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ON A TWO POINT BOUNDARY PROBLEM FOR FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS WITH A DEVIATING ARGUMENT

Abstract. The aim of the paper is to study the question on the existence and uniqueness of a solution of the problem

$$u'(t) = p(t)u(\tau(t)) + q(t), \qquad u(a) + \lambda u(b) = c,$$

where $p, q : [a, b] \to R$ are Lebesgue integrable functions, $\tau : [a, b] \to [a, b]$ is a measurable function, and $\lambda, c \in R$. More precisely, some solvability conditions established in [5, 6, 8] are refined for the special case, where τ maps the segment $[a, b]$ into some subsegment $[\tau_0, \tau_1] \subseteq [a, b]$.

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რეზიუმე. ნაშრომის მიზანია შესწავლილ იქნას

$$u'(t) = p(t)u(\tau(t)) + q(t), \qquad u(a) + \lambda u(b) = c$$

ამოცანის ამონახსნის არსებობის და ერთადერთობის საკითხი, სადაც $p, q : [a, b] \to R$ ლებეგის აზრით ინტეგრებადი ფუნქციებია, $\tau : [a, b] \to [a, b]$ ზომადი ფუნქციაა, ხოლო $\lambda, c \in R$. უფრო ზუსტად, [5, 6, 8]-ში დადგენილი ამოხსნადობის ზოგიერთი პირობა დაზუსტებულია იმ კონკრეტული შემთხვევისათვის, როცა τ $[a, b]$ სეგმენტს თავის რაიმე $[\tau_0, \tau_1] \subseteq [a, b]$ ქვესეგმენტში ასახავს.

Introduction

The following notation is used throughout.

R is the set of all real numbers, $R_+ = [0, +\infty[$.

$$[x]_+ = \frac{|x| + x}{2}, \qquad [x]_- = \frac{|x| - x}{2}.$$

$\widetilde{C}([a,b]; D)$, where $D \subseteq R$, is the set of absolutely continuous functions $u : [a,b] \to D$.

$\mathcal{M}_{ab}$ is the set of measurable functions $\tau : [a,b] \to [a,b]$.

$L([a,b]; R)$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to R$ with the norm

$$\|p\|_L = \int_a^b |p(s)| ds.$$

By a solution of the equation

$$u'(t) = p(t)u(\tau(t)) + q(t), \tag{0.1}$$

where $p, q \in L([a,b]; R)$ and $\tau \in \mathcal{M}_{ab}$, we understand a function $u \in \widetilde{C}([a,b]; R)$ satisfying the equality (0.1) almost everywhere in $[a,b]$. Note also that throughout the paper the equalities and inequalities with integrable functions are understood to hold almost everywhere.

Consider the problem on the existence and uniqueness of a solution of the equation (0.1) satisfying the boundary condition

$$u(a) - \lambda u(b) = c, \tag{0.2}$$

resp.

$$u(a) + \lambda u(b) = c, \tag{0.3}$$

where $\lambda \in R_+$ and $c \in R$.

The theory of differential equations with deviating arguments was begun to develop in 50's of the 20th century (see, e.g., [9,11] and references therein). Throughout the second half of the 20th century, a lot was done to construct the general theory of boundary value problems for functional differential equations (see, e.g., [1, 2, 10, 12] and references therein). In spite of this, there are only few effective criteria for the solvability of special types of boundary value problems for functional differential equations.

In [5,8] and [6], there were established sufficient conditions for the unique solvability of the problem (0.1), (0.2) and (0.1), (0.3), respectively. Moreover, those results are, in general, nonimprovable (see the examples constructed in [5, 6, 8]). On the other hand, if τ maps the segment $[a,b]$ into some subsegment $[\tau_0, \tau_1] \subseteq [a,b]$, the above–mentioned results can be improved in a certain way. In the present paper, some results from [5,6,8] are refined for such special case of the equation (0.1). For this purpose, put

$$\tau_0 = \text{ess inf}\,\{\tau(t) : t \in [a,b]\}, \qquad \tau_1 = \text{ess sup}\,\{\tau(t) : t \in [a,b]\}.$$

It is clear that $\tau_0, \tau_1 \in [a,b]$, $\tau_0 \leq \tau_1$, and $\tau(t) \in [\tau_0, \tau_1]$ for almost all $t \in [a,b]$.

The paper is organized as follows. Sections 1 and 2 deal with the problem (0.1), (0.2) and (0.1), (0.3), respectively. Section 3 is devoted to the examples verifying the optimality of the obtained results.

The Cauchy problem is a special case of the discussed boundary value problem (for $\lambda = 0$). In that case, the below formulated theorems coincide with the results obtained in [4].

The following result is well–known from the general theory of the boundary value problems for functional differential equations (see [1–3, 10, 12]).

Theorem 0.1. *The problem* (0.1), (0.2), *resp.* (0.1), (0.3), *is uniquely solvable if and only if the corresponding homogeneous problem*

$$u'(t) = p(t)u(\tau(t)), \tag{0.1$_0$}$$

$$u(a) - \lambda u(b) = 0, \tag{0.2$_0$}$$

resp.

$$u'(t) = p(t)u(\tau(t)),$$
$$u(a) + \lambda u(b) = 0, \tag{0.3$_0$}$$

has only the trivial solution.

1. Problem (0.1), (0.2)

1.1. Main Results.

Theorem 1.1. *Let* $\lambda \in [0,1]$ *and*

$$A \stackrel{\text{def}}{=} \int_a^{\tau_1} [p(s)]_+ ds + \lambda \int_{\tau_1}^b [p(s)]_+ ds\,. \tag{1.1}$$

If

$$A < 1\,, \tag{1.2}$$

$$\left(\int_a^{\tau_0} [p(s)]_- ds + \lambda \int_{\tau_0}^b [p(s)]_- ds\right)\left(1 - \int_{\tau_0}^{\tau_1} [p(s)]_+ ds\right) > \lambda - 1 + A, \tag{1.3}$$

$$(1 - A)\left(1 + \int_{\tau_0}^{\tau_1} [p(s)]_- ds\right) > \lambda - \int_a^{\tau_0} [p(s)]_- ds - \lambda \int_{\tau_1}^b [p(s)]_- ds\,, \tag{1.4}$$

and either

$$\int_a^{\tau_0} [p(s)]_- ds + \lambda \int_{\tau_1}^b [p(s)]_- ds < \lambda + \sqrt{1 - A}\,, \tag{1.5}$$

$$\int_a^{\tau_1} [p(s)]_- ds + \lambda \int_{\tau_1}^b [p(s)]_- ds < 1 + \lambda + 2\sqrt{1 - A} \tag{1.6}$$

or

$$\int_a^{\tau_0}[p(s)]_-\,ds+\lambda\int_{\tau_1}^b[p(s)]_-\,ds\geq\lambda+\sqrt{1-A}\,, \tag{1.7}$$

$$\int_{\tau_0}^{\tau_1}[p(s)]_-\,ds<1+\frac{1-A}{\int_a^{\tau_0}[p(s)]_-\,ds+\lambda\int_{\tau_1}^b[p(s)]_-\,ds-\lambda}\,, \tag{1.8}$$

then the problem (0.1), (0.2) *has a unique solution.*

Remark 1.1. Theorem 1.1 is nonimprovable in the sense that neither of the strict inequalities (1.3), (1.4), (1.6), and (1.8) can be replaced by the nonstrict one (see Examples 3.1–3.4).

Example 1.1. On the segment $[0,1]$ consider the boundary value problem

$$\begin{gathered}u'(t)=k(1-3t)u\left(\frac{(3r-1)t+1}{3}\right)+q(t),\\ u(0)-\lambda u(1)=c,\end{gathered} \tag{1.9}$$

where $\lambda\in\left[\frac14,1\right]$, $k\in\,]0,6[\,$, $q\in L([a,b];R)$, $c\in R$, and $r\in\left[\frac23,1\right]$ is such that $3(1+3r)(1-r)\geq1$. Put

$$\alpha=\frac{(1+3r)(1-r)}{2}\,,\qquad\beta=\frac{(3r-1)^2}{6}\,. \tag{1.10}$$

It is not difficult to verify that $\tau_0=\frac13$, $\tau_1=r$, and the condition (1.2) holds. Obviously, if

$$k<\frac{3(4\lambda-1)}{4\lambda}+\sqrt{\left[\frac{3(4\lambda-1)}{4\lambda}\right]^2+9\,\frac{1-\lambda}{\lambda}}\,,$$

then the conditions (1.3) and (1.4) are fulfilled. If

$$k<\frac1\alpha+\frac{-1+\sqrt{1+24\lambda^2\alpha(6\alpha-1)}}{12\lambda^2\alpha^2}\,,$$

then the condition (1.5) is satisfied. If

$$k\leq\frac{6(1+\lambda)}{(3r-1)^2+3\lambda(1+3r)(1-r)}\,,$$

then the condition (1.6) holds. If

$$k\geq\frac1\alpha+\frac{-1+\sqrt{1+24\lambda^2\alpha(6\alpha-1)}}{12\lambda^2\alpha^2}\,,$$

then the condition (1.7) is fulfilled. If

$$k<\frac{6\lambda(\alpha+\beta)-1+\sqrt{[6\lambda(\alpha+\beta)-1]^2+144\lambda\alpha\beta(1-\lambda)}}{12\lambda\alpha\beta}\,,$$

then the condition (1.8) is satisfied.

In particular, if $k\in\,]0,3.02[\,\cup[3.65,4.52[\,$, $\lambda=0.95$, and $r=\frac{17}{24}$, then, according to Theorem 1.1, the problem (1.9) has a unique solution.

Theorem 1.2. *Let* $\lambda \in]0,1]$ *and*

$$B \overset{\text{def}}{=} \int_a^{\tau_1} [p(s)]_- ds + \lambda \int_{\tau_1}^b [p(s)]_- ds\,. \tag{1.11}$$

If

$$B < \lambda\,, \tag{1.12}$$

$$(\lambda - B)\left(1 + \int_{\tau_0}^{\tau_1} [p(s)]_+ ds\right) > 1 - \int_a^{\tau_0} [p(s)]_+ ds - \lambda \int_{\tau_1}^b [p(s)]_+ ds\,, \tag{1.13}$$

$$\left(\int_a^{\tau_0} [p(s)]_+ ds + \lambda \int_{\tau_0}^b [p(s)]_+ ds\right)\left(1 - \int_{\tau_0}^{\tau_1} [p(s)]_- ds\right) >$$
$$> 1 - \lambda + \int_a^{\tau_0} [p(s)]_- ds + \lambda \int_{\tau_0}^b [p(s)]_- ds\,, \tag{1.14}$$

and either

$$\int_a^{\tau_0} [p(s)]_+ ds + \lambda \int_{\tau_1}^b [p(s)]_+ ds < 1 + \sqrt{\lambda - B}\,, \tag{1.15}$$

$$\int_a^{\tau_1} [p(s)]_+ ds + \lambda \int_{\tau_1}^b [p(s)]_+ ds < 2 + 2\sqrt{\lambda - B} \tag{1.16}$$

or

$$\int_a^{\tau_0} [p(s)]_+ ds + \lambda \int_{\tau_1}^b [p(s)]_+ ds \geq 1 + \sqrt{\lambda - B}\,, \tag{1.17}$$

$$\int_{\tau_0}^{\tau_1} [p(s)]_+ ds < 1 + \frac{\lambda - B}{\int_a^{\tau_0} [p(s)]_+ ds + \lambda \int_{\tau_1}^b [p(s)]_+ ds - 1}\,, \tag{1.18}$$

then the problem (0.1), (0.2) *has a unique solution.*

Remark 1.2. Theorem 1.2 is nonimprovable in the sense that neither of the strict inequalities (1.13), (1.2), (1.16), and (1.18) can be replaced by the nonstrict one (see Examples 3.5–3.8).

Note also that if $\tau_0 = a$ and $\tau_1 = b$, then from Theorems 1.1 and 1.2 we obtain Corollary 1.1 in [5].

Example 1.2. On the segment $[0,1]$ consider the boundary value problem (1.9), where $\lambda \in \left]\frac{3}{4}, 1\right]$, $k \in]-6\lambda, 0[$, $q \in L([a,b];R)$, $c \in R$, and $r \in \left[\frac{2}{3}, 1\right]$ is such that $3\lambda^2(1+3r)(1-r) \geq 1$. Define the numbers α and β by (1.10).

It is not difficult to verify that $\tau_0 = \frac{1}{3}$, $\tau_1 = r$, and the condition (1.12) holds. Obviously, if

$$|k| > \frac{3(4\lambda - 1)}{4} - \sqrt{\left[\frac{3(4\lambda - 1)}{4}\right]^2 - 9(1 - \lambda)}$$

and

$$|k| < \frac{3(4\lambda - 1)}{4} + \sqrt{\left[\frac{3(4\lambda - 1)}{4}\right]^2 - 9(1 - \lambda)}\,,$$

then the conditions (1.13) and (1.2) are fulfilled. If

$$|k| < \frac{1}{\lambda\alpha} + \frac{-1 + \sqrt{1 + 24\lambda\alpha(6\lambda^2\alpha - 1)}}{12\lambda^2\alpha^2} \,,$$

then the condition (1.15) is satisfied. If

$$|k| \le \frac{12}{(3r-1)^2 + 3\lambda(1+3r)(1-r)} \,,$$

then the condition (1.16) holds. If

$$|k| \ge \frac{1}{\lambda\alpha} + \frac{-1 + \sqrt{1 + 24\lambda\alpha(6\lambda^2\alpha - 1)}}{12\lambda^2\alpha^2} \,,$$

then the condition (1.17) is fulfilled. If

$$|k| > \frac{6(\lambda\alpha + \beta) - 1 - \sqrt{[6(\lambda\alpha + \beta) - 1]^2 - 144\lambda\alpha\beta(1-\lambda)}}{12\lambda\alpha\beta}$$

and

$$|k| < \frac{6(\lambda\alpha + \beta) - 1 + \sqrt{[6(\lambda\alpha + \beta) - 1]^2 - 144\lambda\alpha\beta(1-\lambda)}}{12\lambda\alpha\beta} \,,$$

then the condition (1.18) is satisfied.

In particular, if $k \in [-4.08, -3.66] \cup]-3.10, -0.12[$, $\lambda = 0.95$, and $r = \frac{17}{24}$, then, according to Theorem 1.2, the problem (1.9) has a unique solution.

Remark 1.3. Let $\lambda \in [0,1]$. Denote by H_1 the set of 6−tuples $(x_i)_{i=1}^6 \in R_+^6$ satisfying

$$x_1 + x_2 + \lambda x_3 < 1 \,,$$
$$\big(x_4 + \lambda(x_5 + x_6)\big)(1 - x_2) > \lambda - 1 + x_1 + x_2 + \lambda x_3 \,,$$
$$(1 - x_1 - x_2 - \lambda x_3)(1 + x_5) > \lambda - x_4 - \lambda x_6 \,,$$

and either

$$x_4 + \lambda x_6 < \lambda + \sqrt{1 - x_1 - x_2 - \lambda x_3} \,,$$
$$x_4 + x_5 + \lambda x_6 < 1 + \lambda + 2\sqrt{1 - x_1 - x_2 - \lambda x_3}$$

or

$$x_4 + \lambda x_6 \ge \lambda + \sqrt{1 - x_1 - x_2 - \lambda x_3} \,,$$
$$(x_4 + \lambda x_6 - \lambda)(x_5 - 1) < 1 - x_1 - x_2 - \lambda x_3 \,.$$

Further, denote by H_2 the set of 6−tuples $(x_i)_{i=1}^6 \in R_+^6$ satisfying

$$x_4 + x_5 + \lambda x_6 < \lambda \,,$$
$$(\lambda - x_4 - x_5 - \lambda x_6)(1 + x_2) > 1 - x_1 - \lambda x_3 \,,$$
$$\big(x_1 + \lambda(x_2 + x_3)\big)(1 - x_5) > 1 - \lambda + x_4 + \lambda(x_5 + x_6) \,,$$

and either

$$x_1 + \lambda x_3 < 1 + \sqrt{\lambda - x_4 - x_5 - \lambda x_6}\,,$$
$$x_1 + x_2 + \lambda x_3 < 2 + 2\sqrt{\lambda - x_4 - x_5 - \lambda x_6}$$

or

$$x_1 + \lambda x_3 \geq 1 + \sqrt{\lambda - x_4 - x_5 - \lambda x_6}\,,$$
$$(x_1 + \lambda x_3 - 1)(x_2 - 1) < \lambda - x_4 - x_5 - \lambda x_6\,.$$

Now, according to Theorems 1.1 and 1.2, if $p \in L([a,b];R)$ is such that the point

$$\left(\int\limits_a^{\tau_0}[p(s)]_+ds, \int\limits_{\tau_0}^{\tau_1}[p(s)]_+ds, \int\limits_{\tau_1}^{b}[p(s)]_+ds, \int\limits_a^{\tau_0}[p(s)]_-ds, \int\limits_{\tau_0}^{\tau_1}[p(s)]_-ds, \int\limits_{\tau_1}^{b}[p(s)]_-ds\right)$$

belongs to the set $H_1 \cup H_2$, then the problem (0.1), (0.2) has a unique solution.

On Fig. 1.1 one can see the intersection of the set $H_1 \cup H_2$ and the 3−dimensional subspace $\{(0,x,0,y,z,0) : x,y,z \in R\}$ of the space R^6, i.e., the sets

$$G_1 = \{(x,y,z) \in R^3_+ :\ (0,x,0,y,z,0) \in H_1\},$$
$$G_2 = \{(x,y,z) \in R^3_+ :\ (0,x,0,y,z,0) \in H_2\}.$$

Theorem 1.3. *Let* $\lambda \in [0,1]$,

$$\int_{\tau_0}^{\tau_1}[p(s)]_+ds < 1\,, \qquad \int_{\tau_0}^{\tau_1}[p(s)]_-ds < 1\,, \tag{1.19}$$

and either

$$\int_a^{\tau_1}[p(s)]_+ds + \lambda\int_{\tau_1}^{b}[p(s)]_+ds - \left(\int_a^{\tau_0}[p(s)]_-ds + \lambda\int_{\tau_0}^{b}[p(s)]_-ds\right)\big(1-T\big) < 1-\lambda \tag{1.20}$$

or

$$\left(\int_a^{\tau_0}[p(s)]_+ds + \lambda\int_{\tau_0}^{b}[p(s)]_+ds\right)\big(1-T\big) - \int_a^{\tau_1}[p(s)]_-ds - \lambda\int_{\tau_1}^{b}[p(s)]_-ds > \big(1-\lambda\big)\big(1-T\big)\,, \tag{1.21}$$

where

$$T = \max\left\{\int_{\tau_0}^{\tau_1}[p(s)]_+ds,\ \int_{\tau_0}^{\tau_1}[p(s)]_-ds\right\}\,. \tag{1.22}$$

Then the problem (0.1), (0.2) *has a unique solution.*

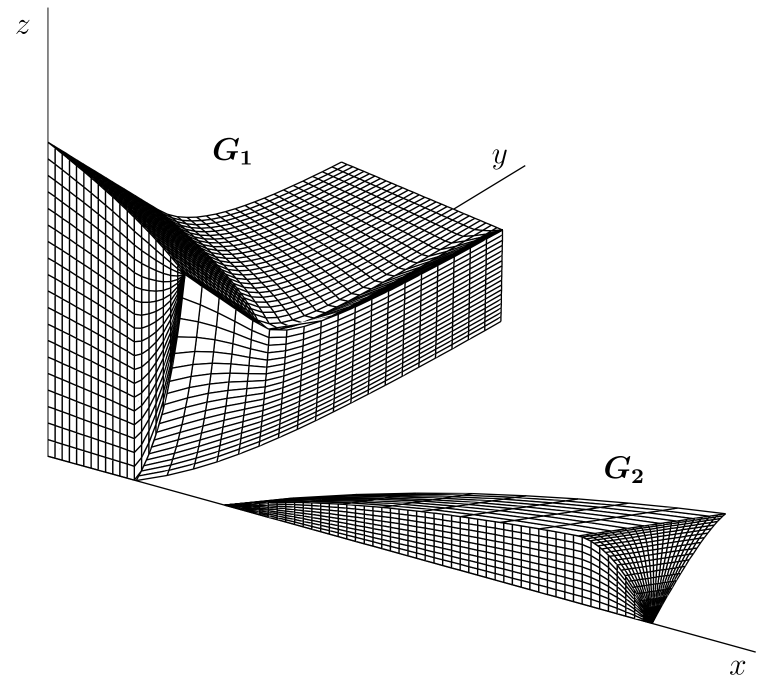

Fig. 1.1.

Remark 1.4. Theorem 1.3 is nonimprovable in the sense that the strict inequalities (1.20) and (1.21) cannot be replaced by the nonstrict ones (see Examples 3.9 and 3.10).

Note also that if the segment $[\tau_0, \tau_1]$ degenerates to a point $c \in [a, b]$, i.e., $\tau(t) = c$ for $t \in [a, b]$, then $T = 0$ and the inequalities (1.20) and (1.21) can be rewritten as

$$\int_a^c p(s)ds + \lambda \int_c^b p(s)ds \neq 1 - \lambda\,.$$

On the other hand, the last relation is sufficient and necessary for the unique solvability of the problem (0.1), (0.2) with $\tau(t) = c$ for $t \in [a, b]$.

Theorem 1.4. *Let $\lambda \in [0, 1[$ and let there exist $\gamma \in \widetilde{C}([a, b];]0, +\infty[)$ such that*

$$\gamma'(t) \geq [p(t)]_+ \gamma(\tau(t)) + [p(t)]_- \quad \text{for} \quad t \in [a, b]\,, \tag{1.23}$$

$$\gamma(a) > \lambda \gamma(b)\,, \tag{1.24}$$

and either

$$\gamma(\tau_0) - \gamma(a) + \lambda\big(\gamma(b) - \gamma(\tau_1)\big) < 1 + \lambda\,, \tag{1.25}$$

$$\lambda\big(\gamma(b) - \gamma(\tau_1)\big) + \gamma(\tau_1) - \gamma(a) < 3 + \lambda \tag{1.26}$$

or

$$\gamma(\tau_0)-\gamma(a)+\lambda\big(\gamma(b)-\gamma(\tau_1)\big)\geq 1+\lambda\,, \tag{1.27}$$

$$\gamma(\tau_1)-\gamma(\tau_0)<1+\frac{1}{\gamma(\tau_0)-\gamma(a)+\lambda\big(\gamma(b)-\gamma(\tau_1)\big)-\lambda}\,. \tag{1.28}$$

Then the problem (0.1), (0.2) *has a unique solution.*

Remark 1.5. Theorem 1.4 is nonimprovable in the sense that the strict inequalities (1.26) and (1.28) cannot be replaced by the nonstrict ones (see Examples 3.3 and 3.4).

Note also that if $\tau_0 = a$ and $\tau_1 = b$, then from Theorem 1.4 we obtain Theorem 1.1 in [8].

Remark 1.6. Let $\lambda \in [1,+\infty[$ and let functions $\widetilde{p}$, $\widetilde{\tau}$, and $\widetilde{q}$ be defined by

$$\begin{gathered}\widetilde{p}(t)\stackrel{\rm def}{=}-p(a+b-t), \qquad \widetilde{\tau}(t)\stackrel{\rm def}{=}a+b-\tau(a+b-t),\\ \widetilde{q}(t)\stackrel{\rm def}{=}-q(a+b-t)\quad\text{for}\quad t\in[a,b].\end{gathered} \tag{1.29}$$

Put $\mu = 1/\lambda$ and $\widetilde{c} = -\mu c$. It is clear that if u is a solution of the problem (0.1), (0.2), then the function v, defined by $v(t) \stackrel{\rm def}{=} u(a+b-t)$ for $t \in [a,b]$, is a solution of the problem

$$v'(t)=\widetilde{p}(t)v(\widetilde{\tau}(t))+\widetilde{q}(t), \qquad v(a)-\mu v(b)=\widetilde{c}\,, \tag{1.30}$$

and vice versa, if v is a solution of the problem (1.30), then the function u, defined by $u(t) \stackrel{\rm def}{=} v(a+b-t)$ for $t \in [a,b]$, is a solution of the problem (0.1), (0.2).

Therefore, Theorems 1.1–1.4 immediately yield the following assertions.

Theorem 1.5. *Let* $\lambda \in [1,+\infty[$ *and*

$$\widetilde{B}\stackrel{\rm def}{=}\frac{1}{\lambda}\int_a^{\tau_0}[p(s)]_-ds+\int_{\tau_0}^b[p(s)]_-ds\,.$$

If

$$\widetilde{B}<1\,,$$

$$\left(\frac{1}{\lambda}\int_a^{\tau_1}[p(s)]_+ds+\int_{\tau_1}^b[p(s)]_+ds\right)\left(1-\int_{\tau_0}^{\tau_1}[p(s)]_-ds\right)>\frac{1}{\lambda}-1+\widetilde{B}\,,$$

$$\left(1-\widetilde{B}\right)\left(1+\int_{\tau_0}^{\tau_1}[p(s)]_+ds\right)>\frac{1}{\lambda}-\frac{1}{\lambda}\int_a^{\tau_0}[p(s)]_+ds-\int_{\tau_1}^b[p(s)]_+ds\,,$$

and either

$$\frac{1}{\lambda}\int_a^{\tau_0}[p(s)]_+ds+\int_{\tau_1}^b[p(s)]_+ds<\frac{1}{\lambda}+\sqrt{1-\widetilde{B}}\,,$$

$$\frac{1}{\lambda}\int_a^{\tau_0}[p(s)]_+ds+\int_{\tau_0}^b[p(s)]_+ds<1+\frac{1}{\lambda}+2\sqrt{1-\widetilde{B}}$$

or

$$\frac{1}{\lambda}\int_a^{\tau_0}[p(s)]_+ds+\int_{\tau_1}^b[p(s)]_+ds\ge\frac{1}{\lambda}+\sqrt{1-\widetilde{B}}\,,$$

$$\int_{\tau_0}^{\tau_1}[p(s)]_+ds<1+\frac{1-\widetilde{B}}{\frac{1}{\lambda}\int_a^{\tau_0}[p(s)]_+ds+\int_{\tau_1}^b[p(s)]_+ds-\frac{1}{\lambda}}\,,$$

then the problem (0.1), (0.2) *has a unique solution.*

Theorem 1.6. *Let* $\lambda\in[1,+\infty[$ *and*

$$\widetilde{A}\overset{\text{def}}{=}\frac{1}{\lambda}\int_a^{\tau_0}[p(s)]_+ds+\int_{\tau_0}^b[p(s)]_+ds\,.$$

If

$$\widetilde{A}<\frac{1}{\lambda}\,,$$

$$\left(\frac{1}{\lambda}-\widetilde{A}\right)\left(1+\int_{\tau_0}^{\tau_1}[p(s)]_-ds\right)>1-\frac{1}{\lambda}\int_a^{\tau_0}[p(s)]_-ds-\int_{\tau_1}^b[p(s)]_-ds,$$

$$\left(\frac{1}{\lambda}\int_a^{\tau_1}[p(s)]_-ds+\int_{\tau_1}^b[p(s)]_-ds\right)\left(1-\int_{\tau_0}^{\tau_1}[p(s)]_+ds\right)>$$

$$>1-\frac{1}{\lambda}+\frac{1}{\lambda}\int_a^{\tau_1}[p(s)]_+ds+\int_{\tau_1}^b[p(s)]_+ds\,,$$

and either

$$\frac{1}{\lambda}\int_a^{\tau_0}[p(s)]_-ds+\int_{\tau_1}^b[p(s)]_-ds<1+\sqrt{\frac{1}{\lambda}-\widetilde{A}}\,,$$

$$\frac{1}{\lambda}\int_a^{\tau_0}[p(s)]_-ds+\int_{\tau_0}^b[p(s)]_-ds<2+2\sqrt{\frac{1}{\lambda}-\widetilde{A}}$$

or

$$\frac{1}{\lambda}\int_a^{\tau_0}[p(s)]_-ds+\int_{\tau_1}^b[p(s)]_-ds\ge1+\sqrt{\frac{1}{\lambda}-\widetilde{A}}\,,$$

$$\int_{\tau_0}^{\tau_1}[p(s)]_-ds<1+\frac{\frac{1}{\lambda}-\widetilde{A}}{\frac{1}{\lambda}\int_a^{\tau_0}[p(s)]_-ds+\int_{\tau_1}^b[p(s)]_-ds-1}\,,$$

then the problem (0.1), (0.2) *has a unique solution.*

Theorem 1.7. *Let* $\lambda\in[1,+\infty[$, *the condition* (1.19) *be fulfilled, and let either*

$$\frac{1}{\lambda}\int_a^{\tau_0}[p(s)]_-ds+\int_{\tau_0}^b[p(s)]_-ds-$$

$$-\left(\frac{1}{\lambda}\int_a^{\tau_1}[p(s)]_+ds+\int_{\tau_1}^b[p(s)]_+ds\right)\big(1-T\big)<1-\frac{1}{\lambda}$$

or

$$\left(\frac{1}{\lambda}\int_a^{\tau_1}[p(s)]_-ds+\int_{\tau_1}^b[p(s)]_-ds\right)\left(1-T\right)-$$
$$-\frac{1}{\lambda}\int_a^{\tau_0}[p(s)]_+ds-\int_{\tau_0}^b[p(s)]_+ds>\left(1-\frac{1}{\lambda}\right)\left(1-T\right),$$

where T is defined by (1.22). *Then the problem* (0.1), (0.2) *has a unique solution.*

Theorem 1.8. *Let $\lambda\in\,]1,+\infty[$ and let there exist $\gamma\in\widetilde{C}([a,b];]0,+\infty[)$ such that*

$$-\gamma'(t)\ge[p(t)]_-\gamma(\tau(t))+[p(t)]_+\quad for\quad t\in[a,b],$$
$$\gamma(a)<\lambda\gamma(b),$$

and either

$$\gamma(a)-\gamma(\tau_0)+\lambda\big(\gamma(\tau_1)-\gamma(b)\big)<1+\lambda,$$
$$\frac{1}{\lambda}\big(\gamma(a)-\gamma(\tau_0)\big)+\gamma(\tau_0)-\gamma(b)<3+\frac{1}{\lambda}$$

or

$$\gamma(a)-\gamma(\tau_0)+\lambda\big(\gamma(\tau_1)-\gamma(b)\big)\ge1+\lambda,$$
$$\gamma(\tau_0)-\gamma(\tau_1)<1+\frac{\lambda}{\gamma(a)-\gamma(\tau_0)+\lambda\big(\gamma(\tau_1)-\gamma(b)\big)-1}.$$

Then the problem (0.1), (0.2) *has a unique solution.*

Remark 1.7. According to Remarks 1.1–1.6 , Theorems 1.5–1.8 are nonimprovable in an appropriate sense.

1.2. **Proofs.** According to Theorem 0.1, to prove Theorems 1.1–1.4 it is sufficient to show that the homogeneous problem (0.1_0), (0.2_0) has only the trivial solution. First introduce the following notation

$$\begin{aligned}A_1&=\int_a^{\tau_0}[p(s)]_+ds, & A_2&=\int_{\tau_0}^{\tau_1}[p(s)]_+ds, & A_3&=\int_{\tau_1}^{b}[p(s)]_+ds,\\ B_1&=\int_a^{\tau_0}[p(s)]_-ds, & B_2&=\int_{\tau_0}^{\tau_1}[p(s)]_-ds, & B_3&=\int_{\tau_1}^{b}[p(s)]_-ds.\end{aligned}\tag{1.31}$$

Proof of Theorem 1.1. Assume that the problem (0.1_0), (0.2_0) has a nontrivial solution u.

First suppose that u does not change its sign in $[\tau_0,\tau_1]$. Without loss of generality we can assume that

$$u(t)\ge0\quad\text{for}\quad t\in[\tau_0,\tau_1].\tag{1.32}$$

Put

$$M=\max\{u(t):t\in[\tau_0,\tau_1]\},\qquad m=\min\{u(t):t\in[\tau_0,\tau_1]\},\tag{1.33}$$

and choose $t_M, t_m \in [\tau_0, \tau_1]$ such that

$$u(t_M) = M, \qquad u(t_m) = m. \tag{1.34}$$

Furthermore, let

$$\alpha_0 = \min\{t_M, t_m\}, \qquad \alpha_1 = \max\{t_M, t_m\}, \tag{1.35}$$

$$\begin{aligned} A_{21} &= \int_{\tau_0}^{\alpha_0} [p(s)]_+ ds, \quad A_{22} = \int_{\alpha_0}^{\alpha_1} [p(s)]_+ ds, \quad A_{23} = \int_{\alpha_1}^{\tau_1} [p(s)]_+ ds, \\ B_{21} &= \int_{\tau_0}^{\alpha_0} [p(s)]_- ds, \quad B_{22} = \int_{\alpha_0}^{\alpha_1} [p(s)]_- ds, \quad B_{23} = \int_{\alpha_1}^{\tau_1} [p(s)]_- ds. \end{aligned} \tag{1.36}$$

It is clear that

$$m \geq 0, \qquad M > 0, \tag{1.37}$$

since if $M = 0$, then, in view of (0.1_0), (1.32), and (1.33), we obtain $u(\tau_0) = 0$ and $u'(t) = 0$ for $t \in [a, b]$, i.e., $u \equiv 0$. Obviously, either

$$t_M < t_m \tag{1.38}$$

or

$$t_M \geq t_m. \tag{1.39}$$

First suppose that (1.38) holds. The integrations of (0.1_0) from a to t_M, from t_M to t_m, from t_m to τ_1, and from τ_1 to b, on account of (1.31), (1.33)–(1.36), and the assumption $\lambda \in [0, 1]$, result in

$$\begin{aligned} M - u(a) = \int_a^{t_M} [p(s)]_+ u(\tau(s)) ds - \int_a^{t_M} [p(s)]_- u(\tau(s)) ds \leq \\ \leq M\big(A_1 + A_{21}\big) - m\big(B_1 + B_{21}\big), \end{aligned} \tag{1.40}$$

$$\begin{aligned} m - M = \int_{t_M}^{t_m} [p(s)]_+ u(\tau(s)) ds - \int_{t_M}^{t_m} [p(s)]_- u(\tau(s)) ds \leq \\ \leq MA_{22} - mB_{22}, \end{aligned} \tag{1.41}$$

$$\begin{aligned} \lambda\big(u(\tau_1) - m\big) \leq u(\tau_1) - m = \int_{t_m}^{\tau_1} [p(s)]_+ u(\tau(s)) ds - \\ - \int_{t_m}^{\tau_1} [p(s)]_- u(\tau(s)) ds \leq MA_{23} - mB_{23}, \end{aligned} \tag{1.42}$$

$$\begin{aligned} u(b) - u(\tau_1) = \int_{\tau_1}^{b} [p(s)]_+ u(\tau(s)) ds - \int_{\tau_1}^{b} [p(s)]_- u(\tau(s)) ds \leq \\ \leq MA_3 - mB_3. \end{aligned} \tag{1.43}$$

Multiplying both sides of (1.43) by λ, summing with (1.40) and (1.42), and taking into account (0.2_0) and the assumption $\lambda \in [0, 1]$, we get

$$M - \lambda m \leq M\big(A_1 + A_{21} + A_{23} + \lambda A_3\big) - m\big(B_1 + B_{21} + B_{23} + \lambda B_3\big).$$

Hence, by virtue of (1.1), (1.2), (1.31), (1.36), and (1.37), the last inequality implies

$$0 < M\big(1 - A_1 - A_{21} - A_{23} - \lambda A_3\big) \leq \\ \leq m\big(\lambda - B_1 - B_{21} - B_{23} - \lambda B_3\big). \tag{1.44}$$

On the other hand, by (1.36) and (1.37), (1.41) results in

$$0 \leq m\big(1 + B_{22}\big) \leq M\big(1 + A_{22}\big). \tag{1.45}$$

Thus, it follows from (1.44) and (1.45) that

$$\big(1 - A_1 - A_{21} - A_{23} - \lambda A_3\big)\big(1 + B_{22}\big) \leq \\ \leq \big(\lambda - B_1 - B_{21} - B_{23} - \lambda B_3\big)\big(1 + A_{22}\big). \tag{1.46}$$

Obviously, on account of (1.1), (1.31), (1.36), and the assumption $\lambda \in [0, 1]$, we find

$$\big(1 - A_1 - A_{21} - A_{23} - \lambda A_{23}\big)\big(1 + B_{22}\big) = \\ = \big(1 - A_1 - A_2 - \lambda A_3\big)\big(1 + B_2\big) - \big(1 - A_1 - A_2 - \lambda A_3\big)\big(B_{21} + B_{23}\big) + \\ + A_{22}\big(1 + B_{22}\big) \geq \big(1 - A\big)\big(1 + B_2\big) - \big(B_{21} + B_{23}\big) + A_{22}$$

and

$$\big(\lambda - B_1 - B_{21} - B_{23} - \lambda B_3\big)\big(1 + A_{22}\big) = \\ = \lambda + \lambda A_{22} - \big(B_1 + \lambda B_3\big)\big(1 + A_{22}\big) - \big(B_{21} + B_{23}\big)\big(1 + A_{22}\big) \leq \\ \leq \lambda - B_1 - \lambda B_3 + A_{22} - \big(B_{21} + B_{23}\big).$$

By virtue of the last two inequalities, (1.46) yields

$$\big(1 - A\big)\big(1 + B_2\big) \leq \lambda - B_1 - \lambda B_3,$$

which, in view of (1.31), contradicts (1.4).

Now suppose that (1.39) is fulfilled. The integrations of (0.1_0) from a to t_m, from t_m to t_M, and from t_M to b, on account of (1.31) and (1.33)–(1.36), result in

$$m - u(a) = \int_a^{t_m} [p(s)]_+ u(\tau(s))ds - \int_a^{t_m} [p(s)]_- u(\tau(s))ds \leq \\ \leq M\big(A_1 + A_{21}\big) - m\big(B_1 + B_{21}\big), \tag{1.47}$$

$$M - m = \int_{t_m}^{t_M} [p(s)]_+ u(\tau(s))ds - \int_{t_m}^{t_M} [p(s)]_- u(\tau(s))ds \leq \\ \leq MA_{22} - mB_{22}, \tag{1.48}$$

$$u(b) - M = \int_{t_M}^{b} [p(s)]_+ u(\tau(s))ds - \int_{t_M}^{b} [p(s)]_- u(\tau(s))ds \leq \\ \leq M\big(A_{23} + A_3\big) - m\big(B_{23} + B_3\big). \tag{1.49}$$

Multiplying both sides of (1.49) by λ, summing with (1.47), and taking into account (0.2_0) and the assumption $\lambda \in [0,1]$, we get

$$m-\lambda M \le M\big(A_1+A_{21}+\lambda A_{23}+\lambda A_3\big)-m\big(B_1+B_{21}+\lambda B_{23}+\lambda B_3\big). \quad (1.50)$$

Hence, by virtue of (1.1), (1.2), (1.31), (1.36), and (1.37), it follows from (1.48) and (1.50) that

$$0 < M\big(1-A_{22}\big) \le m\big(1-B_{22}\big),$$
$$0 \le m\big(1+B_1+B_{21}+\lambda B_{23}+\lambda B_3\big) \le M\big(\lambda+A_1+A_{21}+\lambda A_{23}+\lambda A_3\big).$$

Thus,

$$\big(1+B_1+B_{21}+\lambda B_{23}+\lambda B_3\big)\big(1-A_{22}\big) \le$$
$$\le \big(\lambda+A_1+A_{21}+\lambda A_{23}+\lambda A_3\big)\big(1-B_{22}\big). \quad (1.51)$$

Obviously, in view of (1.1), (1.2), (1.31), (1.36), and the assumption $\lambda \in [0,1]$, we obtain

$$\big(1+B_1+B_{21}+\lambda B_{23}+\lambda B_3\big)\big(1-A_{22}\big) = 1-A_{22}+$$
$$+\big(B_1+B_{21}+\lambda B_{22}+\lambda B_{23}+\lambda B_3\big)\big(1-A_{22}\big)-\lambda B_{22}\big(1-A_{22}\big) \ge$$
$$\ge 1-A_{22}+\big(B_1+\lambda B_2+\lambda B_3\big)\big(1-A_2\big)-\lambda B_{22}$$

and

$$\big(\lambda+A_1+A_{21}+\lambda A_{23}+\lambda A_3\big)\big(1-B_{22}\big) =$$
$$= \lambda-\lambda B_{22}+\big(A_1+A_{21}+\lambda A_{23}+\lambda A_3\big)\big(1-B_{22}\big) \le$$
$$\le \lambda-\lambda B_{22}+A_1+A_{21}+A_{22}+\lambda A_{23}+\lambda A_3-A_{22} \le$$
$$\le \lambda-\lambda B_{22}+A-A_{22}.$$

By virtue of the last two inequalities, (1.51) implies

$$\big(B_1+\lambda B_2+\lambda B_3\big)\big(1-A_2\big) \le \lambda-1+A,$$

which, in view of (1.31), contradicts (1.3).

Now suppose that u changes its sign in $[\tau_0,\tau_1]$. Put

$$m_0 = -\min\{u(t) : t \in [\tau_0,\tau_1]\}, \qquad M_0 = \max\{u(t) : t \in [\tau_0,\tau_1]\} \quad (1.52)$$

and choose $\alpha_0, \alpha_1 \in [\tau_0,\tau_1]$ such that

$$u(\alpha_0) = -m_0, \qquad u(\alpha_1) = M_0. \quad (1.53)$$

It is clear that

$$M_0 > 0, \qquad m_0 > 0, \quad (1.54)$$

and without loss of generality we can assume that $\alpha_0 < \alpha_1$. Furthermore, define numbers A_{2i}, B_{2i} $(i = 1,2,3)$ by (1.36) and put

$$g(x) \overset{\text{def}}{=} \frac{1-A}{x+B_1+\lambda B_3-\lambda}+x \quad \text{for} \quad x > \lambda-B_1-\lambda B_3, \quad (1.55)$$

where A is given by (1.1).

The integrations of (0.1_0) from a to α_0, from α_0 to α_1, and from α_1 to b, in view of (1.31), (1.36), (1.52), and (1.53), result in

$$u(a)+m_0=\int_a^{\alpha_0}[p(s)]_-u(\tau(s))ds-\int_a^{\alpha_0}[p(s)]_+u(\tau(s))ds\le$$
$$\le M_0\big(B_1+B_{21}\big)+m_0\big(A_1+A_{21}\big), \tag{1.56}$$
$$M_0+m_0=\int_{\alpha_0}^{\alpha_1}[p(s)]_+u(\tau(s))ds-\int_{\alpha_0}^{\alpha_1}[p(s)]_-u(\tau(s))ds\le$$
$$\le M_0A_{22}+m_0B_{22}, \tag{1.57}$$
$$M_0-u(b)=\int_{\alpha_1}^{b}[p(s)]_-u(\tau(s))ds-\int_{\alpha_1}^{b}[p(s)]_+u(\tau(s))ds\le$$
$$\le M_0\big(B_{23}+B_3\big)+m_0\big(A_{23}+A_3\big). \tag{1.58}$$

Multiplying both sides of (1.58) by λ, summing with (1.56), and taking into account (0.2_0), (1.54), and the assumption $\lambda\in[0,1]$, we get

$$\lambda M_0+m_0\le M_0\big(B_1+B_{21}+B_{23}+\lambda B_3\big)+m_0\big(A_1+A_{21}+A_{23}+\lambda A_3\big). \tag{1.59}$$

Due to (1.1), (1.2), (1.31), and (1.36), we have

$$A_1+A_{21}+A_{23}+\lambda A_3<1, \qquad A_{22}<1.$$

Thus, it follows from (1.54), (1.57), and (1.59) that

$$B_{22}>1, \qquad B_1+B_{21}+B_{23}+\lambda B_3>\lambda, \tag{1.60}$$

and

$$B_{22}\ge 1+\frac{M_0}{m_0}\big(1-A_{22}\big)\,, \tag{1.61}$$
$$\frac{M_0}{m_0}\ge\frac{1-A_1-A_{21}-A_{23}-\lambda A_3}{B_1+B_{21}+B_{23}+\lambda B_3-\lambda}\,. \tag{1.62}$$

According to (1.60) and the fact that

$$\big(1-A_{22}\big)\big(1-A_1-A_{21}-A_{23}-\lambda A_3\big)\ge 1-A_1-A_{21}-A_{22}-A_{23}-\lambda A_3=1-A,$$

from (1.61) and (1.62) we get

$$B_{22}\ge 1+\frac{1-A}{B_1+B_{21}+B_{23}+\lambda B_3-\lambda}\,. \tag{1.63}$$

First suppose that (1.5) and (1.6) are satisfied. By virtue of (1.60), from (1.63) we have

$$1-A\le\big(B_{22}-1\big)\big(B_1+B_{21}+B_{23}+\lambda B_3-\lambda\big)\le$$
$$\le\frac{1}{4}\big(B_1+B_{21}+B_{22}+B_{23}+\lambda B_3-1-\lambda\big)^2=\frac{1}{4}\big(B_1+B_2+\lambda B_3-1-\lambda\big)^2,$$

which, in view of (1.2), (1.31), (1.36), and (1.60), contradicts (1.6).

Now suppose that (1.7) and (1.8) are fulfilled. It is not difficult to verify that, on account of (1.7) and (1.31), the function g defined by (1.55) is nondecreasing in $[0,+\infty[$. Therefore, from (1.63) we obtain

$$B_{21}+B_{22}+B_{23} \geq 1+\frac{1-A}{B_1+B_{21}+B_{23}+\lambda B_3-\lambda}+B_{21}+B_{23}=$$
$$=1+g\big(B_{21}+B_{23}\big) \geq 1+g(0)=1+\frac{1-A}{B_1+\lambda B_3-\lambda},$$

which, in view of (1.31) and (1.36), contradicts (1.8). □

Proof of Theorem 1.2. Assume that the problem (0.1_0), (0.2_0) has a nontrivial solution u.

First suppose that u does not change its sign in $[\tau_0,\tau_1]$. Without loss of generality we can assume that (1.32) is fulfilled. Define the numbers M and m by (1.33) and choose $t_M,t_m \in [\tau_0,\tau_1]$ such that (1.34) holds. Furthermore, define the numbers α_0, α_1 and A_{2i},B_{2i} $(i=1,2,3)$ by (1.35) and (1.36), respectively. It is clear that (1.37) is satisfied, since if $M=0$, then, in view of (0.1_0), (1.32), and (1.33), we obtain $u(\tau_0)=0$ and $u'(t)=0$ for $t \in [a,b]$, i.e., $u \equiv 0$. It is also evident that either (1.38) or (1.39) is fulfilled.

First suppose that (1.38) holds. The integrations of (0.1_0) from t_M to t_m, from a to τ_0, and from τ_0 to b, in view of (1.31)–(1.34), result in

$$M-m=\int_{t_M}^{t_m}[p(s)]_-u(\tau(s))ds-\int_{t_M}^{t_m}[p(s)]_+u(\tau(s))ds \leq MB_2, \tag{1.64}$$

$$u(a)-u(\tau_0)=\int_a^{\tau_0}[p(s)]_-u(\tau(s))ds-\int_a^{\tau_0}[p(s)]_+u(\tau(s))ds \leq$$
$$\leq MB_1-mA_1, \tag{1.65}$$

$$u(\tau_0)-u(b)=\int_{\tau_0}^{b}[p(s)]_-u(\tau(s))ds-\int_{\tau_0}^{b}[p(s)]_+u(\tau(s))ds \leq$$
$$\leq M\big(B_2+B_3\big)-m\big(A_2+A_3\big). \tag{1.66}$$

Multiplying both sides of (1.66) by λ, summing with (1.65), and taking into account (0.2_0), (1.33), (1.37), and the assumption $\lambda \in\,]0,1]$, we get

$$M(\lambda-1) \leq u(\tau_0)(\lambda-1) \leq M\big(B_1+\lambda B_2+\lambda B_3\big)-m\big(A_1+\lambda A_2+\lambda A_3\big),$$

i.e.,

$$0 \leq m\big(A_1+\lambda A_2+\lambda A_3\big) \leq M\big(1-\lambda+B_1+\lambda B_2+\lambda B_3\big). \tag{1.67}$$

On the other hand, due to (1.11), (1.12), (1.31), (1.37), and the assumption $\lambda \in\,]0,1]$, (1.64) yields

$$0<M\big(1-B_2\big) \leq m. \tag{1.68}$$

Thus, it follows from (1.67) and (1.68) that

$$\big(A_1+\lambda A_2+\lambda A_3\big)\big(1-B_2\big) \leq 1-\lambda+B_1+\lambda B_2+\lambda B_3,$$

which, on account of (1.31), contradicts (1.2).

Now suppose that (1.39) is fulfilled. The integrations of (0.1_0) from a to t_m, from t_m to t_M, and from t_M to b, on account of (1.31) and (1.33)–(1.36), yield

$$u(a) - m = \int_a^{t_m} [p(s)]_- u(\tau(s))ds - \int_a^{t_m} [p(s)]_+ u(\tau(s))ds \leq \\ \leq M\big(B_1 + B_{21}\big) - m\big(A_1 + A_{21}\big), \tag{1.69}$$

$$m - M = \int_{t_m}^{t_M} [p(s)]_- u(\tau(s))ds - \int_{t_m}^{t_M} [p(s)]_+ u(\tau(s))ds \leq \\ \leq MB_{22} - mA_{22}, \tag{1.70}$$

$$M - u(b) = \int_{t_M}^{b} [p(s)]_- u(\tau(s))ds - \int_{t_M}^{b} [p(s)]_+ u(\tau(s))ds \leq \\ \leq M\big(B_{23} + B_3\big) - m\big(A_{23} + A_3\big). \tag{1.71}$$

Multiplying both sides of (1.71) by λ, summing with (1.69), and taking into account (0.2_0) and the assumption $\lambda \in\,]0, 1]$, we get

$$\lambda M - m \leq M\big(B_1 + B_{21} + \lambda B_{23} + \lambda B_3\big) - m\big(A_1 + A_{21} + \lambda A_{23} + \lambda A_3\big).$$

Hence, by virtue of (1.11), (1.12), (1.31), (1.36), (1.37), and the assumption $\lambda \in\,]0, 1]$, the last inequality results in

$$0 < M\big(\lambda - B_1 - B_{21} - \lambda B_{23} - \lambda B_3\big) \leq m\big(1 - A_1 - A_{21} - \lambda A_{23} - \lambda A_3\big). \tag{1.72}$$

On the other hand, due to (1.36) and (1.37), (1.70) implies

$$0 \leq m\big(1 + A_{22}\big) \leq M\big(1 + B_{22}\big). \tag{1.73}$$

Thus, it follows from (1.72) and (1.73) that

$$\big(\lambda - B_1 - B_{21} - \lambda B_{23} - \lambda B_3\big)\big(1 + A_{22}\big) \leq \\ \leq \big(1 - A_1 - A_{21} - \lambda A_{23} - \lambda A_3\big)\big(1 + B_{22}\big). \tag{1.74}$$

Obviously, on account of (1.11), (1.31), (1.36), and the assumption $\lambda \in\,]0, 1]$, we obtain

$$\big(\lambda - B_1 - B_{21} - \lambda B_{23} - \lambda B_3\big)\big(1 + A_{22}\big) = \\ = \big(\lambda - B_1 - B_{21} - B_{22} - \lambda B_{23} - \lambda B_3\big)\big(1 + A_2\big) + B_{22}\big(1 + A_2\big) - \\ - \big(\lambda - B_1 - B_{21} - \lambda B_{23} - \lambda B_3\big)\big(A_{21} + A_{23}\big) \geq \\ \geq \big(\lambda - B\big)\big(1 + A_2\big) + B_{22} - \lambda\big(A_{21} + A_{23}\big)$$

and

$$\big(1 - A_1 - A_{21} - \lambda A_{23} - \lambda A_3\big)\big(1 + B_{22}\big) = \\ = 1 - A_1 - \lambda A_3 - \big(A_{21} + \lambda A_{23}\big) + \big(1 - A_1 - A_{21} - \lambda A_{23} - \lambda A_3\big)B_{22} \leq \\ \leq 1 - A_1 - \lambda A_3 - \lambda\big(A_{21} + A_{23}\big) + B_{22}.$$

By virtue of the last two inequalities, (1.74) yields

$$\big(\lambda - B\big)\big(1 + A_2\big) \leq 1 - A_1 - \lambda A_3,$$

which, in view of (1.31), contradicts (1.13).

Now suppose that u changes its sign in $[\tau_0, \tau_1]$. Define the numbers m_0 and M_0 by (1.52) and choose $\alpha_0, \alpha_1 \in [\tau_0, \tau_1]$ such that (1.53) holds. It is clear that (1.54) is satisfied and without loss of generality we can assume that $\alpha_0 < \alpha_1$. Moreover, define the numbers A_{2i}, B_{2i} $(i = 1, 2, 3)$ by (1.36) and put

$$g(x) \stackrel{\text{def}}{=} \frac{\lambda - B}{x + A_1 + \lambda A_3 - 1} + x \quad \text{for} \quad x > 1 - A_1 - \lambda A_3, \tag{1.75}$$

where B is given by (1.11).

In a similar manner as in the second part of the proof of Theorem 1.1, it can be shown that the inequalities (1.57) and (1.59) hold. Due to (1.11), (1.12), (1.31), (1.36), and the assumption $\lambda \in \,]0, 1]$, we have

$$B_1 + B_{21} + B_{23} + \lambda B_3 < \lambda, \qquad B_{22} < 1.$$

Thus, by virtue of (1.54), it follows from (1.57) and (1.59) that

$$A_{22} > 1, \qquad A_1 + A_{21} + A_{23} + \lambda A_3 > 1, \tag{1.76}$$

and

$$A_{22} \geq 1 + \frac{m_0}{M_0}\big(1 - B_{22}\big)\,, \tag{1.77}$$

$$\frac{m_0}{M_0} \geq \frac{\lambda - B_1 - B_{21} - B_{23} - \lambda B_3}{A_1 + A_{21} + A_{23} + \lambda A_3 - 1}\,. \tag{1.78}$$

According to (1.76), the assumption $\lambda \in \,]0, 1]$, and the fact that

$$\big(1 - B_{22}\big)\big(\lambda - B_1 - B_{21} - B_{23} - \lambda B_3\big) \geq \lambda - B_1 - B_{21} - \lambda B_{22} - B_{23} - \lambda B_3 \geq \lambda - B,$$

from (1.77) and (1.78) we get

$$A_{22} \geq 1 + \frac{\lambda - B}{A_1 + A_{21} + A_{23} + \lambda A_3 - 1}\,. \tag{1.79}$$

First suppose that (1.15) and (1.16) are satisfied. By virtue of (1.76), from (1.79) we have

$$\lambda - B \leq \big(A_{22} - 1\big)\big(A_1 + A_{21} + A_{23} + \lambda A_3 - 1\big) \leq$$
$$\leq \frac{1}{4}\big(A_1 + A_{21} + A_{22} + A_{23} + \lambda A_3 - 2\big)^2 = \frac{1}{4}\big(A_1 + A_2 + \lambda A_3 - 2\big)^2,$$

which, in view of (1.12), (1.31), (1.36), and (1.76), contradicts (1.16).

Now suppose that (1.17) and (1.18) are fulfilled. It is not difficult to verify that, on account of (1.17) and (1.31), the function g defined by (1.75) is nondecreasing in $[0,+\infty[$. Therefore, from (1.79) we obtain

$$A_{21}+A_{22}+A_{23}\geq 1+\frac{\lambda-B}{A_1+A_{21}+A_{23}+\lambda A_3-1}+A_{21}+A_{23}=$$
$$=1+g\big(A_{21}+A_{23}\big)\geq 1+g(0)=1+\frac{\lambda-B}{A_1+\lambda A_3-1}\,,$$

which, in view of (1.31) and (1.36), contradicts (1.18). □

Proof of Theorem 1.3. Assume that the problem (0.1_0), (0.2_0) has a nontrivial solution u.

First suppose that u has a zero in $[\tau_0,\tau_1]$. Define the numbers m_0 and M_0 by (1.52) and choose $\alpha_0,\alpha_1\in[\tau_0,\tau_1]$ such that (1.53) holds. Obviously,

$$m_0\geq 0,\qquad M_0\geq 0,\qquad m_0+M_0>0 \tag{1.80}$$

since, if $m_0=0$ and $M_0=0$, then, in view of (0.1_0) and (1.52), we obtain $u(\tau_0)=0$ and $u'(t)=0$ for $t\in[a,b]$, i.e., $u\equiv 0$. It is also evident that without loss of generality we can assume that $\alpha_0<\alpha_1$.

The integration of (0.1_0) from α_0 to α_1, on account of (1.52), (1.53), and (1.80), yields

$$\begin{aligned}M_0+m_0&=\int_{\alpha_0}^{\alpha_1}[p(s)]_+u(\tau(s))ds-\int_{\alpha_0}^{\alpha_1}[p(s)]_-u(\tau(s))ds\leq\\&\leq M_0\int_{\tau_0}^{\tau_1}[p(s)]_+ds+m_0\int_{\tau_0}^{\tau_1}[p(s)]_-ds,\end{aligned} \tag{1.81}$$

which, by virtue of (1.19) and (1.80), results in $M_0+m_0<M_0+m_0$, a contradiction.

Now suppose that u has no zero in $[\tau_0,\tau_1]$. Without loss of generality we can assume that $u(t)>0$ for $t\in[\tau_0,\tau_1]$. Define the numbers M and m by (1.33) and choose $t_M,t_m\in[\tau_0,\tau_1]$ such that (1.34) holds. Furthermore, let

$$\begin{aligned}f_+(t)&\overset{\text{def}}{=}\int_a^t[p(s)]_+ds+\lambda\int_t^b[p(s)]_+ds\quad\text{for}\quad t\in[a,b],\\f_-(t)&\overset{\text{def}}{=}\int_a^t[p(s)]_-ds+\lambda\int_t^b[p(s)]_-ds\quad\text{for}\quad t\in[a,b].\end{aligned} \tag{1.82}$$

It is obvious that

$$M>0,\qquad m>0, \tag{1.83}$$

and either (1.38) or (1.39) is satisfied.

If (1.38) holds, then the integration of (0.1_0) from t_M to t_m, on account of (1.33), (1.34), and (1.83), results in

$$M-m=\int_{t_M}^{t_m}[p(s)]_-u(\tau(s))ds-\int_{t_M}^{t_m}[p(s)]_+u(\tau(s))ds\leq M\int_{\tau_0}^{\tau_1}[p(s)]_-ds.$$

If (1.39) holds, then the integration of (0.1_0) from t_m to t_M, in view of (1.33), (1.34), and (1.83), results in

$$M - m = \int_{t_m}^{t_M} [p(s)]_+ u(\tau(s))ds - \int_{t_m}^{t_M} [p(s)]_- u(\tau(s))ds \leq M \int_{\tau_0}^{\tau_1} [p(s)]_+ ds.$$

Therefore, due to (1.19) and (1.83), in both cases (1.38) and (1.39) we have

$$0 < M\big(1 - T\big) \leq m, \tag{1.84}$$

where T is defined by (1.22).

First suppose that (1.20) holds with T given by (1.22). The integrations of (0.1_0) from a to t_M and from t_M to b, on account of (1.33) and (1.34), imply

$$\begin{aligned} M - u(a) = \int_a^{t_M} [p(s)]_+ u(\tau(s))ds - \int_a^{t_M} [p(s)]_- u(\tau(s))ds \leq \\ \leq M \int_a^{t_M} [p(s)]_+ ds - m \int_a^{t_M} [p(s)]_- ds, \end{aligned} \tag{1.85}$$

$$\begin{aligned} u(b) - M = \int_{t_M}^b [p(s)]_+ u(\tau(s))ds - \int_{t_M}^b [p(s)]_- u(\tau(s))ds \leq \\ \leq M \int_{t_M}^b [p(s)]_+ ds - m \int_{t_M}^b [p(s)]_- ds. \end{aligned} \tag{1.86}$$

Multiplying both sides of (1.86) by λ, summing with (1.85), and taking into account (0.2_0), (1.82), and the assumption $\lambda \in [0, 1]$, we get

$$\begin{aligned} M(1 - \lambda) \leq M \left(\int_a^{t_M} [p(s)]_+ ds + \lambda \int_{t_M}^b [p(s)]_+ ds \right) - \\ -m \left(\int_a^{t_M} [p(s)]_- ds + \lambda \int_{t_M}^b [p(s)]_- ds \right) = M f_+(t_M) - m f_-(t_M). \end{aligned} \tag{1.87}$$

It is easy to verify that, in view of the assumption $\lambda \in [0, 1]$, the functions f_+ and f_- defined by (1.82) are nondecreasing in $[a, b]$ and thus, due to (1.31) and (1.82), it follows from (1.87) that

$$\begin{aligned} M(1 - \lambda) \leq M f_+(t_M) - m f_-(t_M) \leq M f_+(\tau_1) - m f_-(\tau_0) = \\ = M\big(A_1 + A_2 + \lambda A_3\big) - m\big(B_1 + \lambda B_2 + \lambda B_3\big). \end{aligned} \tag{1.88}$$

By virtue of (1.84), (1.88) yields

$$M(1 - \lambda) \leq M\big(A_1 + A_2 + \lambda A_3\big) - M\big(B_1 + \lambda B_2 + \lambda B_3\big)\big(1 - T\big),$$

which, in view of (1.31) and (1.83), contradicts (1.20).

Now suppose that (1.21) holds with T given by (1.22). The integrations of (0.1_0) from a to t_m and from t_m to b, on account of (1.33) and (1.34), imply

$$m - u(a) = \int_a^{t_m} [p(s)]_+ u(\tau(s))ds - \int_a^{t_m} [p(s)]_- u(\tau(s))ds \geq$$
$$\geq m \int_a^{t_m} [p(s)]_+ ds - M \int_a^{t_m} [p(s)]_- ds, \tag{1.89}$$
$$u(b) - m = \int_{t_m}^{b} [p(s)]_+ u(\tau(s))ds - \int_{t_m}^{b} [p(s)]_- u(\tau(s))ds \geq$$
$$\geq m \int_{t_m}^{b} [p(s)]_+ ds - M \int_{t_m}^{b} [p(s)]_- ds. \tag{1.90}$$

Multiplying both sides of (1.90) by λ, summing with (1.89), and taking into account (0.2_0), (1.82), and the assumption $\lambda \in [0,1]$, we obtain

$$m(1-\lambda) \geq m \left(\int_a^{t_m} [p(s)]_+ ds + \lambda \int_{t_m}^{b} [p(s)]_+ ds \right) -$$
$$-M \left(\int_a^{t_m} [p(s)]_- ds + \lambda \int_{t_m}^{b} [p(s)]_- ds \right) = mf_+(t_m) - Mf_-(t_m). \tag{1.91}$$

As above, in view of the assumption $\lambda \in [0,1]$, the functions f_+ and f_- defined by (1.82) are nondecreasing in $[a,b]$ and thus, due to (1.31) and (1.82), it follows from (1.91) that

$$m(1-\lambda) \geq mf_+(t_m) - Mf_-(t_m) \geq mf_+(\tau_0) - Mf_-(\tau_1) =$$
$$= m\big(A_1 + \lambda A_2 + \lambda A_3\big) - M\big(B_1 + B_2 + \lambda B_3\big). \tag{1.92}$$

By virtue of (1.19), (1.22), and (1.84), (1.92) implies

$$m(1-\lambda)(1-T) \geq m\big(A_1 + \lambda A_2 + \lambda A_3\big)\big(1-T\big) - m\big(B_1 + B_2 + \lambda B_3\big),$$

which, in view of (1.31) and (1.83), contradicts (1.21). $\square$

Proof of Theorem 1.4. Assume that the problem (0.1_0), (0.2_0) has a nontrivial solution u.

According to (1.23), (1.24), and Theorem 1.1 in [7], u changes its sign in $[\tau_0, \tau_1]$. Define the numbers m_0 and M_0 by (1.52) and choose $\alpha_0, \alpha_1 \in [\tau_0, \tau_1]$ such that (1.53) holds. Obviously, (1.54) is satisfied and without loss of generality we can assume that $\alpha_1 < \alpha_0$. From (0.1_0), (0.2_0), (1.23), and (1.24), due to (1.52) and (1.54), we obtain

$$\big(M_0\gamma(t) + u(t)\big)' \geq$$
$$\geq [p(t)]_+\big(M_0\gamma(\tau(t)) + u(\tau(t))\big) + [p(t)]_-\big(M_0 - u(\tau(t))\big) \geq \tag{1.93}$$
$$\geq [p(t)]_+\big(M_0\gamma(\tau(t)) + u(\tau(t))\big) \quad \text{for} \quad t \in [a,b],$$
$$M_0\gamma(a) + u(a) - \lambda\big(M_0\gamma(b) + u(b)\big) > 0,$$

and

$$\begin{gathered}\big(m_0\gamma(t)-u(t)\big)'\geq\\ \geq [p(t)]_+\big(m_0\gamma(\tau(t))-u(\tau(t))\big)+[p(t)]_-\big(m_0+u(\tau(t))\big)\geq\\ \geq [p(t)]_+\big(m_0\gamma(\tau(t))-u(\tau(t))\big)\quad\text{for}\quad t\in[a,b],\\ m_0\gamma(a)-u(a)-\lambda\big(m_0\gamma(b)-u(b)\big)>0.\end{gathered}\tag{1.94}$$

Hence, according to (1.23), (1.24), Theorem 1.1 in [7], and Remark 0.1 in [7], we get

$$M_0\gamma(t)+u(t)\geq 0,\qquad m_0\gamma(t)-u(t)\geq 0\quad\text{for}\quad t\in[a,b].$$

By virtue of the last inequalities, it follows from (1.93) and (1.94) that

$$\big(M_0\gamma(t)+u(t)\big)'\geq 0,\qquad \big(m_0\gamma(t)-u(t)\big)'\geq 0\quad\text{for}\quad t\in[a,b].\tag{1.95}$$

The integration of the first inequality in (1.95) from α_1 to α_0, in view of (1.53) and (1.54), yields

$$M_0\gamma(\alpha_0)-m_0-M_0\gamma(\alpha_1)-M_0\geq 0,$$

i.e.,

$$\gamma(\alpha_0)-\gamma(\alpha_1)\geq 1+\frac{m_0}{M_0}\,.\tag{1.96}$$

On the other hand, the integrations of the second inequality in (1.95) from a to α_1 and from α_0 to b, on account of (1.53), imply

$$m_0\gamma(\alpha_1)-M_0-m_0\gamma(a)+u(a)\geq 0,\tag{1.97}$$

$$m_0\gamma(b)-u(b)-m_0\gamma(\alpha_0)-m_0\geq 0.\tag{1.98}$$

Multiplying both sides of (1.98) by λ, summing with (1.97), and taking into account (0.2_0), (1.54), and the assumption $\lambda\in[0,1[$, we get

$$\gamma(\alpha_1)-\gamma(a)+\lambda\big(\gamma(b)-\gamma(\alpha_0)\big)\geq\lambda+\frac{M_0}{m_0}\,.\tag{1.99}$$

First suppose that (1.25) and (1.26) are fulfilled. Summing (1.96) and (1.99) and taking into account (1.54), we obtain

$$\gamma(\alpha_0)-\gamma(a)+\lambda\big(\gamma(b)-\gamma(\alpha_0)\big)\geq 1+\lambda+\frac{M_0}{m_0}+\frac{m_0}{M_0}\geq 3+\lambda.\tag{1.100}$$

On the other hand, by virtue of the fact that the function γ is nondecreasing in $[a,b]$ and the assumption $\lambda\in[0,1[$, we get

$$\lambda\gamma(b)+(1-\lambda)\gamma(\tau_1)-\gamma(a)\geq\lambda\gamma(b)+(1-\lambda)\gamma(\alpha_0)-\gamma(a),$$

which, together with (1.100), contradicts (1.26).

Now suppose that (1.27) and (1.28) are satisfied. According to (1.27), (1.54), and the fact that the function γ is nondecreasing in $[a,b]$, it follows from (1.99) that

$$\frac{m_0}{M_0}\geq\frac{1}{\gamma(\alpha_1)-\gamma(a)+\lambda\big(\gamma(b)-\gamma(\alpha_0)\big)-\lambda}$$

and thus, (1.96) implies

$$\gamma(\alpha_0)-\gamma(\alpha_1)\geq 1+\frac{1}{\gamma(\alpha_1)-\gamma(a)+\lambda\big(\gamma(b)-\gamma(\alpha_0)\big)-\lambda}\,. \tag{1.101}$$

Let

$$g(x)\overset{\text{def}}{=}\frac{1}{x-\lambda}+x\quad\text{for}\quad x>\lambda. \tag{1.102}$$

By virtue of (1.101), (1.102), the assumption $\lambda\in[0,1[$, and the fact that the function γ is nondecreasing in $[a,b]$, we get

$$\begin{aligned}
&\gamma(\tau_1)-\gamma(\tau_0)=\\
&=\gamma(\alpha_0)-\gamma(\alpha_1)+\gamma(\tau_1)-\gamma(\alpha_0)+\gamma(\alpha_1)-\gamma(\tau_0)\geq\\
&\geq 1+\frac{1}{\gamma(\alpha_1)-\gamma(a)+\lambda\big(\gamma(b)-\gamma(\alpha_0)\big)-\lambda}+\\
&+\lambda\big(\gamma(\tau_1)-\gamma(\alpha_0)\big)+\gamma(\alpha_1)-\gamma(\tau_0)=\\
&=1+g\Big(\gamma(\alpha_1)-\gamma(a)+\lambda\big(\gamma(b)-\gamma(\alpha_0)\big)\Big)+\\
&+\gamma(a)-\gamma(\tau_0)+\lambda\big(\gamma(\tau_1)-\gamma(b)\big).
\end{aligned} \tag{1.103}$$

It is easy to verify that the function g is nondecreasing in $[1+\lambda,+\infty[$ and thus, according to (1.27) and the fact that the function γ is nondecreasing in $[a,b]$, we find

$$g\Big(\gamma(\alpha_1)-\gamma(a)+\lambda\big(\gamma(b)-\gamma(\alpha_0)\big)\Big)\geq g\Big(\gamma(\tau_0)-\gamma(a)+\lambda\big(\gamma(b)-\gamma(\tau_1)\big)\Big).$$

Therefore, (1.103) yields

$$\begin{aligned}
\gamma(\tau_1)-\gamma(\tau_0)\geq 1+g\Big(\gamma(\tau_0)-\gamma(a)+\lambda\big(\gamma(b)-\gamma(\tau_1)\big)\Big)+&\\
+\gamma(a)-\gamma(\tau_0)+\lambda\big(\gamma(\tau_1)-\gamma(b)\big)=&\\
=1+\frac{1}{\gamma(\tau_0)-\gamma(a)+\lambda\big(\gamma(b)-\gamma(\tau_1)\big)-\lambda}&\,,
\end{aligned}$$

which contradicts (1.28). □

2. Problem (0.1), (0.3)

2.1. **Main Results.**

Theorem 2.1. *Let* $\lambda\in R_+$, *the condition* (1.19) *be fulfilled, and let either*

$$\begin{aligned}
&\int_a^{\tau_1}[p(s)]_+ds+\lambda\int_{\tau_0}^{b}[p(s)]_-ds-\\
&-\left(\int_a^{\tau_0}[p(s)]_-ds+\lambda\int_{\tau_1}^{b}[p(s)]_+ds\right)\big(1-T\big)<1+\lambda
\end{aligned} \tag{2.1}$$

or

$$\left(\int_a^{\tau_0}[p(s)]_+ds+\lambda\int_{\tau_1}^b[p(s)]_-ds\right)\bigl(1-T\bigr)-$$
$$-\int_a^{\tau_1}[p(s)]_-ds-\lambda\int_{\tau_0}^b[p(s)]_+ds>\bigl(1+\lambda\bigr)\bigl(1-T\bigr)\,, \tag{2.2}$$

where T *is defined by* (1.22). *Then the problem* (0.1), (0.3) *has a unique solution.*

Remark 2.1. Theorem 2.1 is nonimprovable in the sense that the strict inequalities (2.1) and (2.2) cannot be replaced by the nonstrict ones (see Examples 3.11 and 3.12).

Note also that if the segment $[\tau_0,\tau_1]$ degenerates to a point $c\in[a,b]$, i.e., $\tau(t)=c$ for $t\in[a,b]$, then $T=0$ and the inequalities (2.1) and (2.2) can be rewritten as

$$\int_a^c p(s)ds-\lambda\int_c^b p(s)ds\neq 1+\lambda\,.$$

On the other hand, the last relation is sufficient and necessary for the unique solvability of the problem (0.1), (0.3) with $\tau(t)=c$ for $t\in[a,b]$.

The following theorem can be understood as a supplement of the previous one for the case $T\geq 1$, where T is given by (1.22).

Theorem 2.2. *Let* $\lambda\in[0,1]$,

$$D\overset{\text{def}}{=}\int_a^{\tau_1}[p(s)]_+ds+\lambda\int_{\tau_1}^b[p(s)]_-ds\,, \tag{2.3}$$

and let one of the following items be fulfilled:

a)

$$\int_{\tau_0}^{\tau_1}[p(s)]_+ds\geq 1\,, \tag{2.4}$$

$$\int_a^{\tau_1}[p(s)]_+ds+\lambda\int_{\tau_0}^b[p(s)]_-ds+$$
$$+\left(\int_a^{\tau_0}[p(s)]_-ds+\lambda\int_{\tau_1}^b[p(s)]_+ds\right)\left(\int_{\tau_0}^{\tau_1}[p(s)]_+ds-1\right)<1+\lambda\,; \tag{2.5}$$

b)

$$\int_{\tau_0}^{\tau_1}[p(s)]_-ds\geq 1\,, \tag{2.6}$$

$$D\geq 1-\lambda^2\,, \tag{2.7}$$

$$\int_a^{\tau_1}[p(s)]_+ds+\lambda\int_{\tau_0}^b[p(s)]_-ds+$$

$$+\left(\int_a^{\tau_0}[p(s)]_-ds+\lambda\int_{\tau_1}^b[p(s)]_+ds\right)\left(\int_{\tau_0}^{\tau_1}[p(s)]_-ds-1\right)<1+\lambda\,; \quad (2.8)$$

c) *the condition* (2.6) *holds,*

$$D<1-\lambda^2\,, \qquad (2.9)$$

and either

$$\int_a^{\tau_0}[p(s)]_-ds+\lambda\int_{\tau_1}^b[p(s)]_+ds<-\lambda+\sqrt{1-D}\,, \qquad (2.10)$$

$$\int_a^{\tau_1}[p(s)]_-ds+\lambda\int_{\tau_1}^b[p(s)]_+ds<1-\lambda+2\sqrt{1-D} \qquad (2.11)$$

or

$$\int_a^{\tau_0}[p(s)]_-ds+\lambda\int_{\tau_1}^b[p(s)]_+ds\geq-\lambda+\sqrt{1-D} \qquad (2.12)$$

and the condition (2.8) *holds.*

Then the problem (0.1), (0.3) *has a unique solution.*

Remark 2.2. Theorem 2.2 is nonimprovable in the sense that neither of the strict inequalities (2.5), (2.8), and (2.11) can be replaced by the nonstrict one (see Examples 3.13–3.16).

Note also that if $\tau_0=a$ and $\tau_1=b$, then from Theorems 2.1 and 2.2 we obtain Corollary 1.1 in [6].

Example 2.1. On the segment $[0,1]$ consider the boundary value problem

$$\begin{gathered}u'(t)=k\left(\frac{|2t-1|}{2}-\frac{1}{4}\right)u\left(\frac{2t+1}{4}\right)+q(t),\\ u(0)+\lambda u(1)=c,\end{gathered} \qquad (2.13)$$

where $\lambda\in[0,1]$, $k\in R$, $q\in L([a,b];R)$, and $c\in R$.

It is not difficult to verify that $\tau_0=\frac{1}{4}$ and $\tau_1=\frac{3}{4}$. Obviously, if $k\leq-16$, then the condition (2.4) holds. If

$$8(1+\lambda)-\sqrt{[8(1+\lambda)]^2+512(1+\lambda)}<k\leq 0,$$

then the condition (2.5) is fulfilled. If $k\geq 16$, then the condition (2.6) is satisfied. If $k\geq 32(1-\lambda^2)$, then the condition (2.7) holds. If

$$0\leq k<-\frac{8(1+\lambda)}{\lambda}+\sqrt{\left[\frac{8(1+\lambda)}{\lambda}\right]^2+512\,\frac{1+\lambda}{\lambda}}\,,$$

then the condition (2.8) is fulfilled. If $0\leq k<32(1-\lambda^2)$, then the condition (2.9) is satisfied. If

$$0\leq k<-\frac{16(1+2\lambda^2)}{\lambda^2}+16\sqrt{\left[\frac{1+2\lambda^2}{\lambda^2}\right]^2+4\,\frac{1-\lambda^2}{\lambda^2}}\,,$$

then the condition (2.10) holds. If

$$0 \le k < 32\,\frac{(2+\lambda)(1-\lambda)-2}{(2+\lambda)^2} + 32\sqrt{\left[\frac{(2+\lambda)(1-\lambda)-2}{(2+\lambda)^2}\right]^2 + \frac{3+\lambda^2}{(2+\lambda)^2}}\ ,$$

then the condition (2.11) is satisfied. If

$$k \ge -\frac{16(1+2\lambda^2)}{\lambda^2} + 16\sqrt{\left[\frac{1+2\lambda^2}{\lambda^2}\right]^2 + 4\,\frac{1-\lambda^2}{\lambda^2}}\ ,$$

then the condition (2.12) is fulfilled.

In particular if $k \in\,]-17.4, -16[\, \cup [16, 22.21[\, \cup [23.38, 23.81[$ and $\lambda = 0.3$, then, according to Theorem 2.2, the problem (2.13) has a unique solution.

Remark 2.3. Let $\lambda \in [0,1]$. Denote by H the set of $6-$tuples $(x_i)_{i=1}^6 \in R_+^6$ satisfying one of the following items:

a)

$$x_2 < 1\,, \qquad x_5 < 1\,,$$

and either

$$x_1 + x_2 + \lambda(x_5 + x_6) - (x_4 + \lambda x_3)(1 - T) < 1 + \lambda$$

or

$$(x_1 + \lambda x_6)(1 - T) - x_4 - x_5 - \lambda(x_2 + x_3) > (1 + \lambda)(1 - T)\,,$$

where $T = \max\{x_2, x_5\}$;

b)

$$x_2 \ge 1\,,$$
$$x_1 + x_2 + \lambda(x_5 + x_6) + (x_4 + \lambda x_3)(x_2 - 1) < 1 + \lambda\,;$$

c)

$$x_5 \ge 1\,,$$
$$x_1 + x_2 + \lambda x_6 \ge 1 - \lambda^2\ ,$$
$$x_1 + x_2 + \lambda(x_5 + x_6) + (x_4 + \lambda x_3)(x_5 - 1) < 1 + \lambda\,;$$

d)

$$x_5 \ge 1\,,$$
$$x_1 + x_2 + \lambda x_6 < 1 - \lambda^2\ ,$$

and either

$$x_4 + \lambda x_3 < -\lambda + \sqrt{1 - x_1 - x_2 - \lambda x_6}\ ,$$
$$x_4 + x_5 + \lambda x_3 < 1 - \lambda + 2\sqrt{1 - x_1 - x_2 - \lambda x_6}$$

or

$$x_4 + \lambda x_3 \ge -\lambda + \sqrt{1 - x_1 - x_2 - \lambda x_6}\ ,$$
$$x_1 + x_2 + \lambda(x_5 + x_6) + (x_4 + \lambda x_3)(x_5 - 1) < 1 + \lambda\,.$$

Now, according to Theorems 2.1 and 2.2, if $p \in L([a,b];R)$ is such that the point

$$\left(\int_a^{\tau_0}[p(s)]_+ds, \int_{\tau_0}^{\tau_1}[p(s)]_+ds, \int_{\tau_1}^{b}[p(s)]_+ds, \int_a^{\tau_0}[p(s)]_-ds, \int_{\tau_0}^{\tau_1}[p(s)]_-ds, \int_{\tau_1}^{b}[p(s)]_-ds\right)$$

belongs to the set H, then the problem (0.1), (0.3) has a unique solution.

On Fig. 2.1 one can see an intersection of the set H and the 3−dimensional subspace $\{(0,x,0,y,z,0) : x,y,z \in R\}$ of the space R^6, i.e., the set

$$G = \{(x,y,z) \in R^3_+ : \ (0,x,0,y,z,0) \in H\}.$$

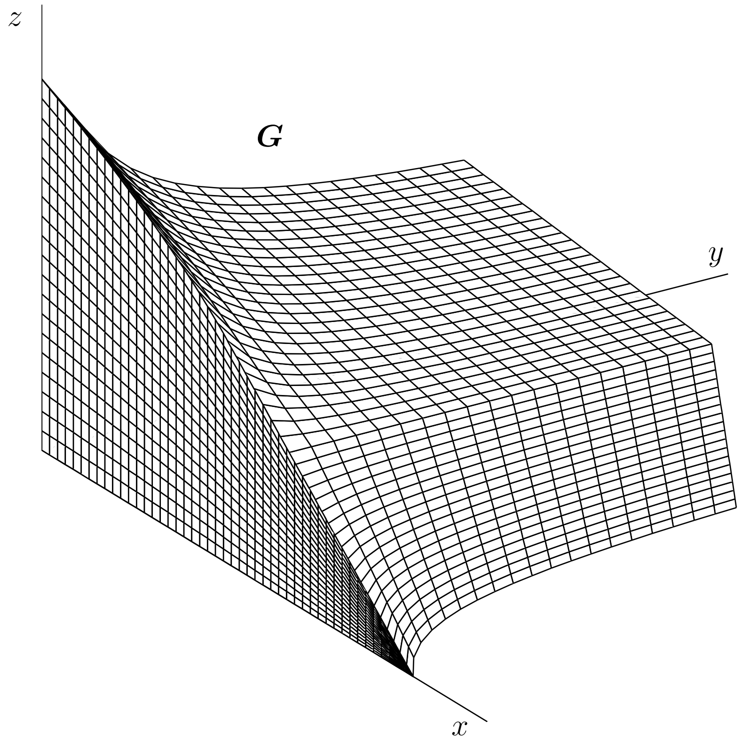

Fig. 2.1.

Remark 2.4. Let $\lambda \in [1,+\infty[$ and let the functions $\widetilde{p}$, $\widetilde{\tau}$, and $\widetilde{q}$ be defined by (1.29). Put $\mu = 1/\lambda$ and $\widetilde{c} = \mu c$.

It is clear that if u is a solution of the problem (0.1), (0.3), then the function v, defined by $v(t) \stackrel{\text{def}}{=} u(a+b-t)$ for $t \in [a,b]$, is a solution of the problem

$$v'(t) = \widetilde{p}(t)v(\widetilde{\tau}(t)) + \widetilde{q}(t), \qquad v(a) + \mu v(b) = \widetilde{c}, \tag{2.14}$$

and vice versa, if v is a solution of the problem (2.14), then the function u, defined by $u(t) \stackrel{\text{def}}{=} v(a+b-t)$ for $t \in [a,b]$, is a solution of the problem (0.1), (0.3).

Therefore, Theorem 2.2 immediately implies the following assertion.

Theorem 2.3. *Let* $\lambda \in [1, +\infty[$,

$$\widetilde{D} \stackrel{\text{def}}{=} \frac{1}{\lambda} \int_a^{\tau_0} [p(s)]_+ ds + \int_{\tau_0}^b [p(s)]_- ds \,,$$

and let one of the following items be fulfilled:

a)

$$\int_{\tau_0}^{\tau_1} [p(s)]_- ds \geq 1$$

and the condition (2.8) *holds;*

b)

$$\int_{\tau_0}^{\tau_1} [p(s)]_+ ds \geq 1 \,, \qquad \widetilde{D} \geq 1 - \frac{1}{\lambda^2} \,,$$

and the condition (2.5) *holds;*

c)

$$\int_{\tau_0}^{\tau_1} [p(s)]_+ ds \geq 1 \,, \qquad \widetilde{D} < 1 - \frac{1}{\lambda^2} \,,$$

and either

$$\frac{1}{\lambda} \int_a^{\tau_0} [p(s)]_- ds + \int_{\tau_1}^b [p(s)]_+ ds < -\frac{1}{\lambda} + \sqrt{1 - \widetilde{D}} \,,$$

$$\frac{1}{\lambda} \int_a^{\tau_0} [p(s)]_- ds + \int_{\tau_0}^b [p(s)]_+ ds < 1 - \frac{1}{\lambda} + 2\sqrt{1 - \widetilde{D}}$$

or

$$\frac{1}{\lambda} \int_a^{\tau_0} [p(s)]_- ds + \int_{\tau_1}^b [p(s)]_+ ds \geq -\frac{1}{\lambda} + \sqrt{1 - \widetilde{D}}$$

and the condition (2.5) *holds.*

Then the problem (0.1), (0.3) *has a unique solution.*

Remark 2.5. According to Remarks 2.2 and 2.4, Theorem 2.3 is nonimprovable in an appropriate sense.

2.2. **Proofs.** According to Theorem 0.1, to prove Theorems 2.1 and 2.2 it is sufficient to show that the homogeneous problem (0.1_0), (0.3_0) has only the trivial solution. As above, define the numbers A_i, B_i ($i = 1,2,3$) by (1.31).

Proof of Theorem 2.1. Assume that the problem (0.1_0), (0.3_0) has a nontrivial solution u.

First suppose that u has a zero in $[\tau_0, \tau_1]$. Define the numbers m_0 and M_0 by (1.52) and choose $\alpha_0, \alpha_1 \in [\tau_0, \tau_1]$ such that (1.53) holds. Obviously, (1.80) is satisfied since, if $m_0 = 0$ and $M_0 = 0$, then, in view of (0.1_0) and (1.52), we obtain $u(\tau_0) = 0$ and $u'(t) = 0$ for $t \in [a, b]$, i.e., $u \equiv 0$. It is also evident that without loss of generality we can assume that $\alpha_0 < \alpha_1$.

The integration of (0.1_0) from α_0 to α_1, by virtue of (1.52), (1.53), and (1.80), yields the inequality (1.81), which, on account of (1.19) and (1.80), results in $M_0 + m_0 < M_0 + m_0$, a contradiction.

Now suppose that u has no zero in $[\tau_0, \tau_1]$. Without loss of generality we can assume that $u(t) > 0$ for $t \in [\tau_0, \tau_1]$. Define the numbers M and m by (1.33) and choose $t_M, t_m \in [\tau_0, \tau_1]$ such that (1.34) holds. It is obvious that (1.83) is fulfilled and either (1.38) or (1.39) is satisfied. As in the proof of Theorem 1.3, one can show that in both cases (1.38) and (1.39) the inequality (1.84) holds, where T is defined by (1.22).

On the other hand, the integrations of (0.1_0) from a to t_M, from t_M to b, from a to t_m, and from t_m to b, in view of (1.33) and (1.34), yield

$$M - u(a) = \int_a^{t_M} [p(s)]_+ u(\tau(s))ds - \int_a^{t_M} [p(s)]_- u(\tau(s))ds \leq$$
$$\leq M \int_a^{t_M} [p(s)]_+ ds - m \int_a^{t_M} [p(s)]_- ds, \tag{2.15}$$
$$M - u(b) = \int_{t_M}^{b} [p(s)]_- u(\tau(s))ds - \int_{t_M}^{b} [p(s)]_+ u(\tau(s))ds \leq$$
$$\leq M \int_{t_M}^{b} [p(s)]_- ds - m \int_{t_M}^{b} [p(s)]_+ ds, \tag{2.16}$$
$$m - u(a) = \int_a^{t_m} [p(s)]_+ u(\tau(s))ds - \int_a^{t_m} [p(s)]_- u(\tau(s))ds \geq$$
$$\geq m \int_a^{t_m} [p(s)]_+ ds - M \int_a^{t_m} [p(s)]_- ds, \tag{2.17}$$
$$m - u(b) = \int_{t_m}^{b} [p(s)]_- u(\tau(s))ds - \int_{t_m}^{b} [p(s)]_+ u(\tau(s))ds \geq$$
$$\geq m \int_{t_m}^{b} [p(s)]_- ds - M \int_{t_m}^{b} [p(s)]_+ ds. \tag{2.18}$$

Put

$$\begin{aligned} f_1(t) &\overset{\text{def}}{=} M \int_a^t [p(s)]_+ ds - \lambda m \int_t^b [p(s)]_+ ds \quad \text{for} \quad t \in [a, b], \\ f_2(t) &\overset{\text{def}}{=} m \int_a^t [p(s)]_- ds - \lambda M \int_t^b [p(s)]_- ds \quad \text{for} \quad t \in [a, b], \end{aligned} \tag{2.19}$$

$$\begin{aligned} f_3(t) &\overset{\text{def}}{=} m \int_a^t [p(s)]_+ ds - \lambda M \int_t^b [p(s)]_+ ds \quad \text{for} \quad t \in [a, b], \\ f_4(t) &\overset{\text{def}}{=} M \int_a^t [p(s)]_- ds - \lambda m \int_t^b [p(s)]_- ds \quad \text{for} \quad t \in [a, b]. \end{aligned} \tag{2.20}$$

First suppose that (2.1) holds, where T is defined by (1.22). Multiplying both sides of (2.16) by λ, summing with (2.15), and taking into account (0.3_0), (2.19), and the assumption $\lambda \in R_+$, we get

$$M(1+\lambda) \leq M\int_a^{t_M}[p(s)]_+ds - \lambda m\int_{t_M}^b[p(s)]_+ds-$$
$$-\left(m\int_a^{t_M}[p(s)]_-ds - \lambda M\int_{t_M}^b[p(s)]_-ds\right) = f_1(t_M)-f_2(t_M). \qquad (2.21)$$

It is easy to verify that the functions f_1 and f_2 defined by (2.19) are nondecreasing in $[a,b]$ and therefore, due to (1.31) and (2.19), it follows from (2.21) that

$$M(1+\lambda) \leq f_1(t_M)-f_2(t_M) \leq f_1(\tau_1)-f_2(\tau_0) =$$
$$= M\big(A_1+A_2+\lambda B_2+\lambda B_3\big) - m\big(B_1+\lambda A_3\big). \qquad (2.22)$$

Thus, (1.84) and (2.22) imply

$$M(1+\lambda) \leq M\big(A_1+A_2+\lambda B_2+\lambda B_3\big) - M\big(B_1+\lambda A_3\big)\big(1-T\big),$$

which, in view of (1.31) and (1.83), contradicts (2.1).

Now suppose that (2.2) is satisfied, where T is defined by (1.22). Multiplying both sides of (2.18) by λ, summing with (2.17), and taking into account (0.3_0), (2.20), and the assumption $\lambda \in R_+$, we obtain

$$m(1+\lambda) \geq m\int_a^{t_m}[p(s)]_+ds - \lambda M\int_{t_m}^b[p(s)]_+ds-$$
$$-\left(M\int_a^{t_m}[p(s)]_-ds - \lambda m\int_{t_m}^b[p(s)]_-ds\right) = f_3(t_m)-f_4(t_m). \qquad (2.23)$$

It is easy to verify that the functions f_3 and f_4 defined by (2.20) are nondecreasing in $[a,b]$ and thus, due to (1.31) and (2.20), it follows from (2.23) that

$$m(1+\lambda) \geq f_3(t_m)-f_4(t_m) \geq f_3(\tau_0)-f_4(\tau_1) =$$
$$= m\big(A_1+\lambda B_3\big) - M\big(B_1+B_2+\lambda A_2+\lambda A_3\big). \qquad (2.24)$$

By virtue of (1.19) and (1.22), (1.84) and (2.24) yield

$$m(1+\lambda)(1-T) \geq m\big(A_1+\lambda B_3\big)\big(1-T\big) - m\big(B_1+B_2+\lambda A_2+\lambda A_3\big),$$

which, in view of (1.31) and (1.83), contradicts (2.2). □

Proof of Theorem 2.1. Assume that the problem (0.1_0), (0.3_0) has a nontrivial solution u.

First suppose that u changes its sign in $[\tau_0,\tau_1]$. Define numbers m_0 and M_0 by (1.52) and choose $\alpha_0,\alpha_1 \in [\tau_0,\tau_1]$ such that (1.53) holds. It is clear that (1.54) is satisfied and without loss of generality we can assume that $\alpha_0 < \alpha_1$. Furthermore, define the numbers A_{2i}, B_{2i} $(i=1,2,3)$ by (1.36).

The integrations of (0.1_0) from a to α_0, from α_0 to α_1, from α_1 to b, and from τ_1 to b, in view of (1.31), (1.36), (1.52), and (1.53), result in

$$-m_0 - u(a) = \int_a^{\alpha_0} [p(s)]_+ u(\tau(s))ds - \int_a^{\alpha_0} [p(s)]_- u(\tau(s))ds \le$$
$$\le M_0\big(A_1 + A_{21}\big) + m_0\big(B_1 + B_{21}\big), \tag{2.25}$$

$$u(a) + m_0 = \int_a^{\alpha_0} [p(s)]_- u(\tau(s))ds - \int_a^{\alpha_0} [p(s)]_+ u(\tau(s))ds \le$$
$$\le M_0\big(B_1 + B_{21}\big) + m_0\big(A_1 + A_{21}\big), \tag{2.26}$$

$$M_0 + m_0 = \int_{\alpha_0}^{\alpha_1} [p(s)]_+ u(\tau(s))ds - \int_{\alpha_0}^{\alpha_1} [p(s)]_- u(\tau(s))ds \le$$
$$\le M_0 A_{22} + m_0 B_{22}, \tag{2.27}$$

$$M_0 - u(b) = \int_{\alpha_1}^{b} [p(s)]_- u(\tau(s))ds - \int_{\alpha_1}^{b} [p(s)]_+ u(\tau(s))ds \le$$
$$\le M_0\big(B_{23} + B_3\big) + m_0\big(A_{23} + A_3\big), \tag{2.28}$$

$$u(b) - M_0 \le u(b) - u(\tau_1) = \int_{\tau_1}^{b} [p(s)]_+ u(\tau(s))ds -$$
$$- \int_{\tau_1}^{b} [p(s)]_- u(\tau(s))ds \le M_0 A_3 + m_0 B_3. \tag{2.29}$$

Multiplying both sides of (2.28) by λ, summing with (2.25), and taking into account (0.3_0) and the assumption $\lambda \in [0,1]$, we get

$$\lambda M_0 - m_0 \le$$
$$\le M_0\big(A_1 + A_{21} + \lambda B_{23} + \lambda B_3\big) + m_0\big(B_1 + B_{21} + \lambda A_{23} + \lambda A_3\big). \tag{2.30}$$

Analogously, (2.26) and (2.29) imply

$$m_0 - \lambda M_0 \le M_0\big(B_1 + B_{21} + \lambda A_3\big) + m_0\big(A_1 + A_{21} + \lambda B_3\big). \tag{2.31}$$

First suppose that the assumption a) holds. In that case $\lambda \ne 0$. According to (1.36), (2.4), and (2.5), we have $B_{22} < 1$. Consequently, in view of (1.54), (2.27) yields

$$0 < m_0\big(1 - B_{22}\big) \le M_0\big(A_{22} - 1\big). \tag{2.32}$$

Moreover, it follows from (2.5) that

$$A_{22} < 1 + \lambda. \tag{2.33}$$

From (2.2) we get

$$M_0\big(\lambda - A_1 - A_{21} - \lambda B_{23} - \lambda B_3\big) \le m_0\big(B_1 + B_{21} + \lambda A_{23} + \lambda A_3 + 1\big),$$

which, together with (2.32), implies

$$\big(\lambda - A_1 - A_{21} - \lambda B_{23} - \lambda B_3\big)\big(1 - B_{22}\big) \le$$
$$\le \big(B_1 + B_{21} + \lambda A_{23} + \lambda A_3 + 1\big)\big(A_{22} - 1\big). \tag{2.34}$$

Obviously,

$$\big(\lambda - A_1 - A_{21} - \lambda B_{23} - \lambda B_3\big)\big(1 - B_{22}\big) \geq \\ \geq \lambda - A_1 - A_{21} - \lambda\big(B_{22} + B_{23} + B_3\big). \tag{2.35}$$

On the other hand, by virtue of (2.33) and the assumption $\lambda \in]0,1]$, we obtain

$$\big(B_1 + B_{21} + \lambda A_{23} + \lambda A_3 + 1\big)\big(A_{22} - 1\big) = \\ = \big(B_1 + \lambda A_3\big)\big(A_{22} - 1\big) + B_{21}\big(A_{22} - 1\big) + \lambda A_{23}\big(A_{22} - 1\big) + \\ + A_{22} - 1 \leq \big(B_1 + \lambda A_3\big)\big(A_2 - 1\big) + \lambda B_{21} + A_{22} + A_{23} - 1. \tag{2.36}$$

Using (1.31), (1.36), (2.35), and (2.36), (2.34) results in

$$A_1 + A_2 + \lambda\big(B_2 + B_3\big) + \big(B_1 + \lambda A_3\big)\big(A_2 - 1\big) \geq 1 + \lambda,$$

which, in view of (1.31), contradicts (2.5).

Now suppose that the assumption b) holds. In that case $\lambda \neq 0$. According to (1.36), (2.6), and (2.8), we have $A_{22} < 1$. Consequently, on account of (1.54), (2.27) implies

$$0 < M_0\big(1 - A_{22}\big) \leq m_0\big(B_{22} - 1\big). \tag{2.37}$$

Moreover, it follows from (2.3), (2.6)–(2.8), and the assumption $\lambda \in]0,1]$ that

$$B_{22} < 1 + \lambda. \tag{2.38}$$

From (2.31) we obtain

$$m_0\big(1 - A_1 - A_{21} - \lambda B_3\big) \leq M_0\big(B_1 + B_{21} + \lambda A_3 + \lambda\big), \tag{2.39}$$

which, together with (2.37), yields

$$\big(1 - A_1 - A_{21} - \lambda B_3\big)\big(1 - A_{22}\big) \leq \\ \leq \big(B_1 + B_{21} + \lambda A_3 + \lambda\big)\big(B_{22} - 1\big). \tag{2.40}$$

Clearly,

$$\big(1 - A_1 - A_{21} - \lambda B_3\big)\big(1 - A_{22}\big) \geq \\ \geq 1 - A_1 - A_{21} - A_{22} - \lambda B_3\,. \tag{2.41}$$

On the other hand, by virtue of (2.38) and the assumption $\lambda \in]0,1]$, we get

$$\big(B_1 + B_{21} + \lambda A_3 + \lambda\big)\big(B_{22} - 1\big) = \\ = \big(B_1 + \lambda A_3\big)\big(B_{22} - 1\big) + B_{21}\big(B_{22} - 1\big) + \lambda B_{22} - \lambda \leq \\ \leq \big(B_1 + \lambda A_3\big)\big(B_2 - 1\big) + \lambda\big(B_{21} + B_{22}\big) - \lambda. \tag{2.42}$$

Using (1.31), (1.36), (2.41), and (2.42), (2.40) implies

$$A_1 + A_2 + \lambda\big(B_2 + B_3\big) + \big(B_1 + \lambda A_3\big)\big(B_2 - 1\big) \geq 1 + \lambda,$$

which, in view of (1.31), contradicts (2.8).

Finally suppose that the assumption c) holds. According to (1.31), (1.36), (2.3), and (2.9), we have

$$A_{22} < 1, \qquad A_1 + A_{21} + \lambda B_3 < 1.$$

Thus, it follows from (1.54), (2.27), and (2.31) that

$$B_{22} > 1, \qquad B_1 + B_{21} + \lambda A_3 + \lambda > 0, \tag{2.43}$$

and

$$\big(1 - A_1 - A_{21} - \lambda B_3\big)\big(1 - A_{22}\big) \le \big(B_1 + B_{21} + \lambda A_3 + \lambda\big)\big(B_{22} - 1\big). \tag{2.44}$$

According to (2.3), (2.43), and the fact that

$$\big(1 - A_1 - A_{21} - \lambda B_3\big)\big(1 - A_{22}\big) \ge 1 - A_1 - A_{21} - A_{22} - \lambda B_3 \ge 1 - D,$$

from (2.44) we get

$$B_{22} \ge 1 + \frac{1 - D}{B_1 + B_{21} + \lambda A_3 + \lambda}\,. \tag{2.45}$$

First suppose that (2.10) and (2.11) are satisfied. By virtue of (2.43), from (2.45) we obtain

$$1 - D \le \big(B_1 + B_{21} + \lambda A_3 + \lambda\big)\big(B_{22} - 1\big) \le$$
$$\le \frac{1}{4}\big(B_1 + B_{21} + B_{22} + \lambda A_3 - 1 + \lambda\big)^2 \le \frac{1}{4}\big(B_1 + B_2 + \lambda A_3 - 1 + \lambda\big)^2,$$

which, in view of (1.31), (2.9), and (2.43), contradicts (2.11).

Now suppose that (2.8) and (2.12) are fulfilled. Let

$$g(x) \stackrel{\text{def}}{=} \frac{1 - D}{x + B_1 + \lambda A_3 + \lambda} + x \quad \text{for} \quad x > -B_1 - \lambda A_3 - \lambda,$$

where D is given by (2.3). It is not difficult to verify that, on account of (1.31) and (2.12), the function g is nondecreasing in $[0, +\infty[$. Therefore, from (2.45) we obtain

$$B_{21} + B_{22} + B_{23} \ge 1 + \frac{1 - D}{B_1 + B_{21} + \lambda A_3 + \lambda} + B_{21} =$$
$$= 1 + g\big(B_{21}\big) \ge 1 + g(0) = 1 + \frac{1 - D}{B_1 + \lambda A_3 + \lambda}\,,$$

which, in view of (1.31), (1.36), and (2.3), contradicts (2.8).

Now suppose that u does not change its sign in $[\tau_0, \tau_1]$. Without loss of generality we can assume that (1.32) is satisfied. Define the numbers M and m by (1.33) and choose $t_M, t_m \in [\tau_0, \tau_1]$ such that (1.34) holds. It is clear that (1.37) is satisfied since, if $M = 0$, then, in view of (0.1_0), (1.32), and (1.33), we obtain $u(\tau_0) = 0$ and $u'(t) = 0$ for $t \in [a, b]$, i.e., $u \equiv 0$.

The integrations of (0.1_0) from a to t_M, from t_M to b, from a to τ_0, and from τ_1 to b, in view of (1.31)–(1.34), result in (2.15), (2.16),

$$-u(a) \le u(\tau_0) - u(a) =$$

$$= \int_a^{\tau_0} [p(s)]_+ u(\tau(s))ds - \int_a^{\tau_0} [p(s)]_- u(\tau(s))ds \le MA_1, \qquad (2.46)$$

$$-u(b) \le u(\tau_1) - u(b) =$$

$$= \int_{\tau_1}^{b} [p(s)]_- u(\tau(s))ds - \int_{\tau_1}^{b} [p(s)]_+ u(\tau(s))ds \le MB_3. \qquad (2.47)$$

Moreover, from (2.15) and (2.16), in view of (1.31) and (1.37), we find

$$M - u(a) \le M\big(A_1 + A_2\big), \qquad (2.48)$$

$$M - u(b) \le M\big(B_2 + B_3\big). \qquad (2.49)$$

Multiplying both sides of (2.47) by λ, summing with (2.48), and taking into account (0.3_0), (1.37), and the assumption $\lambda \in [0,1]$, we get

$$A_1 + A_2 + \lambda B_3 \ge 1. \qquad (2.50)$$

Analogously, (2.46) and (2.49) yield

$$A_1 + \lambda B_2 + \lambda B_3 \ge \lambda. \qquad (2.51)$$

First suppose that the assumption a) holds. By virtue of (1.31), (2.4), and (2.51), (2.5) results in

$$1 + \lambda > A_1 + A_2 + \lambda\big(B_2 + B_3\big) \ge 1 + \lambda, \qquad (2.52)$$

a contradiction.

Now suppose that the assumption b) holds. With respect to (1.31), (2.6), (2.50), and the assumption $\lambda \in [0,1]$, (2.8) implies (2.52), a contradiction.

Finally suppose that the assumption c) holds. On account of (1.31) and (2.3), (2.50) contradicts (2.9). □

3. Comments and Examples

In the examples below, the functions $p \in L([a,b];R)$ and $\tau \in \mathcal{M}_{ab}$ are constructed such that the homogeneous problem (0.1_0), (0.2_0), resp. (0.1_0), (0.3_0), has a nontrivial solution. Then, according to the Fredholm property (for more general case see, e.g., [1–3,10,12]) of the problem (0.1), (0.2), resp. (0.1), (0.3), there exist $q \in L([a,b];R)$ and $c \in R$ such that the problem (0.1), (0.2), resp. (0.1), (0.3), has no solution.

Example 3.1. Let $\lambda \in [0,1]$ and let $x_i, y_i \in R_+$ $(i = 1,2,3)$ be such that

$$x_1 + x_2 + \lambda x_3 < 1 \qquad (3.1)$$

and

$$\big(y_1 + \lambda y_2 + \lambda y_3\big)\big(1 - x_2\big) = \lambda - 1 + x_1 + x_2 + \lambda x_3.$$

Let, moreover, $a = 0$, $b = 7$,

$$p(t) = \begin{cases} -y_1 & \text{for} \quad t \in [0,1[\\ x_1 & \text{for} \quad t \in [1,2[\\ x_2 & \text{for} \quad t \in [2,3[\\ -y_2 & \text{for} \quad t \in [3,4[\\ 0 & \text{for} \quad t \in [4,5[\\ x_3 & \text{for} \quad t \in [5,6[\\ -y_3 & \text{for} \quad t \in [6,7] \end{cases}, \tag{3.2}$$

and

$$\tau(t) = \begin{cases} 2 & \text{for} \quad t \in [0,1[\cup [3,4[\cup [6,7] \\ 3 & \text{for} \quad t \in [1,3[\cup [5,6[\\ 4 & \text{for} \quad t \in [4,5[\end{cases}.$$

Obviously, $\tau_0 = 2$, $\tau_1 = 4$, and

$$\begin{gathered} \int_a^{\tau_0} [p(s)]_+ ds = x_1, \quad \int_{\tau_0}^{\tau_1} [p(s)]_+ ds = x_2, \quad \int_{\tau_1}^{b} [p(s)]_+ ds = x_3, \\ \int_a^{\tau_0} [p(s)]_- ds = y_1, \quad \int_{\tau_0}^{\tau_1} [p(s)]_- ds = y_2, \quad \int_{\tau_1}^{b} [p(s)]_- ds = y_3. \end{gathered} \tag{3.3}$$

On the other hand, the function

$$u(t) = \begin{cases} y_1(1-x_2)(1-t) + 1 - x_1 - x_2 & \text{for} \quad t \in [0,1[\\ x_1(t-2) + 1 - x_2 & \text{for} \quad t \in [1,2[\\ x_2(t-3) + 1 & \text{for} \quad t \in [2,3[\\ y_2(1-x_2)(3-t) + 1 & \text{for} \quad t \in [3,4[\\ 1 - y_2(1-x_2) & \text{for} \quad t \in [4,5[\\ x_3(t-5) + 1 - y_2(1-x_2) & \text{for} \quad t \in [5,6[\\ y_3(1-x_2)(7-t) + 1 + x_3 - (y_2+y_3)(1-x_2) & \text{for} \quad t \in [6,7] \end{cases}$$

is a nontrivial solution of the problem (0.1_0), (0.2_0).

This example shows that the strict inequality (1.3) in Theorem 1.1 cannot be replaced by the nonstrict one.

Example 3.2. Let $\lambda \in [0,1]$ and let $x_i, y_i \in R_+$ $(i = 1,2,3)$ be such that (3.1) holds and

$$\big(1 - x_1 - x_2 - \lambda x_3\big)\big(1 + y_2\big) = \lambda - y_1 - \lambda y_3.$$

Let, moreover, $a = 0$, $b = 7$, $p \in L([a,b];R)$ be defined by (3.2), and

$$\tau(t) = \begin{cases} 2 & \text{for} \quad t \in [4,5[\\ 3 & \text{for} \quad t \in [1,3[\cup [5,6[\\ 4 & \text{for} \quad t \in [0,1[\cup [3,4[\cup [6,7] \end{cases}.$$

Obviously, $\tau_0 = 2$, $\tau_1 = 4$, and (3.3) is fulfilled.

On the other hand, the function

$$u(t)=\begin{cases} y_1(1-t)+1+y_2-(x_1+x_2)(1+y_2) & \text{for} \quad t\in[0,1[\\ x_1(1+y_2)(t-2)+1+y_2-x_2(1+y_2) & \text{for} \quad t\in[1,2[\\ x_2(1+y_2)(t-3)+1+y_2 & \text{for} \quad t\in[2,3[\\ y_2(4-t)+1 & \text{for} \quad t\in[3,4[\\ 1 & \text{for} \quad t\in[4,5[\\ x_3(1+y_2)(t-5)+1 & \text{for} \quad t\in[5,6[\\ y_3(7-t)+1-y_3+x_3(1+y_2) & \text{for} \quad t\in[6,7] \end{cases}$$

is a nontrivial solution of the problem (0.1_0), (0.2_0).

This example shows that the strict inequality (1.4) in Theorem 1.1 cannot be replaced by the nonstrict one.

Example 3.3. Let $\lambda\in[0,1]$ and let $x_i, y_i \in R_+$ $(i=1,2,3)$ be such that (3.1) holds and

$$y_1+\lambda y_3<\lambda+\sqrt{1-x_1-x_2-\lambda x_3}\,,$$
$$y_1+y_2+\lambda y_3\geq 1+\lambda+2\sqrt{1-x_1-x_2-\lambda x_3}\,.$$

Put $\alpha=\sqrt{1-x_1-x_2-\lambda x_3}$ and $k=\lambda+\alpha-y_1-\lambda y_3$. Obviously, $k>0$ and $y_2\geq 1+\alpha+k$. Let, moreover, $a=0$, $b=10$,

$$p(t)=\begin{cases} -y_1 & \text{for} \quad t\in[0,1[\\ x_1 & \text{for} \quad t\in[1,2[\\ x_2 & \text{for} \quad t\in[2,3[\\ -k & \text{for} \quad t\in[3,4[\\ -1 & \text{for} \quad t\in[4,5[\\ -\big(y_2-1-\alpha-k\big) & \text{for} \quad t\in[5,6[\\ -\alpha & \text{for} \quad t\in[6,7[\\ 0 & \text{for} \quad t\in[7,8[\\ x_3 & \text{for} \quad t\in[8,9[\\ -y_3 & \text{for} \quad t\in[9,10] \end{cases}, \tag{3.4}$$

and

$$\tau(t)=\begin{cases} 7 & \text{for} \quad t\in[0,1[\,\cup\,[3,4[\,\cup\,[9,10]\\ 4 & \text{for} \quad t\in[1,3[\,\cup\,[4,5[\,\cup\,[6,7[\,\cup\,[8,9[\\ 5 & \text{for} \quad t\in[5,6[\\ 2 & \text{for} \quad t\in[7,8[\end{cases}. \tag{3.5}$$

Obviously, $\tau_0=2$, $\tau_1=7$, and (3.3) is satisfied.

On the other hand, the function

$$u(t)=\begin{cases}\alpha y_1(1-t)+\alpha k+x_1+x_2-1 & \text{for} \quad t\in[0,1[\\ x_1(2-t)+\alpha k+x_2-1 & \text{for} \quad t\in[1,2[\\ x_2(3-t)+\alpha k-1 & \text{for} \quad t\in[2,3[\\ \alpha k(4-t)-1 & \text{for} \quad t\in[3,4[\\ t-5 & \text{for} \quad t\in[4,5[\\ 0 & \text{for} \quad t\in[5,6[\\ \alpha(t-6) & \text{for} \quad t\in[6,7[\\ \alpha & \text{for} \quad t\in[7,8[\\ x_3(8-t)+\alpha & \text{for} \quad t\in[8,9[\\ \alpha y_3(10-t)+\alpha-x_3-\alpha y_3 & \text{for} \quad t\in[9,10]\end{cases} \tag{3.6}$$

is a nontrivial solution of the problem (0.1_0), (0.2_0).

In the sequel, let $\lambda\in[0,1[$ and $x_i=0$ $(i=1,2,3)$. Put

$$\gamma(t)=\delta+\int_a^t [p(s)]_- ds \quad \text{for} \quad t\in[a,b], \tag{3.7}$$

where $\delta>\frac{\lambda}{1-\lambda}(y_1+y_2+y_3)$ and $p\in L([a,b];R)$ is defined by (3.4). Obviously, γ satisfies (1.23) with $\tau\in\mathcal{M}_{ab}$ given by (3.5), (1.24), and

$$\gamma(\tau_0)-\gamma(a)=y_1, \quad \gamma(\tau_1)-\gamma(\tau_0)=y_2, \quad \gamma(b)-\gamma(\tau_1)=y_3. \tag{3.8}$$

Thus, (1.25) is fulfilled and

$$\lambda\big(\gamma(b)-\gamma(\tau_1)\big)+\gamma(\tau_1)-\gamma(a)=y_1+y_2+\lambda y_3\geq 3+\lambda.$$

On the other hand, as we have shown, the problem (0.1_0), (0.2_0) has a nontrivial solution u given by (3.6).

This example shows that the strict inequalities (1.6) in Theorem 1.1 and (1.26) in Theorem 1.4 cannot be replaced by the nonstrict ones.

Example 3.4. Let $\lambda\in[0,1]$ and let $x_i,y_i\in R_+$ $(i=1,2,3)$ be such that (3.1) holds and

$$y_1+\lambda y_3\geq\lambda+\sqrt{1-x_1-x_2-\lambda x_3}\,,$$
$$y_2\geq 1+\frac{1-x_1-x_2-\lambda x_3}{y_1+\lambda y_3-\lambda}\,.$$

Put $\alpha = 1 - x_1 - x_2 - \lambda x_3$ and $\beta = y_1 + \lambda y_3 - \lambda$. Obviously, $\alpha > 0$, $\beta > 0$, and $y_2 \geq 1 + \frac{\alpha}{\beta}$. Let, moreover, $a = 0$, $b = 9$,

$$p(t) = \begin{cases} x_1 & \text{for} \quad t \in [0,1[\\ -y_1 & \text{for} \quad t \in [1,2[\\ x_2 & \text{for} \quad t \in [2,3[\\ -1 & \text{for} \quad t \in [3,4[\\ -\big(y_2 - 1 - \frac{\alpha}{\beta}\big) & \text{for} \quad t \in [4,5[\\ -\frac{\alpha}{\beta} & \text{for} \quad t \in [5,6[\\ 0 & \text{for} \quad t \in [6,7[\\ -y_3 & \text{for} \quad t \in [7,8[\\ x_3 & \text{for} \quad t \in [8,9] \end{cases}, \tag{3.9}$$

and

$$\tau(t) = \begin{cases} 3 & \text{for} \quad t \in [0,1[\cup [2,4[\cup [5,6[\cup [8,9] \\ 6 & \text{for} \quad t \in [1,2[\cup [7,8[\\ 4 & \text{for} \quad t \in [4,5[\\ 2 & \text{for} \quad t \in [6,7[\end{cases}. \tag{3.10}$$

Obviously, $\tau_0 = 2$, $\tau_1 = 6$, and (3.3) is satisfied.

On the other hand, the function

$$u(t) = \begin{cases} \beta x_1(1-t) + y_1\alpha - (1-x_2)\beta & \text{for} \quad t \in [0,1[\\ \alpha y_1(2-t) - (1-x_2)\beta & \text{for} \quad t \in [1,2[\\ \beta x_2(3-t) - \beta & \text{for} \quad t \in [2,3[\\ \beta(t-4) & \text{for} \quad t \in [3,4[\\ 0 & \text{for} \quad t \in [4,5[\\ \alpha(t-5) & \text{for} \quad t \in [5,6[\\ \alpha & \text{for} \quad t \in [6,7[\\ \alpha y_3(7-t) + \alpha & \text{for} \quad t \in [7,8[\\ \beta x_3(9-t) + \alpha(1-y_3) - \beta x_3 & \text{for} \quad t \in [8,9] \end{cases} \tag{3.11}$$

is a nontrivial solution of the problem (0.1_0), (0.2_0).

In the sequel, let $\lambda \in [0,1[$ and $x_i = 0$ $(i = 1,2,3)$. Define the function $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ by (3.7), where $\delta > \frac{\lambda}{1-\lambda}(y_1 + y_2 + y_3)$ and $p \in L([a,b];R)$ is given by (3.9). Obviously, γ satisfies (1.23) with $\tau \in \mathcal{M}_{ab}$ given by (3.10), (1.24), and (3.8). Thus, (1.27) is fulfilled and

$$\gamma(\tau_1) - \gamma(\tau_0) = y_2 \geq 1 + \frac{\alpha}{\beta} = 1 + \frac{1}{\gamma(\tau_0) - \gamma(a) + \lambda\big(\gamma(b) - \gamma(\tau_1)\big) - \lambda}.$$

On the other hand, as we have shown, the problem (0.1_0), (0.2_0) has a nontrivial solution u given by (3.11).

This example shows that the strict inequalities (1.8) in Theorem 1.1 and (1.28) in Theorem 1.4 cannot be replaced by the nonstrict ones.

Example 3.5. Let $\lambda \in \,]0,1]$ and let $x_i, y_i \in R_+$ $(i = 1,2,3)$ be such that

$$y_1 + y_2 + \lambda y_3 < \lambda \tag{3.12}$$

and

$$\big(\lambda - y_1 - y_2 - \lambda y_3\big)\big(1 + x_2\big) = 1 - x_1 - \lambda x_3.$$

Let, moreover, $a = 0$, $b = 7$,

$$p(t) = \begin{cases} -y_1 & \text{for} \quad t \in [0,1[\\ x_1 & \text{for} \quad t \in [1,2[\\ 0 & \text{for} \quad t \in [2,3[\\ -y_2 & \text{for} \quad t \in [3,4[\\ x_2 & \text{for} \quad t \in [4,5[\\ x_3 & \text{for} \quad t \in [5,6[\\ -y_3 & \text{for} \quad t \in [6,7] \end{cases}, \tag{3.13}$$

and

$$\tau(t) = \begin{cases} 5 & \text{for} \quad t \in [0,1[\cup [3,4[\cup [6,7] \\ 4 & \text{for} \quad t \in [1,2[\cup [4,6[\\ 3 & \text{for} \quad t \in [2,3[\end{cases}.$$

Obviously, $\tau_0 = 3$, $\tau_1 = 5$, and (3.3) is fulfilled.

On the other hand, the function

$$u(t) = \begin{cases} y_1(1+x_2)(1-t) + 1 - x_1 + y_2(1+x_2) & \text{for} \quad t \in [0,1[\\ x_1(t-2) + 1 + y_2(1+x_2) & \text{for} \quad t \in [1,2[\\ 1 + y_2(1+x_2) & \text{for} \quad t \in [2,3[\\ y_2(1+x_2)(4-t) + 1 & \text{for} \quad t \in [3,4[\\ x_2(t-5) + 1 + x_2 & \text{for} \quad t \in [4,5[\\ x_3(t-5) + 1 + x_2 & \text{for} \quad t \in [5,6[\\ y_3(1+x_2)(7-t) + 1 + x_2 + x_3 - y_3(1+x_2) & \text{for} \quad t \in [6,7] \end{cases}$$

is a nontrivial solution of the problem (0.1_0), (0.2_0).

This example shows that the strict inequality (1.13) in Theorem 1.2 cannot be replaced by the nonstrict one.

Example 3.6. Let $\lambda \in \,]0,1]$ and let $x_i, y_i \in R_+$ $(i = 1,2,3)$ be such that (3.12) holds and

$$\big(x_1 + \lambda x_2 + \lambda x_3\big)\big(1 - y_2\big) = 1 - \lambda + y_1 + \lambda\big(y_2 + y_3\big).$$

Let, moreover, $a = 0$, $b = 7$, $p \in L([a,b];R)$ be defined by (3.13), and

$$\tau(t) = \begin{cases} 3 & \text{for} \quad t \in [0,1[\cup [3,4[\cup [6,7] \\ 4 & \text{for} \quad t \in [1,2[\cup [4,6[\\ 5 & \text{for} \quad t \in [2,3[\end{cases}.$$

Obviously, $\tau_0 = 3$, $\tau_1 = 5$, and (3.3) is fulfilled.

On the other hand, the function

$$u(t) = \begin{cases} y_1(1-t) + 1 - x_1(1-y_2) & \text{for} \quad t \in [0,1[\\ x_1(1-y_2)(t-2) + 1 & \text{for} \quad t \in [1,2[\\ 1 & \text{for} \quad t \in [2,3[\\ y_2(3-t) + 1 & \text{for} \quad t \in [3,4[\\ x_2(1-y_2)(t-4) + 1 - y_2 & \text{for} \quad t \in [4,5[\\ x_3(1-y_2)(t-5) + 1 - y_2 + x_2(1-y_2) & \text{for} \quad t \in [5,6[\\ y_3(7-t) + 1 - y_2 - y_3 + (x_2+x_3)(1-y_2) & \text{for} \quad t \in [6,7] \end{cases}$$

is a nontrivial solution of the problem (0.1_0), (0.2_0).

This example shows that the strict inequality (1.2) in Theorem 1.2 cannot be replaced by the nonstrict one.

Example 3.7. Let $\lambda \in]0,1]$ and let $x_i, y_i \in R_+$ $(i = 1,2,3)$ be such that (3.12) holds and

$$x_1 + \lambda x_3 < 1 + \sqrt{\lambda - y_1 - y_2 - \lambda y_3}\,,$$
$$x_1 + x_2 + \lambda x_3 \geq 2 + 2\sqrt{\lambda - y_1 - y_2 - \lambda y_3}\,.$$

Put $\alpha = \sqrt{\lambda - y_1 - y_2 - \lambda y_3}$ and $k = 1 + \alpha - x_1 - \lambda x_3$. Obviously, $k > 0$ and $x_2 \geq 1 + \alpha + k$. Let, moreover, $a = 0$, $b = 10$,

$$p(t) = \begin{cases} x_1 & \text{for} \quad t \in [0,1[\\ -y_1 & \text{for} \quad t \in [1,2[\\ -y_2 & \text{for} \quad t \in [2,3[\\ k & \text{for} \quad t \in [3,4[\\ \alpha & \text{for} \quad t \in [4,5[\\ x_2 - 1 - \alpha - k & \text{for} \quad t \in [5,6[\\ 1 & \text{for} \quad t \in [6,7[\\ 0 & \text{for} \quad t \in [7,8[\\ x_3 & \text{for} \quad t \in [8,9[\\ -y_3 & \text{for} \quad t \in [9,10] \end{cases},$$

and

$$\tau(t) = \begin{cases} 4 & \text{for} \quad t \in [0,1[\cup [3,4[\cup [8,9[\\ 7 & \text{for} \quad t \in [1,3[\cup [4,5[\cup [6,7[\cup [9,10] \\ 5 & \text{for} \quad t \in [5,6[\\ 2 & \text{for} \quad t \in [7,8[\end{cases}.$$

Obviously, $\tau_0 = 2$, $\tau_1 = 7$, and (3.3) is satisfied.

On the other hand, the function

$$u(t) = \begin{cases} \alpha x_1(1-t) + y_1 + y_2 + \alpha(k-1) & \text{for} \quad t \in [0,1[\\ y_1(2-t) + y_2 + \alpha(k-1) & \text{for} \quad t \in [1,2[\\ y_2(3-t) + \alpha(k-1) & \text{for} \quad t \in [2,3[\\ \alpha k(4-t) - \alpha & \text{for} \quad t \in [3,4[\\ \alpha(t-5) & \text{for} \quad t \in [4,5[\\ 0 & \text{for} \quad t \in [5,6[\\ t-6 & \text{for} \quad t \in [6,7[\\ 1 & \text{for} \quad t \in [7,8[\\ \alpha x_3(8-t) + 1 & \text{for} \quad t \in [8,9[\\ y_3(10-t) + 1 - y_3 - \alpha x_3 & \text{for} \quad t \in [9,10] \end{cases}$$

is a nontrivial solution of the problem (0.1_0), (0.2_0).

This example shows that the strict inequality (1.16) in Theorem 1.2 cannot be replaced by the nonstrict one.

Example 3.8. Let $\lambda \in]0,1]$ and let $x_i, y_i \in R_+$ $(i = 1,2,3)$ be such that (3.12) holds and

$$x_1 + \lambda x_3 \geq 1 + \sqrt{\lambda - y_1 - y_2 - \lambda y_3}\,,$$
$$x_2 \geq 1 + \frac{\lambda - y_1 - y_2 - \lambda y_3}{x_1 + \lambda x_3 - 1}\,.$$

Put $\alpha = \lambda - y_1 - y_2 - \lambda y_3$ and $\beta = x_1 + \lambda x_3 - 1$. Obviously, $\alpha > 0$, $\beta > 0$, and $x_2 \geq 1 + \frac{\alpha}{\beta}$. Let, moreover, $a = 0$, $b = 9$,

$$p(t) = \begin{cases} x_1 & \text{for} \quad t \in [0,1[\\ -y_1 & \text{for} \quad t \in [1,2[\\ -y_2 & \text{for} \quad t \in [2,3[\\ \frac{\alpha}{\beta} & \text{for} \quad t \in [3,4[\\ x_2 - 1 - \frac{\alpha}{\beta} & \text{for} \quad t \in [4,5[\\ 1 & \text{for} \quad t \in [5,6[\\ 0 & \text{for} \quad t \in [6,7[\\ -y_3 & \text{for} \quad t \in [7,8[\\ x_3 & \text{for} \quad t \in [8,9] \end{cases},$$

and

$$\tau(t) = \begin{cases} 3 & \text{for} \quad t \in [0,1[\cup [8,9] \\ 6 & \text{for} \quad t \in [1,4[\cup [5,6[\cup [7,8[\\ 4 & \text{for} \quad t \in [4,5[\\ 2 & \text{for} \quad t \in [6,7[\end{cases}.$$

Obviously, $\tau_0 = 2$, $\tau_1 = 6$, and (3.3) is satisfied.

On the other hand, the function

$$u(t) = \begin{cases} \alpha x_1(1-t) + \beta(y_1+y_2) - \alpha & \text{for} \quad t \in [0,1[\\ \beta y_1(2-t) + \beta y_2 - \alpha & \text{for} \quad t \in [1,2[\\ \beta y_2(3-t) - \alpha & \text{for} \quad t \in [2,3[\\ \alpha(t-4) & \text{for} \quad t \in [3,4[\\ 0 & \text{for} \quad t \in [4,5[\\ \beta(t-5) & \text{for} \quad t \in [5,6[\\ \beta & \text{for} \quad t \in [6,7[\\ \beta y_3(7-t) + \beta & \text{for} \quad t \in [7,8[\\ \alpha x_3(9-t) + \beta(1-y_3) - \alpha x_3 & \text{for} \quad t \in [8,9] \end{cases}$$

is a nontrivial solution of the problem (0.1_0), (0.2_0).

This example shows that the strict inequality (1.18) in Theorem 1.2 cannot be replaced by the nonstrict one.

Example 3.9. Let $\lambda \in \,]0,1]$ (for the case $\lambda = 0$ see Example 3.11), $k \in \,]0,1[$, and $\varepsilon \geq 0$. Choose $m > 0$ such that

$$m \leq \min\left\{\lambda,\ \frac{\lambda(1-k)k}{\lambda(1-k)+\varepsilon k}\right\}$$

and put $a = 0$, $b = 3$, and

$$p(t) = \begin{cases} -\frac{\lambda-m}{m} & \text{for} \quad t \in [0,1[\\ \frac{k-m}{k} & \text{for} \quad t \in [1,2[\\ \frac{\lambda(1-k)+\varepsilon k}{\lambda k} & \text{for} \quad t \in [2,3] \end{cases}, \qquad \tau(t) = \begin{cases} 1 & \text{for} \quad t \in [0,1[\\ 2 & \text{for} \quad t \in [1,2[\\ t^* & \text{for} \quad t \in [2,3] \end{cases},$$

where

$$t^* = \begin{cases} 1 + \frac{1}{k-m}\left(\frac{\lambda(1-k)k}{\lambda(1-k)+\varepsilon k} - m\right) & \text{if} \quad m \neq k \\ 2 & \text{if} \quad m = k \end{cases}.$$

It is not difficult to verify that $\tau_0 = 1$, $\tau_1 = 2$, and

$$\int_a^{\tau_1} [p(s)]_+ ds = \int_{\tau_0}^{\tau_1} [p(s)]_+ ds = \frac{k-m}{k}, \quad \int_{\tau_1}^{b} [p(s)]_+ ds = \frac{\lambda(1-k)+\varepsilon k}{\lambda k},$$

$$\int_a^{\tau_0} [p(s)]_- ds = \frac{\lambda - m}{m}, \quad \int_{\tau_0}^{b} [p(s)]_- ds = \int_{\tau_0}^{\tau_1} [p(s)]_- ds = 0\,.$$

Thus, the condition (1.19) holds, $T = \frac{k-m}{k}$, and

$$\int_a^{\tau_1} [p(s)]_+ ds + \lambda \int_{\tau_1}^{b} [p(s)]_+ ds -$$
$$- \left(\int_a^{\tau_0} [p(s)]_- ds + \lambda \int_{\tau_0}^{b} [p(s)]_- ds\right)\big(1-T\big) = 1 - \lambda + \varepsilon\,.$$

On the other hand, the function

$$u(t)=\begin{cases}\lambda-(\lambda-m)t & \text{for} \quad t\in[0,1[\\ (k-m)(t-1)+m & \text{for} \quad t\in[1,2[\\ (1-k)(t-3)+1 & \text{for} \quad t\in[2,3]\end{cases}$$

is a nontrivial solution of the problem (0.1_0), (0.2_0).

This example shows that the strict inequality (1.20) in Theorem 1.3 cannot be replaced by the nonstrict one.

Example 3.10. Let $\lambda\in[0,1]$, $k>1$, and $\varepsilon\in[0,1[$. Choose $M\geq\frac{k+\lambda}{1-\varepsilon}$ and put $a=0$, $b=4$, and

$$p(t)=\begin{cases}\frac{M-\lambda-\varepsilon M}{k} & \text{for} \quad t\in[0,1[\\ -\frac{M-k}{M} & \text{for} \quad t\in[1,2[\\ -\frac{k-1}{M} & \text{for} \quad t\in[2,3[\\ 0 & \text{for} \quad t\in[3,4]\end{cases}, \qquad \tau(t)=\begin{cases}t^* & \text{for} \quad t\in[0,1[\\ 1 & \text{for} \quad t\in[1,3[\\ 2 & \text{for} \quad t\in[3,4]\end{cases},$$

where

$$t^*=\begin{cases}2-\frac{\varepsilon Mk}{(M-k)(M-\lambda-\varepsilon M)} & \text{if} \quad M\neq k\\ 2 & \text{if} \quad M=k\end{cases}.$$

It is not difficult to verify that $\tau_0=1$, $\tau_1=2$, and

$$\int_a^{\tau_0}[p(s)]_+ds=\frac{M-\lambda-\varepsilon M}{k}, \qquad \int_{\tau_0}^{b}[p(s)]_+ds=\int_{\tau_0}^{\tau_1}[p(s)]_+ds=0,$$

$$\int_a^{\tau_1}[p(s)]_-ds=\int_{\tau_0}^{\tau_1}[p(s)]_-ds=\frac{M-k}{M}, \qquad \int_{\tau_1}^{b}[p(s)]_-ds=\frac{k-1}{M}.$$

Thus, the condition (1.19) holds, $T=\frac{M-k}{M}$, and

$$\left(\int_a^{\tau_0}[p(s)]_+ds+\lambda\int_{\tau_0}^{b}[p(s)]_+ds\right)\big(1-T\big)-$$

$$-\int_a^{\tau_1}[p(s)]_-ds-\lambda\int_{\tau_1}^{b}[p(s)]_-ds=\big(1-\lambda\big)\big(1-T\big)-\varepsilon.$$

On the other hand, the function

$$u(t)=\begin{cases}\lambda+(M-\lambda)t & \text{for} \quad t\in[0,1[\\ (M-k)(2-t)+k & \text{for} \quad t\in[1,2[\\ (k-1)(3-t)+1 & \text{for} \quad t\in[2,3[\\ 1 & \text{for} \quad t\in[3,4]\end{cases}$$

is a nontrivial solution of the problem (0.1_0), (0.2_0).

This example shows that the strict inequality (1.21) in Theorem 1.3 cannot be replaced by the nonstrict one.

Example 3.11. Let $\lambda \in [0,+\infty[$, $k>1$, and $\varepsilon \ge 0$. Choose $m>0$ such that

$$m \le \min\left\{1,\ \frac{k(k+\lambda)}{k+\lambda+\varepsilon k}\right\}$$

and put $a=0$, $b=3$, and

$$p(t)=\begin{cases}\frac{k+\lambda}{k}+\varepsilon & \text{for} \quad t\in[0,1[\\ -\frac{k-m}{k} & \text{for} \quad t\in[1,2[\\ \frac{1-m}{m} & \text{for} \quad t\in[2,3]\end{cases}, \quad \tau(t)=\begin{cases}t^* & \text{for} \quad t\in[0,1[\\ 1 & \text{for} \quad t\in[1,2[\\ 2 & \text{for} \quad t\in[2,3]\end{cases},$$

where

$$t^*=\begin{cases}2-\frac{1}{k-m}\left(\frac{k(k+\lambda)}{k+\lambda+\varepsilon k}-m\right) & \text{if} \quad m\ne k\\ 1 & \text{if} \quad m=k\end{cases}.$$

It is not difficult to verify that $\tau_0=1$, $\tau_1=2$, and

$$\int_a^{\tau_1}[p(s)]_+ds=\frac{k+\lambda}{k}+\varepsilon\,, \quad \int_{\tau_1}^b[p(s)]_+ds=\frac{1-m}{m}\,, \quad \int_{\tau_0}^{\tau_1}[p(s)]_+ds=0\,,$$

$$\int_a^{\tau_0}[p(s)]_-ds=0\,, \quad \int_{\tau_0}^b[p(s)]_-ds=\int_{\tau_0}^{\tau_1}[p(s)]_-ds=\frac{k-m}{k}\,.$$

Thus, the condition (1.19) holds, $T=\frac{k-m}{k}$, and

$$\int_a^{\tau_1}[p(s)]_+ds+\lambda\int_{\tau_0}^b[p(s)]_-ds-$$
$$-\left(\int_a^{\tau_0}[p(s)]_-ds+\lambda\int_{\tau_1}^b[p(s)]_+ds\right)\big(1-T\big)=1+\lambda+\varepsilon\,.$$

On the other hand, the function

$$u(t)=\begin{cases}(k+\lambda)t-\lambda & \text{for} \quad t\in[0,1[\\ (k-m)(2-t)+m & \text{for} \quad t\in[1,2[\\ (1-m)(t-3)+1 & \text{for} \quad t\in[2,3]\end{cases}$$

is a nontrivial solution of the problem (0.1_0), (0.3_0).

This example shows that the strict inequality (2.1) in Theorem 2.1 cannot be replaced by the nonstrict one.

Example 3.12. Let $\lambda \in]0,+\infty[$ (for the case $\lambda=0$ see Example 3.10), $k\in]0,\lambda[$, and $\varepsilon \in \left[0,\frac{\lambda-k}{k}\right[$. Choose $M\ge \frac{k(\lambda-k)}{\lambda-k-\varepsilon k}$ and put $a=0$, $b=3$, and

$$p(t)=\begin{cases}-\left(\frac{\lambda-k}{M}+\varepsilon\right) & \text{for} \quad t\in[0,1[\\ \frac{M-k}{M} & \text{for} \quad t\in[1,2[\\ -\frac{M+1}{k} & \text{for} \quad t\in[2,3]\end{cases}, \quad \tau(t)=\begin{cases}t^* & \text{for} \quad t\in[0,1[\\ 2 & \text{for} \quad t\in[1,2[\\ 1 & \text{for} \quad t\in[2,3]\end{cases},$$

where

$$t^* = \begin{cases} 1 + \frac{1}{M-k}\left(\frac{M(\lambda-k)}{\lambda-k+\varepsilon M} - k\right) & \text{if} \quad M \neq k \\ 2 & \text{if} \quad M = k \end{cases}.$$

It is not difficult to verify that $\tau_0 = 1$, $\tau_1 = 2$, and

$$\int_a^{\tau_0} [p(s)]_+ ds = 0\,, \quad \int_{\tau_0}^b [p(s)]_+ ds = \int_{\tau_0}^{\tau_1} [p(s)]_+ ds = \frac{M-k}{M}\,,$$

$$\int_a^{\tau_1} [p(s)]_- ds = \frac{\lambda - k}{M} + \varepsilon\,, \quad \int_{\tau_0}^{\tau_1} [p(s)]_- ds = 0\,, \quad \int_{\tau_1}^b [p(s)]_- ds = \frac{M+1}{k}\,.$$

Thus, the condition (1.19) holds, $T = \frac{M-k}{M}$, and

$$\left(\int_a^{\tau_0} [p(s)]_+ ds + \lambda \int_{\tau_1}^b [p(s)]_- ds\right)\big(1-T\big) -$$

$$- \int_a^{\tau_1} [p(s)]_- ds - \lambda \int_{\tau_0}^b [p(s)]_+ ds = \big(1+\lambda\big)\big(1-T\big) - \varepsilon\,.$$

On the other hand, the function

$$u(t) = \begin{cases} \lambda - (\lambda - k)t & \text{for} \quad t \in [0,1[\\ (M-k)(t-1) + k & \text{for} \quad t \in [1,2[\\ (M+1)(3-t) - 1 & \text{for} \quad t \in [2,3] \end{cases}$$

is a nontrivial solution of the problem (0.1_0), (0.3_0).

This example shows that the strict inequality (2.2) in Theorem 2.1 cannot be replaced by the nonstrict one.

Example 3.13. Let $\lambda \in [0,1]$ and let $x_i, y_i \in R_+$ $(i = 1,2,3)$ be such that

$$x_2 \geq 1$$

and

$$x_1 + x_2 + \lambda\big(y_2 + y_3\big) + \big(y_1 + \lambda x_3\big)\big(x_2 - 1\big) = 1 + \lambda.$$

Let, moreover, $a = 0$, $b = 8$,

$$p(t) = \begin{cases} -y_1 & \text{for} \quad t \in [0,1[\\ x_1 & \text{for} \quad t \in [1,2[\\ x_2 - 1 & \text{for} \quad t \in [2,3[\\ 0 & \text{for} \quad t \in [3,4[\\ 1 & \text{for} \quad t \in [4,5[\\ -y_2 & \text{for} \quad t \in [5,6[\\ -y_3 & \text{for} \quad t \in [6,7[\\ x_3 & \text{for} \quad t \in [7,8] \end{cases},$$

and

$$\tau(t) = \begin{cases} 2 & \text{for} \quad t \in [0,1[\cup [7,8] \\ 5 & \text{for} \quad t \in [1,3[\cup [4,7[\\ 6 & \text{for} \quad t \in [3,4[\end{cases}.$$

Obviously, $\tau_0 = 2$, $\tau_1 = 6$, and (3.3) is satisfied.

On the other hand, the function

$$u(t) = \begin{cases} y_1(x_2-1)(t-1)+1-x_2-x_1 & \text{for} \quad t \in [0,1[\\ x_1(t-2)+1-x_2 & \text{for} \quad t \in [1,2[\\ (x_2-1)(t-3) & \text{for} \quad t \in [2,3[\\ 0 & \text{for} \quad t \in [3,4[\\ t-4 & \text{for} \quad t \in [4,5[\\ y_2(5-t)+1 & \text{for} \quad t \in [5,6[\\ y_3(6-t)+1-y_2 & \text{for} \quad t \in [6,7[\\ x_3(x_2-1)(7-t)+1-y_2-y_3 & \text{for} \quad t \in [7,8] \end{cases}$$

is a nontrivial solution of the problem (0.1_0), (0.3_0).

This example shows that the strict inequality (2.5) in Theorem 2.2 cannot be replaced by the nonstrict one.

Example 3.14. Let $\lambda \in [0,1]$ and let $x_i, y_i \in R_+$ $(i = 1,2,3)$ be such that

$$y_2 \geq 1\,, \qquad x_1 + x_2 + \lambda y_3 \geq 1 - \lambda^2\,,$$

and

$$x_1 + x_2 + \lambda\big(y_2 + y_3\big) + \big(y_1 + \lambda x_3\big)\big(y_2 - 1\big) = 1 + \lambda.$$

Let, moreover, $a = 0$, $b = 8$,

$$p(t) = \begin{cases} x_1 & \text{for} \quad t \in [0,1[\\ -y_1 & \text{for} \quad t \in [1,2[\\ x_2 & \text{for} \quad t \in [2,3[\\ -1 & \text{for} \quad t \in [3,4[\\ 0 & \text{for} \quad t \in [4,5[\\ -(y_2-1) & \text{for} \quad t \in [5,6[\\ x_3 & \text{for} \quad t \in [6,7[\\ -y_3 & \text{for} \quad t \in [7,8] \end{cases},$$

and

$$\tau(t) = \begin{cases} 3 & \text{for} \quad t \in [0,1[\cup [2,4[\cup [5,6[\cup [7,8] \\ 6 & \text{for} \quad t \in [1,2[\cup [6,7[\\ 2 & \text{for} \quad t \in [4,5[\end{cases}.$$

Obviously, $\tau_0 = 2$, $\tau_1 = 6$, and (3.3) is satisfied.

On the other hand, the function

$$u(t)=\begin{cases} x_1(t-1)+1-x_2-y_1(y_2-1) & \text{for} \quad t\in[0,1[\\ y_1(y_2-1)(t-2)+1-x_2 & \text{for} \quad t\in[1,2[\\ x_2(t-3)+1 & \text{for} \quad t\in[2,3[\\ 4-t & \text{for} \quad t\in[3,4[\\ 0 & \text{for} \quad t\in[4,5[\\ (y_2-1)(6-t)+1-y_2 & \text{for} \quad t\in[5,6[\\ x_3(y_2-1)(7-t)+1-y_2-x_3(y_2-1) & \text{for} \quad t\in[6,7[\\ y_3(8-t)+1-y_2-x_3(y_2-1)-y_3 & \text{for} \quad t\in[7,8] \end{cases}$$

is a nontrivial solution of the problem (0.1_0), (0.3_0).

This example shows that the strict inequality (2.8) in the assumption b) of Theorem 2.2 cannot be replaced by the nonstrict one.

Example 3.15. Let $\lambda\in[0,1]$ and let $x_i,y_i\in R_+$ $(i=1,2,3)$ be such that

$$y_2\geq 1\,,\qquad x_1+x_2+\lambda y_3<1-\lambda^2, \tag{3.14}$$
$$y_1+\lambda x_3<-\lambda+\sqrt{1-x_1-x_2-\lambda y_3}\,,$$
$$y_1+y_2+\lambda x_3\geq 1-\lambda+2\sqrt{1-x_1-x_2-\lambda y_3}\,.$$

Put $\alpha=\sqrt{1-x_1-x_2-\lambda y_3}$ and $k=\alpha-\lambda-y_1-\lambda x_3$. Obviously, $k>0$ and $y_2\geq 1+\alpha+k$. Let, moreover, $a=0$, $b=10$, $p\in L([a,b];R)$ be defined by (3.4), and

$$\tau(t)=\begin{cases} 8 & \text{for} \quad t\in[0,1[\,\cup\,[3,4[\,\cup\,[8,9[\\ 4 & \text{for} \quad t\in[1,3[\,\cup\,[4,5[\,\cup\,[6,7[\,\cup\,[9,10]\\ 5 & \text{for} \quad t\in[5,6[\\ 2 & \text{for} \quad t\in[7,8[\end{cases}.$$

Obviously, $\tau_0=2$, $\tau_1=8$, and (3.3) is satisfied.

On the other hand, the function

$$u(t)=\begin{cases} \alpha y_1(1-t)+x_1+x_2+k\alpha-1 & \text{for} \quad t\in[0,1[\\ x_1(2-t)+x_2+k\alpha-1 & \text{for} \quad t\in[1,2[\\ x_2(3-t)+k\alpha-1 & \text{for} \quad t\in[2,3[\\ k\alpha(4-t)-1 & \text{for} \quad t\in[3,4[\\ t-5 & \text{for} \quad t\in[4,5[\\ 0 & \text{for} \quad t\in[5,6[\\ \alpha(t-6) & \text{for} \quad t\in[6,7[\\ \alpha & \text{for} \quad t\in[7,8[\\ \alpha x_3(t-8)+\alpha & \text{for} \quad t\in[8,9[\\ y_3(t-9)+\alpha+\alpha x_3 & \text{for} \quad t\in[9,10] \end{cases}$$

is a nontrivial solution of the problem (0.1_0), (0.3_0).

This example shows that the strict inequality (2.11) in Theorem 2.2 cannot be replaced by the nonstrict one.

Example 3.16. Let $\lambda \in [0,1]$ and let $x_i, y_i \in R_+$ $(i = 1,2,3)$ be such that (3.14) holds and

$$y_1 + \lambda x_3 \geq -\lambda + \sqrt{1 - x_1 - x_2 - \lambda y_3}\,,$$
$$x_1 + x_2 + \lambda\big(y_2 + y_3\big) + \big(y_1 + \lambda x_3\big)\big(y_2 - 1\big) \geq 1 + \lambda.$$

Put $\alpha = 1 - x_1 - x_2 - \lambda y_3$ and $\beta = y_1 + \lambda x_3 + \lambda$. Obviously, $\alpha > 0$, $\beta > 0$, and $y_2 \geq 1 + \frac{\alpha}{\beta}$. Let, moreover, $a = 0$, $b = 9$, $p \in L([a,b];R)$ be defined by (3.9), and

$$\tau(t) = \begin{cases} 3 & \text{for} \quad t \in [0,1[\, \cup\, [2,4[\, \cup\, [5,6[\, \cup\, [7,8[\\ 7 & \text{for} \quad t \in [1,2[\, \cup\, [8,9] \\ 4 & \text{for} \quad t \in [4,5[\\ 2 & \text{for} \quad t \in [6,7[\end{cases}.$$

Obviously, $\tau_0 = 2$, $\tau_1 = 7$, and (3.3) is satisfied.

On the other hand, the function

$$u(t) = \begin{cases} x_1\beta(1-t) + y_1\alpha + x_2\beta - \beta & \text{for} \quad t \in [0,1[\\ y_1\alpha(2-t) + x_2\beta - \beta & \text{for} \quad t \in [1,2[\\ x_2\beta(3-t) - \beta & \text{for} \quad t \in [2,3[\\ \beta(t-4) & \text{for} \quad t \in [3,4[\\ 0 & \text{for} \quad t \in [4,5[\\ \alpha(t-5) & \text{for} \quad t \in [5,6[\\ \alpha & \text{for} \quad t \in [6,7[\\ y_3\beta(t-7) + \alpha & \text{for} \quad t \in [7,8[\\ x_3\alpha(t-8) + \alpha + y_3\beta & \text{for} \quad t \in [8,9] \end{cases}$$

is a nontrivial solution of the problem (0.1_0), (0.3_0).

This example shows that the strict inequality (2.8) in the assumption c) of Theorem 2.2 cannot be replaced by the nonstrict one.

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References

1. N. V. Azbelev, V. P. Maksimov, L. F. Rakhmatullina, ntroduction to the theory of functional differential equations. (Russian) *Nauka, Moscow,* 1991.
2. N. V. Azbelev and L. F. Rakhmatullina, Theory of linear abstract functional differential equations and applications. *Mem. Differential Equations Math. Phys.* **8**(1996), 1–102.

3. E. Bravyi, A note on the Fredholm property of boundary value problems for linear functional differential equations. *Mem. Differential Equations Math. Phys.* **20**(2000), 133–135.
4. R. Hakl and A. Lomtatidze, A note on the Cauchy problem for first order linear differential equations with a deviating argument. *Arch. Math.* (*Brno*) **38**(2002), No. 1, 61–71.
5. R. Hakl, A. Lomtatidze, and J. Šremr, On a periodic type boundary value problem for first order linear functional differential equations. *Nonlinear Oscillations* **5**(2002), No. 3, 416–432.
6. R. Hakl, A. Lomtatidze, and J. Šremr, On an antiperiodic type boundary value problem for first order linear functional differential equations. *Arch. Math.* (*Brno*) **38**(2002), No. 2, 149–160.
7. R. Hakl, A. Lomtatidze, and J. Šremr, On nonnegative solutions of a periodic type boundary value problem for first order scalar functional differential equations. *Adv. Differential Equations* (submitted).
8. R. Hakl, A. Lomtatidze, and J. Šremr, Solvability and unique solvability of a periodic type boundary value problem for first order scalar functional differential equations. *Georgian Math. J.* **9**(2002), No. 3, 525–547.
9. J. Hale, Theory of functional differential equations. Second edition. *Applied Mathematical Sciences,* 3. *Springer-Verlag, New York-Heidelberg,* 1977.
10. I. Kiguradze and B. Půža, On boundary value problems for systems of linear functional differential equations. *Czechoslovak Math. J.* **47**(1997), No. 2, 341–373.
11. V. Kolmanovskii and A. Myshkis, Introduction to the theory and applications of functional differential equations. *Mathematics and its Applications,* 463. *Kluwer Academic Publishers, Dordrecht,* 1999.
12. Š. Schwabik, M. Tvrdý, and O. Vejvoda, Differential and integral equations. Boundary value problems and adjoints. *D. Reidel Publishing Co., Dordrecht-Boston, Mass.-London, in co-ed. with Academia, Publishing House of the Czechoslovak Academy of Sciences, Praha*, 1979.

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