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FORMULAS OF VARIATION OF SOLUTION FOR NON-LINEAR CONTROLLED DELAY DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS INITIAL CONDITION

Abstract. Formulas of variation of solution of a non-linear controlled differential equation with variable delays and with discontinuous initial condition are proved. The discontinuous initial condition means that at the initial moment the values of the initial function and the trajectory, generally speaking, do not coincide. The obtained ones, in contrast to the well-known formulas, contain new terms which are connected with the variation of the initial moment and discontinuity of the initial condition.

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Introduction

The formulas of variation of solution play an important role in proving necessary conditions of optimality for optimal problems [1]–[5]. In the present work we prove the formulas of variation of solution for controlled differential equations with variable delays and discontinuous initial condition. These formulas are analogous to those given in [6] and their proof carried out by the method given in [7].

1. Formulation of Main Results

Let $\mathcal{J}=[a,b]$ be a finite interval; $\mathcal{O}\subset R^n$, $G\subset R^r$ be open sets. The function $f:\mathcal{J}\times\mathcal{O}^s\times G\to R^n$ satisfies the following conditions: for almost all $t\in\mathcal{J}$, the function $f(t,\cdot):\mathcal{O}^s\times G\to R^n$ is continuously differentiable; for any $(x_1,\dots,x_s,u)\in\mathcal{O}^s\times G$, the functions $f(t,x_1,\dots,x_s,u)$, $f_{x_i}(t,x_1,\dots,x_s,u)$, $i=1,\dots,s$, $f_u(t,x_1,\dots,x_s,u)$ are measurable on $\mathcal{J}$; for arbitrary compacts $K\subset\mathcal{O}$, $M\subset G$ there exists a function $m_{K,M}(\cdot)\in L(\mathcal{J},R_+)$, $R_+=[0,+\infty)$, such that for any $(x_1,\dots,x_s,u)\in K^s\times M$ and for almost all $t\in\mathcal{J}$, the following inequality is fulfilled

$$|f(t,x_1,\dots,x_s,u)|+\sum_{i=1}^{s}|f_{x_i}(\cdot)|+|f_u(\cdot)|\le m_{K,M}(t).$$

Let the scalar functions $\tau_i(t)$, $i=1,\dots,s$, $t\in R$, be absolutely continuous satisfying the conditions: $\tau_i(t)\le t$, $\dot\tau_i(t)>0$, $i=1,\dots,s$. Let Φ be the set of piecewise continuous functions $\varphi:\mathcal{J}_1=[\tau,b]\to\mathcal{O}$ with a finite number of discontinuity points of the first kind, satisfying the conditions $\operatorname{cl}\varphi(\mathcal{J}_1)\subset\mathcal{O}$, $\tau=\min\{\tau_1(a),\dots,\tau_s(a)\}$, $\|\varphi\|=\sup\{|\varphi(t)|,\ t\in\mathcal{J}_1\}$; Ω be the set of measurable functions $u:\mathcal{J}\to G$, satisfying condition $\operatorname{cl}\{u(t):\ t\in\mathcal{J}\}$ is a compact lying in G, $\|u\|=\sup\{|u(t)|\ :\ t\in\mathcal{J}\}$.

To every element $\wp=(t_0,x_0,\varphi,u)\in A=\mathcal{J}\times\mathcal{O}\times\Phi\times\Omega$, let us correspond the differential equation

$$\dot x(t)=f(t,x(\tau_1(t)),\dots,x(\tau_s(t)),u(t)) \tag{1.1}$$

with discontinuous initial condition

$$x(t)=\varphi(t),\quad t\in[\tau,t_0),\quad x(t_0)=x_0. \tag{1.2}$$

Definition 1.1. Let $\wp=(t_0,x_0,\varphi,u)\in A$, $t_0<b$. A function $x(t)=x(t;\wp)\in\mathcal{O}$, $t\in[\tau,t_1]$, $t_1\in(t_0,b]$, is said to be a solution of the equation (1.1) with the initial condition (1.2), or a solution corresponding to the element $\wp\in A$, defined on the interval $[\tau,t_1]$, if on the interval $[\tau,t_0]$ the function $x(t)$ satisfies the condition (1.2), while on the interval $[t_0,t_1]$ it is absolutely continuous and almost everywhere satisfies the equation (1.1).

Let us introduce the set of variation

$$\begin{aligned}V=\{\delta\wp=(\delta t_0,\delta x_0,\delta\varphi,\delta u)\ :\ \delta\varphi\in\Phi-\widetilde\varphi,\ \ \delta u\in\Omega-\widetilde u,\\ |\delta t_0|\le\alpha,\ \ |\delta x_0|\le\alpha,\ \ \|\delta\varphi\|\le\alpha,\ \ \|\delta u\|\le\alpha\},\end{aligned} \tag{1.3}$$

where $\widetilde{\varphi} \in \Phi$, $\widetilde{u} \in \Omega$ are fixed functions, $\alpha > 0$ is a fixed number.

Let $\widetilde{x}(t)$ be a solution corresponding to the element $\widetilde{\wp} = (\widetilde{t}_0, \widetilde{x}_0, \widetilde{\varphi}, \widetilde{u}) \in A$, defined on the interval $[\tau, \widetilde{t}_1]$, $\widetilde{t}_i \in (a, b)$, $i = 0, 1$. There exist numbers $\varepsilon_1 > 0$, $\delta_1 > 0$ such that for an arbitrary $(\varepsilon, \delta\wp) \in [0, \varepsilon_1] \times V$, to the element $\widetilde{\wp} + \varepsilon\delta\wp \in A$ there corresponds a solution $x(t; \widetilde{\wp} + \varepsilon\delta\wp)$, defined on $[\tau, \widetilde{t}_1 + \delta_1]$.

Due to uniqueness, the solution $x(t, \widetilde{\wp})$ is a continuation of the solution $\widetilde{x}(t)$ on the interval $[\tau, \widetilde{t}_1 + \delta_1]$. Therefore the solution $\widetilde{x}(t)$ in the sequel is assumed to be defined on the interval $[\tau, \widetilde{t}_1 + \delta_1]$, (see Lemma 2.2)

Let us define the increment of the solution $\widetilde{x}(t) = x(t; \widetilde{\wp})$

$$\begin{gathered} \Delta x(t) = \Delta x(t; \varepsilon\delta\wp) = x(t; \widetilde{\wp} + \varepsilon\delta\wp) - \widetilde{x}(t), \\ (t, \varepsilon, \delta\wp) \in [\tau, \widetilde{t}_1 + \delta_1] \times [0, \varepsilon_1] \times V. \end{gathered} \tag{1.4}$$

In order to formulate the main results, we will need the following notation

$$\begin{aligned} \sigma_i^- &= \big(\widetilde{t}_0, \underbrace{\widetilde{x}_0, \dots, \widetilde{x}_0}_{i\text{-times}}, \underbrace{\widetilde{\varphi}(\widetilde{t}_0^-), \dots, \widetilde{\varphi}(\widetilde{t}_0^-)}_{(p-i)\text{-times}}, \widetilde{\varphi}(\tau_{p+1}(\widetilde{t}_0^-)), \dots, \widetilde{\varphi}(\tau_s(\widetilde{t}_0^-))\big), \\ &\qquad i = 0, \dots, p, \\ \sigma_i^- &= \big(\gamma_i, \widetilde{x}(\tau_1(\gamma_i)), \dots, \widetilde{x}(\tau_{i-1}(\gamma_i)), \widetilde{x}_0, \\ &\qquad \widetilde{\varphi}(\tau_{i+1}(\gamma_i^-)), \dots, \widetilde{\varphi}(\tau_s(\gamma_i^-))\big), \\ \overset{\circ}{\sigma}{}_i^- &= \big(\gamma_i, \widetilde{x}(\tau_1(\gamma_i)), \dots, \widetilde{x}(\tau_{i-1}(\gamma_i)), \widetilde{\varphi}(\widetilde{t}_0^-), \\ &\qquad \widetilde{\varphi}(\tau_{i+1}(\gamma_i^-)), \dots, \widetilde{\varphi}(\tau_s(\gamma_i^-))\big), \quad i = p+1, \dots, s, \\ \gamma_i &= \gamma_i(\widetilde{t}_0); \quad \gamma_i(t) = \tau_i^{-1}(t); \quad \dot{\gamma}_i^- = \dot{\gamma}_i(\widetilde{t}_0^-); \\ \omega &= (t, x_1, \dots, x_s), \quad \widetilde{f}[\omega] = f(\omega, \widetilde{u}(t)), \\ \widetilde{f}_{x_i}[t] &= f_{x_i}(t, \widetilde{x}(\tau_1(t)), \dots, \widetilde{x}(\tau_s(t)), \widetilde{u}(t)). \end{aligned} \tag{1.5}$$

Theorem 1. *Let the following conditions be fulfilled:*

1.1. $\gamma_i = \widetilde{t}_0$, $\quad i = 1, \dots, p$, $\quad \widetilde{t}_0 < \gamma_{p+1} < \dots < \gamma_s < \widetilde{t}_1$;

1.2. *There exists a number $\delta > 0$ such that*

$$t \le \gamma_1(t) \le \dots \le \gamma_p(t), \quad t \in (\widetilde{t}_0 - \delta, \widetilde{t}_0];$$

1.3. *There exist the finite limits*

$$\begin{gathered} \dot{\gamma}_i^-, \quad i = 1, \dots, s, \\ \lim_{\omega \to \sigma_i^-} \widetilde{f}[\omega] = f_i^-, \quad \omega \in (\widetilde{t}_0 - \delta, \widetilde{t}_0] \times \mathcal{O}^s, \quad i = 0, \dots, p, \\ \lim_{(\omega_1, \omega_2) \to (\sigma_i^-, \overset{\circ}{\sigma}{}_i^-)} \big[\widetilde{f}[\omega_1] - \widetilde{f}[\omega_2]\big] = f_i^-, \quad \omega_1, \omega_2 \in (\gamma_i - \delta, \gamma_i) \times \mathcal{O}^s, \\ i = p+1, \dots, s. \end{gathered}$$

Then there exist numbers $\varepsilon_2 > 0$, $\delta_2 > 0$ such that for an arbitrary $(t, \varepsilon, \delta\wp) \in [\widetilde{t}_1 - \delta_2, \widetilde{t}_1 + \delta_2] \times [0, \varepsilon_2] \times V^-$, $V^- = \{\delta\wp \in V : \delta t_0 \le 0\}$, the *formula*

$$\Delta x(t;\varepsilon\delta\wp)=\varepsilon\delta x(t;\delta\wp)+o(t;\varepsilon\delta\wp)^1 \tag{1.6}$$

is valid, where

$$\delta x(t;\delta\wp)=\Big\{Y(\widetilde{t}_0;t)\sum_{i=0}^{p}(\widehat{\gamma}_{i+1}^{-}-\widehat{\gamma}_i^{-})f_i^{-}-\\ -\sum_{i=p+1}^{s}Y(\gamma_i;t)f_i^{-}\dot{\gamma}_i^{-}\Big\}\delta t_0+\beta(t;\delta\wp), \tag{1.7}$$

$$\widehat{\gamma}_0^{-}=1,\quad \widehat{\gamma}_i^{-}=\dot{\gamma}_i^{-},\quad i=1,\dots,p,\quad \widehat{\gamma}_{p+1}^{-}=0,$$

$$\beta(t;\delta\wp)=Y(\widetilde{t}_0;t)\delta x_0+\sum_{i=p+1}^{s}\int_{\tau_i(\widetilde{t}_0)}^{\widetilde{t}_0}Y(\gamma_i(\xi);t)\widetilde{f}_{x_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\delta\varphi(\xi)d\xi+\\ +\int_{\widetilde{t}_0}^{t}Y(\xi;t)\widetilde{f}_u[\xi]\delta u(\xi)d\xi.$$

$Y(\xi;t)$ *is the matrix-function satisfying the equation*

$$\frac{\partial Y(\xi;t)}{\partial\xi}=-\sum_{i=1}^{s}Y(\gamma_i(\xi);t)\widetilde{f}_{x_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi),\quad \xi\in[\widetilde{t}_0,t], \tag{1.8}$$

and the condition

$$Y(\xi;t)=\begin{cases}I, & \xi=t,\\ \Theta, & \xi>t.\end{cases} \tag{1.9}$$

Here I *is the identity matrix,* Θ *is the zero matrix.*

Theorem 2. *Let the condition* 1.1 *and the following conditions be fulfilled:*

1.4. *There exists a number* $\delta>0$ *such that*

$$t\le\gamma_1(t)\le\dots\le\gamma_p(t),\quad t\in[\widetilde{t}_0,\widetilde{t}_0+\delta);$$

1.5. *There exist the finite limits*

$$\dot{\gamma}_i^{+},\quad i=1,\dots,s,$$

$$\lim_{\omega\to\sigma_i^{+}}\widetilde{f}[\omega]=f_i^{+},\quad \omega\in[\widetilde{t}_0,\widetilde{t}_0+\delta)\times\mathcal{O}^s,\quad i=0,\dots,p,$$

$$\lim_{(\omega_1,\omega_2)\to(\sigma_i^{+},\overset{\circ}{\sigma}{}_i^{+})}\big[\widetilde{f}[\omega_1]-\widetilde{f}[\omega_2]\big]=f_i^{+},\quad \omega_1,\omega_2\in[\gamma_i,\gamma_i+\delta)\times\mathcal{O}^s,$$

$$i=p+1,\dots,s\ \ (\textit{see}\ (1.5)).$$

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[1]Here and in the sequel, the values (scalar or vector) which have the corresponding order of smallness uniformly for $(t,\delta\wp)$, will be denoted by $O(t;\varepsilon\delta\wp)$, $o(t;\varepsilon\delta\wp)$.

Then there exist numbers $\varepsilon_2 > 0$, $\delta_2 > 0$ *such that for an arbitrary* $(t,\varepsilon,\delta\wp) \in [\widetilde{t}_1 - \delta_2, \widetilde{t}_1 + \delta_2] \times [0,\varepsilon_2] \times V^+$, $V^+ = \{\delta\wp \in V : \delta t_0 \geq 0\}$, *the formula* (1.6) *is valid, where* $\delta x(t;\delta\wp)$ *has the form*

$$\begin{gathered}\delta x(t;\delta\wp) = \bigg\{ Y(\widetilde{t}_0;t)\sum_{i=0}^{p}(\widehat{\gamma}_{i+1}^{+} - \widehat{\gamma}_i^{+})f_i^{+} - \\ - \sum_{i=p+1}^{s} Y(\gamma_i;t)f_i^{+}\dot{\gamma}_i^{+}\bigg\}\delta t_0 + \beta(t;\delta\wp), \\ \widehat{\gamma}_0^{-} = 1, \quad \widehat{\gamma}_i^{+} = \dot{\gamma}_i^{+}, \quad i = 1,\dots,p, \quad \widehat{\gamma}_{p+1}^{+} = 0.\end{gathered} \tag{1.10}$$

The following theorem is a result of Theorems 1 and 2.

Theorem 3. *Let the conditions of Theorems* 1 *and* 2 *be fulfilled. Moreover, let*

$$\sum_{i=0}^{p}(\widehat{\gamma}_{i+1}^{-} - \widehat{\gamma}_i^{-})f_i^{-} = \sum_{i=0}^{p}(\widehat{\gamma}_{i+1}^{+} - \widehat{\gamma}_i^{+})f_i^{+} = f_0,$$
$$f_i^{-}\dot{\gamma}_i^{-} = f_i^{+}\dot{\gamma}_i^{+} = f_i, \quad i = p+1,\dots,s.$$

Then there exist numbers $\varepsilon_2 > 0$, $\delta_2 > 0$ *such that for an arbitrary* $(t,\varepsilon,\delta\wp) \in [\widetilde{t}_1 - \delta_2, \widetilde{t}_1 + \delta_2] \times [0,\varepsilon_2] \times V$ *the formula* (1.6) *is valid, where* $\delta x(t;\varepsilon\delta\wp)$ *has the form*

$$\delta x(t;\delta\wp) = \bigg\{ Y(\widetilde{t}_0,t)f_0 - \sum_{i=p+1}^{s} Y(\gamma_i;t)f_i\bigg\}\delta t_0 + \beta(t;\delta\wp).$$

2. Auxiliary Lemmas

To every element $\wp = (t_0, x_0, \varphi, u) \in A$, let us correspond the functional-differential equation

$$\dot{y} = f(t_0,\varphi,u,y)(t) = f\big(t, h(t_0,\varphi,y)(\tau_1(t)),\dots,h(t_0,\varphi,y)(\tau_s(t)),u(t)\big) \tag{2.1}$$

with the initial condition

$$y(t_0) = x_0, \tag{2.2}$$

where the operator $h(\cdot)$ is defined by the formula

$$h(t_0,\varphi,y)(t) = \begin{cases} \varphi(t), & t \in [\tau, t_0), \\ y(t), & t \in [t_0, b]. \end{cases} \tag{2.3}$$

Definition 2.1. Let $\wp = (t_0, x_0, \varphi, u) \in A$. An absolutely continuous function $y(t) = y(t;\wp) \in \mathcal{O}$, $t \in [r_1, r_2] \subset \mathcal{J}$, is said to be a solution of the equation (2.1) with the initial condition (2.2), or a solution corresponding to the element $\wp \in A$, defined on the interval $[r_1, r_2]$, if $t_0 \in [r_1, r_2]$, $y(t_0) = x_0$ and the function $y(t)$ satisfies the equation (2.1) almost everywhere on $[r_1, r_2]$.

Remark 2.1. Let $y(t;\wp)$, $t\in[r_1,r_2]$, $\wp\in A$, be a solution of the equation (2.1) with the initial condition (2.2). Then the function

$$x(t;\wp)=h(t_0,\varphi,y(\cdot;\wp))(t),\quad t\in[r_1,r_2], \tag{2.4}$$

is a solution of the equation (1.1) with the initial condition (1.2) (see Definition 1.1, (2.3)).

Lemma 2.1. *Let $\widetilde{y}(t)$, $t\in[r_1,r_2]\subset(a,b)$, be a solution corresponding to the element $\widetilde{\wp}\in A$; let $K_1\subset\mathcal{O}$ be a compact which contains some neighborhood of the set $\operatorname{cl}\widetilde{\varphi}(\mathcal{J}_1)\cup\widetilde{y}([r_1,r_2])$ and let $M_1\subset G$ be a compact which contains some neighborhood of the set $\operatorname{cl}\widetilde{u}(\mathcal{J})$. Then there exist numbers $\varepsilon_1>0$, $\delta_1>0$ such that for an arbitrary $(\varepsilon,\delta\wp)\in[0,\varepsilon_1]\times V$, to the element $\widetilde{\wp}+\varepsilon\delta\wp\in A$ there corresponds a solution $y(t;\widetilde{\wp}+\varepsilon\delta\wp)$ defined on $[r_1-\delta_1,r_2+\delta_1]\subset\mathcal{J}$. Moreover,*

$$\begin{aligned}
&\varphi(t)=\widetilde{\varphi}(t)+\varepsilon\delta\varphi(t)\in K_1,\quad t\in\mathcal{J}_1,\\
&u(t)=\widetilde{u}(t)+\varepsilon\delta u(t)\in M_1,\quad t\in\mathcal{J},\\
&y(t;\widetilde{\wp}+\varepsilon\delta\wp)\in K_1,\quad t\in[r_1-\delta_1,r_2+\delta_1],\\
&\lim_{\varepsilon\to0}y(t;\widetilde{\wp}+\varepsilon\delta\wp)=y(t;\widetilde{\wp})\\
&\qquad\qquad \textit{uniformly for } (t,\wp)\in[r_1-\delta_1,r_2+\delta_1]\times V.
\end{aligned} \tag{2.5}$$

This lemma is analogous of Lemma 2.1 in [7, p. 21] and it is proved analogously.

Lemma 2.2. *Let $\widetilde{x}(t)$, $t\in[\tau,\widetilde{t}_1]$ be a solution corresponding to the element $\widetilde{\wp}\in A$, $\widetilde{t}_i\in(a,b)$, $i=0,1$; let $K_1\subset\mathcal{O}$ be a compact which contains some neighborhood of the set $\operatorname{cl}\widetilde{\varphi}(\mathcal{J}_1)\cup\widetilde{x}([\widetilde{t}_0,\widetilde{t}_1])$ and let $M_1\subset G$ be a compact which contains some neighborhood of the set $\operatorname{cl}\widetilde{u}(\mathcal{J})$. Then there exist numbers $\varepsilon_1>0$, $\delta_1>0$ such that for any $(\varepsilon,\delta\wp)\in[0,\varepsilon_1]\times V$, to the element $\widetilde{\wp}+\varepsilon\delta\wp\in A$ there corresponds the solution $x(t;\widetilde{\wp}+\varepsilon\delta\wp)$, $t\in[\tau,\widetilde{t}_1+\delta_1]\subset\mathcal{J}_1$. Moreover,*

$$\begin{aligned}
&x(t;\widetilde{\wp}+\varepsilon\delta\wp)\in K_1,\quad t\in[\tau,\widetilde{t}_1+\delta_1],\\
&u(t)=\widetilde{u}(t)+\varepsilon\delta u(t)\in M_1,\quad t\in\mathcal{J}.
\end{aligned} \tag{2.6}$$

It is easy to see that if in Lemma 2.1 $r_1=\widetilde{t}_0$, $r_2=\widetilde{t}_1$, then $\widetilde{y}(t)=\widetilde{x}(t)$, $t\in[\widetilde{t}_0,\widetilde{t}_1]$; $x(t;\widetilde{\wp}+\varepsilon\delta\wp)=h(t_0,\varphi,y(\cdot;\widetilde{\wp}+\varepsilon\delta\wp))(t)$, $(t,\varepsilon,\delta\wp)\in[\tau,\widetilde{t}_1+\delta_1]\times[0,\varepsilon_1]\times V$ (see (2.4)).

Thus Lemma 2.2 is a simple corollary (see (2.5)) of Lemma 2.1.

Due to uniqueness, the solution $y(t;\widetilde{\wp})$ on the interval $[r_1-\delta_1,r_2+\delta_1]$ is a continuation of the solution $\widetilde{y}(t)$; therefore the solution $\widetilde{y}(t)$ in the sequel is assumed to be defined on the whole interval $[r_1-\delta_1,r_2+\delta_1]$.

Let us define the increment of the solution $\widetilde{y}(t)=y(t;\widetilde{\wp})$,

$$\begin{aligned}
\Delta y(t)=\Delta y(t;\varepsilon\delta\wp)&=y(t;\widetilde{\wp}+\varepsilon\delta\wp)-\widetilde{y}(t),\\
&(t,\varepsilon,\delta\wp)\in[r_1-\delta_1,r_2+\delta_1]\times[0,\varepsilon_1]\times V.
\end{aligned} \tag{2.7}$$

It is obvious (see Lemma 2.1) that

$$\lim_{\varepsilon\to 0}\Delta y(t;\varepsilon\delta\wp)=0 \ \text{ uniformly for } \ (t,\delta\wp)\in[r_1-\delta_1,r_2+\delta_1]\times V. \tag{2.8}$$

Lemma 2.3 ([7, p. 35])**.** *For arbitrary compacts $K\subset\mathcal{O}$, $M\subset G$ there exists a function $L_{K,M}(\cdot)\in L(\mathcal{J},R_+)$ such that for an arbitrary $x_i',x_i''\in K$, $i=1,\dots,s$, $u',u''\in M$ and for almost all $t\in\mathcal{J}$, the inequality*

$$\begin{aligned}\big|f(t,x_1',\dots,x_s',u')-f(t,x_1'',\dots,x_s'',u'')\big|\le\\ \le L_{K,M}(t)\bigg(\sum_{i=1}^{s}|x_i'-x_i''|+|u'-u''|\bigg)\end{aligned} \tag{2.9}$$

is valid.

Lemma 2.4. *Let $\gamma_i=\widetilde{t}_0$, $i=1,\dots,p$, $\gamma_{p+1}<\cdots<\gamma_s\le r_2$ and let the conditions* 1.2 *and* 1.3 *of Theorem* 1 *be fulfilled, then there exists a number $\varepsilon_2>0$ such that for any $(\varepsilon,\delta\wp)\in[0,\varepsilon_2]\times V^-$ the inequality*

$$\max_{t\in[\widetilde{t}_0,r_2+\delta_1]}|\Delta y(t)|\le O(\varepsilon\delta\wp) \tag{2.10}$$

is valid. Moreover,

$$\Delta y(\widetilde{t}_0)=\varepsilon\bigg\{\delta x_0+\bigg[\sum_{i=0}^{p}\big(\widehat{\gamma}_{i+1}^--\widehat{\gamma}_i^-\big)f_i^-\bigg]\delta t_0\bigg\}+o(\varepsilon\delta\wp). \tag{2.11}$$

Proof. Let $\varepsilon_2\in(0,\varepsilon_1]$ be so small that for an any $(\varepsilon,\delta\wp)\in[0,\varepsilon_2]\times V^-$ the following relations are fulfilled:

$$\begin{gathered}t_0=\widetilde{t}_0+\varepsilon\delta t_0\in(\widetilde{t}_0-\delta,\widetilde{t}_0],\\ \widetilde{t}_0<\gamma_{p+1}(t_0)<\gamma_{p+1}<\gamma_{p+2}(t_0)<\cdots<\gamma_{s-1}<\gamma_s(t_0).\end{gathered} \tag{2.12}$$

The function $\Delta y(t)$ on the interval $[\widetilde{t}_0,r_2+\delta_1]$ satisfies the equation

$$\dot{\Delta}y(t)=a(t;\varepsilon\delta\wp), \tag{2.13}$$

where

$$a(t;\varepsilon\delta\wp)=f(t_0,\varphi,u,\widetilde{y}+\Delta y)(t)-f(\widetilde{t}_0,\widetilde{\varphi},\widetilde{u},\widetilde{y})(t). \tag{2.14}$$

Now let us rewrite the equation (2.13) in the integral form

$$\Delta y(t)=\Delta y(\widetilde{t}_0)+\int_{\widetilde{t}_0}^{t}a(\xi,\varepsilon\delta\wp)\,d\xi,\quad t\in[\widetilde{t}_0,r_2+\delta_1].$$

Hence

$$|\Delta y(t)|\le|\Delta y(\widetilde{t}_0)|+\int_{\widetilde{t}_0}^{t}|a(\xi;\varepsilon\delta\wp)|\,d\xi=\Delta y(\widetilde{t}_0)+a_1(t;\varepsilon\delta\wp). \tag{2.15}$$

Now let us prove the equality (2.11). It is easy to see that

$$\Delta y(\widetilde{t}_0) = y(\widetilde{t}_0;\widetilde{\wp}+\varepsilon\delta\wp) - \widetilde{y}(\widetilde{t}_0) = \varepsilon\delta x_0 + \int\limits_{t_0}^{\widetilde{t}_0} f(t_0,\varphi,u,\widetilde{y}+\Delta y)(t)\,dt. \quad (2.16)$$

Transform the integral addend of (2.16):

$$\int\limits_{t_0}^{\widetilde{t}_0} f(t_0,\varphi,u,\widetilde{y}+\Delta y)(t)\,dt =$$

$$= \sum_{i=0}^{p} \int\limits_{\rho_i(t_0)}^{\rho_{i+1}(t_0)} f(t,\widetilde{y}(\tau_1(t))+\Delta y(\tau_1(t)),\dots,\widetilde{y}(\tau_i(t))+\Delta y(\tau_i(t)),$$

$$\varphi(\tau_{i+1}(t)),\dots,\varphi(\tau_s(t)),u(t))dt = \sum_{i=0}^{p} I_i, \quad (2.17)$$

$$\rho_0(t)=t, \quad \rho_i=\gamma_i(t), \quad i=1,\dots,p, \quad \rho_{p+1}(t_0)=\widetilde{t}_0.$$

It is obvious that

$$I_0 = \varepsilon(\dot{\gamma}_1^- - 1)f_0^-\delta t_0 + \int\limits_{t_0}^{\gamma_1(t_0)} \big[f(t,\varphi(\tau_1(t)),\dots,\varphi(\tau_s(t)),u(t)) - f_0^-\big]\,dt =$$

$$= \varepsilon(\dot{\gamma}_1^- - 1)f_0^-\delta t_0 + \alpha(\varepsilon\delta\wp). \quad (2.18)$$

We now show that

$$\alpha(\varepsilon\delta\wp) = o(\varepsilon\delta\wp). \quad (2.19)$$

On account of the condition 1.3 and (1.3), we have

$$\lim_{\varepsilon\to 0} \sup_{t\in[t_0,\gamma_1(t_0)]} \big|f(t,\varphi(\tau_1(t)),\dots,\varphi(\tau_s(t)),u(t)) - f_0^-\big| =$$

$$= \lim_{\omega\to\sigma_0^-} \big|\widetilde{f}[\omega] - f_0^-\big| = 0, \quad \omega\in(\widetilde{t}_0-\delta,\widetilde{t}_0]\times\mathcal{O}^s,$$

from which immediately follows (2.19). Analogously, the equalities

$$\begin{aligned} I_i &= \varepsilon(\dot{\gamma}_{i+1}^- - \dot{\gamma}_i^-)f_i^-\delta t_0 + o(\varepsilon\delta\wp), \quad i=1,\dots,p-1,\\ I_p &= -\varepsilon\dot{\gamma}_p f_p^-\delta t_0 + o(\varepsilon\delta\wp) \end{aligned} \quad (2.20)$$

are proved. By virtue of, (2.17)–(2.20) it follows (2.11). Before proving (2.10), let us remark that if $i=p+1,\dots,s$, $\xi\in[\gamma_i(t_0),\gamma_i]$, then

$$\lim_{\varepsilon\to 0}(\xi,\widetilde{y}(\tau_1(\xi))+\Delta y(\tau_1(\xi)),\dots,\widetilde{y}(\tau_i(\xi))+\Delta y(\tau_i(\xi)),$$

$$\varphi(\tau_{i+1}(\xi)),\dots,\varphi(\tau_s(\xi))) = \sigma_i^-,$$

$$\lim_{\varepsilon\to 0}(\xi,\widetilde{y}(\tau_1(\xi))+\Delta y(\tau_1(\xi)),\dots,\widetilde{y}(\tau_{i-1}(\xi)),\varphi(\tau_i(\xi)),\dots,\varphi(\tau_s(\xi))) = \overset{\circ}{\sigma}{}_i^-$$

(see (1.5), (2.8)).

Thus, by virtue of the condition 1.3 for a sufficiently small $\varepsilon_2 \in (0, \varepsilon_1]$, the functions

$$\sup_{t\in[t_0,\widetilde{t}_0]} \dot{\gamma}_i(t), \qquad \sup_{t\in[\gamma_i(t_0),\gamma_i]} |a(t;\varepsilon\delta\wp)|, \quad i = p+1, \dots, s,$$

are bounded on the set $[0, \varepsilon_2] \times V^-$.

Hence for any $(\varepsilon, \delta\wp] \in [0, \varepsilon_2) \times V^-$ the estimation

$$\int\limits_{\gamma_i(t_0)}^{\gamma_i} |a(t;\varepsilon\delta\wp)|\, dt \le O(\varepsilon\delta\wp), \quad i = p+1, \dots, s, \tag{2.21}$$

is valid.

Now estimate $a_1(t;\varepsilon\delta\wp)$, $t \in [\widetilde{t}_0, r_2 + \delta_1]$. We consider several cases.

Let $t \in [\widetilde{t}_0, \gamma_{p+1}(t_0)]$. Then on the basis of the inequality (2.9) and the form of the operator $h(\cdot)$ we get

$$\begin{aligned}
a_1(t;\varepsilon\delta\wp) = \int\limits_{\widetilde{t}_0}^{t} \big[f(\xi, \widetilde{y}(\tau_1(\xi)) + \Delta y(\tau_1(\xi)), \dots, \widetilde{y}(\tau_p(\xi)) + \Delta y(\tau_p(\xi)),&\\
\varphi(\tau_{p+1}(\xi)), \dots, \varphi(\tau_s(\xi)), u(\xi))-&\\
- f(\xi, \widetilde{y}(\tau_1(\xi)), \dots, \widetilde{y}(\tau_p(\xi)), \widetilde{\varphi}(\tau_{p+1}(\xi)), \dots, \widetilde{\varphi}(\tau_s(\xi)), \widetilde{u}(\xi))\big] d\xi \le&\\
\le \int\limits_{\widetilde{t}_0}^{t} L_{K_1,M_1}(\xi)\Big[\sum_{i=1}^{p} |\Delta y(\tau_i(\xi))| + \varepsilon \sum_{i=p+1}^{s} |\delta\varphi(\tau_i(\xi))| + \varepsilon|\delta u(\xi)|\Big] d\xi \le&\\
\le O(\varepsilon\delta\wp) + \int\limits_{\widetilde{t}_0}^{t} L(\xi)|\Delta y(\xi)|\, d\xi&
\end{aligned} \tag{2.22}$$

(see (2.14)), where

$$L(\xi) = \sum_{i=1}^{s} \chi(\gamma_i(\xi)) L_{K_1,M_1}(\gamma_i(\xi))\dot{\gamma}_i(\xi), \tag{2.23}$$

$\chi(t)$ is the characteristic function of the interval $\mathcal{J}$. When $\gamma_i(\xi) > b$, we assume that $\chi(\gamma_i(\xi))L_{K_1,M_1}(\gamma_i(\xi)) = 0$.

If $t \in [\gamma_{p+1}(t_0), \gamma_{p+1}]$, then on the basis of (2.21) and (2.22) we obtain:

$$\begin{aligned}
a_1(t;\varepsilon\delta\wp) = a_1(\gamma_{p+1}(t_0);\varepsilon\delta\wp) + \int\limits_{\gamma_{p+1}(t_0)}^{\gamma_{p+1}} |a(\xi;\varepsilon\delta\wp)|\, d\xi \le&\\
\le O(\varepsilon\delta\wp) + \int\limits_{\widetilde{t}_0}^{t} L(\xi)|\Delta y(\xi)|\, d\xi.&
\end{aligned}$$

Thus the estimate (2.22) is valid on the whole interval $[\widetilde{t}_0, \gamma_{p+1}]$.

Let $t \in [\gamma_{p+1}, \gamma_{p+2}(t_0)]$, then

$$a_1(t;\varepsilon\delta\wp) \le a_1(\gamma_{p+1};\varepsilon\delta\wp)+$$
$$+\int\limits_{\gamma_{p+1}}^{t} L_{K_1,M_1}(\xi)\Big[\sum_{i=1}^{p+1}|\Delta y(\tau_i(\xi))| + \varepsilon\sum_{i=p+2}^{s}|\delta\varphi(\xi)| + \varepsilon|\delta u(\xi)|\Big]\,d\xi \le$$
$$\le a_1(\gamma_{p+1};\varepsilon\delta\wp) + O(\varepsilon\delta\wp) + \sum_{i=1}^{p+1}\int\limits_{\tau_i(\gamma_{p+1})}^{\tau_i(t)} L_{K_1,M_1}(\gamma_i(\xi))\dot{\gamma}_i(\xi)|\Delta y(\xi)|\,d\xi.$$

As $\tau_i(\gamma_{p+1}) \ge \widetilde{t}_0$, $\tau_i(t) \le t$, $i = 1,\dots,p+1$, we can rewrite the obtained inequality in the form

$$a_1(t;\varepsilon\delta\wp) \le O(\varepsilon\delta\wp) + a_1(\gamma_{p+1};\varepsilon\delta\wp) + \int\limits_{\widetilde{t}_0}^{t} L(\xi)|\Delta y(\xi)|\,d\xi.$$

Thus, when $t \in [\widetilde{t}_0, \gamma_{p+2}(t_0)]$, the estimate

$$a_1(t;\varepsilon\delta\wp) \le O(\varepsilon\delta\wp) + 2\int\limits_{\widetilde{t}_0}^{t} L(\xi)|\Delta y(\xi)|\,d\xi \tag{2.24}$$

is valid. By virtue of (2.21), we can analogously prove the validity of (2.24) on the interval $[\widetilde{t}_0, \gamma_{p+2}]$. If we continue this process, we obtain

$$a_1(t;\varepsilon\delta\wp) \le O(\varepsilon\delta\wp) + (i+1)\int\limits_{\widetilde{t}_0}^{t} L(\xi)|\Delta y(\xi)|\,d\xi,$$
$$t \in [\widetilde{t}_0, \gamma_{p+i+1}], \quad i = 2,\dots,s-p-1.$$

Let $t \in [\gamma_s, r_2 + \delta_1]$. Then

$$a_1(t;\varepsilon\delta\wp) \le a_1(\gamma_s;\varepsilon\delta\wp) + \sum_{i=1}^{s}\int\limits_{\gamma_s}^{t} L_{K_1,M_1}(\xi)|\Delta y(\tau_i(\xi))|\,d\xi =$$
$$= a_1(\gamma_s;\varepsilon\delta\wp) + \sum_{i=1}^{s}\int\limits_{\tau_i(\gamma_s)}^{\tau_i(t)} L_{K_1,M_1}(\gamma_i(\xi))\dot{\gamma}_i(\xi)|\Delta y(\xi)|\,d\xi.$$

As $\tau_i(\gamma_s) \ge \widetilde{t}_0$, $i = 1,\dots,s$, we have

$$a_1(t;\varepsilon\delta\wp) \le a_1(\gamma_s;\varepsilon\delta\wp) + \int\limits_{\widetilde{t}_0}^{t} L(\xi)|\Delta y(\xi)|\,d\xi = O(\varepsilon\delta\wp)+$$

$$+ (s-p+1)\int\limits_{\widetilde{t}_0}^{t} L(\xi)|\Delta y(\xi)|\,d\xi, \quad t\in[\widetilde{t}_0, r_2+\delta_1]. \tag{2.25}$$

Taking into account (2.11), (2.25), from the inequality (2.15) immediately follows

$$|\Delta y(t)| \le O(\varepsilon\delta\wp) + (s-p+1)\int\limits_{\widetilde{t}_0}^{t} L(\xi)|\Delta y(\xi)|\,d\xi, \quad t\in[\widetilde{t}_0, r_2+\delta_1].$$

By virtue of Gronwall's lemma, we obtain(2.10). □

Lemma 2.5. *Let $\gamma_i = \widetilde{t}_0$, $i=1,\dots,p$; $\gamma_{p+1} < \dots < \gamma_s \le r_2$ and let the conditions* 1.4 *and* 1.5 *of Theorem* 2 *be fulfilled. Then there exists a number $\varepsilon_2 > 0$ such that for any $(\varepsilon,\delta\wp)\in[0,\varepsilon_2]\times V^+$ the inequality*

$$\max_{t\in[t_0, r_1+\delta_1]} |\Delta y(t)| \le O(\varepsilon\delta\wp) \tag{2.26}$$

is valid. Moreover,

$$\Delta y(t_0) = \varepsilon\big[\delta x_0 - f_p^{+}\delta t_0\big] + o(\varepsilon\delta\wp). \tag{2.27}$$

Proof. Let $\varepsilon_2\in(0,\varepsilon_1]$ be so small that for any $(\varepsilon,\delta\wp)\in[0,\varepsilon_2]\times V^+$ the following relations are fulfilled:

$$\begin{gathered} t_0\in[\widetilde{t}_0, \widetilde{t}_0+\delta), \\ \gamma_p(t_0) < \gamma_{p+1} < \gamma_{p+1}(t_0) < \gamma_{p+2} < \dots < \gamma_s < \gamma_s(t_0) < r_2+\delta_1. \end{gathered} \tag{2.28}$$

The function $\Delta y(t)$ on the interval $[t_0, r_1+\delta_1]$ satisfies the equation (2.13) which we can rewrite in the integral form

$$\Delta y(t) = \Delta y(t_0) + \int\limits_{t_0}^{t} a(\xi;\varepsilon\delta\wp)\,d\xi, \quad t\in[t_0, r_2+\delta_1].$$

Hence

$$|\Delta y(t)| \le |\Delta y(t_0)| + \int\limits_{t_0}^{t} a(\xi;\varepsilon\delta\wp)\,d\xi = |\Delta y(t_0)| + a_2(t;\varepsilon\delta\wp). \tag{2.29}$$

Now prove (2.27):

$$\Delta y(t_0) = \varepsilon\delta x_0 - \int\limits_{\widetilde{t}_0}^{t} f(\widetilde{t}_0, \widetilde{\varphi}, \widetilde{u}, \widetilde{y})(t)\,dt =$$

$$= \varepsilon\delta x_0 - \int\limits_{\widetilde{t}_0}^{t} f(t, \widetilde{y}(\tau_1(t)),\dots,\widetilde{y}(\tau_p(t)), \widetilde{\varphi}(\tau_{p+1}(t)),\dots,\widetilde{\varphi}(\tau_s(t)), \widetilde{u}(t))\,dt =$$

$$= \varepsilon[\delta x_0 - f_p^{+}\delta x_0] + o(\varepsilon\delta\wp).$$

By virtue of the condition 1.5 for a sufficiently small $\varepsilon_2 \in (0, \varepsilon_1]$, the functions

$$\sup_{t\in[\widetilde{t}_0,t_0]} \dot{\gamma}_i(t), \quad i = 1,\dots,s, \qquad \sup_{t\in[\gamma_{i-1}(t_0),\gamma_i(t_0)]} |a(t;\varepsilon\delta\wp)|, \quad i = 1,\dots,p,$$
$$\gamma_0(t_0) = t_0, \qquad \sup_{t\in[\gamma_i,\gamma_i(t_0)]} |a(t;\varepsilon\delta\wp)|, \quad i = p+1,\dots,s,$$

are bounded on the set $[0, \varepsilon_2] \times V^+$.

It is obvious that if $i = 1,\dots,p$, then

$$|\gamma_i(t_0) - \gamma_{i-1}(t_0)| \le |\gamma_i(t_0) - \gamma_i(\widetilde{t}_0)| + |\gamma_{i-1}(\widetilde{t}_0) - \gamma_{i-1}(t_0)| \le O(\varepsilon\delta\wp),$$
$$\gamma_0(\widetilde{t}_0) = \widetilde{t}_0.$$

From these conditions it follows that for an arbitrary $(\varepsilon, \delta\wp) \in [0, \varepsilon_2] \times V^+$ the estimates

$$\int_{\gamma_{i-1}(t_0)}^{\gamma_i(t_0)} |a(\xi;\varepsilon\delta\wp)|\, d\xi \le O(\varepsilon\delta\wp), \quad i = 1,\dots,p,$$
$$\int_{\gamma_i}^{\gamma_{i-1}(t_0)} |a(\xi;\varepsilon\delta\wp)|\, d\xi \le O(\varepsilon\delta\wp), \quad i = p+1,\dots,s. \tag{2.30}$$

are valid.

Now estimate $a_2(t;\varepsilon\delta\wp)$ on the interval $[t_0, r_1 + \delta_1]$. We consider several cases.

Let $t \in [t_0, \gamma_p(t_0)]$. Then

$$a_2(t;\varepsilon\delta\wp) \le \sum_{i=1}^{p} \int_{\gamma_{i-1}(t_0)}^{\gamma_i(t_0)} |a(\xi;\varepsilon\delta\wp)|, d\xi \le O(\varepsilon\delta\wp) \tag{2.31}$$

(see (2.30)).

Let $t \in [\gamma_p(t_0), \gamma_{p+1}]$. Then

$$a_2(t;\varepsilon\delta\wp) \le a_2(\gamma_p(t_0);\varepsilon\delta\wp)+$$
$$+\int_{\gamma_p(t_0)}^{t} L_{K_1,M_1}(\xi)\biggl(\sum_{i=1}^{p} |\Delta y(\tau_i(\xi))| + \varepsilon \sum_{i=p+1}^{s} |\delta\varphi(\tau_i(\xi))| + \varepsilon|\delta u(\xi)|\biggr)\, d\xi \le$$
$$\le O(\varepsilon\delta\wp) + \sum_{i=1}^{p} \int_{\tau_i(\gamma_p(t_0))}^{\tau_i(t)} L_{K_1,M_1}(\gamma_i(\xi))\dot{\gamma}_i(\xi)|\Delta y(\xi)|\, d\xi.$$

As $\tau_i(\gamma_p(t_0)) > \tau_i(\gamma_i(t_0)) = t_0$, $\tau_i(t) \le t$, $i = 1,\dots,p$,

$$a_2(t;\varepsilon\delta\wp) \le O(\varepsilon\delta\wp) + \int_{t_0}^{t} L(\xi)|\Delta y(\xi)|\, d\xi, \quad t \in [t_0, \gamma_{p+1}],$$

(see (2.23), (2.31)).

Using (2.27), (2.30), it can be analogously proved that (see proof of Lemma 2.4)

$$|\Delta y(t)| \le O(\varepsilon\delta\wp) + (s-p+1)\int\limits_{t_0}^{t} L(\xi)|\Delta y(\xi)|\,d\xi, \quad t \in [t_0, r_2+\delta_2].$$

(see (2.29).

By virtue of Gronwall's lemma, we obtain (2.26). □

3. Proof of Theorem 1

Let $r_1 = \widetilde{t}_0$, $r_2 = \widetilde{t}_1$. Then for any $(\varepsilon, \delta\wp) \in [0, \varepsilon_1] \times V^-$ the solution $y(t; \widetilde{\wp} + \varepsilon\delta\wp)$ is defined on the interval $[\widetilde{t}_0 - \delta_1, \widetilde{t}_1 + \delta_1]$ and the solution $x(t; \widetilde{\wp} + \varepsilon\delta\wp)$ is defined on the interval $[\tau, \widetilde{t}_1 + \delta_1]$. Moreover,

$$y(t; \widetilde{\wp} + \varepsilon\delta\wp) = x(t; \widetilde{\wp} + \varepsilon\delta\wp), \quad t \in [t_0, \widetilde{t}_1 + \delta_1],$$

(see Lemmas 2.1 and 2.2 and (2.4)).

Thus

$$\Delta x(t) = \begin{cases} \varepsilon\delta\varphi(t), & t \in [\tau, t_0), \\ y(t; \widetilde{\wp} + \varepsilon\delta\wp) - \widetilde{\varphi}(t), & t \in [t_0, \widetilde{t}_0], \\ \Delta y(t), & t \in [\widetilde{t}_0, \widetilde{t}_1 + \delta_1] \end{cases} \tag{3.1}$$

(see (1.4), (2.7)).

Let $\delta_2 \in (0, \min(\delta_1, \widetilde{t}_1 - \gamma_s))$. By virtue of Lemma 2.4, there exists a number $\varepsilon_2 \in (0, \varepsilon_1]$ such that

$$|\Delta x(t)| \le O(\varepsilon\delta\wp) \quad \forall (t, \varepsilon, \delta\wp) \in [\widetilde{t}_0, \widetilde{t}_1 + \delta_2] \times [0, \varepsilon_2] \times V^-, \tag{3.2}$$

$$\Delta x(\widetilde{t}_0) = \varepsilon\bigg\{\delta x_0 + \bigg[\sum_{i=0}^{p} (\widehat{\gamma}_{i+1}^- - \widehat{\gamma}_i^-) f_i^-\bigg]\delta t_0\bigg\} + o(\varepsilon\delta\wp) \tag{3.3}$$

(see (3.1).

The function $\Delta x(t)$ on the interval $[\widetilde{t}_0, t_1 + \delta_2]$ satisfies the following equation

$$\dot{\Delta} x(t) = \sum_{i=1}^{s} \widetilde{f}_{x_i}[t]\Delta x(\tau_i(t)) + \varepsilon \widetilde{f}_u[t]\delta u(t) + R(t; \varepsilon\delta\wp), \tag{3.4}$$

where

$$R(t;\varepsilon\delta\wp) = f(t, \widetilde{x}(\tau_1(t)) + \Delta x(\tau_1(t)), \dots, \widetilde{x}(\tau_s(t)) + \Delta x(\tau_s(t)), \widetilde{u}(t) + \varepsilon\delta u(t)) - \\ - \widetilde{f}[t] - \sum_{i=1}^{s} \widetilde{f}_{x_i}[t]\Delta x(\tau_i(t)) - \varepsilon \widetilde{f}_u[t]\delta u(t). \tag{3.5}$$

By means of the Cauchy formula, the solution of the equation (3.4) can be represented in the form

$$\Delta x(t) = Y(\widetilde{t}_0;t)\Delta x(\widetilde{t}_0) + \varepsilon \int_{\widetilde{t}_0}^{t} Y(\xi;t)\widetilde{f}_u[\xi]\delta u(\xi)\, d\xi + \\ + \sum_{i=0}^{1} h_i(t;\widetilde{t}_0, \varepsilon\delta\wp), \quad t \in [\widetilde{t}_0, \widetilde{t}_1 + \delta_2], \tag{3.6}$$

where

$$\begin{cases} h_0(t;\widetilde{t}_0, \varepsilon\delta\wp) = \displaystyle\sum_{i=p+1}^{s} \int_{\tau_i(\widetilde{t}_0)}^{\widetilde{t}_0} Y(\gamma_i(\xi);t)\widetilde{f}_{x_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\Delta x(\xi)\, d\xi, \\ h_1(t;\widetilde{t}_0, \varepsilon\delta\wp) = \displaystyle\int_{\widetilde{t}_0}^{t} Y(\xi;t)R(\xi;\varepsilon\delta\wp)\, d\xi. \end{cases} \tag{3.7}$$

The matrix function $Y(\xi;t)$ satisfies the equation (1.8) and the condition (1.9).

By virtue of Lemma 3.4 [7, p. 37], the function $Y(\xi;t)$ is continuous on the set $\Pi = \{(\xi,t)\ :\ a \le \xi \le t,\, t \in \mathcal{J}\}$. Hence

$$Y(\widetilde{t}_0;t)\Delta x(\widetilde{t}_0) = \varepsilon Y(\widetilde{t}_0;t)\Big\{\delta x_0 + \Big[\sum_{i=0}^{p}(\widehat{\gamma}_{i+1}^{-} - \widehat{\gamma}_i^{-})f_i^{-}\Big]\delta t_0\Big\} + o(t;\varepsilon\delta\wp) \tag{3.8}$$

(see (3.3)).

Now we transform $h_0(t;\widetilde{t}_0, \varepsilon\delta\wp)$. We have

$$h_0(t;\widetilde{t}_0, \varepsilon\delta\wp) = \sum_{i=p+1}^{s} \Big[\varepsilon \int_{\tau_i(\widetilde{t}_0)}^{t_0} Y(\gamma_i(\xi);t)\widetilde{f}_{x_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\delta\varphi(\xi)\, d\xi + \\ + \int_{t_0}^{\widetilde{t}_0} Y(\gamma_i(\xi);t)\widetilde{f}_{x_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\Delta x(\xi)\, d\xi\Big] = \\ = \varepsilon \sum_{i=p+1}^{s} \int_{\tau_i(\widetilde{t}_0)}^{\widetilde{t}_0} Y(\gamma_i(\xi);t)\widetilde{f}_{x_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\delta\varphi(\xi)\, d\xi + \\ + \sum_{i=p+1}^{s} \int_{\gamma_i(t_0)}^{\gamma_i} Y(\xi;t)\widetilde{f}_{x_i}[\xi]\Delta x(\tau_i(\xi))\, d\xi + o(t;\varepsilon\delta\wp) \tag{3.9}$$

(see (2.12)).

Owing to the inequality (2.12), the expression $h_1(t;\widetilde{t}_0,\varepsilon\delta\wp)$ with $[\widetilde{t}_1-\delta_2,\widetilde{t}_1+\delta_2]$ can be represented as

$$h_1(t;\widetilde{t}_0,\varepsilon\delta\wp)=\sum_{k=1}^{4}\alpha_k(t;\varepsilon\delta\wp),$$

where

$$\alpha_1(t;\varepsilon\delta\wp)=\int\limits_{\widetilde{t}_0}^{\gamma_{p+1}(t_0)}\omega(\xi;t,\varepsilon\delta\wp)\,d\xi,\quad \alpha_2(t;\varepsilon\delta\wp)=\sum_{i=p+1}^{s}\int\limits_{\gamma_i(t_0)}^{\gamma_i}\omega(\xi;t,\varepsilon\delta\wp)\,d\xi,$$

$$\alpha_3(t;\varepsilon\delta\wp)=\sum_{i=p+1}^{s}\int\limits_{\gamma_i}^{\gamma_{i+1}(t_0)}\omega(\xi;t,\varepsilon\delta\wp)\,d\xi,\quad \alpha_4(t;\varepsilon\delta\wp)=\int\limits_{\gamma_s}^{t}\omega(\xi;t,\varepsilon\delta\wp)\,d\xi,$$

$$\omega(\xi;t,\varepsilon\delta\wp)=Y(\xi;t)R(t;\varepsilon\delta\wp).$$

Let us estimate $\alpha_1(t;\varepsilon\delta\wp)$. (see (3.1)). We have:

$$|\alpha_1(t;\varepsilon\delta\wp)|\le\|Y\|\int\limits_{\widetilde{t}_0}^{\gamma_{p+1}(t_0)}\Big|f(t,\widetilde{x}(\tau_1(t))+\Delta x(\tau_1(t)),\dots,$$
$$\widetilde{x}(\tau_p(t))+\Delta x(\tau_p(t)),\varphi(\tau_{p+1}(t)),\dots,\varphi(\tau_s(t)),u(t))-$$
$$-f(t,\widetilde{x}(\tau_1(t)),\dots,\widetilde{x}(\tau_p(t)),\widetilde{\varphi}(\tau_{p+1}(t)),\dots,\widetilde{\varphi}(\tau_s(t)),\widetilde{u}(t))-$$
$$-\sum_{i=1}^{p}\widetilde{f}_{x_i}[t]\Delta x(\tau_i(t))-\varepsilon\sum_{i=p+1}^{s}\widetilde{f}_{x_i}[t]\delta\varphi(\tau_i(t))-\varepsilon\widetilde{f}_u[t]\delta u(t)\Big|dt\le$$
$$\le\|Y\|\int\limits_{\widetilde{t}_0}^{\gamma_{p+1}(t_0)}\bigg\{\bigg|\int\limits_0^1\bigg[\frac{d}{d\xi}f(t,\widetilde{x}(\tau_1(t))+\xi\Delta x(\tau_1(t)),\dots,$$
$$\widetilde{x}(\tau_p(t))+\xi\Delta x(\tau_p(t)),\varphi(\tau_{p+1}(t))+\xi\varepsilon\delta\varphi(\tau_{p+1}(t)),\dots,$$
$$\varphi(\tau_s(t))+\xi\varepsilon\delta\varphi(\tau_s(\xi)),u(t)+\xi\varepsilon\delta u(t))-$$
$$-\sum_{i=1}^{p}\widetilde{f}_{x_i}[t]\Delta x(\tau_i(t))-\varepsilon\sum_{i=p+1}^{s}\widetilde{f}_{x_i}[t]\delta\varphi(\tau_i(t))-\varepsilon\widetilde{f}_u[t]\delta u(t)\bigg]\bigg|d\xi\bigg\}dt\le$$
$$\le\|Y\|\int\limits_{\widetilde{t}_0}^{\widetilde{t}_1+\delta_2}\bigg\{\int\limits_0^1\bigg[\sum_{i=1}^{p}|f_{x_i}(t,\widetilde{x}(\tau_1(t))+\xi\Delta x(\tau_1(t)),\dots,)-\widetilde{f}_{x_i}[t]|\times$$
$$\times|\Delta x(\tau_i(t))|+$$
$$+\varepsilon\sum_{i=p+1}^{s}\big|f_{x_i}(t,\widetilde{x}(\tau_1(t))+\xi\Delta x(\tau_1(t)),\dots,)-\widetilde{f}_{x_i}[t]\big|\cdot|\delta\varphi(\tau_i(t))|+$$
$$+\varepsilon|f_u(t,\widetilde{x}(\tau_1(t))+\xi\Delta x(\tau_1(t)),\dots,)-\widetilde{f}_u[t]|\cdot|\delta u(t)|\bigg]d\xi\bigg\}dt\le$$

$$\leq \|Y\| \Big(O(\varepsilon\delta\wp) \sum_{i=1}^{p} \sigma_i(\varepsilon\delta\wp) + \varepsilon\alpha \sum_{i=p+1}^{s} \sigma_i(\varepsilon\delta\wp) + \varepsilon\alpha\sigma(\varepsilon\delta\wp) \Big), \qquad (3.10)$$

where

$$\|Y\| = \sup_{(\xi,t)\in\Pi} |Y(\xi;t)|,$$

$$\sigma_i(\varepsilon\delta\wp) = \int_{\widetilde{t}_0}^{\widetilde{t}_1+\delta_2} \Big[\int_0^1 \big| f_{x_i}(t, \widetilde{x}(\tau_1(t)) + \xi\Delta x(\tau_1(t)), \dots) - \widetilde{f}_{x_i}[t] \big| \, d\xi \Big] dt,$$

$$\sigma(\varepsilon\delta\wp) = \int_{\widetilde{t}_0}^{\widetilde{t}_1+\delta_2} \Big[\int_0^1 \big| f_u(t, \widetilde{x}(\tau_1(t)) + \xi\Delta x(\tau_1(t)), \dots) - \widetilde{f}_u[t] \big| \, d\xi \Big] dt.$$

As $\varepsilon \to 0$, then $\widetilde{\varphi}(t) + \xi\varepsilon\delta\varphi(t) \to \widetilde{\varphi}(t)$, $\widetilde{u}(t) + \xi\varepsilon\delta u(t) \to \widetilde{u}(t)$, $\Delta x(\tau_i(t)) \to 0$, $i = 1, \dots, p$, uniformly for $(\xi, t, \delta\wp) \in [0,1] \times [\widetilde{t}_0, \widetilde{t}_1 + \delta_2] \times V^-$. Thus, by Lebesgue's theorem $\lim_{\varepsilon\to 0} \sigma_i(\varepsilon\delta\wp) = 0$, $i = 1, \dots, s$, $\lim_{\varepsilon\to 0} \sigma(\varepsilon\delta\wp) = 0$ uniformly for $\delta\wp \in V^-$. (see (2.6)). Thus

$$\alpha_1(t; \varepsilon\delta\wp) = o(t; \varepsilon\delta\wp).$$

Now we transform $\alpha_2(t; \varepsilon\delta\mu)$. Let us note that if $t \in [\gamma_i(t_0), \gamma_i]$, then $|\Delta x(\tau_j(t))| \leq O(\varepsilon\delta\wp)$, $j = 1, \dots, i-1$, $\Delta x(\tau_j(t)) = \varepsilon\delta\varphi(\tau_j(t))$, $j = i+1, \dots, s$, $i = p+1, \dots, s$.(see (3.1) (3.2)). Hence

$$\int_{\gamma_i(t_0)}^{\gamma_i} \omega(\xi; \varepsilon\delta\wp) \, d\xi = \int_{\gamma_i(t_0)}^{\gamma_i} Y(\xi;t)\beta_i(\xi) \, d\xi -$$

$$- \int_{\gamma_i(t_0)}^{\gamma_i} Y(\xi;t)\widetilde{f}_{x_i}[\xi]\Delta x(\tau_i(\xi)) \, d\xi + o(t; \varepsilon\delta\wp),$$

where

$$\beta_i(\xi) = f(\xi, \widetilde{x}(\tau_1(\xi)) + \Delta x(\tau_i(\xi)), \dots, \widetilde{x}(\tau_1(\xi)) + \Delta x(\tau_i(\xi)),$$
$$\varphi(\tau_{i+1}(\xi)), \dots, \varphi(\tau_s(\xi)), u(\xi)) -$$
$$- f(\xi, \widetilde{x}(\tau_1(\xi)), \dots, \widetilde{x}(\tau_{i-1}(\xi)), \varphi(\tau_i(\xi)), \dots, \varphi(\tau_s(\xi)), \widetilde{u}(\xi)),$$

$$o(t; \varepsilon\delta\wp) = - \sum_{j=1}^{i-1} \int_{\gamma_i(t_0)}^{\gamma_i} Y(\xi;t)\widetilde{f}_{x_j}[\xi]\Delta x(\tau_j(\xi)) \, d\xi -$$

$$-\varepsilon \sum_{j=i+1}^{s} \int_{\gamma_i(t_0)}^{\gamma_i} Y(\xi;t)\widetilde{f}_{x_i}[\xi]\delta\varphi(\tau_j(\xi)) \, d\xi - \varepsilon \int_{\gamma_i(t_0)}^{\gamma_i} \widetilde{f}_u[\xi]\delta u(\xi) \, d\xi.$$

It is obvious that

$$\int\limits_{\gamma_i(t_0)}^{\gamma_i} Y(\xi;t)\beta_i(\xi)\,d\xi = \int\limits_{\gamma_i(t_0)}^{\gamma_i} Y(\xi;t)[\beta_i(\xi)-f_i^-]d\xi + \int\limits_{\gamma_i(t_0)}^{\gamma_i} Y(\xi;t)f_i^-d\xi =$$
$$= \alpha_5(t;\varepsilon\delta\wp)+\alpha_6(t;\varepsilon\delta\wp).$$

Next, when $\xi\in[\gamma_i(t_0),\gamma_i]$, $\tau_j(\xi)\ge \widetilde{t}_0$, $j=1,\dots,i-1$ $(i=p+1,\dots,s)$; hence

$$\lim_{\varepsilon\to 0}\big(x(\tau_j(t))+\Delta x(\tau_j(t))\big)=\lim_{\varepsilon\to 0}\widetilde{x}(\tau_j(\xi))=\widetilde{x}(\tau_j(\gamma_i)),\quad j=1,\dots,i-1.$$

When $\xi\in[\gamma_i(t_0),\gamma_i]$, then $\tau_i(\xi)\in[t_0,\widetilde{t}_0]$; hence

$$\widetilde{x}(\tau_i(\xi))+\Delta x(\tau_i(\xi))=x(\tau_i(\xi);\widetilde{\wp}+\varepsilon\delta\wp)=y(\tau_i(\xi);\widetilde{\wp}+\varepsilon\delta\wp)=$$
$$=\widetilde{y}(\tau_i(\xi))+\Delta y(\tau_i(\xi))\quad \text{(see (2.7))}.$$

Thus, taking into consideration the continuity of the function $\widetilde{y}(t)$ on the interval $[\widetilde{t}_0-\delta_2,\widetilde{t}_1+\delta_2]$, the inequality (2.8) and the condition $y(\widetilde{t}_0)=\widetilde{x}_0$, we obtain

$$\lim_{\varepsilon\to 0}\widetilde{x}(\tau_i(\xi))+\Delta x(\tau_i(\xi))=\lim_{\xi\to\gamma_i}\widetilde{y}(\tau_i(\xi))=\widetilde{x}_0.$$

Using the relations obtained above, we can conclude that when $i=p+1,\dots,s$, $\xi\in[\gamma_i(t_0),\gamma_i]$

$$\lim_{\varepsilon\to 0}\big(\xi,\widetilde{x}(\tau_1(\xi))+\Delta x(\tau_1(\xi)),\dots,\widetilde{x}(\tau_i(\xi))+\Delta x(\tau_i(\xi)),$$
$$\varphi(\tau_{i+1}(\xi)),\dots,\varphi(\tau_s(\xi))\big)=\sigma_i^-,\quad i=p+1,\dots,s.$$

On the other hand,

$$\lim_{\varepsilon\to 0}\big(\xi,\widetilde{x}(\tau_1(\xi)),\dots,\widetilde{x}(\tau_{i-1}(\xi)),\widetilde{\varphi}(\tau_i(\xi)),\dots,\widetilde{\varphi}(\tau_s(\xi))\big)=\overset{\circ}{\sigma}{}_i^-.$$
$$i=p+1,\dots,s.$$

Thus

$$\lim_{\varepsilon\to 0}\sup_{\xi\in[\gamma_i(t_0),\gamma_i]}|\beta_i(\xi)-f_i^-|=0\quad\text{uniformly for}\quad \delta\wp\in V^-.$$

The functions $Y(\xi;t)$ are continuous on the set $[\gamma_i(t_0),\gamma_i]\times[\widetilde{t}_1-\delta_2,\widetilde{t}_1+\delta_2]\subset\Pi$. Moreover,

$$\gamma_i-\gamma_i(t_0)=\varepsilon\dot{\gamma}_i^-\delta t_0+o(\varepsilon\delta\wp).$$

Hence $\alpha_5(t;\varepsilon\delta\wp)$ has the order $o(t;\varepsilon\delta\wp)$ and $\alpha_6(t;\varepsilon\delta\wp)$ has the form

$$\alpha_6(t;\varepsilon\delta\wp)=-\varepsilon Y(\gamma_i;t)f_i^-\dot{\gamma}_i^-\delta t_0+o(t;\varepsilon\delta\wp).$$

Thus

$$\alpha_2(t;\varepsilon\delta\wp)=-\varepsilon\sum_{i=p+1}^{s}Y(\gamma_i;t)f_i^-\dot{\gamma}_i^-\delta t_0-$$
$$\sum_{i=p+1}^{s}\int\limits_{\gamma_i(t_0)}^{\gamma_i}Y(\xi;t)\widetilde{f}_{x_i}[\xi]\Delta x(\tau_i(\xi))\,d\xi+o(t;\varepsilon\delta\wp).$$

The equalitys $\alpha_i(t;\varepsilon\delta\wp) = o(t;\varepsilon\delta\wp)$, $i = 3, 4$, are proved analogously (see (3.10)).

Now $h_1(t;\widetilde{t}_0,\varepsilon\delta\wp)$ is represented by the form

$$h_1(t;\widetilde{t}_0,\varepsilon\delta\wp) = -\varepsilon\sum_{i=p+1}^{s} Y(\gamma_i;t)f_i^-\dot{\gamma}_i^-\delta t_0 -$$
$$-\sum_{i=p+1}^{s}\int\limits_{\gamma_i(t_0)}^{\gamma_i} Y(\xi;t)\widetilde{f}_{x_i}[\xi]\Delta x(\tau_i(\xi))\,d\xi + o(t;\varepsilon\delta\wp). \quad (3.11)$$

From (3.6), taking into account (3.8), (3.9), (3.11), we obtain (1.6), where $\delta x(t;\delta\wp)$ has the form (1.7).

4. Proof of Theorem 2

Let $r_1 = \widetilde{t}_0$, $r_2 = \widetilde{t}_1$. Then for any $(\varepsilon,\delta\wp) \in [0,\varepsilon_1]\times V^+$, the solution $y(t;\widetilde{\wp}+\varepsilon\delta\wp)$ is defined on the interval $[\widetilde{t}_1-\delta_1,\widetilde{t}_1+\delta_1]$, and the solution $x(t;\widetilde{\wp}+\varepsilon\delta\wp)$ is defined on the interval $[\tau,\widetilde{t}_1+\delta_1]$. Moreover,

$$y(t;\widetilde{\wp}+\varepsilon\delta\wp) = x(t;\widetilde{\wp}+\varepsilon\delta\wp), \quad t\in[t_0,\widetilde{t}_1+\delta_1]$$

(see Lemmas 2.1, 2,2 and (2.4)). Thus

$$\Delta x(t) = \begin{cases} \varepsilon\delta\varphi(t), & t\in[\tau,\widetilde{t}_0],\\ \varphi(t)-\widetilde{x}(t), & t\in[\widetilde{t}_0,t_0),\\ \Delta y(t), & t\in(t_0,\widetilde{t}_1+\delta_1). \end{cases} \quad (4.1)$$

Let the numbers $\delta_2\in(0,\delta_1]$, $\varepsilon_2\in[0,\varepsilon_1]$, the existence of which are proved in Lemma 2.5, be so small that for any $(\varepsilon,\delta\wp)\in(0,\varepsilon_2]\times V^+$ the inequality

$$\gamma_s(t_0) < \widetilde{t}_1 - \delta_2$$

is valid. From Lemma 2.5 and (4.1) we have

$$|\Delta x(t)| \le O(\varepsilon\delta\wp) \quad \forall(t,\varepsilon\delta\wp)\in[t_0,\widetilde{t}_1+\delta_1]\times(0,\varepsilon_2]\times V^+, \quad (4.2)$$
$$\Delta x(t_0) = \varepsilon[\delta x_0 - f_p^+\delta t_0] + o(\varepsilon\delta\wp). \quad (4.3)$$

The function $\Delta x(t)$ satisfies the equation (3.4) on the interval $[t_0,\widetilde{t}_1+\delta_2]$.

By means of the Cauchy formula, the solution $\Delta x(t)$ can be represented in the form

$$\Delta x(t) = Y(t_0;t)\Delta x(t_0) + \varepsilon\int\limits_{t_0}^{t} Y(\xi,t)\widetilde{f}_u[\xi]\delta u(\xi)\,d\xi +$$
$$+\sum_{i=0}^{1} h_i(t;t_0,\varepsilon\delta\wp), \quad t\in[t_0,\widetilde{t}_1+\delta_2], \quad (4.4)$$

where

$$h_0(t;t_0,\varepsilon\delta\wp)=\sum_{i=1}^{s}\int\limits_{\tau_i(t_0)}^{t_0}Y(\gamma_i(\xi);t)\widetilde{f}_{x_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\Delta x(\xi)\,d\xi, \tag{4.5}$$

and $h_1(t;t_0,\varepsilon\delta\wp)$ has the form (3.7).

By virtue of Lemma 3.4 [7, p. 37]. the function $Y(\xi;t)$ is continuous on the set $[\widetilde{t}_0,\tau_s(\widetilde{t}_1-\delta_2)]\times[\widetilde{t}_1-\delta_2,\widetilde{t}_1+\delta_2]$. It is obvious that $t_0\in[\widetilde{t}_0,\tau_s(\widetilde{t}_1-\delta_2)]$, hence

$$Y(t_0;t)\Delta x(t_0)=\varepsilon Y(\widetilde{t}_0;t)[\delta x_0-f_p^+\delta t_0]+o(t;\varepsilon\delta\wp). \tag{4.6}$$

(see (4.3)).

Now let us transform $h_0(t;t_0,\varepsilon\delta\wp)$, (see (4.5)). We have

$$\begin{aligned}h_0(t;t_0,\varepsilon\delta\wp)&=\sum_{i=1}^{p}\int\limits_{\tau_i(t_0)}^{t_0}Y(\gamma_i(\xi);t)\widetilde{f}_{x_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\Delta x(\xi)\,d\xi+\\&+\sum_{i=p+1}^{s}\Big[\varepsilon\int\limits_{\tau_i(t_0)}^{\widetilde{t}_0}Y(\gamma_i(\xi);t)\widetilde{f}_{x_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\delta\varphi(\xi)\,d\xi+\\&+\int\limits_{\widetilde{t}_0}^{t_0}Y(\gamma_i(\xi);t)\widetilde{f}_{x_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\Delta x(\xi)\,d\xi\Big]=\\&=\sum_{i=1}^{p}\int\limits_{t_0}^{\gamma_i(t_0)}Y(\xi;t)\widetilde{f}_{x_i}[\xi]\Delta x(\tau_i(\xi))\,d\xi+\\&+\varepsilon\sum_{i=p+1}^{s}\int\limits_{\tau_i(\widetilde{t}_0)}^{\widetilde{t}_0}Y(\gamma_i(\xi);t)\widetilde{f}_{x_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\delta\varphi(\xi)\,d\xi+\\&+\sum_{i=p+1}^{s}\int\limits_{\gamma_i}^{\gamma_i(t_0)}Y(\xi;t)\widetilde{f}_{x_i}[\xi]\Delta x(\tau_i(\xi))\,d\xi+o(t;\varepsilon\delta\wp).\end{aligned} \tag{4.7}$$

After elementary transformations, we can easily prove the following equality

$$\begin{aligned}&\sum_{i=1}^{p}\int\limits_{t_0}^{\gamma_i(t_0)}Y(\xi;t)\widetilde{f}_{x_i}[\xi]\Delta x(\tau_i(\xi))\,d\xi=\\&=\sum_{i=1}^{p}\sum_{j=0}^{i-1}\int\limits_{\gamma_j(t_0)}^{\gamma_{j+1}(t_0)}Y(\xi;t)\widetilde{f}_{x_i}[\xi]\Delta x(\tau_i(\xi))\,d\xi=\end{aligned}$$

$$= \sum_{i=0}^{p-1} \sum_{j=i+1}^{p} \int\limits_{\gamma_i(t_0)}^{\gamma_{i+1}(t_0)} Y(\xi;t)\widetilde{f}_{x_j}[\xi]\Delta x(\tau_j(\xi))\,d\xi, \quad \gamma_0(t_0)=t_0. \tag{4.8}$$

Owing to the inequality (2.28), the expression $h_1(t;t_0,\varepsilon\delta\wp)$ with $t \in [\widetilde{t}_1 - \delta_2, \widetilde{t}_1 + \delta_2]$ can be represented as

$$h_1(t;t_0,\varepsilon\delta\wp) = \sum_{k=1}^{5} \beta_k(t;\varepsilon\delta\wp), \tag{4.9}$$

where

$$\beta_1(t;\varepsilon\delta\wp) = \sum_{i=0}^{p-1} \int\limits_{\gamma_i(t_0)}^{\gamma_{i+1}(t_0)} \omega(\xi;t,\varepsilon\delta\wp)\,d\xi, \quad \beta_2(t;\varepsilon\delta\wp) = \int\limits_{\gamma_p(t_0)}^{\gamma_{p+1}} \omega(\xi;t,\varepsilon\delta\wp)\,d\xi,$$

$$\beta_3(t;\varepsilon\delta\wp) = \sum_{i=p+1}^{s} \int\limits_{\gamma_i}^{\gamma_i(t_0)} \omega(\xi;t,\varepsilon\delta\wp)\,d\xi, \quad \beta_4(t;\varepsilon\delta\wp) = \sum_{i=p+1}^{s-1} \int\limits_{\gamma_i(t_0)}^{\gamma_{i+1}} \omega(\xi;t,\varepsilon\delta\wp)\,d\xi,$$

$$\beta_5(t;\varepsilon\delta\wp) = \int\limits_{\gamma_s(t_0)}^{t} \omega(\xi;t,\varepsilon\delta\wp)\,d\xi, \qquad \omega(\xi;t,\varepsilon\delta\wp) = Y(\xi;t)R(\xi;\varepsilon\delta\wp).$$

For $\beta_1(t;\varepsilon\delta\wp)$ we have

$$\beta_1(t;\varepsilon\delta\wp) = \sum_{i=0}^{p-1} \int\limits_{\gamma_i(t_0)}^{\gamma_{i+1}(t_0)} Y(\xi;t)\big[f(\xi,\widetilde{x}(\tau_1(\xi)) + \Delta x(\tau_1(\xi)),\dots,$$
$$\widetilde{x}(\tau_i(\xi)) + \Delta x(\tau_i(\xi)), \varphi(\tau_{i+1}(\xi)),\dots,\varphi(\tau_s(\xi)), u(\xi)) -$$
$$-f(\xi,\widetilde{x}(\tau_1(\xi)),\dots,\widetilde{x}(\tau_p(\xi)),\widetilde{\varphi}(\tau_{p+1}(\xi)),\dots,\widetilde{\varphi}(\tau_s(\xi)),\widetilde{u}(\xi))\big]d\xi -$$
$$-\sum_{i=0}^{p-1} \int\limits_{\gamma_i(t_0)}^{\gamma_{i+1}(t_0)} Y(\xi;t)\sum_{j=1}^{s} \widetilde{f}_{x_j}[\xi]\Delta x(\tau_j(\xi))\,d\xi - \varepsilon \int\limits_{\gamma_i(t_0)}^{\gamma_{i+1}(t_0)} Y(\xi;t)\widetilde{f}_u[\xi]\delta u(\xi)\,d\xi =$$
$$= \beta_{11}(t;\varepsilon\delta\wp) - \beta_{12}(t;\varepsilon\delta\wp) - \beta_{13}(t;\varepsilon\delta\wp). \tag{4.10}$$

When $\xi \in [\gamma_i(t_0), \gamma_{i+1}(t_0)]$, then $\tau_j(\xi) \ge t_0$, $j=1,\dots,i$, $\tau_j(\xi) \le t_0$, $j = i+1,\dots,p$, $\tau_j(\xi) \le \widetilde{t}_0$, $j=p+1,\dots,s$; hence

$$|\Delta x(\tau_j(\xi))| \le O(\varepsilon\delta\wp), \quad j=1,\dots,i;$$
$$|\Delta x(\tau_j(\xi))| = \varepsilon\delta\varphi(\tau_j(\xi)), \quad j=p+1,\dots,s,$$

(see(4.1), (4.2)).

For each $i=0,\dots,p-1$, $\gamma_{i+1}(t_0) - \gamma_i(t_0) \to 0$ as $\varepsilon \to 0$. Consequently,

$$\beta_{12}(t;\varepsilon\delta\wp) = \sum_{i=0}^{p-1} \sum_{j=i+1}^{p} \int\limits_{\gamma_i(t_0)}^{\gamma_{i+1}(t_0)} Y(\xi;t)\widetilde{f}_{x_j}[\xi]\Delta x(\tau_j(\xi))\,d\xi + o(t;\varepsilon\delta\wp). \tag{4.11}$$

Further,

$$\lim_{\varepsilon\to 0}\sup_{\xi\in[\gamma_i(t_0),\gamma_{i+1}(t_0)]}\big|f(\xi,\widetilde{x}(\tau_1(\xi))+\Delta x(\tau_1(\xi)),\dots,\widetilde{x}(\tau_i(\xi))+\Delta x(\tau_i(\xi)),$$
$$\varphi(\tau_{i+1}(\xi)),\dots,\varphi(\tau_s(\xi)),\widetilde{u}(\xi)+\varepsilon\delta u(\xi))-f_i^+ + f_p^+-$$
$$-f(\xi,\widetilde{x}(\tau_1(\xi)),\dots,\widetilde{x}(\tau_p(\xi)),\widetilde{\varphi}(\tau_{p+1}(\xi)),\dots,\varphi(\tau_s(\xi)),\widetilde{u}(\xi))\big|=0,$$
$$i=0,\dots,p-1,$$

uniformly for $\delta\wp\in V^+$.

From the properties of the functions $Y(\xi;t)$ and $\gamma_i(t)$, $i=1,\dots,p$, immediately follow the equalities

$$\lim_{\varepsilon\to 0}\sup_{\xi\in[\gamma_i(t_0),\gamma_{i+1}(t_0)]}|Y(\xi;t)-Y(\widetilde{t}_0;t)|=0,\quad i=0,\dots,p-1,$$

uniformly for $t\in[\widetilde{t}_1-\delta_2,\widetilde{t}_1+\delta_2]$,

$$\gamma_{i+1}(t_0)-\gamma_i(t_0)=\varepsilon(\dot{\gamma}_{i+1}^+-\gamma_i^+)\delta t_0+o(\varepsilon\delta\wp),\quad i=0,\dots,p-1,\quad \dot{\gamma}_0=1.$$

These equalities allow us to write the following

$$\beta_{11}(t;\varepsilon\delta\wp)=\varepsilon Y(\widetilde{t}_0;t)\sum_{i=0}^{p}(f_i^+-f_p^+)(\dot{\gamma}_{i+1}^+-\dot{\gamma}_i^+)\delta t_0+o(t;\varepsilon\delta\wp),\qquad(4.12)$$

$$\beta_{13}(t;\varepsilon\delta\wp)=o(t;\varepsilon\delta\wp).\qquad(4.13)$$

From (4.10), taking into consideration (4.11)–(4.13), we obtain

$$\beta_1(t;\varepsilon\delta\wp)=\varepsilon Y(\widetilde{t}_0;t)\bigg[\sum_{i=0}^{p}(\widehat{\gamma}_{i+1}^+-\widehat{\gamma}_i^+)f_i^++f_p^+\bigg]\delta t_0-$$
$$-\sum_{i=0}^{p-1}\sum_{j=i+1}^{p}\int\limits_{\gamma_i(t_0)}^{\gamma_{i+1}(t_0)}Y(\xi;t)\widetilde{f}_{x_j}[\xi]\Delta x(\tau_j(\xi))\,d\xi+o(t;\varepsilon\delta\wp).\qquad(4.14)$$

Further,

$$\beta_2(t;\varepsilon\delta\wp)=\int\limits_{\gamma_p(t_0)}^{\gamma_{p+1}}Y(\xi;t)\bigg[f(\xi,\widetilde{x}(\tau_1(\xi))+\Delta x(\tau_1(\xi)),\dots,$$
$$\widetilde{x}(\tau_p(\xi))+\Delta x(\tau_p(\xi)),\varphi(\tau_{p+1}(\xi)),\dots,\varphi(\tau_s(\xi)),u(\xi))-$$
$$-f(\xi,\widetilde{x}(\tau_1(\xi)),\dots,\widetilde{x}(\tau_p(\xi)),\widetilde{\varphi}(\tau_{p+1}(\xi)),\dots,\widetilde{\varphi}(\tau_s(\xi)),\widetilde{u}(\xi))-$$
$$-\sum_{j=1}^{p}\widetilde{f}_{x_j}[\xi]\Delta x(\tau_j(\xi))-\varepsilon\sum_{j=p+1}^{s}\widetilde{f}_{x_j}[\xi]\delta\varphi(\tau_j(\xi))-\varepsilon\widetilde{f}_u[\xi]\delta u(\xi)\bigg]d\xi.$$

In a standard way it can be proved (see (3.10)) that

$$|\beta_2(t;\varepsilon\delta\wp)|\leq\|Y\|\bigg(O(\varepsilon\delta\wp)\sum_{i=1}^{p}\sigma_i(t_0;\varepsilon\delta\wp)+$$

$$+\varepsilon\alpha\sum_{i=p+1}^{s}\sigma_i(t_0;\varepsilon\delta\wp)+\varepsilon\alpha\sigma(t_0;\varepsilon\delta\wp)\Big),$$

where

$$\sigma_i(t_0;\varepsilon\delta\wp)=\int\limits_{\gamma_p(t_0)}^{\gamma_{p+1}}\Big[\int\limits_0^1\big|f_{x_i}(t,\widetilde{x}(\tau_1(t)+\xi\Delta x(\tau_1(t)),\dots)-\widetilde{f}_{x_i}[t]\big|d\xi\Big]\,dt,$$

$$\sigma(t_0;\varepsilon\delta\wp)=\int\limits_{\gamma_p(t_0)}^{\gamma_{p+1}}\Big[\int\limits_0^1\big|f_u(t,\widetilde{x}(\tau_1(t)+\xi\Delta x(\tau_1(t)),\dots)-\widetilde{f}_u[t]\big|d\xi\Big]\,dt,$$

when $t\in[\gamma_p(t_0),\gamma_{p+1}]$, $\tau_j(t)\geq t_0$, $j=1,\dots,p$. Hence (see (4.1))

$$\Delta x(\tau_j(t))=\Delta y(\tau_j(t)).$$

Thus

$$\sigma_i(t_0;\varepsilon\delta\wp)\leq\int\limits_{\widetilde{t}_0}^{\gamma_{p+1}}\Big[\int\limits_0^1\big|f_{x_i}(t,\widetilde{x}(\tau_1(t))+\xi\Delta y(\tau_i(t)),\dots,$$
$$\widetilde{x}(\tau_p(\xi))+\xi\Delta y(\tau_p(t)),\varphi(\tau_{p+1}(t)),\dots,\varphi(\tau_s(t)),u(t))-$$
$$-f_{x_i}(t,\widetilde{x}(\tau_1(t)),\dots,\widetilde{x}(\tau_s(t)),\widetilde{u}(t))\big|d\xi\Big]dt,$$

$$\sigma(t_0;\varepsilon\delta\wp)\leq\int\limits_{\widetilde{t}_0}^{\gamma_{p+1}}\Big[\int\limits_0^1\big|f_u(t,\widetilde{x}(\tau_1(t))+\xi\Delta y(\tau_i(t)),\dots,$$
$$\widetilde{x}(\tau_p(\xi))+\xi\Delta y(\tau_p(t)),\varphi(\tau_{p+1}(t)),\dots,\varphi(\tau_s(t)),u(t))-$$
$$-f_u(t,\widetilde{x}(\tau_1(t)),\dots,\widetilde{x}(\tau_s(t)),\widetilde{u}(t))\big|d\xi\Big]dt,$$

whence (see (2.8))

$$\lim_{\varepsilon\to0}\sigma_i(t_0;\varepsilon\delta\wp)=0,\quad\lim_{\varepsilon\to0}\sigma(t_0;\varepsilon\delta\wp)=0\quad\text{uniformly for }\ \delta\wp\in V^+.$$

Thus

$$\beta_2(t;\varepsilon\delta\wp)=o(t;\varepsilon\delta\wp).\tag{4.15}$$

Now let us transform $\beta_3(t;\varepsilon\delta\wp)$.

$$\beta_3(t;\varepsilon\delta\wp)=\sum_{i=p+1}^{s}\int\limits_{\gamma_i}^{\gamma_i(t_0)}Y(\xi;t)\big[f(\xi,\widetilde{x}(\tau_1(\xi))+\Delta x(\tau_1(\xi)),\dots,$$
$$\widetilde{x}(\tau_{i-1}(\xi))+\Delta x(\tau_{i-1}(\xi)),\varphi(\tau_i(\xi)),\dots,\varphi(\tau_s(\xi)),u(\xi))-$$
$$-f(\xi,\widetilde{x}(\tau_1(\xi)),\dots,\widetilde{x}(\tau_i(\xi)),\widetilde{\varphi}(\tau_{i+1}(\xi)),\dots,\widetilde{\varphi}(\tau_s(\xi)),\widetilde{u}(\xi))\big]d\xi-$$

$$-\sum_{i=p+1}^{s}\Bigg[\sum_{j=1}^{i-1}\int\limits_{\gamma_i}^{\gamma_i(t_0)} Y(\xi;t)\widetilde{f}_{x_j}[\xi]\Delta x(\tau_j(\xi))\,d\xi+$$

$$+\int\limits_{\gamma_i}^{\gamma_i(t_0)} Y(\xi;t)\widetilde{f}_{x_i}[\xi]\Delta x(\tau_i(\xi))\,d\xi+\varepsilon\sum_{j=i+1}^{s}\int\limits_{\gamma_i}^{\gamma_i(t_0)} Y(\xi;t)\widetilde{f}_{x_j}[\xi]\delta\varphi(\tau_j(\xi))\,d\xi\Bigg]-$$

$$-\varepsilon\int\limits_{\gamma_i}^{\gamma_i(t_0)} Y(\xi;t)\widetilde{f}_{u}[\xi]\delta u(\xi)\,d\xi \tag{4.16}$$

According to 1.5, we have

$$\lim_{\varepsilon\to 0}\sup_{\xi\in[\gamma_i,\gamma_i(t_0)]}\big|f(\xi,\widetilde{x}(\tau_1(\xi))+\Delta x(\tau_1(\xi)),\dots,\widetilde{x}(\tau_{i-1}(\xi))+\Delta x(\tau_{i-1}(\xi)),$$
$$\varphi(\tau_i(\xi)),\dots,\varphi(\tau_s(\xi)),u(\xi))-$$
$$-f(\xi,\widetilde{x}(\tau_1(\xi)),\dots,\widetilde{x}(\tau_i(\xi)),\widetilde{\varphi}(\tau_{i+1}(\xi)),\dots,\widetilde{\varphi}(\tau_s(\xi)),\widetilde{u}(\xi))+f_i^+\big|=0,$$
$$i=p+1,\dots,s,\quad \text{uniformly for }\ \delta\wp\in V^+.$$

Further,

$$|\Delta x(\tau_j(\xi))|\le O(\varepsilon\delta\wp),\quad j=1,\dots,i-1,\quad \xi\in[\gamma_i,\gamma_i(t_0)],$$
$$\lim_{\varepsilon\to 0}\sup_{\xi\in[\gamma_i,\gamma_i(t_0)]}|Y(\xi;t)-Y(\gamma_i;t)|=0,$$
$$i=p+1,\dots,s,\quad t\in[\widetilde{t}_1-\delta_1,\widetilde{t}_1+\delta_2].$$

On the basis of these relations we obtain

$$\beta_3(t;\varepsilon\delta\wp)=-\varepsilon\sum_{i=p+1}^{s}Y(\gamma_i;t)f_i^+\dot{\gamma}_i^+\delta t_0-$$
$$-\sum_{i=p+1}^{s}\int\limits_{\gamma_i}^{\gamma_i(t_0)} Y(\xi;t)\widetilde{f}_{x_i}[\xi]\Delta x(\tau_i(\xi))\,d\xi+o(t;\varepsilon\delta\wp).$$

Analogously (see (3.10)), it can be proved that

$$\beta_i(t;\varepsilon\delta\wp)=o(t;\varepsilon\delta\wp),\quad i=4,5. \tag{4.17}$$

By virtue of (4.14)–(4.17) for the expression $h_1(t;t_0,\varepsilon\delta\wp)$ we obtain

$$h_1(t;t_0,\varepsilon\delta\wp)=\varepsilon\Bigg\{Y(\widetilde{t}_0,t)\Bigg[\sum_{i=0}^{p}(\widehat{\gamma}_{i+1}^+-\widehat{\gamma}_i^+)f_i^++f_p^+\Bigg]-$$

$$-\sum_{i=p+1}^{s}Y(\gamma_i;t)\dot{\gamma}_i^+f_i^+\Bigg\}\delta t_0-\sum_{i=0}^{p-1}\sum_{j=i+1}^{p}\int\limits_{\gamma_i(t_0)}^{\gamma_{i+1}(t_0)} Y(\xi;t)\widetilde{f}_{x_j}[\xi]\Delta x(\tau_j(\xi))\,d\xi-$$

$$-\sum_{i=p+1}^{s}\int\limits_{\gamma_i}^{\gamma_i(t_0)} Y(\xi;t)\widetilde{f}_{x_i}[\xi]\Delta x(\tau_i(\xi))\,d\xi + o(t;\varepsilon\delta\wp) \tag{4.18}$$

(see (4.9)).

Finally we note that with $t \in [\tilde{t} - \delta_2, \tilde{t_1} + \delta_2]$

$$\varepsilon\int\limits_{t_0}^{t} Y(\xi;t)\tilde{f}_u[\xi]\delta u(\xi)d\xi = \varepsilon\int\limits_{t_0}^{t} Y(\xi;t)\tilde{f}_u[\xi]\delta u(\xi)d\xi + o(t;\varepsilon\delta\wp) \tag{4.19}$$

From (4.4), taking into consideration (4.6), (4.7), (4.8), (4.18), (4.19), we obtain (1.6), where $\delta x(t;\delta\wp)$ has the form (1.10).

References

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