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Ravi P. Agarwal, Haishen Lü and Donal O'Regan

AN UPPER AND LOWER SOLUTION METHOD FOR THE ONE-DIMENSIONAL SINGULAR p–LAPLACIAN

Abstract. The singular boundary value problem

$$\begin{cases} (\varphi_p(y'))' + q\,(t)\,f(t,y) = 0, & \text{for } t \in (0,1), \\ y(0) = y(1) = 0 \end{cases}$$

is studied in this paper with $\varphi_p(s) = |s|^{p-2}\,s$, $p > 1$. The nonlinearity may be singular at $y = 0, t = 0$ and $t = 1$, and the function f may change sign. An upper and lower solution approach is presented.

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რეზიუმე. ნაშრომში შესწავლილია სინგულარული სასაზღვრო ამოცანა

$$\begin{cases} (\varphi_p(y'))' + q\,(t)\,f(t,y) = 0, & \text{როცა } t \in (0,1), \\ y(0) = y(1) = 0, \end{cases}$$

სადაც $\varphi_p(s) = |s|^{p-2}\,s$, $p > 1$. არაწრფივობა შეიძლება სინგულარული იყოს $y = 0$, $t = 0$ და $t = 1$ წერტილებში, ამას გარდა f ფუნქცია შეიძლება ნიშანცვლადი იყოს. გამოყენებული ზედა და ქვედა ამონახსნების მეთოდი.

1. Introduction

In this paper we study the singular boundary value problem

$$\begin{cases} (\varphi_p(y'))' + q\,(t)\, f(t, y) = 0, & \text{for } t \in (0,1), \\ y(0) = y(1) = 0 \end{cases} \tag{1.1}$$

where $\varphi_p(s) = |s|^{p-2}\, s, p > 1$. The singularity may occur at $y = 0, t = 0$ and $t = 1$, and the function f is allowed to change sign.

The boundary value problem (1.1) has been discussed extensively in the literature; see $[3-8]$, and the references therein. In almost all of these papers qf is allowed to be positive. As a result the solutions are concave. When $p = 2$ the authors in $[1, 2]$ studied the case when f is allowed to change sign.

In this paper we note in particular that q is not necessarily in $L^1\,[0,1]$. Also f may not be a Carathéodory function because of the singular behaviour of the y variable. The ideas presented here were motivated by the papers $[1-2]$ where the case $p = 2$ is considered. Finally we remark that equations of the form (1.1) occur in non-Newtonial fluid theory, and in the study of turbulent flow of a gas in a porous medium[3].

To conclude the introduction we state a general existence principle [8], which will be needed in section 2, for the singular Dirichlet boundary value problem

$$\begin{cases} (\varphi_p(y'))' + g(t, y) = 0, \ 0 < t < 1, \\ y(0) = 0 = y(1). \end{cases} \tag{1.2}$$

Lemma 1.1. *Suppose the following conditions are satisfied:*
$(H1)$ *$g : (0,1) \times R \to R$ is continuous,*
$(H2)$ *there exists $q \in C\,(0,1)$ with $q > 0$ on $(0,1)$ and*

$$\int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\int_s^{\frac{1}{2}} q\,(r)\,dr\right) ds + \int_{\frac{1}{2}}^1 \varphi_p^{-1}\left(\int_{\frac{1}{2}}^s q\,(r)\,dr\right) ds < \infty$$

such that $|g\,(t, y)| \leq q\,(t)$ for a.e. $t \in (0,1)$ and $y \in R$. Then (1.2) has a solution $y \in C\,[0,1] \cap C^1\,(0,1)$ with $\varphi_p\,(y') \in AC\,(0,1)$.

Notice $\varphi_p^{-1}\,(s) = |s|^{1/(p-1)}\, sign\,(s)$ is the inverse function to $\varphi_p\,(s)$.

2. Existence Results

In this section we discuss the Dirichlet singular boundary value problem

$$\begin{cases} (\varphi_p(y'))' + q\,(t)\, f(t, y) = 0, \ 0 < t < 1, \\ y(0) = y(1) = 0, \end{cases} \tag{2.1}$$

where our nonlinearity f may change sign. We begin with our main result.

Theorem 2.1. *Let $n_0 \in \{3, 4, \ldots\}$ be fixed and suppose the following conditions are satisfied:*

$$f : [0,1] \times (0, \infty) \to R \text{ is continuous,} \tag{2.2}$$

$$\begin{cases} \text{let } n \in \{n_0, n_0 + 1, \ldots\} \text{ and associated with each } n \text{ we} \\ \text{have a constant } \rho_n \text{ such that } \{\rho_n\} \text{ is a nonincreasing} \\ \text{sequence with } \lim_{n\to\infty} \rho_n = 0 \text{ and such that for} \\ \frac{1}{2^{n+1}} \le t < 1 \text{ we have } q(t) f(t, \rho_n) \ge 0, \end{cases} \tag{2.3}$$

$$\begin{cases} q \in C(0,1) \text{ with } q > 0 \text{ on } (0,1) \text{ and} \\ \int_0^{\frac{1}{2}} \varphi_p^{-1} \left(\int_s^{\frac{1}{2}} q(r)\, dr \right) ds + \int_{\frac{1}{2}}^1 \varphi_p^{-1} \left(\int_{\frac{1}{2}}^s q(r)\, dr \right) ds < \infty, \end{cases} \tag{2.4}$$

$$\begin{cases} \text{there exists a function } \alpha \in C[0,1] \cap C^1(0,1),\ \varphi_p(\alpha') \in C^1(0,1), \\ \text{with } \alpha(0) = \alpha(1) = 0,\ \alpha(t) > 0 \text{ on } (0,1) \text{ such that} \\ q(t) f(t, \alpha(t)) + (\varphi_p(\alpha'(t)))' \ge 0 \text{ for } t \in (0,1) \end{cases} \tag{2.5}$$

and

$$\begin{cases} \text{there exists a function } \beta \in C[0,1] \cap C^1(0,1), \\ \varphi_p(\beta') \in C^1(0,1),\ \text{with } \beta(t) \ge \alpha(t) \text{ and } \beta(t) \ge \rho_{n_0} \text{ on } [0,1] \\ \text{such that } q(t) f(t, \beta(t)) + (\varphi_p(\beta'(t)))' \le 0 \text{ for } t \in (0,1) \\ \text{with } q(t) f(\frac{1}{2^{n_0+1}}, \beta(t)) + (\varphi_p(\beta'(t)))' \le 0 \text{ for } t \in (0, \frac{1}{2^{n_0+1}}). \end{cases} \tag{2.6}$$

Then (2.1) has a solution in $y \in C[0,1] \cap C^1(0,1)$ with $\varphi_p(y') \in C^1(0,1)$ with $y(t) \ge \alpha(t)$ for $t \in [0,1]$.

Proof. For $n = n_0, n_0 + 1, \ldots,$ let

$$e_n = \left[\frac{1}{2^{n+1}}, 1 \right] \quad \text{and} \quad \theta_n(t) = \max \left\{ \frac{1}{2^{n+1}}, t \right\},\ 0 \le t \le 1$$

and

$$f_n(t,x) = \max \{ f(\theta_n(t), x), f(t,x) \}.$$

Next we define inductively

$$g_{n_0}(t,x) = f_{n_0}(t,x)$$

and

$$g_n(t,x) = \min \{ f_{n_0}(t,x), \ldots, f_n(t,x) \},\ n = n_0 + 1, n_0 + 2, \ldots.$$

Notice

$$f(t,x) \le \cdots \le g_{n+1}(t,x) \le g_n(t,x) \le \cdots \le g_{n_0}(t,x)$$

for $(t,x) \in (0,1) \times (0,\infty)$ and

$$g_n(t,x) = f(t,x) \quad \text{for } (t,x) \in e_n \times (0,\infty).$$

Without loss of generality assume $\rho_{n_0} \le \min_{t \in [\frac{1}{3}, \frac{2}{3}]} \alpha(t)$. Fix $n \in \{n_0, n_0 + 1, \ldots\}$. Let $t_n \in \left[0, \frac{1}{3}\right]$ and $s_n \in \left[\frac{2}{3}, 1\right]$ be such that

$$\alpha(t_n) = \alpha(s_n) = \rho_n \text{ and } \alpha(t) \le \rho_n \text{ for } t \in [0, t_n] \cup [s_n, 1].$$

Define

$$\alpha_n(t) = \begin{cases} \rho_n & \text{if } t \in [0, t_n] \cup [s_n, 1] \\ \alpha(t) & \text{if } t \in\in (t_n, s_n). \end{cases}$$

Consider the boundary value problem

$$\begin{cases} (\varphi_p(y'))' + q(t)\, g^*_{n_0}(t, y) = 0, \ 0 < t < 1, \\ y(0) = y(1) = \rho_{n_0}; \end{cases} \tag{2.7}$$

here

$$g^*_{n_0}(t, y) = \begin{cases} g_{n_0}(t, \beta(t)) + r(\beta(t) - y), \ y > \beta(t), \\ g_{n_0}(t, y), \qquad \alpha_{n_0}(t) \le y \le \beta(t) \\ g_{n_0}(t, \alpha_{n_0}(t)) + r(\alpha_{n_0}(t) - y), \ y < \alpha_{n_0}(t) \end{cases}$$

where $r : R \to [-1, 1]$ is the radial retraction defined by

$$r(x) = \begin{cases} x, & |x| \le 1 \\ \frac{x}{|x|}, & |x| > 1. \end{cases}$$

From Lemma 1.1 we know that (2.7) has a solution $y_{n_0} \in C[0,1] \cap C^1(0,1)$ with $\varphi_p\left(y'_{n_0}\right) \in C^1(0,1)$. We first show

$$y_{n_0}(t) \ge \alpha_{n_0}(t), t \in [0, 1]. \tag{2.8}$$

Suppose (2.8) is not true. Then $y_{n_0} - \alpha_{n_0}$ has a negative absolute minimum at $\tau \in (0,1)$. Now since $y_{n_0}(0) - \alpha_{n_0}(0) = 0 = y_{n_0}(1) - \alpha_{n_0}(1)$ there exists $\tau_0, \tau_1 \in [0,1]$ with $\tau \in (\tau_0, \tau_1)$ and

$y_{n_0}(\tau_0) - \alpha_{n_0}(\tau_0) = y_{n_0}(\tau_1) - \alpha_{n_0}(\tau_1) = 0$ and $y_{n_0}(t) - \alpha_{n_0}(t) < 0, \ t \in (\tau_0, \tau_1)$.

We now claim

$$\left(\varphi_p\left(y'_{n_0}(t)\right)\right)' \le \left(\varphi_p\left(\alpha'_{n_0}(t)\right)\right)' \ \text{ for a.e. } t \in (\tau_0, \tau_1). \tag{2.9}$$

We first show that if (2.9) is true then (2.8) will follow. Let

$$w(t) = y_{n_0}(t) - \alpha_{n_0}(t) < 0, \ \text{for } t \in (\tau_0, \tau_1).$$

Then

$$\int_{\tau_0}^{\tau_1} \left((\varphi_p(y'_{n_0}(t)))' - (\varphi_p\left(\alpha'_{n_0}(t)\right))'\right) w(t)\, dt \ge 0. \tag{2.10}$$

On the other hand, the inequality

$$(\varphi_p(b) - \varphi_p(a))(b - a) \ge 0, \ \text{for } a, b \in R,$$

yields

$$\begin{aligned} & \int_{\tau_0}^{\tau_1} \left((\varphi_p(y'_{n_0}(t)))' - (\varphi_p\left(\alpha'_{n_0}(t)\right))'\right) w(t)\, dt \\ = & \ -\int_{\tau_0}^{\tau_1} \left(\varphi_p(y'_{n_0}(t)) - \varphi_p\left(\alpha'_{n_0}(t)\right)\right)\left(y'_{n_0}(t) - \alpha'_{n_0}(t)\right) dt \\ < & \ \ 0, \end{aligned}$$

a contradiction. As a result if we show that (2.9) is true then (2.8) will follow. To see (2.9) we will in fact prove more i.e. we will prove

$$(\varphi_p(y_{n_0}(t)))' \le (\varphi_p(\alpha_{n_0}(t)))' \text{ for } t \in (\tau_0, \tau_1) \ \text{ provided } t \ne t_{n_0} \text{ or } t \ne s_{n_0}. \tag{2.11}$$

Fix $t \in (\tau_0, \tau_1)$ and assume $t \neq t_{n_0}$ or $t \neq s_{n_0}$. Then

$$\begin{aligned}
& \left(\varphi_p\left(y_{n_0}'(t)\right)\right)' - \left(\varphi_p\left(\alpha_{n_0}'(t)\right)\right)' \\
= \; & -\left[q(t)\{g_{n_0}(t,\alpha_{n_0}(t)) + r(\alpha_{n_0}(t) - y_{n_0}(t))\} + (\varphi_p(\alpha_{n_0}(t)))'\right] \\
= \; & \begin{cases} -\left[q(t)\{g_{n_0}(t,\alpha(t)) + r(\alpha(t) - y_{n_0}(t))\} + (\varphi_p(\alpha'(t)))'\right] \\ \qquad\qquad\qquad\qquad \text{if } t \in (t_{n_0}, s_{n_0}) \\ -q(t)\{g_{n_0}(t,\rho_{n_0}) + r(\rho_{n_0} - y_{n_0}(t))\} \text{ if } t \in (0,t_{n_0}) \cup (s_{n_0},1). \end{cases}
\end{aligned}$$

Case (i) . $t \geq \frac{1}{2^{n_0+1}}$.

Then since $g_{n_0}(t,x) = f(t,x)$ for $x \in (0,\infty)$ we have

$$\begin{aligned}
& \left(\varphi_p\left(y_{n_0}'(t)\right)\right)' - \left(\varphi_p\left(\alpha_{n_0}'(t)\right)\right)' \\
= \; & \begin{cases} -\left[q(t)\{f(t,\alpha(t)) + r(\alpha(t) - y_{n_0}(t))\} + (\varphi_p(\alpha'(t)))'\right] \\ \qquad\qquad\qquad\qquad \text{if } t \in (t_{n_0}, s_{n_0}) \\ -q(t)\{f(t,\rho_{n_0}) + r(\rho_{n_0} - y_{n_0}(t))\} \text{ if } t \in (0,t_{n_0}) \cup (s_{n_0},1) \end{cases} \\
< \; & 0,
\end{aligned}$$

from (2.3) and (2.5).

Case (ii) . $t \in \left(0, \frac{1}{2^{n_0+1}}\right)$.

Then since

$$g_{n_0}(t,x) = \max\left\{f\left(\frac{1}{2^{n_0+1}}, x\right), f(t,x)\right\}$$

we have $g_{n_0}(t,x) \geq f(t,x)$ and $g_{n_0}(t,x) \geq f\left(\frac{1}{2^{n_0+1}}, x\right)$ for $x \in (0,\infty)$. Thus we have

$$\begin{aligned}
& \left(\varphi_p\left(y_{n_0}'(t)\right)\right)' - \left(\varphi_p\left(\alpha_{n_0}'(t)\right)\right)' \\
\leq \; & \begin{cases} -\left\{q(t)\left[f(t,\alpha(t)) + r(\alpha(t) - y_{n_0}(t))\right] + (\varphi_p(\alpha'(t)))'\right\} \\ \qquad\qquad\qquad\qquad \text{if } t \in (t_{n_0}, s_{n_0}) \\ -q(t)\left[f\left(\frac{1}{2^{n_0+1}}, \rho_{n_0}\right) + r(\rho_{n_0} - y_{n_0}(t))\right] \text{ if } t \in (0,t_{n_0}) \cup (s_{n_0},1) \end{cases} \\
< \; & 0,
\end{aligned}$$

from (2.3) and (2.5).

Consequently (2.9) (and so (2.8)) holds and now since $\alpha(t) \leq \alpha_{n_0}(t)$ for $t \in [0,1]$ we have

$$\alpha(t) \leq \alpha_{n_0}(t) \leq y_{n_0}(t) \text{ for } t \in [0,1]. \tag{2.12}$$

Next we show

$$y_{n_0}(t) \leq \beta(t) \text{ for } t \in [0,1]. \tag{2.13}$$

If (2.13) is not true then $y_{n_0} - \beta$ would have a positive absolute maximum at say $t_0 \in (0,1)$, in which case $y_{n_0}'(t_0) = \beta'(t_0)$. It is easy to check (see [10]) that $(\varphi_p(y_{n_0}'))'(t_0) - (\varphi_p(\beta'))'(t_0) \leq 0$. There are two cases to consider, namely $t_0 \in [\frac{1}{2^{n_0+1}}, 1)$ and $t_0 \in \left(0, \frac{1}{2^{n_0+1}}\right)$.

Case (i). $t_0 \in [\frac{1}{2^{n_0+1}}, 1)$.

Then $y_{n_0}(t_0) > \beta(t_0)$ together with $g_{n_0}(t_0, x) = f(t_0, x)$ for $x \in (0, \infty)$ gives

$$\begin{aligned} & (\varphi_p(y'_{n_0}))'(t_0) - (\varphi_p(\beta'))'(t_0) \\ = & -q(t_0)\left[g_{n_0}(t_0, \beta(t_0)) + r(\beta(t_0) - y_{n_0}(t_0))\right] - (\varphi_p(\beta'))'(t_0) \\ = & -q(t_0)\left[f(t_0, \beta(t_0)) + r(\beta(t_0) - y_{n_0}(t_0))\right] - (\varphi_p(\beta'))'(t_0) \\ > & \ 0 \end{aligned}$$

from (2.6), a contradiction.
Case (ii). $t_0 \in \left(0, \frac{1}{2^{n_0+1}}\right)$.
Then $y_{n_0}(t_0) > \beta(t_0)$ together with

$$g_{n_0}(t_0, x) = \max\left\{f\left(\frac{1}{2^{n_0+1}}, x\right), f(t_0, x)\right\}$$

for $x \in (0, \infty)$ gives

$$\begin{aligned} & (\varphi_p(y_{n_0}))'(t_0) - (\varphi_p(\beta'))'(t_0) \\ = & -q(t_0)\left\{\max\left[f\left(\frac{1}{2^{n_0+1}}, \beta(t_0)\right), f(t_0, \beta(t_0))\right] + r(\beta(t_0) - y_{n_0}(t_0))\right\} \\ & -(\varphi_p(\beta'))'(t_0) \\ > & \ 0, \end{aligned}$$

form (2.6), a contradiction.
Thus (2.13) holds, so we have

$$\alpha(t) \le \alpha_{n_0}(t) \le y_{n_0}(t) \le \beta(t) \ \text{ for } t \in [0, 1].$$

Next we consider the boundary value problem

$$\begin{cases} (\varphi_p(y'))' + q(t)\, g^*_{n_0+1}(t, y) = 0, \ 0 < t < 1, \\ y(0) = y(1) = \rho_{n_0+1}; \end{cases} \tag{2.14}$$

here

$$g^*_{n_0+1}(t, y) = \begin{cases} g_{n_0+1}(t, y_{n_0}(t)) + r(y_{n_0}(t) - y), \ y > y_{n_0}(t), \\ g_{n_0+1}(t, y), \qquad \alpha_{n_0+1}(t) \le y \le y_{n_0+1}(t) \\ g_{n_0+1}(t, \alpha_{n_0+1}(t)) + r(\alpha_{n_0+1}(t) - y), \ y < \alpha_{n_0+1}(t). \end{cases}$$

From Lemma 1.1 we know that (2.14) has a solution $y_{n_0+1} \in C[0, 1] \cap C^1(0, 1)$ with $\varphi_p\left(y'_{n_0+1}\right) \in C^1(0, 1)$. We first show

$$y_{n_0+1}(t) \ge \alpha_{n_0+1}(t), t \in [0, 1]. \tag{2.15}$$

Suppose (2.15) is not true. Then $y_{n_0+1} - \alpha_{n_0+1}$ has a negative absolute minimum at $\tau \in (0, 1)$. Now since $y_{n_0+1}(0) - \alpha_{n_0+1}(0) = 0 = y_{n_0+1}(1) - \alpha_{n_0+1}(1)$ there exists $\tau_0, \tau_1 \in [0, 1]$ with $\tau \in (\tau_0, \tau_1)$ and

$$y_{n_0+1}(\tau_0) - \alpha_{n_0+1}(\tau_0) = y_{n_0+1}(\tau_1) - \alpha_{n_0+1}(\tau_1) = 0$$

and

$$y_{n_0+1}(t) - \alpha_{n_0+1}(t) < 0, \ t \in (\tau_0, \tau_1).$$

If we show

$$\left(\varphi_p\left(y_{n_0+1}(t)\right)\right)' \leq \left(\varphi_p\left(\alpha_{n_0+1}(t)\right)\right)' \quad \text{for a.e. } t \in (\tau_0, \tau_1), \tag{2.16}$$

then as before (2.15) is true. Fix $t \in (\tau_0, \tau_1)$ and assume $t \neq t_{n_0+1}$ or $t \neq s_{n_0+1}$. Then

$$\begin{aligned}&\left(\varphi_p\left(y'_{n_0+1}(t)\right)\right)' - \left(\varphi_p\left(\alpha'_{n_0+1}(t)\right)\right)' \\ = &\begin{cases} -\left\{q(t)\left[g_{n_0+1}(t,\alpha(t)) + r(\alpha(t) - y_{n_0+1}(t))\right] + \left(\varphi_p(\alpha'(t))\right)'\right\} \\ \qquad\qquad\qquad\qquad \text{if } t \in (t_{n_0+1}, s_{n_0+1}) \\ -q(t)\left[g_{n_0+1}(t,\rho_{n_0+1}) + r(\rho_{n_0+1} - y_{n_0+1}(t))\right] \\ \qquad\qquad\qquad\qquad \text{if } t \in (0, t_{n_0+1}) \cup (s_{n_0+1}, 1). \end{cases}\end{aligned}$$

Case (i). $t \geq \frac{1}{2^{n_0+2}}$.

Then since $g_{n_0+1}(t,x) = f(t,x)$ for $x \in (0,\infty)$ we have

$$\begin{aligned}&\left(\varphi_p\left(y_{n_0+1}(t)\right)\right)' - \left(\varphi_p\left(\alpha_{n_0+1}(t)\right)\right)' \\ = &\begin{cases} -\left\{q(t)\left[f(t,\alpha(t)) + r(\alpha(t) - y_{n_0+1}(t))\right] + \left(\varphi_p(\alpha'(t))\right)'\right\} \\ \qquad\qquad\qquad\qquad \text{if } t \in (t_{n_0+1}, s_{n_0+1}) \\ -q(t)\left[f(t,\rho_{n_0+1}) + r(\rho_{n_0+1} - y_{n_0+1}(t))\right] \\ \qquad\qquad\qquad\qquad \text{if } t \in (0, t_{n_0+1}) \cup (s_{n_0+1}, 1). \end{cases} \\ < &\ 0,\end{aligned}$$

from (2.3) and (2.5).

Case (ii) $t \in \left(0, \frac{1}{2^{n_0+2}}\right)$.

Then since

$$\begin{aligned}&g_{n_0+1}(t,x) \\ &= \min\left\{\max\left\{f\left(\frac{1}{2^{n_0+1}}, x\right), f(t,x)\right\}, \max\left\{f\left(\frac{1}{2^{n_0+2}}, x\right), f(t,x)\right\}\right\}\end{aligned}$$

we have

$$g_{n_0+1}(t,x) \geq f(t,x)$$

and

$$g_{n_0+1}(t,x) \geq \min\left\{f\left(\frac{1}{2^{n_0+1}}, x\right), f\left(\frac{1}{2^{n_0+2}}, x\right)\right\}$$

for $x \in (0,\infty)$. Thus we have

$$\begin{aligned}&\left(\varphi_p\left(y'_{n_0+1}(t)\right)\right)' - \left(\varphi_p\left(\alpha'_{n_0+1}(t)\right)\right)' \\ \leq &\begin{cases} -\left\{q(t)\left[f(t,\alpha(t)) + r(\alpha(t) - y_{n_0+1}(t))\right] + \left(\varphi_p(\alpha''(t))\right)'\right\} \\ \qquad\qquad\qquad\qquad \text{if } t \in (t_{n_0+1}, s_{n_0+1}) \\ -q(t)\left\{\min\left\{f\left(\frac{1}{2^{n_0+1}}, \rho_{n_0+1}\right), f\left(\frac{1}{2^{n_0+2}}, \rho_{n_0+1}\right)\right\}\right. \\ \qquad \left.+r(\rho_{n_0+1} - y_{n_0+1}(t))\right\} \text{ if } t \in (0, t_{n_0+1}) \cup (s_{n_0+1}, 1). \end{cases} \\ < &\ 0,\end{aligned}$$

from (2.3) and (2.5). (note $f\left(\frac{1}{2^{n_0+1}}, \rho_{n_0+1}\right) \geq 0$ since $f(t,\rho_{n_0+1}) \geq 0$ for $t \in \left[\frac{1}{2^{n_0+2}}, 1\right]$ and $\frac{1}{2^{n_0+1}} \in \left[\frac{1}{2^{n_0+2}}, 1\right]$).

Consequently (2.15) is true so

$$\alpha(t) \leq \alpha_{n_0+1}(t) \leq y_{n_0+1}(t) \text{ for } t \in [0,1]. \tag{2.17}$$

Next we show

$$y_{n_0+1}(t) \leq y_{n_0}(t) \text{ for } t \in [0,1]. \tag{2.18}$$

If (2.18) is not true then $y_{n_0+1} - y_{n_0}$ would have a positive absolute maximum at say $t_0 \in (0,1)$, in which case

$$y'_{n_0+1}(t_0) = y'_{n_0}(t_0) \text{ and } (\varphi_p(y'_{n_0+1}))'(t_0) - (\varphi_p(y'_{n_0}))'(t_0) \leq 0.$$

Then $y_{n_0+1}(t_0) > y_{n_0}(t_0)$ together with $g_{n_0}(t_0, x) \geq g_{n_0+1}(t_0, x)$ for $x \in (0, \infty)$ gives

$$\begin{aligned}
& (\varphi_p(y'_{n_0+1}))'(t_0) - (\varphi_p(y'_{n_0}))'(t_0) \\
= \; & -q(t_0)\left[g_{n_0+1}(t_0, y_{n_0}(t)) + r(y_{n_0}(t) - y_{n_0+1}(t_0))\right] - (\varphi_p(y'_{n_0}))'(t_0) \\
\geq \; & -q(t_0)\left[g_{n_0}(t_0, y_{n_0}(t)) + r(y_{n_0}(t_0) - y_{n_0+1}(t_0))\right] - (\varphi_p(y'_{n_0}))'(t_0) \\
= \; & -q(t_0)\left[r(y_{n_0}(t_0) - y_{n_0+1}(t_0))\right] \\
> \; & 0,
\end{aligned}$$

a contradiction.

Now proceed inductively to construct $y_{n_0+2}, y_{n_0+3}, \ldots$ as follows. Suppose we have y_k for some $k \in \{n_0+1, n_0+2, \ldots\}$ with $\alpha_k(t) \leq y_k(t) \leq y_{k-1}(t)$ for $t \in [0,1]$. Then consider the boundary value problem

$$\begin{cases} (\varphi_p(y'))' + q(t)\, g^*_{k+1}(t, y) = 0, \; 0 < t < 1, \\ y(0) = y(1) = \rho_{k+1}; \end{cases} \tag{2.19}$$

here

$$g^*_{k+1}(t, y) = \begin{cases} g_{k+1}(t, y_k(t)) + r(y_k(t) - y), \; y > y_k(t) \\ g_{k+1}(t, y), \qquad \alpha_{k+1}(t) \leq y \leq y_k(t) \\ g_{k+1}(t, \alpha_{k+1}(t)) + r(\alpha_{k+1}(t) - y), \; y < \alpha_{k+1}(t). \end{cases}$$

Now Lemma 1.1 guarantees (2.19) has a solution $y_{k+1} \in C[0,1] \cap C^1(0,1)$ with $\varphi_p(y_{k+1}) \in C^1(0,1)$, and essentially the same reasoning as above yields

$$\alpha(t) \leq \alpha_{k+1}(t) \leq y_{k+1}(t) \leq y_k(t) \text{ for } t \in [0,1]. \tag{2.20}$$

Thus for each $n \in \{n_0+1, \ldots\}$ we have

$$\alpha(t) \leq y_n(t) \leq y_{n-1}(t) \leq \ldots\ldots \leq y_{n_0}(t) \leq \beta(t) \text{ for } t \in [0,1]. \tag{2.21}$$

Now lets look at the internal $\left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right]$. Let

$$R_{n_0} = \sup\left\{|q(t)\, f(t,x)| : t \in \left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right] \text{ and } \alpha(t) \leq x \leq y_{n_0}(t)\right\}.$$

The mean value theorem implies that there exists $\tau \in \left(\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right)$ with $|y'_n(\tau)| \leq 2 \sup_{[0,1]} y_{n_0}(t)$. Hence for $t \in \left(\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right)$,

$$|y'_n(t)| \leq \varphi_p^{-1}\left(\varphi_p(|y'_n(\tau)|) + \left|\int_\tau^t (\varphi_p(y'_n(\tau)))' dx\right|\right).$$

As a result

$$\{y_n\}_{n=n_0}^{\infty} \text{ is a bounded, equicontinuous family on } \left[\frac{1}{2^{n_0+1}}, 1-\frac{1}{2^{n_0+1}}\right]. \tag{2.22}$$

The Arzela-Ascoli theorem guarantees the existence of a subsequence N_{n_0} of integers and a function $z_{n_0} \in C\left[\frac{1}{2^{n_0+1}}, 1-\frac{1}{2^{n_0+1}}\right]$ with y_n converging uniformly to z_{n_0} on $\left[\frac{1}{2^{n_0+1}}, 1-\frac{1}{2^{n_0+1}}\right]$ as $n \to \infty$ through N_{n_0}. Similarly

$$\{y_n\}_{n=n_0+1}^{\infty} \text{ is a bounded, equicontinuous family on } \left[\frac{1}{2^{n_0+2}}, 1-\frac{1}{2^{n_0+2}}\right],$$

so there is a subsequence N_{n_0+1} of N_{n_0} and a function

$$z_{n_0+1} \in C\left[\frac{1}{2^{n_0+2}}, 1-\frac{1}{2^{n_0+2}}\right]$$

with y_n converging uniformly to z_{n_0+1} on $\left[\frac{1}{2^{n_0+2}}, 1-\frac{1}{2^{n_0+2}}\right]$ as $n \to \infty$ through N_{n_0+1}. Note $z_{n_0+1} = z_{n_0}$ on $\left[\frac{1}{2^{n_0+1}}, 1-\frac{1}{2^{n_0+1}}\right]$ since $N_{n_0+1} \subseteq N_{n_0}$. Proceed inductively to obtain subsequence on integers

$$N_{n_0} \supseteq N_{n_0+1} \supseteq \ldots \supseteq N_k \supseteq \ldots$$

and functions

$$z_k \in C\left[\frac{1}{2^{k+1}}, 1-\frac{1}{2^{k+1}}\right]$$

with

$$y_n \text{ converging uniformly to } z_k \text{ on } \left[\frac{1}{2^{k+1}}, 1-\frac{1}{2^{k+1}}\right] \text{ as } n\to\infty \text{ through } N_k$$

and

$$z_k = z_{k-1} \text{ on } \left[\frac{1}{2^k}, 1-\frac{1}{2^k}\right].$$

Define a function $y : [0,1] \to [0,\infty)$ by $y(x) = z_k(x)$ on $\left[\frac{1}{2^{k+1}}, 1-\frac{1}{2^{k+1}}\right]$ and $y(0) = y(1) = 0$. Notice y is well defined and $\alpha(t) \le y(t) \le y_{n_0}(t)\ (\le \beta(t))$ for $t \in (0,1)$. Next we prove y is a solution of (1.1). Fix $t \in (0,1)$ and let $m \in \{n_0, n_0+1, \ldots\}$ be such that $\frac{1}{2^m} < t < 1-\frac{1}{2^m}$. Let $N_m^+ = \{n \in N_m : n \ge m\}$. Let $y_n, n \in N_m^+$, and let $a = \frac{1}{2^m}, b = 1-\frac{1}{2^m}$.

Define the operator, $L : C[a,b] \to C[a,b]$ by

$$(Lu)(t) = u(a) + \int_a^t \varphi_p^{-1}(A_u + \int_s^b q(\tau)\,(g_n^*(\tau, u(\tau)))d\tau)ds$$

where A_u satisfy

$$\int_a^b \varphi_p^{-1}(A_u + \int_s^b q(\tau)\,(g_n^*(\tau, u(\tau)))d\tau)ds = u(a) - u(b).$$

Let $u_m \to u$ uniformly on $[a,b]$. As in the proof of Theorem 2.4[4], if we show $\lim_{m\to\infty} A_{u_m} = A_u$, then this together with φ_p^{-1} continuous, implies

$L : C[a,b] \to C[a,b]$ is continuous, (here A_{u_m} is associated with u_m). First notice

$$\int_a^b \left(\varphi_p^{-1}(A_{u_m} + \int_s^b q(\tau)\,(g_n^*(\tau, u_m(\tau)))d\tau) \right.$$
$$\left. -\varphi_p^{-1}(A_u + \int_s^b q(\tau)\,(g_n^*(\tau, u(\tau)))d\tau) \right) ds$$
$$= u_m(b) - u_m(a) - u(b) + u(a).$$

The Mean Value theorem for integrals implies that there exists $\eta_n \in [0,1]$ with

$$\varphi_p^{-1}(A_{u_m} + \int_{\eta_m}^b q(\tau)\,(g_n^*(\tau, u_m(\tau)))d\tau)$$
$$-\varphi_p^{-1}(A_u + \int_{\eta_m}^b q(\tau)\,(g_n^*(\tau, u(\tau)))d\tau)$$
$$= \frac{u_m(b) - u_m(a) - u(b) + u(a)}{b-a},$$

and now since $u_m \to u$ uniformly on $[a,b]$ we have $\lim_{m\to\infty} A_{u_m} = A_u$.

Now y_m converging uniformly on $[a,b]$ to y as $m \to \infty$ and $Ly_m = y_m$, yields $Ly = y$, i. e.

$$(\varphi_p(y'(t)))' + q(t)(g_n^*(t, y(t)) = 0,\ a \le t \le b.$$

Thus

$$(\varphi_p(y'(t)))' + q(t)(f(t, y(t)) = 0,\ a \le t \le b.$$

We can do this argument for each $t \in (0,1)$ and so $(\varphi_p(y'(t)))' + q(t)(f(t,y(t)) = 0$ for $t \in (0,1)$. It remains to show y is continuous at 0 and 1.

Let $\varepsilon > 0$ be given. Now since $\lim_{m\to\infty} y_m(0) = 0$ there exists $m_1 \in \{m_0, m_0+1, \ldots\}$ with $y_{m_1}(0) < \frac{\varepsilon}{2}$. Since $y_{m_1} \in C[0,1]$ there exists $\delta_{m_1} > 0$ with

$$y_{m_1}(t) < \frac{\varepsilon}{2} \text{ for } t \in [0, \delta_{m_1}].$$

Now for $m \ge m_1$ we have, since $\{y_m(t)\}$ is nonincreasing for each $t \in [0,1]$,

$$\alpha(t) \le y_m(t) \le y_{m_1}(t) < \frac{\varepsilon}{2} \text{ for } t \in [0, \delta_{m_1}].$$

Consequently

$$\alpha(t) \le y(t) \le \frac{\varepsilon}{2} < \varepsilon \text{ for } t \in [0, \delta_{m_1}],$$

and so y is continuous at 0. Similarly y is continuous at 1. As a result, we have shown $y \in C[0,1]$.

Suppose (2.2)−(2.5) hold and in addition assume the following conditions are satisfied:

$$q\,(t)\,f(t,y) + (\varphi_p(\alpha'(t)))^{'} > 0 \quad \text{for } (t,y) \in (0,1) \times \{y \in (0,\infty) : y < \alpha\,(t)\} \tag{2.23}$$

and

$$\begin{cases} \text{there exists s function } \beta \in C[0,1] \cap C^1(0,1), \\ \varphi_p(\beta') \in C^1(0,1) \text{ with } \beta\,(t) \geq \rho_{n_0} \text{ on } [0,1] \\ \text{such that } q\,(t)\,f(t,\beta\,(t)) + (\varphi_p(\beta'(t)))^{'} \leq 0 \text{ for } t \in (0,1) \\ \text{with } q\,(t)\,f(\frac{1}{2^{n_0+1}},\beta\,(t)) + (\varphi_p(\beta'(t)))^{'} \leq 0 \text{ for } t \in (0,\frac{1}{2^{n_0+1}}). \end{cases} \tag{2.24}$$

Then the result in Theorem 2.1 is true. This follows immediately from Theorem 2.1 once we show (2.6) holds i.e. once we show $\beta\,(t) \geq \alpha\,(t)$ for $t \in [0,1]$ .Suppose it is false. Then $\alpha - \beta$ would have a positive absolute maximum at say $t_0 \in (0,1)$, so $(\alpha - \beta)'\,(t_0) = 0$ and $(\varphi_p(\alpha'))^{'}(t_0) \leq (\varphi_p(\beta'))^{'}(t_0)$. Now $\alpha\,(t_0) > \beta\,(t_0)$ and (2.23) implies

$$q\,(t_0)\,f\,(t_0,\beta\,(t_0)) + (\varphi_p(\alpha'))^{'}(t_0) > 0.$$

This together with (2.24) yields

$$(\varphi_p(\alpha'))^{'}(t_0) - (\varphi_p(\beta'))^{'}(t_0) \geq (\varphi_p(\alpha'))^{'}(t_0) + q\,(t_0)\,f\,(t_0,\beta\,(t_0)) > 0,$$

a contradiction. □

Thus we have

Corollary 2.1. *Let $n_0 \in \{1,2,\ldots\}$ be fixed and suppose* (2.2) − (2.5) *,* (2.23) *and* (2.24) *hold. Then* (2.1) *has a solution $y \in C\,[0,1] \cap C^1\,(0,1)$ and $\varphi_p(y') \in C^1\,(0,1)$ with $y\,(t) \geq \alpha\,(t)$ for $t \in [0,1]$.*

Remark 2.1. If in (2.3) we replace $\frac{1}{2^{n+1}} \leq t < 1$ with $0 < t \leq 1 - \frac{1}{2^{n+1}}$ then one would replace (2.6) with

$$\begin{cases} \text{there exists s function } \beta \in C[0,1] \cap C^1(0,1), \\ \varphi_p(\beta') \in C^1(0,1) \text{ with } \beta\,(t) \geq \alpha\,(t)\,, \text{ and } \beta\,(t) \geq \rho_{n_0} \text{ on } [0,1] \\ \text{such that } q\,(t)\,f(t,\beta\,(t)) + (\varphi_p(\beta'(t)))^{'} \leq 0 \text{ for } t \in (0,1) \\ \text{with } q\,(t)\,f(1-\frac{1}{2^{n_0+1}},\beta\,(t)) + (\varphi_p(\beta'(t)))^{'} \leq 0 \text{ for } t \in (1-\frac{1}{2^{n_0+1}},1). \end{cases}$$

If in (2.3) we replace $\frac{1}{2^{n+1}} \leq t < 1$ with $\frac{1}{2^{n+1}} \leq t \leq 1 - \frac{1}{2^{n+1}}$ then essentially the same reasoning as in Theorem 2.1 establishes the following results.

Theorem 2.2. *Let $n_0 \in \{3,4,\ldots\}$ be fixed and suppose* (2.2) , (2.4) , (2.5) *and the following hold:*

$$\begin{cases} \textit{let } n \in \{n_0, n_0+1,\ldots\} \textit{ and associated with each } n \textit{ we} \\ \textit{have a constant } \rho_n \textit{ such that } \{\rho_n\} \textit{ is a nonincreasing} \\ \textit{sequence with } \lim_{n\to\infty}\rho_n = 0 \textit{ and such that for} \\ \frac{1}{2^{n+1}} \leq t \leq 1 - \frac{1}{2^{n+1}} \textit{ we have } q\,(t)\,f\,(t,\rho_n) \geq 0 \end{cases} \tag{2.25}$$

and

$$\begin{cases} \text{there exists a function } \beta \in C[0,1] \cap C^1(0,1), \\ \varphi_p(\beta') \in C^1(0,1), \text{ with } \beta(t) \geq \alpha(t) \text{ and } \beta(t) \geq \rho_{n_0} \text{ on } [0,1] \\ \text{such that } q(t)\, f(t, \beta(t)) + (\varphi_p(\beta'(t)))' \leq 0 \text{ for } t \in (0,1) \text{ with} \\ q(t)\, f(\frac{1}{2^{n_0+1}}, \beta(t)) + (\varphi_p(\beta'(t)))' \leq 0 \text{ for } t \in (0, \frac{1}{2^{n_0+1}}) \text{ and} \\ q(t)\, f(1 - \frac{1}{2^{n_0+1}}, \beta(t)) + (\varphi_p(\beta'(t)))' \leq 0 \text{ for } t \in (1 - \frac{1}{2^{n_0+1}}, 1). \end{cases} \tag{2.26}$$

Then (2.1) *has a solution* $y \in C[0,1] \cap C^1(0,1)$ *with* $\varphi_p(y') \in C^1(0,1)$ *with* $y(t) \geq \alpha(t)$ *for* $t \in [0,1]$.

Corollary 2.2. *Let* $n_0 \in \{3, 4, \ldots\}$ *be fixed and suppose* (2.2), (2.4), (2.5), (2.23), (2.25) *and the following hold.*

$$\begin{cases} \text{there exists a function } \beta \in C[0,1] \cap C^1(0,1), \\ \varphi_p(\beta') \in C^1(0,1), \text{ with and } \beta(t) \geq \rho_{n_0} \text{ on } [0,1] \\ \text{such that } q(t)\, f(t, \beta(t)) + (\varphi_p(\beta'(t)))' \leq 0 \text{ for } t \in (0,1) \text{ with} \\ q(t)\, f(\frac{1}{2^{n_0+1}}, \beta(t)) + (\varphi_p(\beta'(t)))' \leq 0 \text{ for } t \in (0, \frac{1}{2^{n_0+1}}) \text{ and} \\ q(t)\, f(1 - \frac{1}{2^{n_0+1}}, \beta(t)) + (\varphi_p(\beta'(t)))' \leq 0 \text{ for } t \in (1 - \frac{1}{2^{n_0+1}}, 1). \end{cases} \tag{2.27}$$

Then (2.1) *has a solution* $y \in C[0,1] \cap C^1(0,1)$ *with* $\varphi_p(y') \in C^1(0,1)$ *with* $y(t) \geq \alpha(t)$ *for* $t \in [0,1]$.

Next we consider how to construct the lower solution α in (2.5) and (2.23). Suppose the following condition is satisfied:

$$\begin{cases} \text{let } n \in \{n_0, n_0+1, \ldots\} \text{ and associated with each } n \text{ we} \\ \text{have a constant } \rho_n \text{ such that } \{\rho_n\} \text{ is a decreasing} \\ \text{sequence with } \lim_{n\to\infty} \rho_n = 0 \text{ and there exists a} \\ \text{constant } k_0 > 0 \text{ such that for } \frac{1}{2^{n+1}} \leq t \leq 1 - \frac{1}{2^{n+1}} \\ \text{and } 0 < y \leq \rho_n \text{ we have } q(t)\, f(t, y) \geq k_0. \end{cases} \tag{2.28}$$

A slight modification of the argument in Q.Yao and H.Lü [7] guarantees that exists a $\alpha \in C[0,1] \cap C^1(0,1)$, $\varphi_p(\alpha') \in C^1(0,1)$ with $\alpha(0) = \alpha(1) = 0$, $\alpha(t) \leq \rho_{n_0}$, for $t \in [0,1]$ with (2.5) and (2.23) holding. We combine this with Corollary 2.1 to obtain our next result.

Theorem 2.3. *Let* $n_0 \in \{1, 2, \ldots\}$ *be fixed and suppose* (2.2), (2.4), (2.26) *and* (2.28) *hold. Then* (2.1) *has a solution* $y \in C[0,1] \cap C^1(0,1)$ *with* $\varphi_p(y') \in C^1(0,1)$ *with* $y(t) > 0$ *for* $t \in (0,1)$.

Corollary 2.3. *Let* $n_0 \in \{1, 2, \ldots\}$ *be fixed and suppose* (2.2), (2.4), (2.24) *and* (2.28) *(with* $\frac{1}{2^{n+1}} \leq t < 1$ *replaced by* $\frac{1}{2^{n+1}} \leq t \leq 1 - \frac{1}{2^{n+1}}$*) hold. Then* (2.1) *has a solution* $y \in C[0,1] \cap C^1(0,1)$ *with* $\varphi_p(y') \in C^1(0,1)$ *with* $y(t) > 0$ *for* $t \in (0,1)$.

Looking at Theorem 2.3 we see that the main difficulty when discussing examples is the construction of the β in (2.26). Our next result replaces (2.26) with another condition.

Theorem 2.4. *Let $0 \in \{1,2,\ldots\}$ be fixed and suppose* (2.2)–(2.5) *hold. Also assume the following two conditions are satisfied:*

$$\begin{cases} |f(t,y)| \le g(y) + h(y) \;\; on \;\; [0,1]\times(0,\infty) \;\; with \\ g > 0 \; continuous \; and \; nonincreasing \; on \;\; (0,\infty), \\ h \ge 0 \; continuous \; on \; [0,\infty), \;\; and \; \frac{h}{g} \\ nondecreasing \; on \;\; (0,\infty) \end{cases} \tag{2.29}$$

and

$$\begin{cases} for \; any \; R > 0, \frac{1}{g} \; is \; differentiable \; on \; (0,R] \; with \\ g' < 0 \; a.e. \;\; on \; (0,R], \; \frac{g'}{g^2} \in L^1[0,R]. \end{cases} \tag{2.30}$$

In addition assume there exists $M > \sup_{t\in[0,1]} \alpha(t)$ with

$$\frac{1}{\varphi_p^{-1}\left(1+\frac{h(M)}{g(M)}\right)} \int_0^M \frac{du}{\varphi_p^{-1}(g(u))} > b_0 \tag{2.31}$$

holding; here

$$b_0 = \max\left\{ \int_0^{\frac12} \varphi_p^{-1}\left(\int_s^{\frac12} q(r)\,dr\right) ds, \int_{\frac12}^1 \varphi_p^{-1}\left(\int_{\frac12}^s q(r)\,dr\right) ds \right\}.$$

Then (2.1) *has a solution $y \in C[0,1]\cap C^1(0,1)$ with $\varphi_p(y') \in C^1(0,1)$ with $y(t) \ge \alpha(t)$ for $t \in [0,1]$.*

Proof. Fix $n \in \{n_0, n_0+1,\ldots\}$. Choose ε, $0<\varepsilon<M$ with

$$\frac{1}{\varphi_p^{-1}\left(1+\frac{h(M)}{g(M)}\right)} \int_\varepsilon^M \frac{du}{\varphi_p^{-1}(g(u))} > b_0 \tag{2.32}$$

Let $m_0 \in \{3,4,\ldots\}$ be chosen so that $\rho_{m_0} < \varepsilon$ and without loss of generality assume $m_0 \le n_0$. Let e_n, θ_n, f_n, g_n and α_n be as in Theorem 2.1. We consider the boundary value problem (2.7) with in this case $g^*_{n_0}$ given by

$$g^*_{n_0}(t,y) = \begin{cases} g_{n_0}(t,M) + r(M-y), \; y > M, \\ g_{n_0}(t,y), \qquad\qquad \alpha_{n_0}(t) \le y \le M \\ g_{n_0}(t,\alpha_{n_0}(t)) + r(\alpha_{n_0}(t) - y), \; y < \alpha_{n_0}(t). \end{cases}$$

Essentially the same reasoning as in Theorem 2.1 implies that (2.7) has a solution $y_{n_0} \in C[0,1]\cap C^1(0,1)$ with $\varphi_p\left(y'_{n_0}\right) \in C^1(0,1)$ with $y_{n_0}(t) \ge \alpha_{n_0}(t) \ge \alpha(t)$ for $t \in [0,1]$. Next we show

$$y_{n_0}(t) \le M \text{ for } t \in [0,1]. \tag{2.33}$$

Suppose (2.33) is false. Now since $y_{n_0}(0) = y_{n_0}(1) = \rho_{n_0}$ there exists either
Case (i). $t_1, t_2 \in (0,1)$ with $\alpha_{n_0}(t) \le y_{n_0}(t) \le M$ for $t \in [0,t_2)$, $y_{n_0}(t_2) = M$ and $y_{n_0}(t) > M$ on (t_2,t_1) with $y'_{n_0}(t_1) = 0$;
or
Case (ii). $t_3, t_4 \in (0,1), t_4 < t_3$ with $\alpha_{n_0}(t) \le y_{n_0}(t) \le M$ for $t \in (t_3,1]$, $y_{n_0}(t_3) = M$ and $y_{n_0}(t) > M$ on (t_4,t_3) with $y'_{n_0}(t_4) = 0$.

We can assume without loss of generality that either $t_1 \leq \frac{1}{2}$ or $t_4 \geq \frac{1}{2}$. Suppose $t_1 \leq \frac{1}{2}$. Notice for $t \in (t_2, t_1)$ that we have

$$-\left(\varphi_p\left(y_{n_0}'\right)\right)' = q(t)\, g_{n_0}^*(t, y_{n_0}(t)) \leq q(t)\left[g(M) + h(M)\right]; \tag{2.34}$$

note if $t \in (t_2, t_1)$ that we have

$$\begin{aligned} g_{n_0}^*(t, y_{n_0}(t)) &= g_{n_0}(t, M) + r(M - y_{n_0}(t)) \\ &\leq \max\left\{ f\left(\frac{1}{2^{n_0+1}}, M\right), f(t, M)\right\}. \end{aligned}$$

Integrate (2.34) from t_2 to t_1 to obtain

$$\varphi_p\left(y_{n_0}'(t_2)\right) \leq \left[g(M) + h(M)\right] \int_{t_2}^{t_1} q(s)\, ds,$$

and this together with $y_{n_0}(t_2) = M$ yields

$$\frac{\varphi_p\left(y_{n_0}'(t_2)\right)}{g(y_{n_0}(t_2))} \leq \left[1 + \frac{h(M)}{g(M)}\right] \int_{t_2}^{t_1} q(s)\, ds. \tag{2.35}$$

Also for $t \in (0, t_2)$ we have

$$\begin{aligned} -\left(\varphi_p\left(y_{n_0}'(t)\right)\right)' &= q(t) \max\left\{ f\left(\frac{1}{2^{n_0+1}}, y_{n_0}(t)\right), f(t, y_{n_0}(t))\right\} \\ &\leq q(t)\left[g(y_{n_0}(t)) + h(y_{n_0}(t))\right] \end{aligned}$$

and so

$$\frac{-\left(\varphi_p\left(y_{n_0}'(t)\right)\right)'}{g(y_{n_0}(t))} \leq q(t)\left\{1 + \frac{h(y_{n_0}(t))}{g(y_{n_0}(t))}\right\} \leq q(t)\left\{1 + \frac{h(M)}{g(M)}\right\}$$

for $t \in (0, t_2)$. Integrate from $t\,(t \in (0, t_2))$ to t_2 to obtain

$$\begin{aligned} &\frac{-\varphi_p\left(y_{n_0}'(t_2)\right)}{g(y_{n_0}(t_2))} + \frac{\varphi_p\left(y_{n_0}'(t)\right)}{g(y_{n_0}(t))} + \int_t^{t_2} \left\{\frac{-g'(y_{n_0}(x))}{g^2(y_{n_0}(x))}\right\} \left|y_{n_0}'(x)\right|^p dx \\ \leq\ & \left\{1 + \frac{h(M)}{g(M)}\right\} \int_t^{t_2} q(s)\, ds, \end{aligned}$$

and this together with (2.35) yields

$$\frac{\varphi_p\left(y_{n_0}'(t)\right)}{g(y_{n_0}(t))} \leq \left\{1 + \frac{h(M)}{g(M)}\right\} \int_t^{t_1} q(s)\, ds. \text{ for } t \in (0, t_2).$$

Integrate from 0 to t_2 to obtain

$$\begin{aligned} \int_\varepsilon^M \frac{du}{\varphi_p^{-1}(g(u))} &\leq \int_{\rho_{n_0}}^M \frac{du}{\varphi_p^{-1}(g(u))} \\ &\leq \varphi_p^{-1}\left(1 + \frac{h(M)}{g(M)}\right) \int_0^{t_2} \varphi_p^{-1}\left(\int_t^{t_1} q(s)\, ds\right) dt. \end{aligned}$$

That is

$$\int_\varepsilon^M \frac{du}{\varphi_p^{-1}(g(u))} \leq b_0 \varphi_p^{-1}\left(1 + \frac{h(M)}{g(M)}\right).$$

This contradicts (2.32) so (2.33) holds (a similar argument yields a contradiction if $t_4 \geq \frac{1}{2}$). Thus we have

$$\alpha(t) \leq \alpha_{n_0}(t) \leq y_{n_0}(t) \leq M \quad \text{for} \quad t \in [0,1].$$

Essentially the same reasoning as in Theorem 2.1 (from (2.14) onwards) completes the proof. □

Similarly we have the following result.

Theorem 2.5. *Let* $n_0 \in \{1, 2, \ldots\}$ *be fixed and suppose* (2.2), (2.4), (2.5), (2.25), (2.28) *and* (2.29) *hold. In addition assume there exists*

$$M > \sup_{t \in [0,1]} \alpha(t)$$

with (2.31) *holding. Then* (2.1) *has a solution* $y \in C[0,1] \cap C^1(0,1)$ *with* $\varphi_p(y') \in C^1(0,1)$ *with* $y(t) \geq \alpha(t)$ *for* $t \in [0,1]$.

Corollary 2.4. *Let* $n_0 \in \{1, 2, \ldots\}$ *be fixed and suppose* (2.2)–(2.5), (2.23), (2.28) *and* (2.29) *hold. In addition assume there exists a constant* $M > 0$ *with*

$$\frac{1}{\varphi_p^{-1}\left(1 + \frac{h(M)}{g(M)}\right)} \int_0^M \frac{du}{\varphi_p^{-1}(g(u))} > b_0 \tag{2.36}$$

holding; here

$$b_0 = \max\left\{ \int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\int_s^{\frac{1}{2}} q(r)\, dr \right) ds, \int_{\frac{1}{2}}^1 \varphi_p^{-1}\left(\int_{\frac{1}{2}}^s q(r)\, dr \right) ds \right\}.$$

Then (2.1) *has a solution* $y \in C[0,1] \cap C^1(0,1)$ *with* $\varphi_p(y') \in C^1(0,1)$ *with* $y(t) \geq \alpha(t)$ *for* $t \in [0,1]$.

Proof. The result follows immediately from Theorem 2.5 once we show $\alpha(t) \leq M$ for $t \in [0,1]$. Suppose this is false. Now since $\alpha(0) = \alpha(1) = 0$ there exists either

Case (i). $t_1, t_2 \in (0,1), t_2 < t_1$ with $0 \leq \alpha(t) \leq M$ for $t \in [0, t_2)$, $\alpha(t_2) = M$ and $\alpha(t) > M$ on (t_2, t_1) with $\alpha'(t_1) = 0$;

or

Case (ii). $t_3, t_4 \in (0,1), t_4 < t_3$ with $0 \leq \alpha(t) \leq M$ for $t \in (t_3, 1]$, $\alpha(t_3) = M$ and $\alpha(t) > M$ on (t_4, t_3) with $\alpha'_{n_0}(t_4) = 0$.

We can assume without loss of generality that either $t_1 \leq \frac{1}{2}$ or $t_4 \geq \frac{1}{2}$. Suppose $t_1 \leq \frac{1}{2}$. Notice for $t \in (t_2, t_1)$ that we have

$$-(\varphi_p(\alpha'))' \leq q(t)[g(M) + h(M)], \tag{2.37}$$

so integrating from t_2 to t_1 yields

$$\frac{\varphi_p(\alpha'(t_2))}{g(\alpha(t_2))} \leq \left[1 + \frac{h(M)}{g(M)}\right] \int_{t_2}^{t_1} q(s)\, ds. \tag{2.38}$$

Also for $t \in (0, t_2)$ we have that

$$-\left(\varphi_p\left(\alpha'(t)\right)\right)' \leq q(t)\, g(\alpha(t)) \left[1 + \frac{h(\alpha(t))}{g(\alpha(t))}\right] \leq q(t)\, g(\alpha(t)) \left[1 + \frac{h(M)}{g(M)}\right].$$

Integrate from $t\,(t \in (0, t_2))$ to t_2 and use (2.38) to obtain

$$\frac{\varphi_p(\alpha'(t))}{g(\alpha(t))} \leq \left\{1 + \frac{h(M)}{g(M)}\right\} \int_t^{t_1} q(s)\, ds. \text{ for } t \in (0, t_2).$$

Finally integrate from 0 to t_2 to obtain

$$\int_0^M \frac{du}{\varphi_p^{-1}(g(u))} \leq b_0 \varphi_p^{-1}\left(1 + \frac{h(M)}{g(M)}\right),$$

a contradiction. □

Corollary 2.5. *Let $n_0 \in \{1, 2, \ldots\}$ be fixed and suppose* (2.2), (2.4), (2.5), (2.21), (2.24), (2.29) *and* (2.30) *hold. In addition assume there is a constant $M > 0$ with* (2.35) *holding. Then* (2.1) *has a solution $y \in C[0,1] \cap C^1(0,1)$ with $\varphi_p(y') \in C^1(0,1)$ with $y(t) \geq \alpha(t)$ for $t \in [0,1]$.*

Combining Corollary 2.4 *with the comments before Theorem* 2.5 *yields the following theorem.*

Theorem 2.6. *Let $n_0 \in \{1, 2, \ldots\}$ be fixed and suppose* (2.2), (2.4), (2.28), (2.29) *and* (2.30) *hold. In addition assume there is a constant $M > 0$ with* (2.36) *holding. Then* (2.1) *has a solution $y \in C[0,1] \cap C^1(0,1)$ with $\varphi_p(y') \in C^1(0,1)$ with $y(t) > 0$ for $t \in (0,1)$.*

Next we present an example which illustrates how easily the theory is applied in practice.

Example 1. Consider the boundary value problem

$$\begin{cases} \left(|y'|^{p-2} y'\right)' + \left(\frac{t}{y^2} + \frac{1}{32} y^2 - \mu^2\right) = 0, \quad 0 < t < 1 \\ y(0) = y(1) = 0 \end{cases} \tag{2.39}$$

with $1.4 \leq p < 5$ and $\mu^2 > 1$. Then (2.39) has a solution $y \in C[0,1] \cap C^1(0,1)$ with $\varphi_p(y') \in C^1(0,1)$ with $y(t) > 0$ for $t \in (0,1)$.

To see this we will apply Corollary 2.3 with that

$$q \equiv 1, \rho_n = \left(\frac{1}{2^{n+1}(\mu^2 + a)}\right)^{\frac{1}{2}} \text{ and } k_0 = a;$$

here $a > 0$ chosen so that $a \leq \frac{1}{8} - \frac{p-1}{2^p}$; note $\frac{1}{8} - \frac{p-1}{2^p} > 0$ since $1.4 \leq p < 5$. Also choose $n_0 \in \{1, 2, \ldots\}$ with $\rho_{n_0} \leq 1$. Clearly (2.2) and (2.4) hold. Notice for $n \in \{1, 2, \ldots\}$, $\frac{1}{2^{n+1}} \leq t < 1$ and $0 < y \leq \rho_n$ that we have

$$q(t)\, f(t, y) \geq \frac{t}{y^2} - \mu^2 \geq \frac{1}{2^{n+1} \rho_n^2} - \mu^2 = \left(\mu^2 + a\right) - \mu^2 = a,$$

so (2.28) (with $\frac{1}{2^{n+1}} \le t < 1$ replaced by $\frac{1}{2^{n+1}} \le t \le 1 - \frac{1}{2^{n+1}}$) is satisfied. It remains to check (2.24) with

$$\beta(t) = \sqrt{t} + \rho_{n_0}.$$

Now $\left(|\beta'(t)|^{p-2}\beta'(t)\right)' = -\frac{p-1}{2^p}t^{-\frac{p+1}{2}}$ and so for $t \in (0,1)$ we have

$$\begin{aligned}
&\left(|\beta'(t)|^{p-2}\beta'(t)\right)' + q(t) f(t, \beta(t)) \\
&\le -\frac{p-1}{2^p}t^{-\frac{p+1}{2}} + \left(\frac{t}{t} + \frac{\left(\sqrt{t}+\rho_{n_0}\right)^2}{32} - \mu^2\right) \\
&\le -\frac{p-1}{2^p} + \left(1 + \frac{1}{8} - \mu^2\right) \\
&\le 0.
\end{aligned}$$

Also for $t \in \left(0, \frac{1}{2^{n_0+1}}\right)$ we have

$$\begin{aligned}
&\left(|\beta'(t)|^{p-2}\beta'(t)\right)' + q(t) f(t, \beta(t)) \\
&\le -\frac{p-1}{2^p}t^{-\frac{p+1}{2}} + \left(\frac{1}{2^{n_0+1}\rho_{n_0}^2} + \frac{\left(\sqrt{t}+\rho_{n_0}\right)^2}{32} - \mu^2\right) \\
&\le -\frac{p-1}{2^p} + \left(\left(\mu^2 + a\right) + \frac{1}{8} - \mu^2\right) \\
&= a + \frac{1}{8} - \frac{p-1}{2^p} \le 0.
\end{aligned}$$

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Authors' addresses:

Ravi P.Agarwal
Department of Mathematical Sciences
Florida Institute of Technology
Melbourne, FL 32901-6975,
U.S.A.

Haishen Lü
Department of Mathematics
Lanzhou University
Lanzhou, 730000, P.R.
China

Donal O'Regan
Department of Mathematics
National University of Ireland
Galway, Ireland