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UNIQUENESS, EXISTENCE, AND INTEGRAL EQUATION FORMULATIONS FOR INTERFACE SCATTERING PROBLEMS

Abstract. We consider a two-dimensional transmission problem in which Helmholtz equations with different wave numbers hold in adjacent non-locally perturbed half-planes having a common boundary which is an infinite, one-dimensional, rough interface line. First a uniqueness theorem for the interface problem is proved provided that the scatterer is a lossy obstacle. Afterwards, by potential methods, the non-homogeneous interface problem is reduced to a system of integral equations and existence results are established.

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რეზიუმე. ჰელმჰოლცის განტოლებისათვის განხილულია ორგანზომილებიანი საკონტაქტო ამოცანა უსასრულო არებისთვის, რომელთაც აქვთ საერთო, უსასრულობაში განფენილი საზღვარი. დამტკიცებულია საკონტაქტო ამოცანის ამონახსნის ერთადერთობის თეორემა. პოტენციალთა მეთოდის გამოყენებით არაერთგვაროვანი საკონტაქტო ამოცანა დაყვანილია ინტეგრალურ განტოლებათა სისტემაზე და დამტკიცებულია ამოცანის არსებობის თეორემა.

1. Introduction

We consider a two-dimensional transmission problem for the Helmholtz equations (reduced wave equations) in non-locally perturbed half-planes Ω_1 and Ω_2 having a common infinite boundary which is assumed to be the graph of a bounded smooth function. These type of mathematical problems model time-harmonic electromagnetic and acoustic scattering by a penetrable unbounded obstacle in an inhomogeneous (piecewise homogeneous) medium. In both domains we look for scattered waves corresponding to different wave numbers and satisfying certain transmission conditions on the interface. In addition, the scattered waves satisfy the so-called *upward* and *downward propagating radiation conditions* (UPRC and DPRC) along with some growth conditions in the x_2 direction, suggested by Chandler-Wilde & Zhang [9], which generalize both the *Sommerfeld radiation condition* and the *Rayleigh expansion condition* for diffraction gratings (see also [24], [4]). In [8], with the help of the appropriate integral equation formulation, it is shown that the Dirichlet problem for a non-locally perturbed half-plane has exactly one solution satisfying the UPRC, provided that the boundary datum is a bounded and continuous function. This result is valid for all wave numbers and holds without any constraints imposed on the maximum boundary amplitude or slope.

An important corollary of these results in the scattering theory is that for a variety of incident fields including the incident plane wave, the Dirichlet boundary-value problem for scattered field has a unique solution (for detail information concerning the history of the problem see, e.g., [8] and references therein.)

In this paper we first prove the uniqueness theorem for the interface problem provided that an obstacle Ω_1 represents a lossy medium, which means that the corresponding wave number is complex. Afterwards we apply the potential method to reduce the non-homogeneous interface problem to the corresponding system of integral equations and establish existence results on the basis of the theory developed in [11] and [1] for a class of systems of second kind integral equations on unbounded domains.

2. Formulation of the Interface Problem. Preliminary Material

2.1. Here we introduce some notation used throughout.

For $h \in \mathbb{R}$, define $\Gamma_h = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 = h\}$ and $U_h^+ = \{x \in \mathbb{R}^2 \mid x_2 > h\}$, $U_h^- = \{x \in \mathbb{R}^2 \mid x_2 < h\}$.

For $V \subset \mathbb{R}^n$ $(n = 1, 2)$ we denote by $BC(V)$ the set of functions bounded and continuous on V, a Banach space under the norm $\|\cdot\|_{\infty,V}$, defined by $\|\psi\|_{\infty,V} := \sup\limits_{x\in V} |\psi(x)|$. We abbreviate $\|\cdot\|_{\infty,\mathbb{R}}$ by $\|\cdot\|_\infty$.

For $0 < \alpha \le 1$, we denote by $BC^{0,\alpha}(V)$ the Banach space of functions $\varphi \in BC(V)$, which are uniformly Hölder continuous with exponent α, with norm $\|\cdot\|_{0,\alpha,V}$ defined by

$$\|\varphi\|_{0,\alpha,V} := \|\varphi\|_{\infty,V} + \sup_{x,y\in V, x\neq y}\left[\frac{|\varphi(x)-\varphi(y)|}{|x-y|^\alpha}\right].$$

Given $v \in L_\infty(V)$ denote by $\partial_j v$, $j = 1, 2$, the (distributional) derivative $\dfrac{\partial v(x)}{\partial x_j}$; $\nabla v = (\partial_1 v, \partial_2 v)$.

We denote by $BC^1(V)$ the Banach space

$$BC^1(V) := \{\varphi \in BC(V) \mid \partial_j\varphi \in BC(V),\ j = 1, 2\}$$

under the norm

$$\|\varphi\|_{1,V} := \|\varphi\|_{\infty,V} + \|\partial_1\varphi\|_{\infty,V} + \|\partial_2\varphi\|_{\infty,V}.$$

Further, let

$$BC^{1,\alpha}(V) := \{\varphi \in BC^1(V) \mid \partial_j\varphi \in BC^{0,\alpha}(V), j = 1, 2\}$$

denote a Banach space under the norm

$$\|\varphi\|_{1,\alpha,V} := \|\varphi\|_{\infty,V} + \|\partial_1\varphi\|_{0,\alpha,V} + \|\partial_2\varphi\|_{0,\alpha,V}.$$

2.2. Given $f \in BC^{1,\alpha}(\mathbb{R})$, $0 < \alpha \leq 1$, with $f_- := \inf_{x_1\in\mathbb{R}} f(x_1) > 0$ and $f_+ := \sup_{x_1\in\mathbb{R}} f(x_1) < +\infty$, define the adjacent two-dimensional regions Ω_1 and Ω_2 by

$$\Omega_1 = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 < f(x_1)\},$$
$$\Omega_2 = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 > f(x_1)\},$$

so that the interface Γ is

$$\partial\Omega_1 = \partial\Omega_2 = \Gamma := \{(x_1, f(x_1)) \mid x_1 \in \mathbb{R}\}.$$

Whenever we wish to denote explicitly the dependence of the regions and interface on the function f we will write $\Omega_{j,f}$ or Ω_j^f for Ω_j ($j = 1, 2$) and Γ_f or Γ^f for Γ.

Further, let $n(x) = (n_1(x), n_2(x))$ stand for the unit normal vector to Γ at the point $x \in \Gamma$ directed out of Ω_1, and $\partial_{n(x)} = \partial/\partial n(x) = n_1(x)\partial_1 + n_2(x)\partial_2$ and $\partial_{\tau(x)} = \partial/\partial\tau(x) = n_2(x)\partial_1 - n_1(x)\partial_2$ denote the usual normal and tangent derivatives with respect to Γ.

2.3. Now we formulate the interface problem which models the scattering of acoustic (or electromagnetic) waves by the penetrable unbounded obstacle Ω_1. The incident plane wave $u^{inc}(x) = e^{ik_2(x\cdot d)}$, $x \in \mathbb{R}^2$, with $d = (d_1, d_2) \in \Sigma_1 := \{(\xi_1, \xi_2) \in \mathbb{R}^2 \mid \xi_1^2 + \xi_2^2 = 1\}$ the propagation direction, will produce a scattered wave u_2 in Ω_2 and a transmitted wave u_1 in Ω_1. Note that one could also consider other types of incident waves, e.g., the so-called point-source waves, rather then plane waves. The waves u_1 and u_2 are annihilated by the Helmholtz operators (reduced wave operators) $\Delta + k_1^2$ and $\Delta + k_2^2$, respectively, i.e.,

$$(\Delta + k_1^2)u_1(x) = 0, \quad x \in \Omega_1, \tag{2.1}$$

$$(\Delta + k_2^2)u_2(x) = 0, \quad x \in \Omega_2, \tag{2.2}$$

and satisfy the so-called conductive interface (transmission) conditions on Γ (cf. [13], [14], [18], [16], [17], [21])

$$u_2(x) + u^{inc}(x) = u_1(x), \quad x \in \Gamma, \tag{2.3}$$

$$\frac{\mu_2^*}{k_2}\,\partial_{n(x)}[u_2(x) + u^{inc}(x)] = \frac{\mu_1^*}{k_1}\,\partial_{n(x)}u_1(x), \quad x \in \Gamma, \tag{2.4}$$

where Δ is the two-dimensional Laplacian and we assume that

$$\begin{gathered} \mu_1^*, \mu_2^*, k_2 \in \mathbb{R}^+ := (0, +\infty), \quad k_1 = \lambda_1 + i\lambda_2, \\ \lambda_1 = \operatorname{Re} k_1 > 0, \quad \lambda_2 = \operatorname{Im} k_1 > 0. \end{gathered} \tag{2.5}$$

We set

$$\begin{aligned} \mu &:= \frac{\mu_1^*}{\mu_2^*}\frac{k_2}{k_1} = \frac{\mu_1^* k_2}{\mu_2^* |k_1|^2}(\lambda_1 - i\lambda_2) = \mu_1 + i\mu_2, \\ \mu_1 &= \frac{\mu_1^* k_2 \lambda_1}{\mu_2^* |k_1|^2} > 0, \ \ \mu_2 = -\frac{\mu_1^* k_2 \lambda_2}{\mu_2^* |k_1|^2} < 0. \end{aligned} \tag{2.6}$$

The functions u_1 and u_2 have to satisfy additional restrictions at infinity which guarantee the uniqueness. To formulate these conditions we introduce some notations and definitions.

Denote by

$$\Phi_k(x, y) := \frac{i}{4}\,H_0^{(1)}(k|x - y|), \ \ (x, y) \in \mathbb{R}^2, \ x \neq y, \tag{2.7}$$

the free-space Green's function (fundamental solution) for the Helmholtz operator $\Delta + k^2$; here $H_m^{(1)}$ is the Hankel function of the first kind of order m.

Definition 2.1. Given a domain $G \subset \mathbb{R}^2$, call $v \in C^2(G) \cap L_\infty(G)$ a radiating solution of the Helmholtz equation in G if $\Delta v + k^2 v = 0$ in G and

$$v(x) = O(r^{-1/2}), \qquad \frac{\partial v(x)}{\partial r} - ikv(x) = o(r^{-1/2}), \ \ r = |x|, \tag{2.8}$$

as $r = |x| \to +\infty$, uniformly in $x/|x|$.

The conditions (2.8) are the classical Sommerfeld radiation conditions. A set of radiating functions corresponding to the domain G and the parameter k we denote by $\operatorname{Som}(G, k)$.

Definition 2.2 ([9])**.** Given a domain $G \subset \mathbb{R}^2$, say that $v : G \to \mathbb{C}$, a solution of the Helmholtz equation $\Delta v + k^2 v = 0$ in G, satisfies the upward (downward) propagating radiation condition – UPRC (DPRC) in G if, for some $h \in \mathbb{R}$ and $\varphi \in L_\infty(\Gamma_h)$, it holds that $U_h^+ \subset G$ ($U_h^- \subset G$) and

$$v(x) = 2\theta \int\limits_{\Gamma_h} \frac{\partial \Phi_k(x,y)}{\partial y_2}\, \varphi(y)\, ds_y, \quad x \in U_h^+ \;\; (x \in U_h^-), \tag{2.9}$$

where $\theta = 1$ for the UPRC and $\theta = -1$ for the DPRC.

We denote the set of functions satisfying the UPRC [DPRC] in G with the parameter k by UPRC$(G;k)$ [DPRC(G,k)].

Note that the existence of the integral (2.9) for arbitrary $\varphi \in L_\infty(\Gamma_h)$ is assured by the bound which follows from the asymptotic behaviour of the Hankel function for small and large real argument

$$\left|\frac{\partial \Phi_k(x,y)}{\partial y_2}\right| \le C\,|x_2 - y_2|\,\Big(|x-y|^{-2} + |x-y|^{-3/2}\Big), \;\; x, y \in \mathbb{R}^2,\; x \ne y,$$

which holds for some $C > 0$ depending only on k.

From now on, along with equations (2.1)–(2.4) we assume that

$$u_1 \in \mathrm{DPRC}(\Omega_1, k_1), \quad u_2 \in \mathrm{UPRC}(\Omega_2, k_2), \tag{2.10}$$

$$\sup_{\Omega_j} |x_2|^\beta |u_j(x)| < \infty, \;\; j = 1, 2, \tag{2.11}$$

for some $\beta \in \mathbb{R}$. Thus, the interface problem we intend to investigate reads as follows.

Problem (P). Given $f_1 \in BC^{1,\alpha}(\Gamma)$ and $f_2 \in BC^{0,\alpha}(\Gamma)$ find $u_1 \in C^2(\Omega_1) \cap BC^1(\overline{\Omega_1} \setminus U_{h_1}^-)$ $(h_1 < f_-)$ and $u_2 \in C^2(\Omega_2) \cap BC^1(\overline{\Omega_2} \setminus U_{h_2}^+)$ $(h_2 > f_+)$, solutions of the Helmholtz equations (2.1) and (2.2), such that (2.10) and (2.11) are fulfilled and

$$\left.\begin{aligned} &[u_1(x)]^- - [u_2(x)]^+ = f_1(x) \\ &\mu\,[\partial_{n(x)} u_1(x)]^- - [\partial_{n(x)} u_2(x)]^+ = f_2(x) \end{aligned}\right\} \text{ on } \Gamma.$$

The symbols $[\cdot]^+$ and $[\cdot]^-$ denote the limits on Γ from Ω_2 and Ω_1, respectively.

The following result states properties of the upward (downward) propagating radiation condition needed later and shows that any radiating solution satisfies the UPRC (DPRC).

Lemma 2.3 ([10]). *Given $H \in \mathbb{R}$ and $v : U_H^+ \to \mathbb{C}$, the following statements are equivalent:*

(i) *$v \in C^2(U_H^+), v \in L_\infty(U_H^+ \setminus U_a^+)$ for all $a > H$, $\Delta v + k^2 v = 0$ in U_H^+, and v satisfies* UPRC *in U_H^+;*

(ii) *there exists a sequence (v_n) of radiating solutions such that $v_n(x) \to v(x)$ uniformly on compact subsets of U_H^+ and*

$$\sup_{x \in U_H^+ \setminus U_a^+,\, n \in \mathbb{N}} |v_n(x)| < +\infty$$

for all $a > H$;

(iii) *v satisfies* (2.9) *for $h = H$ and some $\varphi \in L_\infty(\Gamma_H)$;*

(iv) *$v \in L_\infty(U_H^+ \setminus U_a^+)$ for some $a > H$ and v satisfies* (2.9) *for each $h > H$ with $\varphi = v|_{\Gamma_h}$;*

(v) *$v \in C^2(U_H^+)$, $v \in L_\infty(U_H^+ \setminus U_a^+)$ for all $a > H$, $\Delta v + k^2 v = 0$ in U_H^+, and, for every $h > H$ and radiating solution in U_H^+, w, such that the restriction of w and $\partial_2 w$ to Γ_h are in $L_1(\Gamma_h)$, it holds that*

$$\int\limits_{\Gamma_h} \left(v\frac{\partial w}{\partial n} - w\frac{\partial v}{\partial n} \right) ds = 0.$$

2.4. Let

$$x, y \in U_a^\pm, \ \ a \in \mathbb{R}, \ \ y' = (y_1, 2a - y_2),$$

where y' is a mirror image of $y = (y_1, y_2) \in \mathbb{R}^2$ with respect to the straight line Γ_a.

Denote by $G_k^{\pm(\mathcal{D})}(x, y; a)$ and $G_k^{\pm(\mathcal{I})}(x, y; a)$ the Dirichlet Green's function and the impedance Green's function for the Helmholtz operator $\Delta + k^2$ in the half-planes $U_a^\pm$. It is well-know that (see, e.g., [9], [8])

$$\begin{aligned} G_k^{\pm(\mathcal{D})}(x, y; a) &= \Phi_k(x, y) - \Phi_k(x, y'), \ \ x, y \in U_a^\pm, \\ G_k^{\pm(\mathcal{I})}(x, y; a) &= \Phi_k(x, y) + \Phi_k(x, y') + P_k^{(\pm)}(x - y'), \ \ x, y \in U_a^\pm, \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} P_k^{(\pm)}(z) :&= -\frac{ik}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{\exp\{i[z_1 t \pm z_2\sqrt{k^2 - t^2}]\}}{\sqrt{k^2 - t^2}[\sqrt{k^2 - t^2} + k]}\, dt = \\ &= \frac{|z| e^{ik|z|}}{\pi} \int\limits_0^\infty \frac{t^{-1/2} e^{-k|z|t}\,[|z| \pm z_2(1 + it)]}{\sqrt{t - 2i}\,[|z|t - i(|z| \pm z_2)]^2}\, dt, \quad z \in \overline{U_0^\pm}. \end{aligned}$$

Here and throughout all square roots are taken with non-negative real and imaginary parts.

The functions $G_k^{\pm(\mathcal{D})}(x, y)$ are radiating in $U_a^\pm$ and

$$G_k^{\pm(\mathcal{D})}(x, y; a) = 0 \quad \text{for} \quad x \in \Gamma_a,$$

while $G_k^{\pm(\mathcal{I})}(x, y)$ are radiating functions in $U_a^\pm$ and

$$\frac{\partial}{\partial x_2} G_k^{\pm(\mathcal{I})}(x, y; a) \pm ik\, G_k^{\pm(\mathcal{I})}(x, y; a) = 0 \quad \text{for} \quad x \in \Gamma_a. \tag{2.13}$$

Moreover, for $G(x,y) \in \{G_k^{\pm(\mathcal{D})}(x,y;a), G_k^{\pm(\mathcal{I})}(x,y;a)\}$ there hold the following inequalities

$$\begin{aligned}
&|G(x,y)|,\ |\nabla_x G(x,y)|,\ |\nabla_y G(x,y)| \leq \\
&\quad \leq C\,\frac{(1+|x_2|)(1+|y_2|)}{|x-y|^{3/2}} \quad \text{for} \quad |x-y| \geq 1, \\
&|G(x,y)| \leq C\,(1+|\log|x-y||) \ \text{ for } \ 0<|x-y|\leq 1, \\
&|\nabla_x G(x,y)|,\ |\nabla_y G(x,y)| \leq C\,|x-y|^{-1} \ \text{ for } \ 0<|x-y|\leq 1, \\
&|G(x,y)|, |\nabla_x G(x,y)|, |\nabla_y G(x,y)|, |\nabla_x\, \partial_{n(y)} G(x,y)| \leq \\
&\quad \leq C_1[1+|x_1-y_1|]^{-3/2} \text{ for } |x_2-y_2|\geq\delta>0,\ |x_2|=H,\ y\in\Gamma,
\end{aligned} \tag{2.14}$$

with $C>0$ depending only on a and k, and $C_1>0$ depending only on a, k, δ, Γ, and H (for details see [6], [9], [8]).

Denote by $G^{(j)}(x,y)$, $j=1,2$, the generalized Dirichlet Green's functions for the domains Ω_j:

$$\begin{aligned}
G^{(1)}(x,y) &= G_{k_1}^{-(\mathcal{D})}(x,y;h_2) - V^{(1)}(x,y), \quad y,\, x\in\overline{\Omega_1}, \\
G^{(2)}(x,y) &= G_{k_2}^{+(\mathcal{D})}(x,y;h_1) - V^{(2)}(x,y), \quad y,\, x\in\overline{\Omega_2},
\end{aligned}$$

where $V^{(1)}(\cdot,y)$ $[V^{(2)}(\cdot,y)]$ is a solution to the Helmholtz equation (2.1) [(2.2)] satisfying the DPRC [UPRC] and the boundary condition

$$\begin{aligned}
&V^{(1)}(x,y) = G_{k_1}^{-(\mathcal{D})}(x,y;h_2), \quad y\in\Omega_1,\ x\in\Gamma, \\
&\Big[V^{(2)}(x,y) = G_{k_2}^{+(\mathcal{D})}(x,y;h_1), \quad y\in\Omega_2,\ x\in\Gamma\Big].
\end{aligned} \tag{2.15}$$

Due to the results obtained in [9], [8], and [2] the functions $V^{(j)}(x,y)$ and $G^{(j)}(x,y)$ are determined uniquely, are radiating and admit some bounds similar to (2.14) (see [23])

$$\begin{aligned}
&|G^{(j)}(x,y)|,\ |\nabla_x G^{(j)}(x,y)|,\ |\nabla_y G^{(j)}(x,y)| \leq \\
&\qquad \leq C_j^*\,\frac{(1+|x_2|)(1+|y_2|)}{|x-y|^{3/2}} \quad \text{for} \quad |x-y|\geq 1, \\
&|G^{(j)}(x,y)| \leq C_j^*\,(1+|\log|x-y||) \ \text{ for } \ 0<|x-y|\leq 1, \\
&|\nabla_x G^{(j)}(x,y)|,\ |\nabla_y G^{(j)}(x,y)| < C_j^*\,|x-y|^{-1} \quad \text{for} \ \ 0<|x-y|\leq 1, \\
&|G^{(j)}(x,y)|,\ |\nabla_x G^{(j)}(x,y)|,\ |\nabla_y G^{(j)}(x,y)|,\ |\nabla_x\,\partial_{n(y)} G^{(j)}(x,y)| \leq \\
&\qquad \leq C_j^{**}\,[1+|x_1-y_1|]^{-3/2} \quad \text{for} \ \ x_2=a_j, y\in\Gamma, \\
&\qquad a_1<f_-<f_+<a_2,
\end{aligned} \tag{2.16}$$

with $C_j^*>0$ depending only on h_j, k_j, and Γ, and $C_j^{**}>0$ depending only on h_j, k_j, a_j and Γ.

Lemma 2.4. *Let* $u_j \in C^2(\Omega_j) \cap C^1(\overline{\Omega_j})$ *be a solution to the equation* $(\Delta + k_j^2)u_j(x) = 0$ *in* Ω_j *satisfying the* UPRC *for* $j = 2$ *and the* DPRC *for* $j = 1$. *Then*

$$u_j(x) = (-1)^j \int\limits_{\Gamma} \frac{\partial G^{(j)}(x,y)}{\partial n(y)} [u_j(y)]_\Gamma \, ds, \quad x \in \Omega_j,$$

where $n(x)$ *is a unit normal vector at the point* $x \in \Gamma$ *pointing out of* Ω_1,

$$[u_j(y)]_\Gamma = \lim_{\Omega_j \ni x \to y \in \Gamma} u_j(x).$$

Proof. For definiteness let $j = 2$. On the one hand, by standard arguments we easily derive (cf. [9], [2])

$$u_2(x) = -\int\limits_{\Gamma} \Big\{ [G^{+(\mathcal{D})}_{k_2}(x,y;h_1)]_\Gamma \, [\partial_{n(y)} u_2(y)]_\Gamma - \\ - [\partial_{n(y)} G^{+(\mathcal{D})}_{k_2}(x,y;h_1)]_\Gamma \, [u_2(y)]_\Gamma \Big\} \, ds, \quad x \in \Omega_2.$$

On the other hand,

$$0 = \int\limits_{\Gamma} \{ [V^{(2)}(x,y)]_\Gamma \, [\partial_{n(y)} u_2(y)]_\Gamma - [\partial_{n(y)} V^{(2)}(x,y)]_\Gamma \, [u_2(y)]_\Gamma \} \, ds, \quad x \in \Omega_2,$$

since $u_2 \in$ UPRC(Ω_2, k_2) and $V^{(2)}(x,\cdot) \in$ Som(Ω_2, k_2), and $[V^{(2)}(x,\cdot)]_{\Gamma_h}$, $[\partial y_2 V^{(2)}(x,\cdot)]_{\Gamma_h} \in L_1(\Gamma_h)$ for $h > x_2$ (see Lemma 2.3).

Now, in view of (2.15) and summing these two equations, the proof is complete.

The case $j = 1$ can be treated quite similarly. □

2.5. Here we introduce some definitions which we will employ later, in Section 4 (for details see [12], [8], [1]).

For a sequence $\{\varphi_n\} \subset BC(\mathbb{R})$ and $\varphi \in BC(\mathbb{R})$ we say that $\{\varphi_n\}$ *converges strictly* to φ and write $\varphi_n \xrightarrow{s} \varphi$ if φ_n converges to φ in the Buck's *strict topology* (s-topology) ([3]) which is equivalent to the following: $\{\varphi_n\}$ is bounded in $BC(\mathbb{R})$ and $\varphi_n \to \varphi$ uniformly on every compact subsets of $\mathbb{R}$.

A set $X \subset BC(\mathbb{R})$ is said to be *sequentially compact in the strict topology* if any sequence in X has a subsequence that is convergent in the strict topology with limit in X.

Further, let $k(\cdot,\cdot)$ be measurable, $k(s,\cdot) \in L_1(\mathbb{R})$ and $\mathcal{K}_k \, \psi(\cdot) := \int_{\mathbb{R}} k(\cdot,t) \, \psi(t) \, dt \in L_\infty(\mathbb{R})$ for every $\psi \in L_\infty(\mathbb{R})$. Assume that

$$|||k||| := \operatorname*{ess\,sup}_{s\in\mathbb{R}} \int\limits_{\mathbb{R}} |k(s,t)| \, dt = \operatorname*{ess\,sup}_{s\in\mathbb{R}} \|k(s,\cdot)\|_{L_1(\mathbb{R})} < \infty.$$

Identify $k(\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{C}$ with the mapping $s \mapsto k(s,\cdot)$ in $\mathbf{Z} := L_\infty(\mathbb{R}, L_1(\mathbb{R}))$, which is measurable and essentially bounded with norm $|||k|||$. Let $\mathbf{K}$ denote the set of those functions $k \in \mathbf{Z}$ having the property $\mathcal{K}_k\psi \in C(\mathbb{R})$ for every $\psi \in L_\infty(\mathbb{R})$. Clearly, $\mathbf{Z}$ is a Banach space with the norm $|||\cdot|||$ and $\mathbf{K}$ is a closed subset of $\mathbf{Z}$. Moreover,

$$|||k||| = \sup_{s\in\mathbb{R}} \|k(s,\cdot)\|_{L_1(\mathbb{R})} \quad \text{for} \quad k \in \mathbf{K}.$$

Note that $\mathcal{K}_k : L_\infty(\mathbb{R}) \to C(\mathbb{R})$ and is bounded iff $k \in \mathbf{K}$. In this case $\|\mathcal{K}_k\| = |||k|||$.

For a sequence $\{k_n\} \in \mathbf{K}$ and $k \in \mathbf{K}$ we say that $\{k_n\}$ is *σ-convergent* (converges in the σ-topology) to k and write $k_n \xrightarrow{\sigma} k$ if $\sup\limits_{n\in\mathbb{N}} |||k_n||| < \infty$ and, for all $\psi \in L_\infty(\mathbb{R})$, $\mathcal{K}_{k_n}\psi(s) \to \mathcal{K}_k\psi(s)$, i.e.,

$$\int\limits_{\mathbb{R}} k_n(s,t)\,\psi(t)\,dt \to \int\limits_{\mathbb{R}} k(s,t)\,\psi(t)\,dt \quad \text{as} \quad n \to +\infty,$$

uniformly on every compact subsets of $\mathbb{R}$ with respect to s.

A subset $\mathbf{K}_1 \subset \mathbf{K}$ is said to be *σ-sequentially compact* if each sequence in $\mathbf{K}_1$ has a σ-convergent subsequence with limit in $\mathbf{K}_1$.

A linear operator $\mathcal{K}$ is said to be *sequentially compact with respect to σ-topology* if for any bounded set $X \subset BC(\mathbb{R})$, the set $\mathcal{K}(X)$ is sequentially compact in the strict topology.

A family $\mathcal{Q}$ of linear operators on $BC(\mathbb{R})$ is said to be *collectively sequentially compact with respect to the σ-topology* if for any bounded set $X \subset BC(\mathbb{R})$ the set $\cup_{\mathcal{K}\in\mathcal{Q}}\mathcal{K}(X)$ is sequentially compact in the strict topology.

Finally, for a sequence of linear operators $\{\mathcal{K}_n\}$ and $\mathcal{K}$ on $BC(\mathbb{R})$ we write $\mathcal{K}_n \xrightarrow{\sigma} \mathcal{K}$ if $\mathcal{K}_n\varphi_n \xrightarrow{s} \mathcal{K}\varphi$ for every $\varphi_n \xrightarrow{s} \varphi$.

3. The Uniqueness Result

Here we show that the homogeneous version of the above formulated interface problem possesses only the trivial solution.

First we introduce some notations which are used in the remaining part of the paper. For $A > 0$, $h_1 < f_-$ and $h_2 > f_+$ define

$$\begin{aligned}
&\Omega_j(A) := \{x \in \Omega_j \mid -A < x_1 < A\}, \;\; j = 1,2,\\
&\Gamma_h(A) := \{x \in \Gamma_h \mid -A < x_1 < A\},\\
&\Gamma(A) := \{x \in \Gamma \mid -A < x_1 < A\},\\
&\Omega_{1,h_1} := \Omega_1 \setminus \overline{U^-_{h_1}} = U^+_{h_1} \setminus \overline{\Omega_2},\\
&\Omega_{2,h_2} := \Omega_2 \setminus \overline{U^+_{h_2}} = U^-_{h_2} \setminus \overline{\Omega_1},\\
&\Omega_{j,h_j}(A) := \{x \in \Omega_{j,h_j} \mid -A < x_1 < A\}, \;\; j = 1,2,\\
&\gamma_j(\pm A) = \{x \in \Omega_{j,h_j}(A) \mid x_1 = \pm A\}, \;\; j = 1,2.
\end{aligned} \tag{3.1}$$

Theorem 3.1. *Let*
(i) *for* $h_1 < f_-$ *and* $h_2 > f_+$

$$u_j : \Omega_j \to \mathbb{C},\ j = 1, 2,\ \ u_1 \in C^2(\Omega_1) \cap BC^1(\overline{\Omega_1} \setminus U^-_{h_1}),$$
$$u_2 \in C^2(\Omega_2) \cap BC^1(\overline{\Omega_2} \setminus U^+_{h_2});$$

(ii) u_1 *and* u_2 *solve the equations* (2.1) *and* (2.2), *respectively, and*

$$[u_1(x)]^- = [u_2(x)]^+ \quad on \quad \Gamma, \tag{3.2}$$
$$\mu\, [\partial_{n(x)} u_1(x)]^- = [\partial_{n(x)} u_2(x)]^+ \quad on \quad \Gamma, \tag{3.3}$$

where k_1, k_2, *and* μ *are determined by* (2.5) *and* (2.6)*;*
(iii) $u_1 \in$ DPRC$\,(\Omega_1, k_1)$ *and* $u_2 \in$ UPRC$\,(\Omega_2, k_2)$*;*
(iv) u_1 *and* u_2 *meet the conditions*

$$\sup_{\Omega_j} |x_2|^\beta |u_j(x)| < \infty,\ \ j = 1, 2,$$

for some $\beta \in \mathbb{R}$.
Then $u_j = 0$ *in* $\Omega_j,\ j = 1, 2.$

Proof. We prove the theorem in several steps.
Step 1. Apply Green's first theorem to u_j and its complex conjugate $\overline{u_j}$ in $\Omega_{j,h_j}(A)$ to obtain

$$\int\limits_{\Omega_{1,h_1}(A)} \{|\nabla u_1|^2 - k_1^2 |u_1|^2\}\, dx = \int\limits_{\Gamma(A)} \frac{\partial u_1}{\partial n}\, \overline{u_1}\, ds - \int\limits_{\Gamma_{h_1}(A)} \frac{\partial u_1}{\partial x_2}\, \overline{u_1}\, ds + {} + \Big[\int\limits_{\gamma_1(A)} - \int\limits_{\gamma_1(-A)} \Big] \frac{\partial u_1}{\partial x_1}\, \overline{u_1}\, ds, \tag{3.4}$$

$$- \int\limits_{\Omega_{2,h_2}(A)} \{|\nabla u_2|^2 - k_2^2 |u_2|^2\}\, dx = \int\limits_{\Gamma(A)} \frac{\partial u_2}{\partial n}\, \overline{u_2}\, ds - \int\limits_{\Gamma_{h_2}(A)} \frac{\partial u_2}{\partial x_2}\, \overline{u_2} ds - {} - \Big[\int\limits_{\gamma_2(A)} - \int\limits_{\gamma_2(-A)} \Big] \frac{\partial u_2}{\partial x_1}\, \overline{u_2}\, ds. \tag{3.5}$$

Multiply (3.4) by $-\mu$, add to (3.5), take into consideration the interface conditions (3.2) and (3.3), and take the imaginary part of the equation obtained

$$2 - \operatorname{Im} \mu \int\limits_{\Omega_{1,h_1}(A)} |\nabla u_1|^2\, dx + \operatorname{Im}(\mu k_1^2) \int\limits_{\Omega_{1,h_1}(A)} |u_1|^2\, dx =$$

$$= \operatorname{Im} \Bigg\{ \mu \int\limits_{\Gamma_{h_1}(A)} \frac{\partial u_1}{\partial x_2}\, \overline{u_1}\, ds - \int\limits_{\Gamma_{h_2}(A)} \frac{\partial u_2}{\partial x_2}\, \overline{u_2}\, ds - \mu \mathcal{R}_1(A) - \mathcal{R}_2(A) \Bigg\}, \tag{3.6}$$

where

$$\mathcal{R}_j(A) := \left(\int\limits_{\gamma_j(A)} - \int\limits_{\gamma_j(-A)} \right) \frac{\partial u_j}{\partial x_1} \overline{u_j}\, ds, \ j = 1, 2. \tag{3.7}$$

Note that

$$-\operatorname{Im} \mu = -\mu_2 = \frac{\mu_1^* k_2 \lambda_2}{\mu_2^* |k_1|^2} > 0, \quad \operatorname{Im}(\mu k_1^2) = \frac{\mu_1^*}{\mu_2^*} k_2 \lambda_2 > 0, \tag{3.8}$$

due to (2.5) and (2.6).

Step 2. Here we derive the inequality

$$\frac{f_- - h_1}{\sqrt{1+L^2}} \int\limits_{\Gamma(A)} |u_1(x)|^2\, ds \le 2 \int\limits_{\Omega_{1,h_1}(A)} |u_1(x)|^2\, dx + \\ +2(f_+ - h_1)(f_- - h_1) \int\limits_{\Omega_{1,h_1}(A)} |\partial_2 u_1(x)|^2\, dx, \tag{3.9}$$

where $L = \sup\limits_{x_1 \in \mathbb{R}} |f'(x_1)| < \infty$.

In fact, from the equality

$$u_1(x_1, f(x_1)) = u_1(x_1, b) + \int\limits_b^{f(x_1)} \partial_2 u_1(x_1, x_2)\, dx_2, \quad h_1 \le b \le f_-,$$

using the Cauchy-Schwarz inequality we get

$$|u_1(x_1, f(x_1))|^2 \le 2|u_1(x_1, b)|^2 + 2[f(x_1) - b] \int\limits_b^{f(x_1)} |\partial_2 u_1(x_1, x_2)|^2 dx_2 \le \\ \le 2|u_1(x_1, b)|^2 + 2(f_+ - h_1) \int\limits_{h_1}^{f(x_1)} |\partial_2 u_1(x_1, x_2)|^2\, dx_2.$$

Integrating over the interval $(-A, A)$ with respect to x_1 gives

$$\int\limits_{-A}^{A} |u_1(x_1, f(x_1))|^2\, dx_1 \le 2 \int\limits_{-A}^{A} |u_1(x_1, b)|^2\, dx_1 + \\ + 2(f_+ - h_1) \int\limits_{\Omega_1, h_1(A)} |\partial_2 u_1(x_1, x_2)|^2\, dx.$$

Note that $ds = \sqrt{1+[f'(x_1)]^2}\,dx_1 \le \sqrt{1+L^2}\,dx_1$. Therefore, we have

$$\frac{1}{\sqrt{1+L^2}} \int\limits_{\Gamma(A)} |u_1(x)|^2\,ds \le$$

$$\le 2\int\limits_{-A}^{A} |u_1(x_1,b)|^2\,dx_1 + 2(f_+ - h_1)\int\limits_{\Omega_{1,h_1}(A)} |\partial_2 u_1(x)|^2 dx.$$

Now, integration from h_1 to f_- with respect to b leads to the inequality (3.9). Note that the coefficients in (3.9) do not depend on A.

Now, by virtue of (3.9) it follows from (3.6) that

$$2\delta_0 \int\limits_{\Gamma(A)} |u_1(x)|^2\,ds \le \operatorname{Im}\Big\{\mu \int\limits_{\Gamma_{h_1}(A)} \partial_2 u_1\,\overline{u_1}\,ds\Big\} -$$

$$- \operatorname{Im} \int\limits_{\Gamma_{h_2}(A)} \partial_2 u_2\,\overline{u_2}\,ds - \operatorname{Im}\{\mu\mathcal{R}_1(A)\} - \operatorname{Im}\mathcal{R}_2(A) \quad (3.10)$$

with $\delta_0 > 0$ independent of A (see (3.8))

$$\delta_0 = \frac{\mu_1^* k_2 \lambda_2}{\mu_2^*}\,\frac{f_- - h_1}{\sqrt{1+L^2}}\,\frac{\delta_1}{\delta_2} > 0,$$

$$\delta_1 = \min\{1, |k_1|^{-2}\}, \quad \delta_2 = \max\{2, 2(f_+ - h_1)(f_- - h_1)\}.$$

Step 3. Due to condition (3.2)

$$[u_1(x)]_\Gamma = [u_2(x)]_\Gamma =: E(x), \quad x \in \Gamma, \quad \text{and let } \tilde{E}(x_1) := E(x_1, f(x_1)). \quad (3.11)$$

By Lemma 2.4 we can then represent u_1 and u_2 in the form

$$u_1(x) = -\int\limits_{\Gamma} \frac{\partial G^{(1)}(x,y)}{\partial n(y)} E(y)\,ds, \quad x \in \Omega_1, \quad (3.12)$$

$$u_2(x) = \int\limits_{\Gamma} \frac{\partial G^{(2)}(x,y)}{\partial n(y)} E(y) ds, \quad x \in \Omega_2. \quad (3.13)$$

Let us consider the functions

$$v_1(x;A) = -\int\limits_{\Gamma(A)} \frac{\partial G^{(1)}(x,y)}{\partial n(y)} E(y)\,ds, \quad x \in \Omega_1, \quad (3.14)$$

$$v_2(x;A) = \int\limits_{\Gamma(A)} \frac{\partial G^{(2)}(x,y)}{\partial n(y)} E(y) ds, \quad x \in \Omega_2. \quad (3.15)$$

It is evident that v_1 is radiating in Ω_1 and v_2 is radiating in Ω_2 (due to the compactness of $\overline{\Gamma(A)}$). Due to the bounds (2.16) (cf. [10], Lemma 6.1), for $p \geq 1$

$$v_1(x;A)|_{\Gamma_{h_1}},\ \partial_1 v_1(x;A)|_{\Gamma_{h_1}},\ \partial_2 v_1(x;A)|_{\Gamma_{h_1}} \in L_p(\Gamma_{h_1}) \cap BC(\Gamma_{h_1}),$$
$$v_2(x;A)|_{\Gamma_{h_2}},\ \partial_1 v_2(x;A)|_{\Gamma_{h_2}},\ \partial_2 v_2(x;A)|_{\Gamma_{h_2}} \in L_p(\Gamma_{h_2}) \cap BC(\Gamma_{h_2}).$$

Therefore, due to Lemma 2.3, v_1 and v_2 are representable in the form of double layer potentials

$$v_1(x;A) = -2 \int\limits_{\Gamma_{h_1}} \frac{\partial \Phi_{k_1}(x,y)}{\partial y_2} [v_1(y;A)]_{\Gamma_{h_1}}\, ds, \quad x_2 < h_1,$$
$$v_2(x;A) = 2 \int\limits_{\Gamma_{h_2}} \frac{\partial \Phi_{k_2}(x,y)}{\partial y_2} [v_2(y;A)]_{\Gamma_{h_2}}\, ds, \quad x_2 > h_2.$$

In turn, these representations imply (see [10], Remark 2.15)

$$v_1(x;A) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} \exp\left\{ ix_1\xi_1 - ix_2\sqrt{k_1^2 - \xi_1^2} \right\} g_1(\xi_1)\, d\xi_1, \quad x_2 < h_1,$$
$$v_2(x;A) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} \exp\left\{ ix_1\xi_1 + ix_2\sqrt{k_2^2 - \xi_1^2} \right\} g_2(\xi_1)\, d\xi_1, \quad x_2 > h_2,$$

where

$$g_1(\xi_1) = \mathcal{F}_{x_1\to\xi_1}[\varphi_1(x_1)] \exp\left\{ ih_1\sqrt{k_1^2 - \xi_1^2} \right\} = $$
$$= \hat{\varphi}_1(\xi_1) \exp\left\{ ih_1\sqrt{k_1^2 - \xi_1^2} \right\},$$
$$g_2(\xi_1) = \mathcal{F}_{x_1\to\xi_1}[\varphi_2(x_1)] \exp\left\{ -ih_2\sqrt{k_2^2 - \xi_1^2} \right\} = $$
$$= \hat{\varphi}_2(\xi_1) \exp\left\{ -ih_2\sqrt{k_2^2 - \xi_1^2} \right\},$$

$$\operatorname{Im} \sqrt{k_1^2 - \xi_1^2} > 0, \ \ \operatorname{Re} \sqrt{k_1^2 - \xi_1^2} > 0, \ \ \sqrt{k_2^2 - \xi_1^2} = i\sqrt{\xi_1^2 - k_2^2} \ \text{ for } \ \xi_1^2 > k_2^2,$$
$$\varphi_1(x_1) := [v_1(x)]_{\Gamma_{h_1}}, \quad \varphi_2(x_1) := [v_2(x)]_{\Gamma_{h_2}},$$

$\mathcal{F}^{\pm 1}$ denote the Fourier (direct and inverse) transforms

$$\hat{\varphi}(\xi_1) = \mathcal{F}_{x_1\to\xi_1}[\varphi(x_1)] := \int\limits_{-\infty}^{+\infty} \varphi(x_1) e^{-ix_1\xi_1}\, dx_1,$$
$$\mathcal{F}^{-1}_{\xi_1\to x_1}[\psi(\xi_1)] := \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} \psi(\xi_1) e^{ix_1\xi_1}\, d\xi_1.$$

Applying these relations we derive (cf. [10], Lemma 6.1)

$$\int\limits_{\Gamma_{h_1}} \frac{\partial v_1}{\partial x_2}\,\overline{v_1}\,ds = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \widehat{\frac{\partial v_1}{\partial x_2}\Big|_{\Gamma_{h_1}}}\;\overline{\widehat{v_1|_{\Gamma_{h_1}}}}\,d\xi_1 =$$

$$= -\frac{i}{2\pi}\int\limits_{-\infty}^{+\infty}\sqrt{k_1^2-\xi_1^2}\,|g_1(\xi_1)|^2\,d\xi_1,$$

$$\int\limits_{\Gamma_{h_2}} \frac{\partial v_2}{\partial x_2}\,\overline{v_2}\,ds = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \widehat{\frac{\partial v_2}{\partial x_2}\Big|_{\Gamma_{h_2}}}\;\overline{\widehat{v_2|_{\Gamma_{h_2}}}}\,d\xi_1 =$$

$$= \frac{i}{2\pi}\int\limits_{-\infty}^{+\infty}\sqrt{k_2^2-\xi_1^2}\,|g_2(\xi_1)|^2 d\xi_1,$$

$$\operatorname{Im}\int\limits_{\Gamma_{h_1}} \frac{\partial v_1}{\partial x_2}\,\overline{v_1}\,ds \le 0, \quad \operatorname{Re}\int\limits_{\Gamma_{h_1}} \frac{\partial v_1}{\partial x_2}\,\overline{v_1} ds \ge 0, \tag{3.16}$$

$$\operatorname{Im}\int\limits_{\Gamma_{h_2}} \frac{\partial v_2}{\partial x_2}\,\overline{v_2}\,ds \ge 0, \quad \operatorname{Re}\int\limits_{\Gamma_{h_2}} \frac{\partial v_2}{\partial x_2}\,\overline{v_2}\,ds \le 0. \tag{3.17}$$

In view of (3.16) and (2.6) we see that

$$\operatorname{Im}\left\{\mu\int\limits_{\Gamma_{h_1}} \frac{\partial v_1}{\partial x_2}\overline{v_1}ds\right\} = \mu_2 \operatorname{Re}\int\limits_{\Gamma_{h_1}} \frac{\partial v_1}{\partial x_2}\overline{v_1}ds + \mu_1 \operatorname{Im}\int\limits_{\Gamma_{h_1}} \frac{\partial v_1}{\partial x_2}\overline{v_1}ds \le 0. \tag{3.18}$$

Step 4. Let (cf. (3.11))

$$w(x_1) := u_1(x_1, f(x_1)) = u_1(x)|_\Gamma = E(x) = \tilde{E}(x_1). \tag{3.19}$$

It is evident that $w \in BC(\mathbb{R})$ and

$$\int\limits_{-A}^{A} |w(x_1)|^2\,dx_1 \le \int\limits_{\Gamma(A)} |u(x_1)|^2\,ds \le (1+L^2)^{1/2}\int\limits_{-A}^{A} |w(x_1)|^2\,dx_1 \tag{3.20}$$

with the same L as in (3.9).

Further, we define

$$W_A(x_1) = \int\limits_{-A}^{A} (1+|x_1-y_1|)^{-3/2}\,|w(y_1)|\,dy_1. \tag{3.21}$$

From (3.14) and (3.15) with the help of (2.16) we easily get

$$|v_j(x;A)|,\ |\nabla_x v_j(x;A)| \le c_j(1+L^2)^{1/2} W_A(x_1) \text{ for } x\in\Gamma_{h_j},\ j=1,2. \tag{3.22}$$

For $x \in \Gamma_{h_j}$ we have

$$|u_j(x)| \le c_j(1+L^2)^{1/2} \int\limits_{-\infty}^{+\infty} [1+|x_1-y_1|^{-3/2}]\,|w(y_1)|\,dy_1 = \\ = c_j(1+L^2)^{1/2}\, W_\infty(x_1), \tag{3.23}$$

$$|u_j(x)-v_j(x)|,\ |\nabla u_j(x)-\nabla v_j(x)| \le \\ \le c_j(1+L^2)^{1/2} \int\limits_{\mathbb{R}\setminus[-A,A]} [1+|x_1-y_1|^{-3/2}]\,|w(y_1)|\,dy_1 = \\ = c_j(1+L^2)^{1/2}\,[W_\infty(x_1)-W_A(x_1)]. \tag{3.24}$$

Using the relations (3.17), (3.21), (3.22), (3.23), and (3.24) we derive

$$-\operatorname{Im} \int\limits_{\Gamma_{h_2}(A)} \frac{\partial u_2}{\partial x_2}\,\overline{u_2}\,ds = -\operatorname{Im} \int\limits_{\Gamma_{h_2}(A)} \left[\frac{\partial u_2}{\partial x_2}\,\overline{u_2} - \frac{\partial v_2}{\partial x_2}\,\overline{v_2}\right] ds - \\ -\operatorname{Im}\left[\,\int\limits_{\Gamma_{h_2}(A)} \frac{\partial v_2}{\partial x_2}\,\overline{v_2}\,ds - \int\limits_{\Gamma_{h_2}} \frac{\partial v_2}{\partial x_2}\,\overline{v_2}\,ds\right] - \operatorname{Im}\int\limits_{\Gamma_{h_2}} \frac{\partial v_2}{\partial x_2}\,\overline{v_2}\,ds \le \\ \le -\operatorname{Im} \int\limits_{\Gamma_{h_2}(A)} \left[\frac{\partial u_2}{\partial x_2}\overline{u_2} - \frac{\partial v_2}{\partial x_2}\overline{v_2}\right] ds - \\ -\operatorname{Im}\left[\,\int\limits_{\Gamma_{h_2}(A)} \frac{\partial v_2}{\partial x_2}\overline{v_2}ds - \int\limits_{\Gamma_{h_2}} \frac{\partial v_2}{\partial x_2}\overline{v_2}ds\right] \le \\ \le \int\limits_{\Gamma_{h_2}(A)} \left|\frac{\partial u_2}{\partial x_2}\,\overline{u_2} - \frac{\partial v_2}{\partial x_2}\,\overline{v_2}\right| ds + \int\limits_{\Gamma_{h_2}\setminus\Gamma_{h_2}(A)} \left|\frac{\partial v_2}{\partial x_2}\,\overline{v_2}\right| ds \le \\ \le \int\limits_{\Gamma_{h_2}(A)} \left\{\left|\frac{\partial u_2}{\partial x_2} - \frac{\partial v_2}{\partial x_2}\right| |\overline{u_2}| + \left|\frac{\partial v_2}{\partial x_2}\right|\ |\overline{u_2}-\overline{v_2}|\right\} ds + \\ + \int\limits_{\Gamma_{h_2}\setminus\Gamma_{h_2}(A)} \left|\frac{\partial v_2}{\partial x_2}\overline{v_2}\right| ds \le \\ \le 2c_2^2\,(1+L^2) \int\limits_{-A}^{A} [W_\infty(x_1)-W_A(x_1)]\,W_\infty(x_1)\,dx_1 + \\ + c_2^2(1+L^2) \int\limits_{\mathbb{R}\setminus[-A,A]} |W_A(x_1)|^2\,dx_1, \tag{3.25}$$

with some $c_2 > 0$ independent of A.

By quite the same arguments we obtain

$$\operatorname{Im}\left\{\mu \int\limits_{\Gamma_{h_1}(A)} \frac{\partial u_1}{\partial x_2}\overline{u_1}ds\right\} = \mu_2 \operatorname{Re} \int\limits_{\Gamma_{h_1}(A)} \frac{\partial u_1}{\partial x_2}\overline{u_1}ds + \mu_1 \operatorname{Im} \int\limits_{\Gamma_{h_1}(A)} \frac{\partial u_1}{\partial x_2}\overline{u_1}ds \leq$$

$$\leq 4c_1^2(1+L^2)|\mu| \int\limits_{-A}^{A} [W_\infty(x_1) - W_A(x_1)]\, W_\infty(x_1)\, dx_1 +$$

$$+ 2c_1^2(1+L^2)|\mu| \int\limits_{\mathbb{R}\setminus[-A,A]} |W_A(x_1)|^2\, dx_1 \tag{3.26}$$

with some $c_1 > 0$ independent of A, due to (3.16), (3.18), (3.21), (3.22), (3.23), and (3.24).

Now, from (3.10), (3.20), (3.25), and (3.26) it follows that

$$\int\limits_{-A}^{A} |w(x_1)|^2\, dx_1 \leq c_* \left\{ \int\limits_{-A}^{A} [W_\infty(x_1) - W_A(x_1)]\, W_\infty(x_1)\, dx_1 + \right.$$

$$\left. + \int\limits_{\mathbb{R}\setminus[-A,A]} |W_A(x_1)|^2\, dx_1 \right\} + M(A_0), \quad A_0 < A \leq +\infty, \tag{3.27}$$

$$M(A_0) = \sup_{A > A_0} \left\{ |\mu|\, |\mathcal{R}_1(A)| + |\mathcal{R}_2(A)| \right\}, \tag{3.28}$$

$$c_* = \max\{2c_2^2(1+L^2), 4c_1^2(1+L^2)\};$$

here $A_0 > 0$ is an arbitrarily fixed number.

Applying Lemma A in [9] (see also Lemma 6.2 in [10]) from (3.27) we conclude that $w \in L_2(\mathbb{R})$ and

$$\int\limits_{\Gamma} |w(x_1)|^2\, dx_1 \leq M(A_0).$$

By the item (i) of Theorem 3.1, (3.11) and (3.19) we then have

$$u_1|_\Gamma,\; u_2|_\Gamma \in L_2(\Gamma) \cap BC^1(\Gamma) \tag{3.29}$$

and

$$\int\limits_{\Gamma} |u_j(x)|^2\, ds \leq (1+L^2)^{1/2} M(A_0), \quad j = 1, 2,$$

with $M(A_0)$ given by (3.28). In what follows we will show that $M(A_0)$ tends to zero as $A_0 \to +\infty$.

Step 5. Since $u_j \in BC^1(\overline{\Omega_{j,h_j}})$, $j = 1, 2$ (see (3.1)) there exist positive numbers $N_j < +\infty$ (depending on h_j) such that

$$|u_j(x)|,\; |\nabla u_j(x)| \leq N_j \quad \text{for} \quad x \in \overline{\Omega_{j,h_j}}. \tag{3.30}$$

Therefore, for $\delta_j = \dfrac{\varepsilon_1}{8N_j^2} > 0$ we have

$$\int\limits_{f(x_1)-\delta_1}^{f(x_1)} \left| \frac{\partial u_1}{\partial x_1}\, \overline{u_1} \right| dx_2 \le N_1^2 \delta_1 = \frac{\varepsilon_1}{8|\mu|}, \quad x_1 \in \mathbb{R}, \tag{3.31}$$

$$\int\limits_{f(x_1)}^{f(x_1)+\delta_2} \left| \frac{\partial u_2}{\partial x_1}\, \overline{u_2} \right| dx_2 \le N_2^2 \delta_2 = \frac{\varepsilon_1}{8}, \quad x_1 \in \mathbb{R}, \tag{3.32}$$

where ε_1 is a sufficiently small positive number such that $h_1 < f(x_1) \pm \delta_j < h_2$. For $\delta_j > 0$ let

$$\Omega^*_{1,h_1}(\delta_1) := \{x \in \Omega_{1,h_1} \mid h_1 < x_2 < f(x_1) - \delta_1\},$$
$$\Omega^*_{2,h_2}(\delta_2) = \{x \in \Omega_{2,h_2} \mid f(x_1) + \delta_2 < x_2 < h_2\}.$$

It can be shown that

$$\operatorname{dist}(\Omega^*_{j,h_j}(\delta_j); \Gamma) = \inf_{x \in \Omega^*_{j,h_j}(\delta_j),\, y \in \Gamma} |x - y| \ge \frac{\delta_j}{\sqrt{1 + L^2}} > 0. \tag{3.33}$$

Step 6. From (3.12) and (3.13)

$$|u_j(x)|^2 \le 2I_{1j}(x; A_1) + 2I_{2j}(x; A_1),$$

where

$$I_{1j}(x; A_1) = \left[\int\limits_{\Gamma(A_1)} \frac{\partial G^{(j)}(x,y)}{\partial n(y)}\, E(y)\, ds \right]^2, \tag{3.34}$$

$$I_{2j}(x; A_1) = \left[\int\limits_{\Gamma \setminus \Gamma(A_1)} \frac{\partial G^{(j)}(x,y)}{\partial n(y)}\, E(y)\, ds \right]^2.$$

Assuming that

$$x \in \overline{\Omega^*_{j,h_j}(\delta_j)}, \quad |x_1| > 2A_1, \tag{3.35}$$

we have $|x - y| \ge |x_1 - y_1| \ge |x_1|/2$ for $y \in \Gamma(A_1)$ and due to (2.16) and Cauchy inequality we get

$$I_{1j}(x; A_1) \le c'_j \int\limits_{-A_1}^{A_1} \frac{dy_1}{(1 + |x_1 - y_1|)^3} \int\limits_{\Gamma(A_1)} |E(y)|^2\, ds \le$$
$$\le 2A_1 c'_j \left(\frac{|x_1|}{2} \right)^{-3} \|E\|^2_{L_2(\Gamma)} \le c'_j \|E\|^2_{L_2(\Gamma)} |x_1|^{-2}, \tag{3.36}$$

where c'_j does not depend on A_1 (note that it depends on δ_j).

Further, under the conditions (3.35) (for definiteness let $x_1 > 2A_1$) with the help of (2.16) and (3.33) we derive

$$I_{2j}(x;A_1) \le (1+L^2)\left\{\left[\int_{-\infty}^{-A_1} + \int_{A_1}^{x_1-1} + \int_{x_1-1}^{x_1+1} + \int_{x_1+1}^{\infty}\right]\left|\frac{\partial G^{(j)}(x,y)}{\partial n(y)}\right| |\tilde{E}(y_1)| dy_1\right\}^2 \le$$

$$\le c_j^2(1+L^2)\left\{\left[\int_{-\infty}^{-A_1} + \int_{A_1}^{x_1-1} + \int_{x_1+1}^{\infty}\right]\frac{|\tilde{E}(y_1)|}{(1+|x_1-y_1|)^{3/2}}dy_1 + \right.$$

$$\left. + \int_{x_1-1}^{x_1+1}\frac{|\tilde{E}(y_1)|}{|x-y|}dy_1\right\}^2 \le$$

$$\le c_j^2(1+L^2)\left[\int_{-\infty}^{+\infty}\frac{dt}{1+t^3} + \frac{\sqrt{1+L^2}}{\delta_j}\right]^2 \int_{\mathbb{R}\setminus[-A_1,A_1]} |\tilde{E}(y_1)|^2\, dy_1 \le$$

$$\le c_j''\|E\|^2_{L_2(\Gamma\setminus\Gamma(A_1))}, \tag{3.37}$$

where $c_j'' > 0$ does not depend on A_1 (note that it depends on δ_j).

In view of (3.34), (3.36), and (3.37) under the conditions (3.35) we have

$$|u_j(x)|^2 \le c_j'\,\|E\|^2_{L_2(\Gamma)}|x_1|^{-2} + c_j''\,\|E\|^2_{L_2(\Gamma\setminus\Gamma(A_1))}, \tag{3.38}$$

where c_j' and c_j'' do not depend on A_1. Therefore, due to (3.11), (3.29) and (3.38) we can choose A_1 such that

$$c_j'\,\|E\|^2_{L_2(\Gamma)}A_1^{-2} + c_j''\,\|E\|^2_{L_2(\Gamma\setminus\Gamma(A_1))} < \frac{\varepsilon_1}{4m_j}$$

and, consequently,

$$|u_j(x)|^2 \le \frac{\varepsilon_1}{4m_j} \quad \text{for} \quad x \in \overline{\Omega^*_{j,h_j}(\delta_j)}, \quad |x_1| = A \ge A_1, \tag{3.39}$$

where $m_j = 2\,|\mu|\,N_j\,(h_2 - h_1)$.

Step 7. Applying (3.7), (3.30), (3.31), (3.32), and (3.39) and taking $A \ge A_1 \ge A_0$ we derive

$$|\mu|\,|\mathcal{R}_1(A)| + |\mathcal{R}_2(A)| \le$$

$$\le |\mu|\left\{\int_{h_1}^{f(-A)-\delta_1} + \int_{f(-A)-\delta_1}^{f(-A)} + \int_{h_1}^{f(A)-\delta_1} + \int_{f(A)-\delta_1}^{f(A)}\right\}\left|\frac{\partial u_1}{\partial x_1}\right|\,|u_1|\,dx_2 +$$

$$+ \left\{\int_{f(-A)}^{f(-A)+\delta} + \int_{f(-A)+\delta}^{h_2} + \int_{f(A)}^{f(A)+\delta} + \int_{f(A)+\delta}^{h_2}\right\}\left|\frac{\partial u_2}{\partial x_1}\right|\,|u_2|\,dx_2 \le$$

$$\le |\mu|\left\{N_1[f(-A)-h_1]\frac{\varepsilon_1}{4m_1} + \frac{\varepsilon_1}{8|\mu|} + N_1[f(A)-h_1]\frac{\varepsilon_1}{4m_1} + \frac{\varepsilon_1}{8|\mu|}\right\} +$$

$$+\left\{\frac{\varepsilon_1}{8}+N_2[h_2-f(-A)]\frac{\varepsilon_1}{4m_2}+\frac{\varepsilon_1}{8}+N_2[h_2-f(A)]\frac{\varepsilon_1}{4m_2}\right\}\leq$$
$$\leq\frac{\varepsilon_1}{4}+2|\mu|N_1(h_2-h_1)\frac{\varepsilon_1}{4m_1}+\frac{\varepsilon_1}{4}+2N_2(h_2-h_1)\frac{\varepsilon_1}{4m_2}=\varepsilon_1.$$

Since $\varepsilon_1>0$ is an arbitrary (sufficiently small) number it follows that

$$\lim_{A\to+\infty} M(A)=0, \tag{3.40}$$

where $M(A)$ is determined by (3.28).

In turn, (3.40) along with (3.29) implies: $u_j(x)=0$ for $x\in\Gamma$, $j=1,2$. Now, applying the uniqueness results for the Dirichlet problem (see [9], Theorem 3.4, and [7], Theorem 3.1) we conclude: $u_j(x)=0$ in Ω_j, $j=1,2$. The proof is complete. □

4. Existence of Solution

4.1. Potentials and integral operators. Let us look for a solution of Problem (P) in the form

$$u_1(x)=\mu^{-1}\,W_1(\varphi)(x)+\mu^{-1}\,V_1(\psi)(x),\quad x\in\Omega_1, \tag{4.1}$$
$$u_2(x)=W_2(\varphi)(x)+V_2(\psi)(x),\quad x\in\Omega_2, \tag{4.2}$$

where

$$W_1(\varphi)(x):=\int_\Gamma\left[\frac{\partial}{\partial n(y)}G^{-(\mathcal{I})}_{k_1}(x,y;h_2)\right]\,\varphi(y)\,ds,$$
$$V_1(\psi)(x):=\int_\Gamma G^{-(\mathcal{I})}_{k_1}(x,y;h_2)\,\psi(y)\,ds,$$
$$W_2(\varphi)(x):=\int_\Gamma\left[\frac{\partial}{\partial n(y)}G^{+(\mathcal{I})}_{k_2}(x,y;h_1)\right]\,\varphi(y)\,ds,$$
$$V_2(\psi)(x):=\int_\Gamma G^{+(\mathcal{I})}_{k_2}(x,y;h_1)\,\psi(y)\,ds;$$

here $G^{-(\mathcal{I})}_{k_1}(x,y;h_2)$ and $G^{+(\mathcal{I})}_{k_2}(x,y;h_1)$ are the impedance Green's functions introduced in Subsection 2.3 for the half-planes $U^-_{h_2}$ and $U^+_{h_1}$, respectively, with $h_1<f_-<f_+<h_2$ (see (2.12), (2.13)).

Recall that $n(x)$ denotes the unit normal vector to Γ at the point $x\in\Gamma$ directed outward of Ω_1. Throughout this section we assume that $\Gamma\in C^{1,1}$ if not otherwise stated.

Further, we introduce the integral operators:

$$(\mathcal{K}_j^*\varphi)(x) := \int\limits_\Gamma K_j^*(x,y)\,\varphi(y)\,ds, \quad x\in\Gamma, \tag{4.3}$$

$$(\mathcal{K}_j\varphi)(x) := \int\limits_\Gamma K_j(x,y)\,\varphi(y)\,ds, \quad x\in\Gamma, \tag{4.4}$$

$$(\mathcal{H}_j\varphi)(x) := \int\limits_\Gamma H_j(x,y)\,\varphi(y)\,ds, \quad x\in\Gamma, \tag{4.5}$$

$$(\mathcal{L}_j^\pm\varphi)(x) := \lim_{\delta\to 0+} n(x)\cdot\nabla_x W_j(\varphi)(x\pm\delta n(x)), \tag{4.6}$$

where

$$K_1^*(x,y) = \partial_{n(y)} G_{k_1}^{-(\mathcal{I})}(x,y;h_2), \quad K_2^*(x,y) = \partial_{n(y)} G_{k_2}^{+(\mathcal{I})}(x,y;h_1), \tag{4.7}$$

$$K_1(x,y) = \partial_{n(x)} G_{k_1}^{-(\mathcal{I})}(x,y;h_2), \quad K_2(x,y) = \partial_{n(x)} G_{k_2}^{+(\mathcal{I})}(x,y;h_1), \tag{4.8}$$

$$H_1(x,y) = G_{k_1}^{-(\mathcal{I})}(x,y;h_2), \quad H_2(x,y) = G_{k_2}^{+(\mathcal{I})}(x,y;h_1). \tag{4.9}$$

For $x,y\in\Gamma$ let

$$\begin{aligned} &\tilde\varphi(y_1) = \varphi(y_1, f(y_1)), \quad \varrho(y_1) = \{1+[f'(y_1)]^2\}^{1/2}, \\ &\widetilde{K}_j^*(x_1,y_1) = \varrho(y_1)\,K_j^*(x,y), \quad \widetilde{K}_j(x_1,y_1) = \varrho(y_1)\,K_j(x,y), \\ &\widetilde{H}_j(x_1,y_1) = \varrho(y_1)\,H_j(x,y), \end{aligned} \tag{4.10}$$

with $x=(x_1,f(x_1))$ and $y=(y_1,f(y_1))$, and

$$(\widetilde{\mathcal{K}}_j^*\tilde\varphi)(x_1) := \int\limits_{-\infty}^{+\infty} \widetilde{K}_j^*(x_1,y_1)\,\tilde\varphi(y_1)\,dy_1, \tag{4.11}$$

$$(\widetilde{\mathcal{K}}_j\tilde\varphi)(x_1) := \int\limits_{-\infty}^{+\infty} \widetilde{K}_j(x_1,y_1)\,\tilde\varphi(y_1)\,dy_1, \tag{4.12}$$

$$(\widetilde{\mathcal{H}}_j\tilde\varphi)(x_1) := \int\limits_{-\infty}^{+\infty} \widetilde{H}_j(x_1,y_1)\,\tilde\varphi(y_1)\,dy_1. \tag{4.13}$$

All integrals involved in (4.3)–(4.6) and (4.11)–(4.13) exist as improper integrals, while (4.6) determines a singular integro-differential operator, and, in general, the operators $\mathcal{L}_j^\pm$ are correctly defined for $\varphi\in C^{1,\alpha}(\Gamma)$ (see (4) below).

For some $M>b>a>0$, $\eta\in(0,1]$ and $k\geq 1$ integer, let us define

$$\mathcal{B}(k,\eta,a,b,M) := \{f\in C^{k,\eta}(\mathbb{R}) \mid \inf_{\mathbb{R}} f(x_1)\geq a,\ \sup_{\mathbb{R}} f(x_1)\leq b \text{ and } \|f\|_{k,\eta,\mathbb{R}}\leq M\}.$$

Below, when necessary, we will indicate dependence of a function, an operator, or of a set on the boundary $f \in \mathcal{B}(k,\eta,a,b,M)$ by a sub- or superscript f.

The following results have been proved in [6] as Lemmas 4.1–4.2, in [25] as Lemmas A.1-A.4 and in [1] as Theorems 3.11-3.12.

Lemma 4.1. *Let* $\psi \in BC(\Gamma_f)$ *and* $f \in \mathcal{B}(1,1,a,b,M)$. *Then*

(i) $W_1(\psi), V_1(\psi) \in C^2(\Omega_1 \cup \Omega_{2,h_2}) \cap \mathrm{DPRC}(\Omega_1, k_1)$, $V_1(\psi) \in C(U^-_{h_2})$ *and they solve the Helmholtz equation* (2.1) *in* $\Omega_1 \cup \Omega_{2,h_2}$, *provided* $h_2 > b$;

(ii) $W_2(\psi), V_2(\psi) \in C^2(\Omega_2 \cup \Omega_{1,h_1}) \cap \mathrm{UPRC}(\Omega_2, k_2)$, $V_2(\psi) \in C(U^+_{h_1})$ *and they solve the Helmholtz equation* (2.2) *in* $\Omega_2 \cup \Omega_{1,h_1}$, *provided* $h_1 < a$;

(iii) *for* $x \in \Gamma_f$

$$[W_j(\psi)(x)]^\pm := \lim_{\delta\to 0+} W_j(\psi)(x \pm \delta n(x)) = (\pm 2^{-1} I + \mathcal{K}^*_j)\psi(x), \tag{4.14}$$

$$[V_j(\psi)(x)]^\pm := \lim_{\delta\to 0+} V_j(\psi)(x \pm \delta n(x)) = \mathcal{H}_j\psi(x), \tag{4.15}$$

$$[\partial_{n(x)} V_j(\psi)(x)]^\pm := \\ = \lim_{\delta\to 0+} n(x)\cdot\nabla V_j(\psi)(x \pm \delta n(x)) = (\mp 2^{-1} I + \mathcal{K}_j)\psi(x), \tag{4.16}$$

$$\lim_{\delta\to 0+} n(x)\cdot[\nabla W_j(\psi)(x+\delta n(x)) - \nabla W_j(\psi)(x-\delta n(x))] = 0, \tag{4.17}$$

where all the limits exist uniformly for x *in compact subsets of* Γ_f*;*

(iv) *there exist* $C_j > 0$ *such that*

$$|x_2|^{-1/2}|V_j(\psi)|,\ |x_2|^{-1/2}|W_j(\psi)| \le C_j\, \|\psi\|_{\infty,\Gamma_f}$$

as $|x_2| \to +\infty$.

As we have already remarked, the operators $\mathcal{L}^\pm_j$ are well defined in $\overline{\Omega}_j$ for $BC^{1,\alpha}$-smooth density functions.

To show this, for a unit vector-function $l(x) = (l_1(x), l_2(x))$ let us introduce the differentiation operators

$$D(\partial_x, l(x)) := l_2(x)\,\frac{\partial}{\partial x_1} - l_1(x)\,\frac{\partial}{\partial x_2},\quad \partial_{l(x)} := l_1(x)\,\frac{\partial}{\partial x_1} + l_2(x)\,\frac{\partial}{\partial x_2}.$$

For an arbitrary C^2-smooth function Ψ and arbitrary unit vector-functions $l^{(j)} = (l^{(j)}_1, l^{(j)}_2)$, $j = 1,2$, there holds the identity

$$\partial_{l^{(1)}(x)}\partial_{l^{(2)}(y)}\,\Psi(x-y) = -(l^{(1)}(x)\cdot l^{(2)}(x))\,\Delta_x\Psi(x-y) - \\ -D(\partial_x, l^{(1)}(x))\,D(\partial_y, l^{(2)}(y))\,\Psi(x-y). \tag{4.18}$$

Denote by $\tilde n(x)$ (with $|\tilde n(x)| = 1$) a $BC^{0,1}$-continuous extension from Γ onto $\mathbb{R}^2$ of the unit normal vector $n(x)$, $x \in \Gamma$, and let $\partial_{\tilde\tau(x)} := D(\partial_x, \tilde n(x))$. Note that for $x \in \Gamma$, $\partial_{\tilde\tau(x)} = \partial_{\tau(x)}$ and $\partial_{\tilde n(x)} = \partial_{n(x)}$ are usual tangent and normal differentiation operators at the point $x \in \Gamma$.

Due to (2.7) and (4.18) we have

$$\partial_{\tilde n(x)}\partial_{\tilde n(y)}\Phi_{k_j}(x,y) = -(\tilde n(x)\cdot\tilde n(y))k_j^2\Phi_{k_j}(x,y) - \partial_{\tilde\tau(y)}\partial_{\tilde\tau(x)}\Phi_{k_j}(x,y) \tag{4.19}$$

for $x \neq y$.

Further, we represent $G^{+(\mathcal{I})}_{k_2}(x, y; h_1)$ as

$$G^{+(\mathcal{I})}_{k_2}(x, y; h_1) = \Phi_{k_2}(x, y) + R^{+(\mathcal{I})}_{k_2}(x, y; h_1), \tag{4.20}$$

where $R^{+(\mathcal{I})}_{k_2}(x, y; h_1) = \Phi_{k_2}(x, y') + P^{(+)}_{k_2}(x - y')$ is a C^2-smooth function in $\overline{U}^{+}_{h_1^*}$ for $h_1^* > h_1$ (cf. (2.12)).

Let for some constants A and B $(A < B)$

$$\Omega_j(A, B) := \{x \in \Omega_j \mid A < x_1 < B\}, \quad \Gamma(A, B) := \{x \in \Gamma \mid A < x_1 < B\}.$$

We decompose $W_2(\varphi)(x)$ as follows

$$W_2(\varphi)(x) = Q_1(x) + Q_2(x) + Q_3(x), \ \ x \in \Omega_2 \cup \Omega_{1,h_1},$$

where $\varphi \in BC^{1,\alpha}(\Gamma)$,

$$\begin{aligned} Q_1(x) &:= \int\limits_{\Gamma \setminus \Gamma(A,B)} [\partial_{n(y)} G^{+(\mathcal{I})}_{k_2}(x, y; h_1)]\, \varphi(y)\, ds, \\ Q_2(x) &:= \int\limits_{\Gamma(A,B)} [\partial_{n(y)} R^{+(\mathcal{I})}_{k_2}(x, y; h_1)]\, \varphi(y)\, ds, \\ Q_3(x) &:= \int\limits_{\Gamma(A,B)} [\partial_{n(y)} \Phi_{k_2}(x, y)]\, \varphi(y)\, ds. \end{aligned}$$

It is evident that for $z \in \Gamma(A/2, B/2)$ and $p = 1, 2$, we have

$$\begin{aligned} &\lim_{x \to z} \tilde{n}(x) \cdot \nabla_x Q_p(x) = \partial_{n(z)} Q_p(z) = \\ &= \int\limits_{\Gamma \setminus \Gamma(A,B)} [\partial_{n(z)} \partial_{n(y)} G^{+(\mathcal{I})}_{k_2}(z, y; h_1)]\, \varphi(y) ds, \end{aligned} \tag{4.21}$$

where the integrals exist as improper integrals.

Applying (4.19) and integration by parts we get

$$\tilde{n}(x)\cdot\nabla_x Q_3(x) = \partial_{\tilde{n}(x)} Q_3(x) = \int\limits_{\Gamma(A,B)} [\partial_{\tilde{n}(x)}\partial_{n(y)}\Phi_{k_2}(x,y)]\,\varphi(y)\,ds =$$

$$= -\int\limits_{\Gamma(A,B)} (\tilde{n}(x)\cdot n(y))\Delta_x\Phi_{k_2}(x,y)\,\varphi(y)\,ds -$$

$$-\int\limits_{\Gamma(A,B)} [\partial_{\tau(y)}\partial_{\tilde{\tau}(x)}\Phi_{k_2}(x,y)]\,\varphi(y)\,ds =$$

$$= \int\limits_{\Gamma(A,B)} (\tilde{n}(x)\cdot n(y))\,k_2^2\,\Phi_{k_2}(x,y)\varphi(y)\,ds + [\partial_{\tilde{\tau}(x)}\Phi_{k_2}(x,y_A)]\,\varphi(y_A) -$$

$$-[\partial_{\tilde{\tau}(x)}\Phi_{k_2}(x,y_B)]\,\varphi(y_B) + \int\limits_{\Gamma(A,B)} [\partial_{\tilde{\tau}(x)}\Phi_{k_2}(x,y)]\,\partial_{\tau(y)}\varphi(y)\,ds,$$

where $y_A = (A, f(A)), y_B = (B, f(B))$, and $x_1 \in (A/2, B/2)$, $h_1 < x_2 < h_2$ $(h_1 < f_-, h_2 > f_+)$.

Note that the last summand has no jump on $\Gamma(A/2, B/2)$ (see, e.g., [20], [13]), that is, for $z \in (A/2, B/2)$

$$\lim_{x\to z}\int\limits_{\Gamma(A,B)} \partial_{\tilde{\tau}(x)}\Phi_{k_2}(x,y)\partial_{\tau(y)}\varphi(y)\,ds = \int\limits_{\Gamma(A,B)} \partial_{\tau(z)}\Phi_{k_2}(z,y)\ \partial_{\tau(y)}\varphi(y)\,ds,$$

where the right-hand side is understood as a singular integral in the Cauchy Principal Value sense and is well defined due to the imbedding $\varphi \in BC^{1,\alpha}(\Gamma)$ (see, e.g., [22]).

Thus, we have shown that for arbitrary $\varphi \in BC^{1,\alpha}(\Gamma)$, $0 < \alpha \le 1$, and arbitrary $z \in \Gamma$

$$\lim_{x\to z}\partial_{\tilde{n}(x)} W_2(\varphi)(x) = \int\limits_{\Gamma\setminus\Gamma(A,B)} [\partial_{n(z)}\partial_{n(y)} G_{k_2}^{+(\mathcal{I})}(z,y;h_1)]\varphi(y)\,ds +$$

$$+\int\limits_{\Gamma(A,B)} [\partial_{n(z)}\partial_{n(y)} R_{k_2}^{+(\mathcal{I})}(z,y;h_1)]\varphi(y)ds +$$

$$+\int\limits_{\Gamma(A,B)} (n(z)\cdot n(y))k_2^2\Phi_{k_2}(z,y)\varphi(y)ds +$$

$$+\int\limits_{\Gamma(A,B)} \partial_{\tau(z)}\Phi_{k_2}(z,y)\,\partial_{\tau(y)}\varphi(y)\,ds + [\partial_{\tau(z)}\Phi_{k_2}(z,y_A)]\,\varphi(y_A) -$$

$$-[\partial_{\tau(z)}\Phi_{k_2}(z,y_B)]\,\varphi(y_B),$$

where A and B are arbitrary constants such that $A/2 < z_1 < B/2$. It is evident that the limit exists uniformly for z in compact subsets of Γ.

The similar results are true for the potential $W_1(\varphi)$. As a consequence we obtain (cf. (4.6))

$$\mathcal{L}_j^+\varphi = \mathcal{L}_j^-\varphi =: \mathcal{L}_j\varphi, \;\; j = 1, 2, \;\; \varphi \in BC^{1,\alpha}(\Gamma).$$

This implies that the operator $\mathcal{L}_1 - \mathcal{L}_2$ is well-defined for functions of the space $BC^{1,\alpha}(\Gamma)$. However, this operator can be extended onto the space of bounded continuous functions $BC(\Gamma)$ (cf. [13]).

Lemma 4.2. *The operator $\mathcal{L}_1 - \mathcal{L}_2$ is well-defined and bounded for functions of the space $BC(\Gamma)$.*

Proof. First, we recall the singular behaviour of the Hankel function $H_0^{(1)}$ as $t \to 0$

$$H_0^{(1)}(t) = \frac{2i}{\pi}(\log\frac{t}{2} + C) + 1 + O(t^2 \log t), \tag{4.22}$$

where C denotes Euler's constant.

With the help of definition (4.6), Lemma 4.1.(ii), and equalities (4.20) and (4.22) we easily conclude that

$$\mathcal{L}\varphi := (\mathcal{L}_1 - \mathcal{L}_2)\,\varphi(x) =$$

$$= \int_\Gamma \{\partial_{n(x)}\partial_{n(y)}[G_{k_1}^{-(\mathcal{I})}(x, y; h_2) - G_{k_2}^{+(\mathcal{I})}(x, y; h_1)]\}\,\varphi(y)\,ds \tag{4.23}$$

is well-defined for arbitrary $\varphi \in BC(\Gamma)$.

The kernel function of the integral operator (4)

$$L(x, y) := \partial_{n(x)}\partial_{n(y)}[G_{k_1}^{-(\mathcal{I})}(x, y; h_2) - G_{k_2}^{+(\mathcal{I})}(x, y; h_1)] \tag{4.24}$$

admits the bounds

$$\begin{aligned} &|L(x, y)| \le c'\,(1 + |\,\log|x - y|\,|) \quad \text{for} \quad |x - y| \le 1, \\ &|L(x, y)| \le c'\,|x - y|^{-3/2} \quad \text{for} \quad |x - y| \ge 1, \end{aligned} \tag{4.25}$$

with some constant $c' > 0$, due to (2.14).

The estimate

$$\|\mathcal{L}\varphi\|_\infty \le c''\|\varphi\|_\infty \quad \text{for} \quad \varphi \in BC(\Gamma)$$

with a positive constant c'' independent of φ, can be obtained by standard arguments (see the proof of Lemma 4.2 in [6]). □

The regularity properties of the aforementioned potential type and integral operators are described by the following lemmas.

Lemma 4.3. *Let $f \in \mathcal{B}(1, 1, a, b, M)$. The operators*

$$\begin{aligned} \mathcal{H}_j, \mathcal{K}_j, \mathcal{K}_j^*, \mathcal{L} \;\; &: \;\; BC(\Gamma_f) \to BC^{0,\beta}(\Gamma_f) \quad \forall\beta \in (0, 1), \qquad (4.26) \\ \mathcal{H}_j, \mathcal{K}_j^* \;\; &: \;\; BC^{0,\alpha}(\Gamma_f) \to BC^{1,\alpha}(\Gamma_f) \quad \forall\alpha \in (0, 1), \end{aligned}$$

are uniformly bounded with respect to f, i.e., there hold the uniform estimates

$$\|\mathcal{S}_0\varphi\|_{0,\beta,\Gamma_f} \le c_0\,\|\varphi\|_{\infty,\Gamma_f}, \tag{4.27}$$

$$\|\mathcal{S}_1\varphi\|_{1,\alpha,\Gamma_f} \le c_0\|\varphi\|_{0,\alpha,\Gamma_f}, \tag{4.28}$$

where $\mathcal{S}_0 \in \{\mathcal{H}_j, \mathcal{K}_j, \mathcal{K}_j^*, \mathcal{L}\}$, $\mathcal{S}_1 \in \{\mathcal{H}_j, \mathcal{K}_j^*\}$, c_0 and c_1 are positive constants depending on a, b, M, h_1 and h_2.

Proof. It is verbatim the proofs of Theorems A28, A43, A50 in [15] and Theorems 3.11, 3.12 in [1]. □

Lemma 4.4. *Let $f \in \mathcal{B}(1,1,a,b,M)$.*

(i) *For $\varphi \in BC^{0,\alpha}(\Gamma_f)$, $\alpha \in (0,1)$ the first order derivatives of the single layer potential $V_j^f(\varphi)$ in Ω^f_{1,h_1} and Ω^f_{2,h_2} have $BC^{0,\alpha}$-extensions to $\Omega^f_{1,h_1}\cup\Gamma_f$ and $\Omega^f_{2,h_2}\cup\Gamma_f$, and*

$$\|V_j^f(\varphi)\|_{1,\alpha,\Omega^f_{1,h_1}\cup\Gamma_f\cup\Gamma_{h_1}},\ \ \|V_j^f(\varphi)\|_{1,\alpha,\Omega^f_{2,h_2}\cup\Gamma_f\cup\Gamma_{h_2}} \le c_j'\,\|\varphi\|_{0,\alpha,\Gamma_f},$$

where the constant c_j' depends only on α, a, b, M, h_1, and h_2.

(ii) *For $\varphi \in BC^{1,\alpha}(\Gamma_f)$, $\alpha \in (0,1)$, the double layer potential $W_j^f(\varphi)$ and its first order derivatives in Ω^f_{1,h_1} and Ω^f_{2,h_2} have continuous extensions to $\Omega^f_{1,h_1}\cup\Gamma_f$ and $\Omega^f_{2,h_2}\cup\Gamma_f$, and*

$$\|W_j^f(\varphi)\|_{1,\Omega^f_{1,h_1}\cup\Gamma_f\cup\Gamma_{h_1}},\ \ \|W_j^f(\varphi)\|_{1,\Omega^f_{2,h_2}\cup\Gamma_f\cup\Gamma_{h_2}} \le c_j''\,\|\varphi\|_{1,\alpha,\Gamma_f},$$

where the constant c_j'' depends only on α, a, b, M, h_1, and h_2. (Note, that we keep the same notations for the aforementioned extensions).

Proof. The proof of the item (i) is verbatim the proof of Theorem 3.11.(b) in [1].

To prove the item (ii) we proceed as follows.

Let, for definiteness, $x \in \Omega^f_{2,h_2}$, and consider the first order derivative of the double layer potential $W_2^f(\varphi)$:

$$\frac{\partial}{\partial x_p}W_2^f(\varphi)(x) = \int\limits_{\Gamma_f}\left[\frac{\partial}{\partial x_p}\frac{\partial}{\partial n(y)}G_{k_2}^{+(\mathcal{I})}(x,y;h_1)\right]\varphi(y)\,ds,\ p=1,2. \tag{4.29}$$

Here we have changed the order of differentiation and integration as the kernel function is infinitely smooth for $x \notin \Gamma_f$ and admits the bounds (2.14). Let $\delta > 0$ be a sufficiently small fixed number such that $h_2 - b \ge \delta$.

For $\mathrm{dist}(x,\Gamma_f) \ge \delta$ we have

$$\left|\frac{\partial}{\partial x_p}W_2^f(\varphi)(x)\right| \le c\,\|\varphi\|_{\infty,\Gamma_f}\int\limits_{\Gamma_f}\frac{ds}{|x-y|^{3/2}} \le c_1(\delta)\,\|\varphi\|_{\infty,\Gamma_f} \tag{4.30}$$

due to the bounds (2.14). Here $c_1(\delta)$ does not depend on f (it depends on δ, a, b, M, h_1, and h_2).

Now, let $\operatorname{dist}(x, \Gamma_f) < \delta$ and

$$\Gamma_f(x_1 - 4\delta, x_1 + 4\delta) = \{y \in \Gamma_f \mid x_1 - 4\delta < y_1 < x_1 + 4\delta\}.$$

Rewrite (4.29) as

$$\frac{\partial}{\partial x_p} W_2^f(\varphi)(x) = I_{\delta,p}^{(1)}(x) + I_{\delta,p}^{(2)}(x) + I_{\delta,p}^{(3)}(x), \quad p = 1, 2, \tag{4.31}$$

where (see (4.20))

$$I_{\delta,p}^{(1)}(x) = \int\limits_{\Gamma_f \setminus \Gamma_f(x_1-4\delta, x_1+4\delta)} \left[\frac{\partial}{\partial x_p} \frac{\partial}{\partial n(y)} G_{k_2}^{+(\mathcal{I})}(x, y; h_1)\right] \varphi(y)\, ds,$$
$$I_{\delta,p}^{(2)}(x) = \int\limits_{\Gamma_f(x_1-4\delta, x_1+4\delta)} \left[\frac{\partial}{\partial x_p} \frac{\partial}{\partial n(y)} \Phi_{k_2}(x, y)\right] \varphi(y)\, ds,$$
$$I_{\delta,p}^{(3)}(x) = \int\limits_{\Gamma_f(x_1-4\delta, x_1+4\delta)} \left[\frac{\partial}{\partial x_p} \frac{\partial}{\partial n(y)} R_{k_2}^{+}(x, y; h_1)\right] \varphi(y)\, ds.$$

Taking into consideration that $|x-y| \geq 4\delta$ for $y \in \Gamma_f \setminus \Gamma_f(x_1-4\delta, x_1+4\delta)$ and applying the bounds (2.14) we get that $I_{\delta,p}^{(1)}(\cdot)$ is continuous in $\Omega_{1,h_1}^f \cup \Gamma_f \cup \Omega_{2,h_2}^f$ and

$$|I_{\delta,p}^{(1)}(x)| < c_2(\delta)\, \|\varphi\|_{\infty,\Gamma_f}, \quad p = 1, 2, \tag{4.32}$$

where $c_2(\delta)$ does not depend on f (it depends on δ, a, b, M, h_1, and h_2).

Since all the derivatives of $R_{k_2}^{+}(x, y; h_1)$ are C^∞-regular bounded kernels in the δ-vicinity of the curve Γ_f, we have that $I_{\delta,p}^{(3)}(\cdot)$ is continuous in $\Omega_{1,h_1}^f \cup \Gamma_f \cup \Omega_{2,h_2}^f$ and

$$|I_{\delta,p}^{(3)}(x)| \leq c_3(\delta)\, \|\varphi\|_{\infty,\Gamma_f}, \quad p = 1, 2, \tag{4.33}$$

where $c_3(\delta)$ does not depend on f (it depends on δ, a, b, M, h_1, and h_2).

With the help of the identities

$$\frac{\partial}{\partial x_p} \Phi_{k_2}(x, y) = -\frac{\partial}{\partial y_p} \Phi_{k_2}(x, y),$$
$$\frac{\partial}{\partial x_1} \frac{\partial}{\partial n(y)} \Phi_{k_2}(x, y) = k_2^2 n_1(y) \Phi_{k_2}(x, y) - \partial_{\tau(y)} \frac{\partial \Phi_{k_2}(x, y)}{\partial y_2},$$
$$\frac{\partial}{\partial x_2} \frac{\partial}{\partial n(y)} \Phi_{k_2}(x, y) = k_2^2 n_2(y) \Phi_{k_2}(x, y) + \partial_{\tau(y)} \frac{\partial \Phi_{k_2}(x, y)}{\partial y_1},$$

where $\partial_{\tau(y)} = n_2(y)\partial_1 - n_1(y)\partial_2$, as above, denotes the tangent derivative, and applying the integration by parts formula we arrive at the equality

$$\begin{aligned} I^{(2)}_{\delta,p}(x) = k_2^2 \int\limits_{\Gamma_f(x_1-4\delta,x_1+4\delta)} n_p(y)\Phi_{k_2}(x,y)\,\varphi(y)\,ds - \\ - \int\limits_{\Gamma_f(x_1-4\delta,x_1+4\delta)} \left[\left(\delta_{2p}\frac{\partial}{\partial y_1} - \delta_{1p}\frac{\partial}{\partial y_2}\right)\Phi_{k_2}(x,y)\right]\partial_{\tau(y)}\varphi(y)\,ds + \\ + \left[\left(\delta_{2p}\frac{\partial}{\partial y_1} - \delta_{1p}\frac{\partial}{\partial y_2}\right)\Phi_{k_2}(x,y^{**})\right]\varphi(y^{**}) - \\ - \left[\left(\delta_{2p}\frac{\partial}{\partial y_1} - \delta_{1p}\frac{\partial}{\partial y_2}\right)\Phi_{k_2}(x,y^{*})\right]\varphi(y^{*}), \end{aligned} \tag{4.34}$$

where $y^* = (x_1 - 4\delta, f(x_1 - 4\delta))$, $y^{**} = (x_1 + 4\delta, f(x_1 + 4\delta))$, and δ_{kp} is the Kronecker's delta.

It is evident that the first, the third and the forth summands in the right-hand side of (4.34) are continuous in $\Omega_{1,h_1} \cup \Gamma_f \cup \Omega_{2,h_2}$ and there holds the inequality

$$\begin{aligned} \left| k_2^2 \int\limits_{\Gamma_f(x_1-4\delta,x_1+4\delta)} n_p(y)\Phi_{k_2}(x,y)\,\varphi(y)\,ds \right| + \\ + \left| \left[\left(\delta_{2p}\frac{\partial}{\partial y_1} - \delta_{1p}\frac{\partial}{\partial y_2}\right)\Phi_{k_2}(x,y^{*})\right]\varphi(y^{*}) \right| + \\ + \left| \left[\left(\delta_{2p}\frac{\partial}{\partial y_1} - \delta_{1p}\frac{\partial}{\partial y_2}\right)\Phi_{k_2}(x,y^{**})\right]\varphi(y^{**}) \right| \le c_4'(\delta)\|\varphi\|_{\infty,\Gamma_f}, \quad p=1,2, \end{aligned} \tag{4.35}$$

since $|x - y^*| \ge 4\delta$, $|x - y^{**}| \ge 4\delta$, and

$$\int\limits_{\Gamma_f(x_1-4\delta,x_1+4\delta)} |n_p(y)\Phi_{k_2}(x,y)|\,ds < c_4''(\delta),$$

where $c_4'(\delta)$ and $c_4''(\delta)$ depend only on δ, a, b, M, h_1, and h_2.

For the second term in the right-hand side of (4.34) we have (see (4.22))

$$\begin{aligned} J(x) := \int\limits_{\Gamma_f(x_1-4\delta,x_1+4\delta)} \left[\left(\delta_{2p}\frac{\partial}{\partial y_1} - \delta_{1p}\frac{\partial}{\partial y_2}\right)\Phi_{k_2}(x,y)\right]\partial_{\tau(y)}\varphi(y)\,ds = \\ = J_1(x) + J_2(x) \end{aligned} \tag{4.36}$$

with

$$\begin{aligned} J_1(x) = \frac{2i}{\pi} \int\limits_{\Gamma_f(x_1-4\delta,x_1+4\delta)} \left(\delta_{2p}\frac{y_1 - x_1}{|x-y|^2} - \delta_{1p}\frac{y_2 - x_2}{|x-y|^2}\right)\partial_{\tau(y)}\varphi(y)\,ds, \\ J_2(x) = \int\limits_{\Gamma_f(x_1-4\delta,x_1+4\delta)} Q(x,y)\,\partial_{\tau(y)}\varphi(y)\,ds, \end{aligned} \tag{4.37}$$

where $Q(\cdot,\cdot)$ is $C^{0,\beta}$-regular ($\forall\beta\in(0,1)$) and can be estimated by some constant independent of Γ_f. Therefore,

$$|J_2(x)| = \left| \int\limits_{\Gamma_f(x_1-4\delta,x_1+4\delta)} Q(x,y)\partial_{\tau(y)}\varphi(y)\,ds \right| \le c_5'(\delta)\,\|\partial_{\tau(y)}\varphi\|_{\infty,\Gamma_f}. \quad (4.38)$$

The function $J_1(x)$ given by (4.37) represents a Cauchy type integral in $\Omega_{1.h_1}$ and $\Omega_{2.h_2}$ with the Hölder continuous density $\partial_{\tau(y)}\varphi\in BC^{0,\alpha}(\Gamma_f)$ ($0<\alpha<1$), and, therefore, it has $C^{0,\alpha}$-continuous bounded extensions to $\Omega_{1,h_1}\cup\Gamma_f$ and $\Omega_{2,h_2}\cup\Gamma_f$ (see, e.g., [22], §§15,16,17; [15], Theorem A46).

As a consequence, we have

$$|J_1(x)| \le c_5''(\delta)\,\|\partial_{\tau(y)}\varphi\|_{0,\alpha,\Gamma_f}, \quad (4.39)$$

where $c_5''(\delta)$ does not depend on f.

Further, (4.36), (4.38), and (4.39) imply

$$|J(x)| \le c_5(\delta)\,\|\varphi\|_{1,\alpha,\Gamma_f}, \quad (4.40)$$

where $c_5(\delta)$ depends only on δ, a, b, M, h_1, and h_2 (and does not depend on f).

Applying the estimates (4.35) and (4.40) to (4.34) we obtain

$$|I^{(2)}_{\delta,p}(x)| \le c_6(\delta)\,\|\varphi\|_{1,\alpha,\Gamma_f}, \quad (4.41)$$

with $c_6(\delta)$ depending on δ, a, b, M, h_1, h_2 (and independent on f).

Now, (4.30), (4.31), (4.32), (4.33), and (4.41) complete the proof. □

4.2. Reduction to integral equations. Applying the representations (4.1) and (4.2), and with the help of Lemmas 4.1, 4.2, and 4.3 we reduce the interface Problem (P) to the system of integral equations on Γ:

$$\mu^{-1}\left(-2^{-1}I+\mathcal{K}_1^*\right)\varphi - \left(2^{-1}I+\mathcal{K}_2^*\right)\varphi + \mu^{-1}\mathcal{H}_1\psi - \mathcal{H}_2\psi = f_1, \quad (4.42)$$

$$\mathcal{L}\varphi + \left(2^{-1}I+\mathcal{K}_1\right)\psi - \left(-2^{-1}I+\mathcal{K}_2\right)\psi = f_2, \quad (4.43)$$

where $\mathcal{K}_j^*, \mathcal{K}_j, \mathcal{H}_j$, and $\mathcal{L}$ are determined by (4.3)–(4.5) and (4), φ and ψ are unknown densities from the space $BC(\Gamma_f)$ and

$$f_1\in BC^{1,\alpha}(\Gamma),\quad f_2\in BC^{0,\alpha}(\Gamma),\quad 0<\alpha<1, \quad (4.44)$$

are given functions.

Rewrite (4.42) and (4.43) in the matrix form

$$\mathcal{M}\chi = F \quad (4.45)$$

with

$$\mathcal{M} = \begin{bmatrix} -(1+\mu)(2\mu)^{-1}I + \mu^{-1}\mathcal{K}_1^* - \mathcal{K}_2^* & \mu^{-1}\mathcal{H}_1 - \mathcal{H}_2 \\ \mathcal{L} & I + \mathcal{K}_1 - \mathcal{K}_2 \end{bmatrix},$$

$$\chi = \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = [\varphi,\psi]^\top, \quad F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = [f_1,f_2]^\top,$$

where $\top$ denotes transposition.

Now, we prove the following

Lemma 4.5. *Let conditions* (4.44) *be fulfilled,* $f \in \mathcal{B}(1,1,a,b,M)$*, and* $\chi = [\varphi,\psi]^\top \in [BC(\Gamma_f)]^2$ *be a solution of the equation* (4.45)*. Then* $\varphi \in BC^{1,\alpha}(\Gamma_f)$ *and* $\psi \in BC^{0,\alpha}(\Gamma_f)$ *with the same* α *as in* (4.44)*.*

Proof. First we show that $\varphi \in BC^{0,\alpha}(\Gamma)$. From equation (4.42) we have

$$-(\mu+1)\mu^{-1}\varphi = f_1 - \mu^{-1}\mathcal{K}_1^*\varphi + \mathcal{K}_2^*\varphi - \mu^{-1}\mathcal{H}_1\psi + \mathcal{H}_2\psi. \qquad (4.46)$$

Since $(\mu+1)/\mu \neq 0$, by Lemma 4.3 it follows that the right-hand side function in (4.46) is $BC^{0,\alpha}$-smooth, thus $\varphi \in BC^{0,\alpha}(\Gamma_f)$ and

$$\|\varphi\|_{0,\alpha,\Gamma_f} \leq \|f\|_{0,\alpha,\Gamma_f} + c_0\,(\|\varphi\|_{\infty,\Gamma_f} + \|f\|_{\infty,\Gamma_f}), \qquad (4.47)$$

where the constant $c_0 > 0$ is the same as in (4.27).

Applying again Lemma 4.3 we then get (cf. (4.26), (4.27))

$$\mathcal{L}\varphi \in BC^{0,\alpha}(\Gamma_f) \quad \text{and} \quad \|\mathcal{L}\varphi\|_{0,\alpha,\Gamma_f} \leq c_0\|\varphi\|_{\infty,\Gamma_f}.$$

Further, since (see (4.43))

$$\psi = f_2 - \mathcal{L}\varphi - \mathcal{K}_1\psi + \mathcal{K}_2\psi,$$

we have $\psi \in BC^{0,\alpha}(\Gamma_f)$ and

$$\|\psi\|_{0,\alpha,\Gamma_f} \leq \|f_2\|_{0,\alpha,\Gamma_f} + c_0(\|\varphi\|_{\infty,\Gamma_f} + \|\psi\|_{\infty,\Gamma_f}). \qquad (4.48)$$

Now, with the help of (4.46), (4.47), (4.48), and Lemma 4.3 (see also (4.28)) finally we obtain $\varphi \in BC^{1,\alpha}(\Gamma_f)$ and

$$\|\varphi\|_{1,\alpha,\Gamma_f} \leq \tilde{c}_0(\|f_1\|_{1,\alpha,\Gamma_f} + \|f_2\|_{0,\alpha,\Gamma_f} + \|\varphi\|_{\infty,\Gamma_f} + \|\psi\|_{\infty,\Gamma_f}),$$

where $\tilde{c}_0$ is a positive constant depending on a, b, M, h_1, and h_2. □

Investigation of the solvability of equation (4.45) we start with the uniqueness question.

Lemma 4.6. *The homogeneous version of the equation* (4.45) $(F = 0)$ *has only the trivial solution, i.e., the operator* $\mathcal{M}$ *is injective.*

Proof. Let $\chi = [\varphi,\psi]^\top \in [BC(\Gamma)]^2$ solve the homogeneous equation

$$\mathcal{M}\chi = 0 \quad \text{on} \quad \Gamma, \qquad (4.49)$$

and $u_1(x)$ and $u_2(x)$ be determined by (4.1) and (4.2) with the density functions φ and ψ.

Lemma 4.5 and equation (4.49) yield $\varphi \in BC^{1,\beta}(\Gamma)$ and $\psi \in BC^{0,\beta}(\Gamma)$ for all $\beta \in (0,1)$. By Lemma 4.4 and equation (4.49) we conclude that u_1 and u_2 satisfy the conditions of the uniqueness Theorem 3.1. Therefore,

$$u_j(x) = 0, \quad x \in \Omega_j, \quad j = 1,2. \qquad (4.50)$$

In what follows we will show that (4.50) implies $\varphi = \psi = 0$. To this end consider the same functions u_1 and u_2, i.e., the potentials (4.1) and (4.2), in the domains Ω_{2,h_2} and Ω_{1,h_1}, respectively. Our goal is to show that u_1 and u_2 vanish in Ω_{2,h_2} and Ω_{1,h_1} as well.

With the help of Lemma 4.1 we have (see (4.14)–(4.17)) for $x \in \Gamma_f$

$$[u_1(x)]^+ - [u_1(x)]^- = \mu^{-1}\varphi(x), \ [\partial_n u_1(x)]^+ - [\partial_n u_1(x)]^- = -\mu^{-1}\psi(x), \quad (4.51)$$

$$[u_2(x)]^+ - [u_2(x)]^- = \varphi(x), \ \ [\partial_n u_2(x)]^+ - [\partial_n u_2(x)]^- = -\psi(x). \quad (4.52)$$

From (4.51)–(4.52) by (4.50) it follows that for $x \in \Gamma_f$

$$-\varphi(x) = [u_2(x)]^- = -\mu[u_1(x)]^+, \ \ \psi(x) = [\partial_n u_2(x)]^- = -\mu[\partial_n u_1(x)]^+. \quad (4.53)$$

Introduce the following notation:

$$\begin{aligned} &\Omega_1^* := \Omega_{2,h_2}, \quad \Omega_2^* := \Omega_{1,h_1}, \\ &v_1(x) := -\mu u_1(x) \ \text{ in } \ \Omega_1^*, \quad v_2(x) := u_2(x) \ \text{ in } \ \Omega_2^*. \end{aligned} \quad (4.54)$$

It is easy to see that v_1 and v_2 solve the following interface problem (see (4.53), (4.54), (2.13))

$$\Delta v_j(x) + k_j^2 v_j(x) = 0 \quad \text{in} \quad \Omega_j^*, \ \ j = 1, 2, \quad (4.55)$$

$$[v_1]^+ = [v_2]^-, \ \ [\partial_n v_1]^+ = [\partial_n v_2]^- \quad \text{on} \quad \Gamma, \quad (4.56)$$

$$\partial_{x_2} v_1 = i k_1 v_1 \quad \text{on} \quad \Gamma_{h_2}, \quad (4.57)$$

$$\partial_{x_2} v_2 = -i k_2 v_2 \quad \text{on} \quad \Gamma_{h_1}, \quad (4.58)$$

where n is the unit normal vector to Γ directed out of Ω_2^*, the symbols $[\cdot]^+$ and $[\cdot]^-$ denote the limits from Ω_1^* and Ω_2^*, respectively. Moreover, due to (4.54) and Lemma 4.4

$$v_1 \in BC^1(\overline{\Omega_1^*}) \quad \text{and} \quad v_2 \in BC^1(\overline{\Omega_2^*}). \quad (4.59)$$

Let

$$\Omega_j^*(A, B) := \{x \in \Omega_j^* \mid A < x_1 < B\}, \ \ \ell_j(A) = \{x \in \Omega_j^* \mid x_2 = A\}.$$

From the Green's identities and conditions (4.55)–(4.58) we obtain (cf. (3.4), (3.5), (2.5))

$$2\lambda_1\lambda_2 \int\limits_{\Omega_1^*(A,B)} |v_1|^2\, dx + \lambda_1 \int\limits_{\Gamma_{h_2}(A,B)} |v_1|^2\, ds + k^2 \int\limits_{\Gamma_{h_1}(A,B)} |v_2|^2\, ds =$$
$$= \operatorname{Im}[R^{(1)}(A) - R^{(1)}(B) + R^{(2)}(A) - R^{(2)}(B)], \quad (4.60)$$

where

$$R^{(j)}(P) = \int\limits_{\ell_j(P)} \frac{\partial v_j}{\partial x_1}\, \overline{v_j}\, ds, \quad P = A, B, \ j = 1, 2.$$

Since $\lambda_1 > 0$, $\lambda_2 > 0$, $k_2 > 0$, and $|R^{(j)}(P)|$ are uniformly bounded for $P \in (-\infty, +\infty)$, $j = 1, 2$, we conclude from (4.59) and (4)

$$v_1 \in L_2(\Omega_1^*), \ \ v_1|_{\Gamma_{h_2}} \in L_2(\Gamma_{h_2}), \ \ v_2|_{\Gamma_{h_1}} \in L_2(\Gamma_{h_1}).$$

In turn, these inclusions imply

$$v_1(x) \to 0 \quad \text{as} \quad |x_1| \to +\infty \ \ (\text{uniformly in } \overline{\Omega_1^*}) \quad (4.61)$$

$$v_2(x_1, h_1) \to 0 \quad \text{as} \quad |x_1| \to +\infty \quad (4.62)$$

due to the uniform continuity of v_j in $\overline{\Omega_j^*}$.

In particular,

$$v_1(x_1, h_2) \to 0 \quad \text{as} \quad |x_1| \to +\infty, \tag{4.63}$$

$$-\varphi(x) = v_1(x)|_\Gamma = v_2(x)|_\Gamma \to 0 \quad \text{as} \quad |x_1| \to +\infty, \tag{4.64}$$

due to (4.53), (4.54), (4.56), and (4.61).

From the relations (4.59) and (4.63) it follows that

$$R^{(1)}(P) \to 0 \quad \text{as} \quad |P| \to +\infty,$$

whence from (4) we get that the limits of $R^{(2)}(A)$ as $A \to \pm\infty$ exist and

$$2\lambda_1\lambda_2 \int\limits_{\Omega_1^*} |v_1|^2\, dx + \lambda_1 \int\limits_{\Gamma_{h_2}} |v_1|^2\, ds + k_2 \int\limits_{\Gamma_{h_1}} |v_2|^2\, ds = r_-^{(2)} - r_+^{(2)}, \tag{4.65}$$

where

$$r_\pm^{(2)} = \lim_{A\to\pm\infty} \operatorname{Im} R^{(2)}(A) = \lim_{A\to\pm\infty} \operatorname{Im} \int\limits_{\ell_2(A)} \frac{\partial v_2}{\partial x_1}\, \overline{v_2}\, ds. \tag{4.66}$$

As a next step we will show that

$$v_2(x) \to 0 \quad \text{as} \quad |x| \to +\infty$$

uniformly in $\overline{\Omega_2^*}$.

Note that (see (4.54))

$$v_2(x) = W_2(\varphi)(x) + V_2(\psi)(x), \quad x \in \Omega_2^*,$$

where $\chi = [\varphi, \psi]^\top$ solves equation (4.49).

By the same arguments as in [8] (see the proof of Theorem 5.1, Step III, p. 3779) and applying the decay condition (4.64) we can prove that

$$W_j(\varphi)(x) \to 0 \quad \text{as} \quad |x_1| \to +\infty, \quad j = 1, 2,$$

uniformly in $\overline{\Omega_j^*}$.

To show that the similar decay property holds also for the single layer potential $V_j(\psi)(x)$ we need that the density function ψ vanishes as $|x| \to +\infty$, which we will establish below by contradiction (cf. the proof of Theorem 5.1 in [8]).

Let there exist a number $\varepsilon > 0$ and a sequence $\{x^n := (a_n, f(a_n))\} \subset \Gamma$ such that $|a_n| \to \infty$ as $n \to \infty$ and $|\psi(x^n)| \geq \varepsilon$.

Define a translation operator

$$(T_a g)(s) = g(s - a) \quad \text{for} \quad s \in \mathbb{R}$$

and let

$$f_n := T_{-a_n} f, \ \tilde{\varphi}_n := T_{-a_n} \tilde{\varphi}, \ \tilde{\psi}_n := T_{-a_n} \tilde{\psi}, \ g_n^{(j)} := T_{-a_n} g^{(j)}, \ j = 1, 2, \tag{4.67}$$

where $\chi = [\varphi, \psi]^\top$ is a solution pair of equation (4.49), $\tilde{\varphi}(s) = \varphi(s, f(s))$, $\tilde{\psi}(s) = \psi(s, f(s))$,

$$g^{(1)} := u_1(\cdot, h_2) = -\mu^{-1} v_1(\cdot, h_2), \quad g^{(2)} := u_2(\cdot, h_1) = v_2(\cdot, h_1), \tag{4.68}$$

and $u_j(x)$ are given by formulas (4.1) and (4.2). It is evident that $f_n \in \mathcal{B}(1,1,a,b,M)$ for $f \in \mathcal{B}(1,1,a,b,M)$.

Note that

$$|\tilde{\psi}_n(0)| = |\psi(a_n, f(a_n))| = |\psi(x^n)| \geq \varepsilon > 0. \tag{4.69}$$

From (4.62)–(4.64) it follows that $g_n^{(1)}, g_n^{(2)}, \tilde{\varphi}_n \in BC(\mathbb{R})$ and

$$g_n^{(1)} \xrightarrow{s} 0, \quad g_n^{(2)} \xrightarrow{s} 0, \quad \tilde{\varphi}_n \xrightarrow{s} 0, \text{ as } n \to \infty, \tag{4.70}$$

where the symbol $\xrightarrow{s}$ denotes the strict convergence (see Subsection 2.4).

Equalities (4.68) imply (see (4.1) and (4.2))

$$g^{(1)}(x_1) = u_1(x_1, h_2) = \mu^{-1}\mathcal{P}_f^{(1)**}(\tilde{\varphi})(x_1) + \mu^{-1}\mathcal{P}_f^{(1)*}(\tilde{\psi})(x_1), \tag{4.71}$$

$$g^{(2)}(x_1) = \mathcal{P}_f^{(2)**}(\tilde{\varphi})(x_1) + \mathcal{P}_f^{(2)*}(\tilde{\psi})(x_1), \tag{4.72}$$

where

$$\mathcal{P}_f^{(j)**}(\tilde{\varphi})(x_1) := \int\limits_{-\infty}^{+\infty} P_f^{(j)**}(x_1,t)\tilde{\varphi}(t)\,dt, \quad x_1 \in \mathbb{R},\ j=1,2, \tag{4.73}$$

$$\mathcal{P}_f^{(j)*}(\tilde{\psi})(x_1) := \int\limits_{-\infty}^{+\infty} P_f^{(j)*}(x_1,t)\tilde{\psi}(t)dt, \quad x_1 \in \mathbb{R},\ j=1,2, \tag{4.74}$$

$$P_f^{(1)**}(x_1,t) = \varrho(t)\left[\partial_{n(y)} G_{k_1}^{-(\mathcal{I})}(x,y;h_2)\right], \ x=(x_1,h_2),\ y=(t,f(t)), \tag{4.75}$$

$$P_f^{(1)*}(x_1,t) = \varrho(t)\left[G_{k_1}^{-(\mathcal{I})}(x,y;h_2)\right], \ \ x=(x_1,h_2), \ \ y=(t,f(t)), \tag{4.76}$$

$$P_f^{(2)**}(x_1,t) = \varrho(t)\left[\partial_{n(y)} G_{k_2}^{+(\mathcal{I})}(x,y;h_1)\right], \ x=(x_1,h_1),\ y=(t,f(t)), \tag{4.77}$$

$$P_f^{(2)*}(x_1,t) = \varrho(t)\left[G_{k_2}^{+(\mathcal{I})}(x,y;h_1)\right], \ \ x=(x_1,h_1), \ \ y=(t,f(t)); \tag{4.78}$$

here $\varrho(t) = \sqrt{1+[f'(t)]^2}$.

From these formulas with the help of (2.12) we get

$$g_n^{(1)} = T_{-a_n} g^{(1)} = \mu^{-1}\mathcal{P}_{f_n}^{(1)**}(\tilde{\varphi}_n) + \mu^{-1}\mathcal{P}_{f_n}^{(1)*}(\tilde{\psi}_n), \tag{4.79}$$

$$g_n^{(2)} = T_{-a_n} g^{(2)} = \mathcal{P}_{f_n}^{(2)**}(\tilde{\varphi}_n) + \mathcal{P}_{f_n}^{(2)*}(\tilde{\psi}_n). \tag{4.80}$$

The relations (4.70) then imply

$$\mathcal{P}_{f_n}^{(j)**}(\tilde{\varphi}_n) + \mathcal{P}_{f_n}^{(j)*}(\tilde{\psi}_n) \xrightarrow{s} 0, \ j=1,2. \tag{4.81}$$

From equations (4.49) we have

$$-(\mu+1)(2\mu)^{-1}\tilde{\varphi}_n + \mu^{-1}\widetilde{\mathcal{K}}^*_{1,f_n}\tilde{\varphi}_n - \widetilde{\mathcal{K}}^*_{2,f_n}\tilde{\varphi}_n + \mu^{-1}\widetilde{\mathcal{H}}^*_{1,f_n}\tilde{\psi}_n - \widetilde{\mathcal{H}}^*_{2,f_n}\tilde{\psi}_n = 0, \tag{4.82}$$

$$\widetilde{\mathcal{L}}_{f_n}\tilde{\varphi}_n + \tilde{\psi}_n + \widetilde{\mathcal{K}}_{1,f_n}\tilde{\psi}_n - \widetilde{\mathcal{K}}_{2,f_n}\tilde{\psi}_n = 0, \tag{4.83}$$

where the operators $\widetilde{\mathcal{K}}^*_{j,f_n}$, $\widetilde{\mathcal{K}}_{j,f_n}$, $\widetilde{\mathcal{H}}_{j,f_n}$ are defined by (4.11)–(4.13), and in accordance with (4)–(4.24)

$$(\widetilde{\mathcal{L}}_f\varphi)(x_1):=\int\limits_{-\infty}^{+\infty}\widetilde{L}_f(x_1,y_1)\tilde{\varphi}(y_1)dy_1,\ \ x_1\in\mathbb{R}, \tag{4.84}$$

$$\widetilde{L}_f(x_1,y_1):=\varrho(t)L(x,y)\ \text{ with }\ x=(x_1,f(x_1)),\ y=(y_1,f(y_1)). \tag{4.85}$$

By Lemma 4.6.(i) in [8] there exists a subsequence of $\{f_n\}$ (for simplicity, we rename it again by f_n) and $f^*\in\mathcal{B}(1,1,a,b,M)$ such that

$$f_n\xrightarrow{s}f^* \qquad \text{and} \qquad f_n'\xrightarrow{s}(f^*)'. \tag{4.86}$$

By Lemmas 4.6.(ii) and 4.1 in [8] we conclude

$$\widetilde{\mathcal{K}}^*_{j,f_n}\tilde{\varphi}_n\xrightarrow{s}0,\quad \widetilde{\mathcal{L}}_{f_n}\tilde{\varphi}_n\xrightarrow{s}0, \tag{4.87}$$

since $\tilde{\varphi}_n\xrightarrow{s}0$, $\widetilde{K}^*_{j,f_n}\xrightarrow{\sigma}\widetilde{K}^*_{j,f^*}$, and $\widetilde{L}_{f_n}\xrightarrow{\sigma}\widetilde{L}_{f^*}$, where the symbol $\xrightarrow{\sigma}$ denotes the σ-convergence (see Subsection 2.4). Since the sequence $\{\tilde{\psi}_n\}$ is uniformly bounded, by Corollary 4.5 in [8] there exist $\tilde{\psi}^{**}$ and $\tilde{\psi}^*$ in $BC(\mathbb{R})$ and subsequences of $(\mu^{-1}\widetilde{\mathcal{H}}_{1,f_n}-\widetilde{\mathcal{H}}_{2,f_n})\tilde{\psi}_n$ and $(\widetilde{\mathcal{K}}_{1,f_n}-\widetilde{\mathcal{K}}_{2,f_n})\tilde{\psi}_n$ (renamed by the same symbols) such that

$$(\mu^{-1}\widetilde{\mathcal{H}}_{1,f_n}-\widetilde{\mathcal{H}}_{2,f_n})\tilde{\psi}_n\xrightarrow{s}\tilde{\psi}^{**}, \tag{4.88}$$

$$(\widetilde{\mathcal{K}}_{1,f_n}-\widetilde{\mathcal{K}}_{2,f_n})\tilde{\psi}_n\xrightarrow{s}\tilde{\psi}^{*}. \tag{4.89}$$

After the above manipulations, it is evident that we may assume that all the relations (4.67)–(4.89) hold for one and the same discrete parameter $n\in\mathbb{N}$.

From (4.70), (4.82), (4.83), and (4.87) along with (4.88) and (4.89) it follows that

$$\tilde{\psi}^{**}=0\quad\text{and}\quad\tilde{\psi}_n\xrightarrow{s}\tilde{\psi}^*. \tag{4.90}$$

Further, by (4.90) and since $\widetilde{H}_{j,f_n}\xrightarrow{\sigma}\widetilde{H}_{j,f^*}$ we have (see Lemmas 4.1 and 4.6 in [8])

$$\widetilde{\mathcal{H}}_{j,f_n}\tilde{\psi}_n\xrightarrow{s}\widetilde{\mathcal{H}}_{j,f^*}\tilde{\psi}^*,\quad \widetilde{\mathcal{K}}_{j,f_n}\tilde{\psi}_n\xrightarrow{s}\widetilde{\mathcal{K}}_{j,f^*}\tilde{\psi}^*. \tag{4.91}$$

As a result, from (4.82)–(4.83) with the help of (4.70), (4.87), and (4.91) we then get

$$\begin{aligned}&\mu^{-1}\widetilde{\mathcal{H}}_{1,f^*}\tilde{\psi}^*-\widetilde{\mathcal{H}}_{2,f^*}\tilde{\psi}^*=0,\\&\tilde{\psi}^*+\widetilde{\mathcal{K}}_{1,f^*}\tilde{\psi}^*-\widetilde{\mathcal{K}}_{2,f^*}\tilde{\psi}^*=0.\end{aligned} \tag{4.92}$$

Using the bounds (2.14) and since $f_n\in\mathcal{B}(1,1,a,b,M)$ and $f_n\xrightarrow{s}f^*$, $f_n'\xrightarrow{s}(f^*)'$, it is easy to see that

$$P^{(j)*}_{f_n}\xrightarrow{\sigma}P^{(j)*}_{f^*}\quad\text{and}\quad P^{(j)**}_{f_n}\xrightarrow{\sigma}P^{(j)**}_{f^*}.$$

Consequently,

$$\mathcal{P}^{(j)*}_{f_n}(\tilde{\psi}_n)\xrightarrow{s}\mathcal{P}^{(j)*}_{f^*}(\tilde{\psi}^*),\quad \mathcal{P}^{(j)**}_{f_n}(\tilde{\varphi}_n)\xrightarrow{s}0.$$

Therefore,

$$\overset{(1)}{g}_n = \mu^{-1}\mathcal{P}^{(1)**}_{f_n}(\tilde{\varphi}_n) + \mu^{-1}\mathcal{P}^{(1)*}_{f_n}(\tilde{\psi}_n) \xrightarrow{s} \mu^{-1}\mathcal{P}^{(1)*}_{f^*}(\tilde{\psi}^*) = 0,$$
$$\overset{(2)}{g}_n = \mathcal{P}^{(2)*}_{f_n}(\tilde{\varphi}_n) + \mathcal{P}^{(2)*}_{f_n}(\tilde{\psi}_n) \xrightarrow{s} \mathcal{P}^{(2)*}_{f^*}(\tilde{\psi}^*) = 0,$$

due to (4.70).

Thus,

$$\mathcal{P}^{(1)*}_{f^*}(\tilde{\psi}) = 0, \qquad \mathcal{P}^{(2)*}_{f^*}(\tilde{\psi}^*) = 0. \tag{4.93}$$

Now, let us introduce the functions

$$w_1(x) := \mu^{-1} V_{1,f^*}(\psi^*)(x), \quad x \in U^-_{h_2} \setminus \Gamma_{f^*}, \tag{4.94}$$
$$w_2(x) := V_{2,f^*}(\psi^*)(x), \;\; x \in U^+_{h_1} \setminus \Gamma_{f^*}, \tag{4.95}$$

where

$$\psi^*(y) := \tilde{\psi}^*(y_1), \quad y = (y_1, f^*(y_1)) \in \Gamma_{f^*}. \tag{4.96}$$

Since $\tilde{\psi}^*$ solves the system of integral equations (4.92), it follows that

$$(\Delta + k_j^2) w_j^*(x) = 0 \quad \text{in} \quad \Omega_j^{f^*},$$

$$\left.\begin{aligned} [w_1(x)]^-_{\Gamma_{f^*}} &= [w_2(x)]^+_{\Gamma_{f^*}}, \\ \mu\,[\partial_n w_1(x)]^-_{\Gamma_{f^*}} &= [\partial_n w_2(x)]^+_{\Gamma_{f^*}}, \end{aligned}\right\} \; x \in \Gamma_{f^*} = \{x \in \mathbb{R}^2 \mid x_2 = f^*(x_1)\},$$

where $\Omega_1^{f^*} = \{x \in \mathbb{R}^2 \mid x_2 < f^*(x_1)\}$, $\Omega_2^{f^*} = \{x \in \mathbb{R}^2 \mid x_2 > f^*(x_1)\}$, and w_1 and w_2 satisfy the UPRC and DPRC, respectively.

Moreover, due to the second equation in (4.92) and Lemmas 4.3 and 4.4, we conclude that w_j $(j = 1, 2)$ have bounded continuous first order derivatives in $\overline{U}^+_{h_1} \setminus \Omega_2^{f^*}$ and $\overline{U}^-_{h_2} \setminus \Omega_1^{f^*}$. Therefore, due to the uniqueness Theorem 3.1

$$w_1(x) = 0 \quad \text{in} \quad \Omega_1^{f^*}, \quad w_2(x) = 0 \quad \text{in} \quad \Omega_2^{f^*}.$$

Further, the equations (4.93) show that

$$w_1(x)|_{x \in \Gamma_{h_2}} = 0, \quad w_2(x)|_{x \in \Gamma_{h_1}} = 0. \tag{4.97}$$

Applying the impedance conditions (2.13) and the representations (4.94) and (4.95) we get

$$\left.\frac{\partial w_1(x)}{\partial x_2}\right|_{x_2 = h_2} = 0, \qquad \left.\frac{\partial w_2(x)}{\partial x_2}\right|_{x_2 = h_1} = 0.$$

By Holmgren's uniqueness theorem we then conclude that

$$w_1(x) = 0 \;\text{ for }\; x \in U^-_{h_2} \setminus \Omega_1^{f^*}, \quad w_2(x) = 0 \;\text{ for }\; x \in U^+_{h_1} \setminus \Omega_2^{f^*}.$$

The equations (4.97), (4.94) and (4.95), and Lemma 4.1 then imply

$$\psi^* = 0 \quad \text{on} \quad \Gamma_{f^*}, \tag{4.98}$$

since $(\partial_n w_1)^-_{\Gamma_{f^*}} - (\partial_n w_1)^+_{\Gamma_{f^*}} = \mu^{-1}\psi^*$. The equality (4.98) contradicts to (4.69) since (see (4.90) and (4.96))

$$|\psi^*(0, f^*(0))| = |\tilde{\psi}^*(0)| = \lim_{n\to+\infty} |\psi^*_n(0)| \geq \varepsilon > 0.$$

Thus, we have proven that

$$\lim_{|x_1|\to\infty} \psi(x) = 0. \tag{4.99}$$

Now, from (4.99) it follows that

$$V_j(\psi)(x) \to 0 \quad \text{as} \quad |x_1| \to +\infty, \quad j = 1, 2, \tag{4.100}$$

uniformly in $\overline{\Omega}^*_j$ (cf. the proof of Theorem 5.1, Step III in [8], p. 3779).

In turn, (4.100) then implies (see (4.66)): $r^{(2)}_{\pm} = 0$.

Applying (4.65) and (4.54) we get

$$v_1(x) = -\mu u_1(x) = 0 \quad \text{for} \quad x \in \Omega^*_1. \tag{4.101}$$

Consequently, we have obtained that u_1, which is represented by (4.1), vanishes in Ω_1 and in $\Omega^*_1 = U^-_{h_2} \setminus \Omega_1$ (see (4.50) and (4.101)). This yields $\varphi = \psi = 0$ on Γ, which completes the proof. □

4.3. Existence results. Now we are in the position to prove the unique solvability of the non-homogeneous system (4.42), (4.43) (i.e., the matrix equation (4.45)) which can be equivalently rewritten as the following system of integral equations on $\mathbb{R}$:

$$\Big(-(\mu+1)(2\mu)^{-1}\, I + \mu^{-1}\, \widetilde{\mathcal{K}}^*_{1,f} - \widetilde{\mathcal{K}}^*_{2,f}\Big)\, \widetilde{\varphi}(x_1) + \\ + \Big(\mu^{-1}\, \widetilde{\mathcal{H}}_{1,f} - \widetilde{\mathcal{H}}_{2,f}\Big)\, \widetilde{\psi}(x_1) = \widetilde{f}_1(x_1), \tag{4.102}$$

$$\widetilde{\mathcal{L}}_f\, \widetilde{\varphi}(x_1) + \Big(I + \widetilde{\mathcal{K}}_{1,f} - \widetilde{\mathcal{K}}_{2,f}\Big)\, \widetilde{\psi}(x_1) = \widetilde{f}_2(x_1), \tag{4.103}$$

where $\widetilde{\mathcal{K}}^*_{j,f}$, $\widetilde{\mathcal{K}}_{j,f}$, $\widetilde{\mathcal{H}}_{j,f}$, and $\widetilde{\mathcal{L}}_f$ are integral operators given by (4.11)–(4.13) and (4.84), respectively, $f \in \mathcal{B}(1, 1, a, b, M)$, and

$$\widetilde{\varphi}(x_1) := \varphi(x_1, f(x_1)), \quad \widetilde{\psi}(x_1) := \psi(x_1, f(x_1)), \quad \widetilde{f}_j(x_1) := f_j(x_1, f(x_1)).$$

The corresponding matrix operator we denote by $\widetilde{\mathcal{M}}_f$:

$$\widetilde{\mathcal{M}}_f := \begin{bmatrix} -(\mu+1)(2\mu)^{-1} I + \mu^{-1}\, \widetilde{\mathcal{K}}^*_{1,f} - \widetilde{\mathcal{K}}^*_{2,f} & \mu^{-1}\widetilde{\mathcal{H}}_{1,f} - \widetilde{\mathcal{H}}_{2,f} \\ \widetilde{\mathcal{L}}_f & I + \widetilde{\mathcal{K}}_{1,f} - \widetilde{\mathcal{K}}_{2,f} \end{bmatrix} \tag{4.104}$$

and let

$$\widetilde{\chi} := [\widetilde{\varphi}, \widetilde{\varphi}]^\top, \quad \widetilde{F} := [\widetilde{f}_1, \widetilde{f}_2]^\top.$$

The equations (4.102)–(4.103) then can be written as

$$\widetilde{\mathcal{M}}_f\, \widetilde{\chi}(x_1) = \widetilde{F}(x_1), \quad x_1 \in \mathbb{R}. \tag{4.105}$$

Now we formulate the properties of the integral operators involved in (4.104) needed to apply the theory developed in [11] and [1] for a class of systems of second kind integral equations on unbounded domains.

Lemma 4.7. *Let $\widetilde{\mathcal{K}}$ denote any of the integral operators $\widetilde{\mathcal{K}}^*_{j,f}$, $\widetilde{\mathcal{K}}_{j,f}$, $\widetilde{\mathcal{H}}_{j,f}$, or $\widetilde{\mathcal{L}}_f$, and let $\widetilde{K}(s,t)$ denote the corresponding kernel, such that*

$$\widetilde{\mathcal{K}}\,\nu(s) = \int_{-\infty}^{+\infty} \widetilde{K}(s,t)\,\nu(t)\,dt \quad \text{for} \quad s \in \mathbb{R}.$$

(a) *There exists a function $k(\cdot) \in L_1(\mathbb{R})$ such that*

$$|\widetilde{K}(s,t)| \le k(s-t) \quad \textit{for} \quad s,t \in \mathbb{R}, \quad s \neq t,$$

where $k(s) = O(|s|^{-3/2})$ as $|s| \to +\infty$.

(b) *The kernel $\widetilde{K}$ satisfies the properties*

$$\sup_{s\in\mathbb{R}} \int_{-\infty}^{+\infty} |\widetilde{K}(s,t)|\,dt < +\infty,$$

and for all $s, s' \in \mathbb{R}$,

$$\lim_{s'\to s} \int_{-\infty}^{+\infty} |\widetilde{K}(s',t) - \widetilde{K}(s,t)|\,dt = 0.$$

(c) *$\widetilde{\mathcal{K}}$ is a bounded mapping from $L_\infty(\mathbb{R})$ to $BC(\mathbb{R})$ and from $L_p(\mathbb{R})$ to $L_p(\mathbb{R})$ for any $p \in [1, +\infty)$.*

Proof. It is verbatim the proof of Lemma 5.1 in [1] due to the bounds (2.14) and equalities (4.10) and (4.24) (see also (4.25)). □

Let

$$\widetilde{\mathcal{M}}^{(0)}_f := \begin{bmatrix} \mu^{-1}\,\widetilde{\mathcal{K}}^*_{1,f} - \widetilde{\mathcal{K}}^*_{2,f} & \mu^{-1}\,\widetilde{\mathcal{H}}_{1,f} - \widetilde{\mathcal{H}}_{2,f} \\ \widetilde{\mathcal{L}}_f & \widetilde{\mathcal{K}}_{1,f} - \widetilde{\mathcal{K}}_{2,f} \end{bmatrix} \tag{4.106}$$

and denote by $\widetilde{M}^{(0)}_f(\cdot,\cdot)$ the matrix kernel corresponding to the operator $\widetilde{\mathcal{M}}^{(0)}_f$.

By (4.104) then

$$\widetilde{\mathcal{M}}_f = I^{(0)} + \widetilde{\mathcal{M}}^{(0)}_f \tag{4.107}$$

with

$$I^{(0)} := \begin{bmatrix} -(1+\mu)(2\mu)^{-1} & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.108}$$

Further, let

$$\widetilde{\Lambda} := \left\{ \widetilde{\mathcal{M}}^{(0)}_f \;\mid\; f \in \mathcal{B}(1,1,a,b,M) \right\}.$$

Lemma 4.8. *Let $\widetilde{\mathcal{M}}_f^{(0)} \in \widetilde{\Lambda}$. Then $\widetilde{\mathcal{M}}_f^{(0)}$ is continuous and sequentially compact with respect to the σ-topology.*

Proof. It is verbatim the proof of Corollary 5.14 in [1] due to Lemma 4.7. □

Lemma 4.9. *Assume that $\{\widetilde{\chi}_n\} \subset [BC(\mathbb{R})]^2$ is a bounded sequence and that there are a sequence $\{\widetilde{\mathcal{M}}_{f_n}^{(0)}\} \subset \widetilde{\Lambda}$ and an operator $\widetilde{\mathcal{M}}_f^{(0)} \in \widetilde{\Lambda}$ such that $f_n \xrightarrow{s} f$, $f_n' \xrightarrow{s} f'$.*

Then $\left(\widetilde{\mathcal{M}}_{f_n}^{(0)} - \widetilde{\mathcal{M}}_f^{(0)}\right) \widetilde{\chi}_n \xrightarrow{s} 0$ as $n \to +\infty$.

Proof. It is verbatim the proof of Lemma 5.16 in [1] and Lemma 4.6 in [8]. □

Lemma 4.10. *The set $\widetilde{\Lambda}$ is collectively sequentially compact with respect to the σ-topology. Furthermore, for every $\{\widetilde{\mathcal{M}}_{f_n}^{(0)}\} \subset \widetilde{\Lambda}$, there exist a subsequence $\{\widetilde{\mathcal{M}}_{f_{n_p}}^{(0)}\}$ and $\widetilde{\mathcal{M}}_f^{(0)} \in \widetilde{\Lambda}$ such that $\widetilde{\mathcal{M}}_{f_{n_p}}^{(0)} \widetilde{\chi}_p \xrightarrow{s} \widetilde{\mathcal{M}}_f^{(0)} \widetilde{\chi}$ for arbitrary $\widetilde{\chi}_p \xrightarrow{s} \widetilde{\chi}$ as $p \to +\infty$.*

Proof. It is verbatim the proof of Theorem 5.17 in [1]. □

As above (see the proof of Lemma 4.6), let T_a be a translation operator and

$$\mathcal{T} := \left\{ T_a \;\mid\; [BC(\mathbb{R})]^2 \to [BC(\mathbb{R})]^2, \;\; \widetilde{\chi}(\cdot) \mapsto \widetilde{\chi}(\cdot - a), \;\; a \in \mathbb{R} \right\}. \tag{4.109}$$

Obviously, $\mathcal{T}$ forms a sufficient subgroup of the group of isometries on $[BC(\mathbb{R})]^2$, that is, for some $j \in \mathbb{N}$ and for each $\widetilde{\chi} \in [BC(\mathbb{R})]^2$ there holds

$$\sup_{|s| \le j} T_a \widetilde{\chi}(s) \ge 2^{-1} \|\widetilde{\chi}\|_\infty, \quad a \in \mathbb{R}.$$

Furthermore, since for $f \in \mathcal{B}(1,1,a,b,M)$ there also holds $f(\cdot - a) \in \mathcal{B}(1,1,a,b,M)$, it is not difficult to see that for $\widetilde{\mathcal{M}}_f^{(0)} \in \widetilde{\Lambda}$, $T_a \in \mathcal{T}$, there holds $T_{-a}\widetilde{\mathcal{M}}_f^{(0)} T_a \in \widetilde{\Lambda}$, due to the structure of the kernels of the operators involved in (4.106).

Let now $\widetilde{\mathcal{M}}_f^{(0)} \in \widetilde{\Lambda}$ and denote by κ a $C^\infty(\mathbb{R})$ function with $|\kappa| \le 1$ on $\mathbb{R}$, $\kappa = 0$ for $t \le 0$ and $\kappa = 1$ for $t \ge m$, where m is a positive number. Denote

$$\kappa_n(t) := \begin{cases} \kappa(n+m+t) & \text{for} \quad t < 0, \\ \kappa(n+m-t) & \text{for} \quad t \ge 0, \end{cases} \qquad n \in \mathbb{N},$$

$$\overline{f} := 2^{-1} [\, \sup_{\mathbb{R}} f + \inf_{\mathbb{R}} f \,].$$

We construct the sequence

$$f_n(t) := \kappa_n(t)\, f(t) + [1 - \kappa_n(t)]\, \overline{f} \tag{4.110}$$

and choose m such that $\{f_n\} \subset \mathcal{B}(1,1,a,b,M)$.

It is easy to see that $f_n \xrightarrow{s} f$, $f_n' \xrightarrow{s} f'$ and by Lemma 4.9 also $\widetilde{\mathcal{M}}_{f_n}^{(0)} \xrightarrow{\sigma} \widetilde{\mathcal{M}}_f^{(0)}$.

Note that

$$\widetilde{\mathcal{M}}^{(0)}_{f_n} = \widetilde{\mathcal{M}}^{(0)}_{\overline{f}} + \widetilde{\mathcal{M}}^{(1)}_{f_n}, \tag{4.111}$$

where

$$\widetilde{\mathcal{M}}^{(1)}_{f_n}\,\widetilde{\chi}(s) : = \widetilde{\mathcal{M}}^{(0)}_{f_n}\,\widetilde{\chi}(s) - \widetilde{\mathcal{M}}^{(0)}_{\overline{f}}\,\widetilde{\chi}(s) = \\ = \int_{-\infty}^{+\infty} \Big[\widetilde{M}^{(0)}_{f_n}(s,t) - \widetilde{M}^{(0)}_{\overline{f}}(s,t)\Big]\;\widetilde{\chi}(t)\,dt.$$

Lemma 4.11. *The operators* $\widetilde{\mathcal{M}}^{(1)}_{f_n}$, $n \in \mathbb{N}$, *are compact.*

Proof. Represent $\widetilde{\mathcal{M}}^{(1)}_{f_n}$ in the form

$$\widetilde{\mathcal{M}}^{(1)}_{f_n} = \widetilde{\mathcal{M}}^{(2)}_{f_n} + \widetilde{\mathcal{M}}^{(3)}_{f_n},$$

where

$$\widetilde{\mathcal{M}}^{(2)}_{f_n}\,\widetilde{\chi}(s) := \int_{-(n+m)}^{(n+m)} \Big[\widetilde{M}^{(0)}_{f_n}(s,t) - \widetilde{M}^{(0)}_{\overline{f}}(s,t)\Big]\;\widetilde{\chi}(t)\,dt,$$

$$\widetilde{\mathcal{M}}^{(3)}_{f_n}\,\widetilde{\chi}(s) := \int_{\mathbb{R}\setminus[-(n+m),(n+m)]} \Big[\widetilde{M}^{(0)}_{f_n}(s,t) - \widetilde{M}^{(0)}_{\overline{f}}(s,t)\Big]\;\widetilde{\chi}(t)\,dt,$$

The compactness of the operator $\widetilde{\mathcal{M}}^{(2)}_{f_n}$ can be shown by the word for word arguments given in the proof of Lemma 5.18 in [1].

It remains to show that $\widetilde{\mathcal{M}}^{(3)}_{f_n}$ is compact.

Taking into account that the kernel $\widetilde{M}^{(0)}_{f_n}(\cdot,\cdot)$ depends on f in the following way (see (4.106) and (4.7)–(4.13))

$$\widetilde{M}^{(0)}_{f}(s,t) = M^{(0)}(s, f(s), t, f(t)),$$

we easily conclude

$$\widetilde{M}^{(0)}_{f_n}(s,t) - \widetilde{M}^{(0)}_{\overline{f}}(s,t) = 0 \;\text{ for }\; |s| \geq n+m \;\text{ and }\; |t| \geq n+m$$

due to the equality (4.110). Therefore,

$$\widetilde{\mathcal{M}}^{(3)}_{f_n}\widetilde{\chi}(s) = 0 \;\text{ for }\; |s| \geq n+m.$$

Now the compactness of the operator $\widetilde{\mathcal{M}}^{(3)}_{f_n}$ follows from Lemma 4.7 and the Arzela–Ascoli theorem. □

Lemma 4.12. *The operator* $I^{(0)} + \widetilde{\mathcal{M}}^{(0)}_{\overline{f}}$ *is bijective* (*and thus a Fredholm operator of index zero*) *on* $[BC(\mathbb{R})]^2$.

Proof. Since $\widetilde{\mathcal{M}}^{(0)}_{\overline{f}}$ is a convolution operator with a matrix kernel in $[L_1(\mathbb{R})]^{2\times 2}$, the proof follows from Theorem A.2 in [26]. □

Lemma 4.13. *The operators* $\widetilde{\mathcal{M}}_{f_n} = I^{(0)} + \widetilde{\mathcal{M}}^{(0)}_{f_n}$ *are bijective on* $[BC(\mathbb{R})]^2$.

Proof. Note that due to the equation (4.111) and Lemmas 4.11 and 4.12 the operator $\widetilde{\mathcal{M}}_{f_n}$ is Fredholm and its index equals to 0. Thus, by Lemma 4.6, $\widetilde{\mathcal{M}}_{f_n}$ is bijective. □

Now we can prove the main existence result for the system of integral equations (4.102)–(4.103) which can be written as (see (4.105) and (4.107))

$$\widetilde{\mathcal{M}}_f \widetilde{\chi} = \widetilde{F} \quad \text{or} \quad \left[I^{(0)} + \widetilde{\mathcal{M}}^{(0)}_f\right] \widetilde{\chi} = \widetilde{F} \quad \text{on} \quad \mathbb{R}$$

(see (4.105), (4.107), (4.108)).

Theorem 4.14. *For all* $f \in \mathcal{B}(1,1,a,b,M)$ *the integral operator*

$$\widetilde{\mathcal{M}}_f = I^{(0)} + \widetilde{\mathcal{M}}^{(0)}_f \;:\; [BC(\mathbb{R})]^2 \to [BC(\mathbb{R})]^2$$

is bijective (and so boundedly invertible) with

$$\sup_{f \in \mathcal{B}(1,1,a,b,M)} \|\widetilde{\mathcal{M}}_f^{-1}\| < \infty.$$

Thus the equations (4.102)–(4.103) *have exactly one solution for every* $f \in \mathcal{B}(1,1,a,b,M)$ *and* $F \in [BC(\Gamma_f)]^2$, *with*

$$\|\chi\|_{[BC(\Gamma_f)]^2} = \|\widetilde{\chi}\|_{[BC(\mathbb{R})]^2} \le C\, \|\widetilde{F}\|_{[BC(\mathbb{R})]^2} = C\, \|F\|_{[BC(\Gamma_f)]^2}$$

for some constant $C > 0$ *depending on* $\mathcal{B}(1,1,a,b,M)$ *and wave numbers* k_j $(j = 1,2)$.

Proof. Due to Lemmas 4.6–4.13 it is easy to see that the following conditions are satisfied:

(a) The set $\widetilde{\Lambda}$ is collectively sequentially compact with respect to the σ-topology and for every sequence $\{\widetilde{\mathcal{M}}^{(0)}_{f_n}\} \subset \widetilde{\Lambda}$ there exists a subsequence $\{\widetilde{\mathcal{M}}^{(0)}_{f_{n_p}}\}$ and $\widetilde{\mathcal{M}}^{(0)}_f \in \widetilde{\Lambda}$ such that $\widetilde{\mathcal{M}}^{(0)}_{f_{n_p}} \xrightarrow{\sigma} \widetilde{\mathcal{M}}^{(0)}_f$ as $p \to +\infty$ (see Lemma 4.10).

(b) The set of translation operators (4.109) forms a sufficient subgroup of the group of isometries on $[BC(\mathbb{R})]^2$ and for an arbitrary translation operator $T_a \in \mathcal{T}$ there holds $T_{-a}\widetilde{\Lambda}T_a \subset \widetilde{\Lambda}$.

(c) $\widetilde{\mathcal{M}}_f = I^{(0)} + \widetilde{\mathcal{M}}^{(0)}_f$ is injective for $\widetilde{\mathcal{M}}^{(0)}_f \in \widetilde{\Lambda}$ (see Lemma 4.6).

(d) For every $\widetilde{\mathcal{M}}^{(0)}_f \in \widetilde{\Lambda}$ there exists a sequence $\{\widetilde{\mathcal{M}}^{(0)}_{f_n}\} \subset \widetilde{\Lambda}$ such that $I^{(0)} + \widetilde{\mathcal{M}}^{(0)}_{f_n}$ is bijective and $\widetilde{\mathcal{M}}^{(0)}_{f_n} \xrightarrow{\sigma} \widetilde{\mathcal{M}}^{(0)}_f$ as $n \to +\infty$ (see Lemma 4.13).

By Theorems 5.12 and 5.13 in [1] we then conclude that all the assertions of the theorem are valid. □

The above theorem along with Lemmas 4.4 and 4.5 leads to the following existence results for the original interface problem.

Theorem 4.15. *Interface Problem* (P) *has exactly one solution for arbitrary data* f_1 *and* f_2 *with* $f_1 \in BC^{1,\alpha}(\Gamma)$ *and* $f_2 \in BC^{0,\alpha}(\Gamma)$, *and*

$$u_j \in C^2(\Omega_j) \cap C^1(\overline{\Omega_j}) \cap BC^1(\overline{\Omega_{j,h_j}}), \quad j = 1, 2.$$

Moreover, in Ω_{j,h_j} *the solution depends continuously on* $\|f_1\|_{\infty,\Gamma}$ *and* $\|f_2\|_{\infty,\Gamma}$, *while* ∇u_j *depends continuously on* $\|f_1\|_{1,\alpha,\Gamma}$ *and* $\|f_2\|_{0,\alpha,\Gamma}$.

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