(t) = [(I+R)\,g]\,(t), \quad t \in [0,T]. \tag{115}$$

By (87) and (115) we have

$$z(t) - z_a(t) = [(I+R_a)\,(g \;-\; g_a]\,(t) \;+\; [(R-R_a)\,(g_a)]\,(t), \quad t \in [0,T].$$

Due to the invertible operator theorem, the estimate

$$\|R-R_a\|_{L_p^n[0,T]\to L_p^n[0,T]} \; \leq \; \frac{\delta_0\,(1+\varrho_0)^2}{1-\delta_0\,(1+\varrho_0)}$$

holds. Conclusively we obtain

$$\|z-z_a\|_{L_p^n[0,T]} \; \leq \; z_v,$$

where a rational z_v is defined by the inequality

$$z_v \; \geq \; (1+\varrho_0)\,g_v \; + \; \left\{\|g_a\|_{L_p^n[0,T]} + g_v\right\}\frac{\delta_0\,(1+\varrho_0)^2}{1-\delta_0\,(1+\varrho_0)}. \tag{116}$$

Now the solution y_a of the problem (57) can be defined as

$$y_a(t) = \int_0^t z_a(s)ds \; + \; \alpha_0^a \; + \; \sum_{q=1}^{m} \alpha_q^a \chi_{[t_q,T]}(t), \tag{117}$$

therewith the estimate

$$\|y-y_a\|_{DS_p^n(m)[0,T]} \; \leq \; y_v$$

takes place, where y is the exact solution to (54) and a rational y_v is defined by

$$y_v \geq z_v + \sum_{q=0}^{m} \left\|\alpha_q^v\right\|_{R^n} . \tag{118}$$

6. n-TH ORDER DIFFERENTIAL EQUATION WITH ARBITRARY DEVIATION

Consider in the space $WS_p^n[0,T](m)$, $1 \leq p < \infty$, the PBVP

$$(\mathcal{L}^n y)(t) \equiv y^{(n)}(t) + \sum_{i=0}^{n-1}\sum_{j=1}^{n_i} p_{ij}(t) y^{(i)}[h_{ij}(t)] = f(t), \tag{119}$$

$$y^{(i)}(\xi) = \begin{cases} \varphi_i^0(\xi), & \xi < 0, \\ \varphi_i^T(\xi), & \xi > T, \end{cases} \quad t \in [0,T],$$

$$\Delta^n y = \alpha, \quad \alpha = \mathrm{col}\left\{\alpha_1, \dots, \alpha_{mn+n}\right\},$$

assuming that $p_{ij}, f \in L_p^1[0,T]$, $\alpha_i \in R^1$, the functions $h_{ij}(\cdot)$, $i = 0, \dots, n-1$, $j = 1, \dots, n_i$, are continuous and strictly monotone on each B_q.

The investigation of the problem (119) includes the following principal stages:

- approximating the problem under consideration within the class of computable operators;
- constructing the solution of the approximate problem (in the case where the solution exists);
- the study of the original problem for the solvability (using the solution of the approximate problem);
- computing a guaranteed error bound of the approximate solution (in the case where the unique solvability of the original problem is established).

Let us consider these stages in details.

Approximation of the problem under consideration within the class of computable operators. The components α_i, $i = 1, \dots, mn+n$, of α are approximated by rational numbers α_i^a with rational error bound $\alpha_i^v \geq |\alpha_i - \alpha_i^a|$. Next define

$$\alpha_a = \mathrm{col}\left\{\alpha_1^a, \dots, \alpha_{mn+n}^a\right\}, \qquad \alpha_v = \mathrm{col}\left\{\alpha_1^v, \dots, \alpha_{mn+n}^v\right\}.$$

Fixing $q = 1, \dots, m+1$, approximate the functions p_{ij}, f on each B_q by polynomials ${}^a p_{ij}^q$ and f_q^a with rational coefficients. Next define rational error bounds

$${}^v p_{ij}^q \geq \left\|p_{ij} - {}^a p_{ij}^q\right\|_{L_p^1[t_{q-1},t_q]}, \quad f_q^v \geq \left\|f - f_q^a\right\|_{L_p^1[t_{q-1},t_q]},$$

as well as the functions

$$p_{ij}^{a}(t) = \sum_{q=1}^{m+1} \chi_q(t)^a p_{ij}^{q}(t), \quad t \in [0,T],$$

$$f_a(t) = \sum_{q=1}^{m+1} \chi_q(t) f_q^a(t), \quad t \in [0,T],$$

$i = 0,\dots,n-1$, $j = 1,\dots,n_i$.

Approximating h_{ij}. Fix $i = 0,\dots,n-1$, $j = 1,\dots,n_i$, calculate rational approximations ${}^a\bar{h}_{ij}^q$ of $h_{ij}(t_q)$, $q = 0,\dots,m+1$, define rational h_v such that

$${}^a\bar{h}_{ij}^q - h_v \le h_{ij}(t_q) \le {}^a\bar{h}_{ij}^q + h_v,$$

and construct the collection of ${}^a h_{ij}^\nu$ by the rule

$${}^a h_{ij}^\nu = \min_{0\le q\le m+1} \left\{{}^a\bar{h}_{ij}^q\right\} + \nu \frac{\max\limits_{0\le q\le m+1} \left\{{}^a\bar{h}_{ij}^q\right\} - \min\limits_{0\le q\le m+1} \left\{{}^a\bar{h}_{ij}^q\right\}}{m_{ij}},$$

$\nu = 0,\dots,m_{ij}$, where

$$m_{ij} = {}^\triangle h_{ij}^\nu \, \nu_0, \qquad {}^\triangle h_{ij}^\nu = \left\{{}^a\bar{h}_{ij}^q\right\}_{q=0}^{m+1}.$$

Here the positive integer parameter ν_0 defines the accuracy of approximation to h_{ij}, assuming ${}^a h_{ij}^{\nu-1} + h_v <^a h_{ij}^\nu - h_v$, $\nu = 1,\dots,m_{ij}$. Denote by ν_{ij}^q, $q = 0,\dots,m+1$, a value of ν such that ${}^a h_{ij}^{\nu_{ij}^q} = {}^a\bar{h}_{ij}^q$. For $q = 1,\dots,m+1$, denote by $m_{ij}^q + 1$ the amount of values ${}^a h_{ij}^\nu$ which belong reliably to the inverse image of h_{ij} on the set B_q. Define the elements of the set $\mathcal{I}_{ij}^q = \left\{{}^a\tilde{h}_{ij}^{q\nu}\right\}_{\nu=0}^{m_{ij}^q}$ as follows:

1. $${}^a\tilde{h}_{ij}^{q\nu} = {}^a h_{ij}^{{}^{q-1}\nu_{ij}^k+\nu},$$
   $\nu = 0,\dots,m_{ij}^q$, if h_{ij} is strictly increasing on B_q;
2. $${}^a\tilde{h}_{ij}^{q\nu} = {}^a h_{ij}^{{}^{q}\nu_{ij}^k-\nu},$$
   $\nu = 0,\dots,m_{ij}^q$, if h_{ij} is strictly decreasing on B_q.

By means of the set $\mathcal{I}_{ij}^q$, in the way described in Section 4, construct the set of points $t_{ij}^{q\nu}$, $\nu = 0,\dots,m_{ij}^q$,

$$t_{q-1} = t_{ij}^{q0} < t_{ij}^{q1} < \dots < t_{ij}^{q(m_{ij}^q-1)} < t_{ij}^{q\,m_{ij}^q} = t_q, \tag{120}$$

such that the condition

$${}^a\tilde{h}_{ij}^{q\nu} - h_v \le h(t_{ij}^{q\nu}) \le {}^a\tilde{h}_{ij}^{q\nu} + h_v, \quad \nu = 0,\dots,m_{ij}^q,$$

holds. Next put

$$B_{ij}^{q\nu} = [t_{ij}^{q(\nu-1)}, t_{ij}^{q\nu}), \quad \nu = 1, \ldots, m_{ij}^q - 1,$$
$$B_{ij}^{q\, m_{ij}^q} = \begin{cases} [t_{ij}^{q(m_{ij}^q - 1)}, t_q), & q < m+1, \\ [t_{ij}^{q({}^q m_{ij}^k - 1)}, T], & q = m+1, \end{cases}$$

and denote by $\chi_{ij}^{q\nu}(\cdot)$ the characteristic function of $B_{ij}^{q\nu}$, $\nu = 1, \ldots, m_{ij}^q$. Construct $h_{ij}^a(\cdot)$ by the following rule

$$h_{ij}^a(t) = \sum_{q=1}^{m+1} \sum_{\nu=1}^{m_{ij}^q} \chi_{ij}^{q\nu}(t)\, {}^a\tilde{h}_{ij}^{q(\nu-1)}, \quad t \in [0, T]. \tag{121}$$

It is easy to see that the function h_{ij}^a is computable over the partition (120).

Below we suppose that $\phi_i^0 \in D_p^1[h_0^*, 0]$, $\phi_i^T \in D_p^1[T, h_T^*]$, $i = 0, \ldots, n-1$, and define the constants h_0^*, h_T^* by the equalities

$$h_0^* = \min_{\substack{0 \le i \le n-1 \\ 1 \le j \le n_i}} \left\{ b_{ij}^0 \right\}, \quad b_{ij}^0 = \min \left\{ \min_{t \in [0,T]} \{h_{ij}(t)\}, \min_{0 \le \nu \le m_{ij}} \left\{ {}^a h_{ij}^\nu - h_v \right\} \right\},$$
$$h_T^* = \max_{\substack{0 \le i \le n-1 \\ 1 \le j \le n_i}} \left\{ b_{ij}^T \right\}, \quad b_{ij}^T = \max \left\{ \min_{t \in [0,T]} \{h_{ij}(t)\}, \max_{0 \le \nu \le m_{ij}} \left\{ {}^a h_{ij}^\nu + h_v \right\} \right\}.$$

Approximate the functions ϕ_i^0, ϕ_i^T by polynomials ${}^a\phi_i^0$, ${}^a\phi_i^T$ with rational coefficients and with rational error bounds $\{{}_0^0\phi_i^v, {}_1^0\phi_i^v\}$. $\{{}_0^T\phi_i^v, {}_1^T\phi_i^v\}$:

$${}_0^0\phi_i^v \ge \left| \phi_i^0(h_0^*) - {}^a\phi_i^0(h_0^*) \right|, \quad {}_1^0\phi_i^v \ge \left\| \dot{\phi}_i^0 - {}^a\dot{\phi}_i^0 \right\|_{L_p^1[h_0^*, 0]},$$
$${}_0^T\phi_i^v \ge \left| \phi_i^T(T) - {}^a\phi_i^T(T) \right|, \quad {}_1^T\phi_i^v \ge \left\| \dot{\phi}_i^T - {}^a\dot{\phi}_i^T \right\|_{L_p^1[T, h_T^*]}.$$

Write the PBVP approximating the problem (119):

$$(\mathcal{L}_a^n y)(t) \equiv y^{(n)}(t) + \sum_{i=0}^{n-1} \sum_{j=1}^{n_i} p_{ij}^a(t) y^{(i)}[h_{ij}^a(t)] = f(t), \tag{122}$$
$$y^{(i)}(\xi) = \begin{cases} {}^a\varphi_i^0(\xi), & \xi < 0, \\ {}^a\varphi_i^T(\xi), & \xi > T, \end{cases} \quad t \in [0, T],$$
$$\Delta^n y = \alpha_a.$$

Here $\mathcal{L}_a^n : WS_p^n[0,T](m) \to L_p^1[0,T]$ is a computable operator.

Construction of the solution of the approximate problem. Denote by $\chi_{ij}(\cdot,\cdot)$ and $\chi_{ij}^a(\cdot,\cdot)$ the characteristic functions of the sets

$$\begin{gathered} \{(t,s) \in [0,T] \times [0,T] :\ 0 \le s \le h_{ij}(t) \le T\} \\ \text{and} \\ \left\{(t,s) \in [0,T] \times [0,T] :\ 0 \le s \le h_{ij}^a(t) \le T\right\}, \end{gathered} \tag{123}$$

respectively, $i = 0, \ldots, n-1$, $j = 1, \ldots, n_i$. The superposition $y^{(i)}[h_{ij}(\cdot)]$ is representable in the form

$$\begin{aligned} y^{(i)}[h_{ij}(t)] = & \int_0^T \chi_{ij}(t,s)\,\frac{(t-s)^{n-i-1}}{(n-i-1)!}\,y^{(n)}(s)ds + \\ & + [1 - \chi_{ij}(t,0)]\,\phi_i[h_{ij}(t)] + \\ & + \sum_{q=1}^{m+1}\sum_{k=i}^{n-1}\chi_{ij}(t,t_{q-1})\,\alpha_{m(q-1)+k+1}\,\frac{(t-t_{q-1})^{k-i}}{(k-i)!}, \end{aligned} \tag{124}$$

$$\phi_i[h_{ij}(t)] = \begin{cases} \varphi_i^0[h_{ij}(t)], & h_{ij}(t) < 0, \\ 0, & 0 \le h_{ij}(t) \le T, \\ \varphi_i^T[h_{ij}(t)], & h_{ij}(t) > T, \end{cases}$$

$i = 0, \ldots, n-1$; $j = 1, \ldots, n_i$. Define the functions $\mathcal{K}(\cdot,\cdot)$ and $\mathcal{K}_a(\cdot,\cdot)$ by the equalities

$$\begin{aligned} \mathcal{K}(t,s) &= \sum_{i=0}^{n-1}\sum_{j=1}^{n_i} p_{ij}(t,s)\chi_{ij}(t,s), \qquad (125)\\ \mathcal{K}_a(t,s) &= \sum_{i=0}^{n-1}\sum_{j=1}^{n_i} p_{ij}^a(t,s)\chi_{ij}^a(t,s), \quad i,j = 1,\ldots,n, \\ p_{ij}(t,s) &= -p_{ij}(t)\frac{(t-s)^{n-i-1}}{(n-i-1)!}, \\ p_{ij}^a(t,s) &= -p_{ij}^a(t)\frac{(t-s)^{n-i-1}}{(n-i-1)!}, \end{aligned}$$

$(t,s) \in [0,T] \times [0,T]$. Using both the representation (124) and definitions (125), write the problem (119) in the integral form

$$\begin{aligned} [(I-K)\,z]\,(t) &= g(t), \quad t \in [0,T], \qquad (126)\\ K : L_p^1[0,T] \to L_p^1[0,T], \quad (Kz)\,(t) &= \int_0^T \mathcal{K}(t,s)z(s)ds, \end{aligned}$$

where $z(\cdot) = y^{(n)}(\cdot)$ and the function $g(\cdot)$ is defined by

$$\begin{aligned} g(t) = f(t) & - \sum_{i=0}^{n-1}\sum_{j=1}^{n_i} p_{ij}(t)\,[1 - \chi_{ij}(t,0)]\,\phi_i[h_{ij}(t)] - \\ & - \sum_{i=0}^{n-1}\sum_{j=1}^{n_i} p_{ij}(t)\sum_{q=1}^{m+1}\sum_{k=i}^{n-1}\chi_{ij}(t,t_{q-1})\,\alpha_{m(q-1)+k+1} \times \\ & \times \frac{(t-t_{q-1})^{k-i}}{(k-i)!}, \quad t \in [0,T]. \end{aligned} \tag{127}$$

Analogously, write the problem (122) in the form

$$[(I-K_a)\,z]\,(t) \;=\; g_a(t), \quad t\in[0,T], \tag{128}$$

where $K_a : L_p^1[0,T] \to L_p^1[0,T]$, $(K_a z)\,(t) = \int_0^T \mathcal{K}_a(t,s)z(s)ds$ and the function $g_a(\cdot)$ is defined by

$$\begin{aligned} g_a(t) = f_a(t) \,-\, &\sum_{i=0}^{n-1}\sum_{j=1}^{n_i} p_{ij}^a(t)\left[1 \,-\, \chi_{ij}^a(t,0)\right]\phi_i^a[h_{ij}^a(t)] \,- \\ &- \sum_{i=0}^{n-1}\sum_{j=1}^{n_i} p_{ij}^a(t) \sum_{q=1}^{m+1}\sum_{k=i}^{n-1} \chi_{ij}^a(t,t_{q-1}) \,\times \\ &\times\, \alpha^a_{m(q-1)+k+1}\,\frac{(t-t_{q-1})^{k-i}}{(k-i)!}, \qquad t\in[0,T], \end{aligned} \tag{129}$$

$$\phi_i^a[h_{ij}^a(t)] \;=\; \begin{cases} {}^a\varphi_i^0[h_{ij}^a(t)], & h_{ij}^a(t)<0,\\ 0, & 0\le h_{ij}^a(t)\le T,\\ {}^a\varphi_i^T[h_{ij}^a(t)], & h_{ij}^a(t)>T. \end{cases}$$

For the function $\chi_{ij}^a(\cdot,\cdot)$, there takes place the representation

$$\chi_{ij}^a(t,s) \;=\; \sum_{q=1}^{m+1}\sum_{\nu=1}^{m_{ij}^q} \chi_{ij}^{q\nu}(t)\chi_{[0,{}^a d_{ij}^{q\nu}]}(s), \tag{130}$$

$(t,s)\in[0,T]\times[0,T]$, with

$${}^a d_{ij}^{q\nu} \;=\; \begin{cases} {}^a\tilde h_{ij}^{q(\nu-1)}, & \text{if } {}^a\tilde h_{ij}^{q(\nu-1)}\in[0,T],\\ 0, & \text{otherwise}, \end{cases}$$

$i=0,\dots,n-1$; $j=1,\dots,n_i$. Using (130), we represent $\mathcal{K}(\cdot,\cdot)$ defined by (125) in the form

$$\begin{aligned} \mathcal{K}_a(t,s) = &\sum_{q=1}^{m+1}\sum_{\nu=1}^{m_{ij}^q}\sum_{i=0}^{n-1}\sum_{j=1}^{n_i}\sum_{\tau=0}^{n-i-1} {}^a p_{ij}^q(t)\,\times \\ &\times \left[\frac{(-1)^{\tau+1}\,t^{n-i-1-\tau}}{(n-i-1-\tau)!}\,\chi_{ij}^{q\nu}(t)\right] \times \left[s^\tau \chi_{[0,{}^a d_{ij}^{q\nu}]}(s)\right], \\ &(t,s)\in[0,T]\times[0,T]. \end{aligned} \tag{131}$$

Define the numbers c_{qi}, $q=1,\dots,m+1$, $i=0,\dots,n-1$, by

$$c_{qi} \;=\; \sum_{j=1}^{n_i}\sum_{\nu=1}^{m_{ij}^q} \frac{1}{2}(n-i)(n-i-1).$$

Relate to the set of five $(q_*, i_*, j_*, \nu_*, \tau_*)$ a new index θ_* defined by

$$\theta_* = \sum_{q=1}^{q_*-1} \sum_{i=0}^{n-1} c_{qi} + \sum_{i=0}^{i_*-1} c_{q_* i} +$$
$$+ \sum_{j=1}^{j_*-1} \sum_{\nu=1}^{m_{ij}^q} \frac{1}{2}(n-i)(n-i-1) +$$
$$+ \sum_{\nu=1}^{\nu_*-1} \frac{1}{2}(n-i)(n-i-1) + \tau_* + 1.$$

In its turn, relate to a value of θ the set of five $(q_\theta, i_\theta, j_\theta, \nu_\theta, \tau_\theta)$ by the following algorithm:

a) the cycle of computing q_θ:
   1) $q_\theta = 1$;
   2) in the case $\theta \le \sum_{i=0}^{n-1} c_{q_\theta i}$ go to 4), otherwise go to 3);
   3) $\theta = \theta - \sum_{i=0}^{n-1} c_{q_\theta i}$, $q_\theta = q_\theta + 1$, then go to 2);
   4) end;

b) the cycle of computing i_θ:
   1) $i_\theta = 0$;
   2) in the case $\theta \le c_{q_\theta i_\theta}$ go to 4), otherwise go to 3);
   3) $\theta = \theta - c_{q_\theta i_\theta}$, $i_\theta = i_\theta + 1$, then go to 2);
   4) end;

c) the cycle of computing j_θ:
   1) $j_\theta = 1$;
   2) in the case $\theta \le \sum_{\nu=1}^{m_{i_\theta j_\theta}^{q_\theta}} \frac{1}{2}(n-i_\theta)(n-i_\theta-1)$ go to 4), otherwise go to 3);
   3) $\theta = \theta - \sum_{\nu=1}^{m_{i_\theta j_\theta}^{q_\theta}} \frac{1}{2}(n-i_\theta)(n-i_\theta-1)$, $j_\theta = j_\theta + 1$, then go to 2);
   4) end;

d) $\nu_\theta \stackrel{\text{def}}{=} \left[\frac{2\theta}{(n-i_\theta)(n-i_\theta-1)}\right] + 1$;
   $\theta = \theta - (\nu_\theta - 1)\, \frac{1}{2}(n-i_\theta)(n-i_\theta-1)$;

e) $\tau_\theta \stackrel{\text{def}}{=} \theta - 1$;

here $\theta = 1, \ldots, \tilde{m} = \sum_{q=1}^{m+1} \sum_{i=0}^{n-1} c_{qi}$. Using $\tilde{m}$, rewrite the representation of

$\mathcal{K}_a(\cdot,\cdot)$ in the form

$$\mathcal{K}_a(t,s) = \sum_{\theta=1}^{\tilde{m}} u_\theta(t)\, v_\theta(s), \quad (t,s) \in [0,T] \times [0,T], \tag{132}$$

$$u_\theta(t) = {}^a p^{q_\theta}_{i_\theta j_\theta}(t) \frac{(-1)^{\tau_\theta+1}\, t^{n-i_\theta-1-\tau_\theta}}{(n-i_\theta-1-\tau_\theta)!} \chi^{q_\theta \nu_\theta}_{i_\theta j_\theta}(t),$$

$$v_\theta(s) = s^{\tau_\theta} \chi_{\left[0,\, {}^a d^{q_\theta \nu_\theta}_{i_\theta j_\theta}\right]}(s).$$

Thus the equation (128) is an integral Fredholm equation of the second kind with a degenerate kernel. The solution z_a of (128) is representable in the form

$$z_a(t) = \sum_{\theta=1}^{\tilde{m}} u_\theta(t)\gamma_\theta + g_a(t), \quad t \in [0,T], \tag{133}$$

where the constants $\gamma_\theta \in R^1$ are to be found by solving the linear algebraic system

$$\Lambda\gamma = \beta. \tag{134}$$

Here $\gamma = \{\gamma_\theta\}_{\theta=1}^{\tilde{m}}$, $\beta = \{\beta_\theta\}_{\theta=1}^{\tilde{m}}$, $\beta_\theta = \int\limits_0^T v_\theta(s) g_a(s)\, ds$, $\tilde{m} \times \tilde{m}$-matrix Λ is defined by

$$\Lambda = \{\lambda_{\theta\sigma}\}_{\theta,\sigma=1}^{\tilde{m}}, \quad \lambda_{\theta\sigma} = \begin{cases} 1 - \int\limits_0^T u_\sigma(s) v_\theta(s)\, ds, & \sigma = \theta, \\ -\int\limits_0^T u_\sigma(s) v_\theta(s)\, ds, & \sigma \neq \theta. \end{cases}$$

Let Λ be invertible (i.e. PBVP (122) is uniquely solvable). Denote $W = \Lambda^{-1}$, $W = \{w_{\theta\sigma}\}_{\theta,\sigma=1}^{\tilde{m}}$. Now γ_θ are defined by

$$\gamma_\theta = \sum_{\sigma=1}^{\theta} w_{\theta\sigma}\beta_\sigma, \qquad \theta = 1, \ldots, \tilde{m}. \tag{135}$$

A norm estimate of the resolvent operator. In the case where Λ is invertible, there exists the resolvent operator $R_a : L^1_p[0,T] \to L^1_p[0,T]$, $[I + R_a] = [I - K_a]^{-1}$. Hence the solution z_a to (128) can be written in the form

$$z_a(t) = \left[(I + R_a)\, g_a\right](t), \quad t \in [0,T], \tag{136}$$

with

$$(R_a g_a)(t) = \int_0^T \mathcal{R}_a(t,s) g_a(s) ds, \quad t \in B_q, \tag{137}$$

$$\mathcal{R}_a(t,s) = \sum_{\theta=1}^{\tilde{m}} \sum_{\sigma=1}^{\tilde{m}} u_\theta(t)\, m_{\theta\sigma} v_\sigma(s), \qquad (t,s) \in [0,T] \times [0,T].$$

This implies

$$\|R_a\|_{L_p^1[0,T] \to L_p^1[0,T]} \le \Big\| \|\mathcal{R}_a(\cdot,s)\|_{L_p^1[0,T]} \Big\|_{L_{p'}^1[0,T]} \overset{\text{def}}{\le} \varrho_0. \tag{138}$$

Constructing an estimate of $\|g - g_a\|_{L_p^1[0,T]}$. Using the relationships (127), (129), write the representation of $g(t) - g_a(t)$, $t \in [0,T]$:

$$\begin{aligned}
g(t) - g_a(t) = & \sum_{q=1}^{m+1} \left[f(t) - f_q^a(t) \right] - \\
- \sum_{i=0}^{n-1} \sum_{j=1}^{n_i} \sum_{q=0}^{m} \sum_{\nu=1}^{m_{ij}^q} \sum_{k=i}^{n-1} & \Bigg\{ \left[p_{ij}(t) - {}^a p_{ij}^{q+1}(t) \right] \times \\
& \times \alpha_{mq+k+1}\, \chi_{ij}(t,t_q) + \\
& + {}^a p_{ij}^{q+1}(t)\, \alpha_{mq+k+1}^a \left(\chi_{ij}(t,t_q) - \chi_{ij}^a(t,t_q) \right) + \\
& + {}^a p_{ij}^{q+1}(t) \left(\alpha_{mq+k+1} - \alpha_{mq+k+1}^a \right) \chi_{ij}^a(t,t_q) \Bigg\} \times \\
& \times \chi_{ij}^{(q+1)\nu}(t) \frac{(t-t_q)^{k-i}}{(k-i)!} - \\
- \sum_{i=0}^{n-1} \sum_{j=1}^{n_i} \sum_{q=1}^{m+1} \sum_{\nu=1}^{m_{ij}^q} \sum_{k=i}^{n-1} & \Bigg\{ \left[p_{ij}(t) - {}^a p_{ij}^q(t) \right] \left[1 - \chi_{ij}(t,0) \right] \times \\
& \times \phi_i[h_{ij}(t)] + {}^a p_{ij}^q(t) \Bigg([1 - \chi_{ij}(t,0)] \phi_i[h_{ij}(t)] - \\
& - [1 - \chi_{ij}^a(t,0)] \phi_i^a[h_{ij}^a(t)] \Bigg) \Bigg\} \times \\
& \times \chi_{ij}^{q\nu}(t) \frac{(t-t_{q-1})^{k-i}}{(k-i)!}.
\end{aligned} \tag{139}$$

Using the results of Section 5, by the representation (139) we obtain the estimate

$$\|g - g_a\|_{L_p^1[0,T]} \le g_v, \tag{140}$$

where

$$
\begin{aligned}
g_v \overset{\text{def}}{\geq} & \sum_{q=1}^{m+1} f_q^v + \\
& + \sum_{i=0}^{n-1}\sum_{j=1}^{n_i}\sum_{q=0}^{m}\sum_{k=i}^{n-1}\Bigg\{ {}^v p_{ij}^{q+1} \max_{1\le\nu\le m_{ij}^q} {}^1\gamma_{ij}^{q\nu} \times \\
& \times \left(|\alpha_{mq+k+1}^a| + \alpha_{mq+k+1}^v\right) + \\
& + \left\| {}^a p_{ij}^{q+1} \right\|_{L_p^1[t_q,t_{q+1}]} |\alpha_{mq+k+1}^a| \max_{1\le\nu\le m_{ij}^q} {}^2\gamma_{ij}^{q\nu} + \\
& + \left\| {}^a p_{ij}^{q+1} \right\|_{L_p^1[t_q,t_{q+1}]} \alpha_{mq+k+1}^v \max_{1\le\nu\le m_{ij}^q} {}^3\gamma_{ij}^{q\nu} \Bigg\} \times \\
& \times \frac{(t_{q+1}-t_q)^{k-i}}{(k-i)!} + \\
& + \sum_{i=0}^{n-1}\sum_{j=1}^{n_i}\sum_{q=1}^{m+1}\sum_{k=i}^{n-1}\Bigg\{ {}^v p_{ij}^{q} \max_{1\le\nu\le m_{ij}^q} {}^4\gamma_{ij}^{q\nu} + \\
& + \left\| {}^a p_{ij}^{q+1} \right\|_{L_p^1[t_{q-1},t_q]} \max_{1\le\nu\le m_{ij}^q} {}^5\gamma_{ij}^{q\nu} \Bigg\} \frac{(t_q-t_{q-1})^{k-i}}{(k-i)!},
\end{aligned}
$$

the constants ${}^\sigma\gamma_{ij}^{q\nu}$, $\sigma = 1,2,3$, are calculated analogously to ${}^{q\nu}_{\sigma}\gamma_{ij}^k$, $\sigma = 1,2,3$ (see (92), (93), (94), (95)), and the constants ${}^\sigma\gamma_{ij}^{q\nu}$, $\sigma = 4,5$ are calculated analogously to ${}^{q\nu}_{\sigma}\gamma_{ij}^k$, $\sigma = 4,5$, (see (96), (97), (98), (99), (100), (101), (102)).

Constructing an estimate of $\|K - K_a\|_{L_p^1[0,T]\to L_p^1[0,T]}$. Define the operator K_Δ: $L_p^1[0,T] \to L_p^1[0,T]$ by $K_\Delta = K - K_a$. There takes place the representation

$$
\begin{aligned}
(K_\Delta z)(t) &= \int_0^T \mathcal{K}_\Delta(t,s)z(s)ds, \quad t \in [0,T], \qquad (141)\\
\mathcal{K}_\Delta(t,s) &= -\sum_{i=0}^{n-1}\sum_{j=1}^{n_i}\left\{p_{ij}(t,s)\chi_{t,s} - p_{ij}^a(t,s)\chi_{t,s}^a\right\},\\
& \qquad (t,s) \in [0,t]\times[0,T].
\end{aligned}
$$

Using the way of constructing the estimate (110), by definition (141) we obtain the estimate

$$
\|K_\Delta\|_{L_p^1[0,T]\to L_p^1[0,T]} \le \delta_0, \qquad \delta_0 \ge \sum_{i=0}^{n-1}\sum_{j=1}^{n_i}\delta_{ij}, \qquad (142)
$$

where δ_{ij} are defined analogously to δ_{ij}^k (see (111), (112), (113)).

The study of the original problem for the solvability and constructing an error bound. Let the approximate problem (122) be uniquely solvable and the condition

$$\delta_0 \ < \ \frac{1}{1+\varrho_0} \tag{143}$$

hold, where ϱ_0 is defined by (138), and δ_0 is defined by (142). In this case the estimate

$$\|K_\Delta\|_{L_p^1[0,T]\to L_p^1[0,T]} \ < \ \frac{1}{\|1+R_a\|_{L_p^1[0,T]\to L_p^1[0,T]}}$$

takes place. This implies, due to the invertible operator theorem, the invertibility of $(I-K)$ as well as the existence of the resolvent operator $R : L_p^1[0,T] \to L_p^1[0,T]$, $(I+R) = (I-K)^{-1}$, providing the representation of the solution z to (126):

$$z(t) = [(I+R)\,g]\,(t), \quad t \in [0,T]. \tag{144}$$

Next, we have

$$z(t) - z_a(t) \ = \ [(I+R_a)\,(g \ - \ g_a]\,(t) \ + \ [(R-R_a)\,(g_a)]\,(t),$$

$t \in [0,T]$. As the estimate

$$\|R-R_a\|_{L_p^1[0,T]\to L_p^1[0,T]} \ \le \ \frac{\delta_0\,(1+\varrho_0)^2}{1-\delta_0\,(1+\varrho_0)}$$

holds, we obtain conclusively

$$\|z - z_a\|_{L_p^1[0,T]} \ \le \ z_v,$$

where the rational z_v is defined by

$$z_v \ \ge \ (1+\varrho_0)\,g_v \ + \ \Big\{\|g_a\|_{L_p^n[0,T]} + g_v\Big\}\,\frac{\delta_0\,(1+\varrho_0)^2}{1-\delta_0\,(1+\varrho_0)}. \tag{145}$$

The solution y_a to (122) is defined by

$$\begin{aligned} y_a^{(n-1)}(t) \ &= \ \int_0^t z_a(s)ds \ + \ \alpha_n^a \ + \ \sum_{q=1}^m \alpha_{mq+n}^a \chi_{[t_q,T]}(t), \\ y_a^{(j)}(t) \ &= \ \int_0^t y_a^{(j+1)}(s)ds \ + \ \alpha_{j+1}^a \ + \ \sum_{q=1}^m \alpha_{mq+j+1}^a \chi_{[t_q,T]}(t), \\ & j = n-2, \dots, 0, \quad t \in [0,T], \end{aligned} \tag{146}$$

therewith the estimate

$$\|y - y_a\|_{WS_p^n(m)[0,T]} \ \le \ y_v$$

holds, where y is the exact solution to (119) and the rational y_v is defined by the inequality

$$y_v \geq z_v + \|\alpha_v\|_{R^{mn+n}} . \tag{147}$$

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