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# THE LINEARIZED MAXIMUM PRINCIPLE FOR QUASI-LINEAR NEUTRAL OPTIMAL PROBLEMS WITH DISCONTINUOUS INITIAL CONDITION AND VARIABLE DELAYS IN CONTROLS

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Let $J=[a,b]\subset R$ be a finite interval, $O\subset R^n$, $G\subset R^r$ be open sets; Let the function $f:J\times O^s\times G^\nu\to R^n$ satisfy the following conditions: for almost all $t\in J$ function $f(t,x_1,\dots,x_s,u_1,\dots,u_\nu)$ is continuously differentiable with respect to $x_i\in O$, $i=1,\dots,s,u_m\in G$, $m=1,\dots,\nu$; for any fixed $(x_1,\dots,x_s,u_1,\dots,u_\nu)\in O^s\times G^\nu$ the functions $f(t,x_1,\dots,x_s,u_1,\dots,u_\nu)$, $f_{x_i}(\cdot)$ $i=1,\dots,s$, $f_{u_m}(\cdot)$, $m=1,\dots,\nu$ are measurable on $J$; for arbitrary compacts $K\subset O$, $N\subset G$ there exists a function $m_{K,N}(\cdot)\in L(J,R_+)$, $R_+=[0,\infty)$, such that for any $(x_1,\dots,x_s,u_1,\dots,u_m)\in K^s\times N^m$ and for almost all $t\in J$, the following inequality is fulfilled

$$\mid f(t,x_1,\dots,x_s,u_1,\dots,u_\nu)\mid+\sum_{i=1}^{s}\mid f_{x_i}(\cdot)\mid+\sum_{m=1}^{\nu}\mid f_{u_m}(\cdot)\mid\leq m_{K,N}(t).$$

Let the scalar functions $\tau_i(t)$, $i=1,\dots,s$, $\theta_m(t)$, $m=1,\dots,\nu$, $t\in R$ and $\eta_j(t)$, $j=1,\dots,k$, $t\in R$, be absolutely continuous and continuously differentiable , respectively, and satisfy the conditions: $\tau_i(t)\leq t$, $\dot\tau_i(t)>0$, $i=1,\dots,s$; $\theta_m(t)\leq t$, $\dot\theta_m(t)>0$, $m=1,\dots,\nu$ $\eta_j(t)<t$, $\dot\eta_j(t)>0$, $j=1,\dots,k$. Let $\Phi$ be the set of continuously differentiable functions $\varphi:J_1=[\tau,b]\to M,\tau=\min\{\eta_1(a),\dots,\eta_k(a),\tau_1(a),\dots,\tau_s(a)\}$, where $M\subset O$ is a convex set, $\|\varphi\|=\sup\{\mid\varphi(t)\mid+\mid\dot\varphi(t)\mid: t\in J\}$; $\Omega$ be the set of measurable functions $u:J_2=[\theta,b]\to U$, such that $cl\{u(t):t\in J\}$ is a compact lying in $G$, $\theta=\min\{\theta_1(a),\dots,\theta_\nu(a)\}$, where $U\subset G$ is a convex set, $\|u\|=\sup\{\mid u(t)\mid: t\in J_2\}$; $A_i(t)$, $t\in J$, $i=1,\dots,k$, be continuous $n\times n$ matrix functions. The scalar functions $q^i(t_0,t_1,x_0,x_1)$, $i=1,\dots,l$, are continuously differentiable on the set $J^2\times O^2$.

To every element $\lambda=(t_0,t_1,x_0,\varphi,u)\in E=J^2\times O\times\Phi\times\Omega$ let us correspond the differential equation

$$\dot x(t)=\sum_{j=1}^{k}A_j(t)\dot x(\eta_j(t))+f(t,x(\tau_1(t)),\dots,x(\tau_s(t)),u(\theta_1(t)),\dots,u(\theta_m(t)))\qquad(1)$$

with discontinuous initial condition

$$x(t)=\varphi(t),\quad t\in[\tau,t_0),x(t_0)=x_0.\qquad(2)$$

**Definition 1.** Let $\lambda=(t_0,t_1,x_0,\varphi,u)\in E$, $t_0<b$. The function $x(t)=x(t;\lambda)\in O$, $t\in[\tau,t_1]$, $t_1\in(t_0,b]$ is said to be a solution corresponding to the element $\lambda$, defined on the interval $[\tau,t_1]$, if on the interval $[\tau,t_0]$ the function $x(t)$ satisfies the condition (2), while on the interval $[t_0,t_1]$ it is absolutely continuous and almost everywhere satisfies the equation (1).

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**Definition 2.** The element $\lambda \in E$ is said to be admissible if the corresponding solution $x(t) = x(t;\lambda)$ satisfies the conditions

$$q^i(t_0, t_1, x_0, x(t_1)) = 0, \quad i = 1, \dots, l. \tag{3}$$

The set of admissible elements will be denoted by $E_0$.

**Definition 3.** The element $\widetilde{\lambda} = (\widetilde{t}_0, \widetilde{t}_1, \widetilde{x}_0, \widetilde{\varphi}, \widetilde{u}) \in E_0$ is said to be locally optimal, if there exists a number $\delta > 0$ such that for an arbitrary element $\lambda \in E_0$ satisfying

$$| \widetilde{t}_0 - t_0 | + | \widetilde{t}_1 - t_1 | + | \widetilde{x}_0 - x_0 | + \| \widetilde{\varphi} - \varphi \| + \| \widetilde{u} - u \| \leq \delta$$

the inequality

$$q^0(\widetilde{t}_0, \widetilde{t}_1, \widetilde{x}_0, \widetilde{x}(\widetilde{t}_1)) \leq q^0(t_0, t_1, x_0, x(t_1)) \tag{4}$$

holds, where $\widetilde{x}(t) = x(t; \widetilde{\lambda})$.

The problem (1)-(4) is said to be optimal problem with discontinuous initial condition and it consists in finding a locally optimal element.

In order to formulate the main results, we will introduce the following notation

$$\sigma_i = (\widetilde{t}_0, \underbrace{\widetilde{x}_0, \dots, \widetilde{x}_0}_{i}, \underbrace{\widetilde{\varphi}(\widetilde{t}_0), \dots, \widetilde{\varphi}(\widetilde{t}_0)}_{(p-i)}, \widetilde{\varphi}(\tau_{p+1}(\widetilde{t}_0)), \dots, \widetilde{\varphi}(\tau_s(\widetilde{t}_0))),\ i = 0, \dots, p\ ;$$

$$\sigma_i = (\gamma_i, \widetilde{x}(\tau_1(\gamma_i)), \dots, \widetilde{x}(\tau_{i-1}(\gamma_i)), \widetilde{x}_0, \widetilde{\varphi}(\tau_{i+1}(\gamma_i)), \dots, \widetilde{\varphi}(\tau_s(\gamma_i))),$$

$$\sigma_i^0 = (\gamma_i, \widetilde{x}(\tau_1(\gamma_i)), \dots, \widetilde{x}(\tau_{i-1}(\gamma_i)), \widetilde{\varphi}(\widetilde{t}_0), \widetilde{\varphi}(\tau_{i+1}(\gamma_i)), \dots, \widetilde{\varphi}(\tau_s(\gamma_i))),$$

$$i = p+1, \dots, s; \sigma_{s+1} = (\widetilde{t}_1, \widetilde{x}(\tau_1(\widetilde{t}_1)), \dots, \widetilde{x}(\tau_s(\widetilde{t}_1))), \quad \gamma_i = \gamma_i(\widetilde{t}_0),$$

$$\rho_j = \rho_j(\widetilde{t}_0), \quad \gamma_i(t) = \tau_i^{-1}(t), \quad \rho_j(t) = \eta_j^{-1}(t); \quad \omega = (t, x_1, \dots, x_s),$$

$$\widetilde{f}(\omega) = f(\omega, \widetilde{u}(\theta_1(t)), \dots, \widetilde{u}(\theta_\nu(t))),$$

$$\widetilde{f_{x_i}}[t] = f_{x_i}(t, \widetilde{x}(\tau_1(t)), \dots, \widetilde{x}(\tau_s(t)), \dots, \widetilde{u}(\theta_1(t)), \dots, \widetilde{u}(\theta_\nu(t)));$$

$$Q = (q^0, \dots, q^l)', \quad \widetilde{Q}_{t_0} = Q_{t_0}(\widetilde{t}_0, \widetilde{t}_1, \widetilde{x}_0, \widetilde{x}(\widetilde{t}_1)).$$

**Theorem 1.** *Let the element* $\widetilde{\lambda} = (\widetilde{t}_0, \widetilde{t}_1, \widetilde{x}_0, \widetilde{\varphi},\ \widetilde{u}) \in E_0$, $\widetilde{t}_0 > a$ *be locally optimal and the following conditions be fulfilled:*

1) $\gamma_i = \widetilde{t}_0$, $i = 1, \dots, p$ $\gamma_{p+1} < \dots < \gamma_s < \widetilde{t}_1$, $\rho_j < \widetilde{t}_1$ $j = 1, \dots, k$;

2) *there exists a number* $\delta > 0$ *such that*

$$\gamma_1(t) \leq \dots \leq \gamma_p(t), \quad t \in (\widetilde{t}_0 - \delta, \widetilde{t}_0];$$

3) *there exist the finite limits:* $\dot{\gamma}_i^- = \dot{\gamma}_i(\widetilde{t}_0-)$, $i = 1, \dots, s$, $\dot{\widetilde{x}}(\eta_j(\widetilde{t}_1-))$, $j = 1, \dots, k$;

$$\lim_{\omega \to \sigma_i} \widetilde{f}(\omega) = f_i^-,\ \omega \in (\widetilde{t}_0 - \delta, \widetilde{t}_0] \times O^s, \quad i = 0, \dots, p,$$

$$\lim_{(\omega_1, \omega_2) \to (\sigma_i, \sigma_i^0)} \left[ \widetilde{f}(\omega_1) - \widetilde{f}(\omega_2) \right] = f_i^-, \quad \omega_1, \omega_2 \in (\gamma_i - \delta, \gamma_i] \times O^s, \quad i = p+1, \dots, s,$$

$$\lim_{\omega \to \sigma_{s+1}} \widetilde{f}(\omega) = f_{s+1}^-, \quad \omega \in (\widetilde{t}_1 - \delta, \widetilde{t}_1] \times O^s.$$

*Then there exist a non-zero vector* $\pi = (\pi_0, \dots, \pi_l)$, $\pi_0 \leq 0$, *and a solution* $\chi(t) = (\chi_1(t), \dots, \chi_n(t))$, $\psi(t) = (\psi_1(t), \dots, \psi_n(t))$ *of the system*

$$\begin{cases} \dot{\chi}(t) = -\sum_{i=1}^s \psi(\gamma_i(t)) \widetilde{f}_{x_i}[\gamma_i(t)] \dot{\gamma}_i(t), \\ \psi(t) = \chi(t) + \sum_{j=1}^k \psi(\rho_j(t) A_j(\rho_j(t)) \dot{\rho}_j(t), \quad t \in [\widetilde{t}_0, \widetilde{t}_1], \\ \psi(t) = 0, t > \widetilde{t}_1, \end{cases} \tag{5}$$

*such that the following conditions are fulfilled:*

a) *the linearized maximum principles:*

$$\int_{\widetilde{t}_0}^{\widetilde{t}_1}\psi(t)\sum_{m=1}^{\nu}\widetilde{f}_{u_m}[t]\widetilde{u}(\theta_m(t))dt \geq \int_{\widetilde{t}_0}^{\widetilde{t}_1}\psi(t)\sum_{m=1}^{\nu}\widetilde{f}_{u_m}[t]u(\theta_m(t))dt, \quad \forall u\in\Omega,$$

$$\sum_{i=p+1}^{s}\int_{\tau_i(\widetilde{t}_0)}^{\widetilde{t}_0}\psi(\gamma_i(t))\widetilde{f}_{x_i}[\gamma_i(t)]\dot{\gamma}_i(t)\widetilde{\varphi}(t)dt+\sum_{j=1}^{k}\int_{\eta_j(\widetilde{t}_0)}^{\widetilde{t}_0}\psi(\rho_j(t))A_j[\rho_j(t)]\dot{\rho}_j(t)\dot{\widetilde{\varphi}}(t)dt \geq$$

$$\geq \sum_{i=p+1}^{s}\int_{\tau_i(\widetilde{t}_0)}^{\widetilde{t}_0}\psi(\gamma_i(t))\widetilde{f}_{x_i}[\gamma_i(t)]\dot{\gamma}_i(t)\varphi(t)dt+$$

$$+\sum_{j=1}^{k}\int_{\eta_j(\widetilde{t}_0)}^{\widetilde{t}_0}\psi(\rho_j(t))A_j[\rho_j(t)]\dot{\rho}_j(t)\dot{\varphi}(t)dt, \quad \forall\varphi\in\Phi;$$

b) *the conditions for the moments* $\widetilde{t}_0,\widetilde{t}_1$*:*

$$\pi\widetilde{Q}_{t_0} \geq -\psi(\widetilde{t}_0-)[\dot{\widetilde{\varphi}}(\widetilde{t}_0)-\sum_{j=1}^{k}A_j(\widetilde{t}_0)\dot{\widetilde{\varphi}}(\eta_j(\widetilde{t}_0))+\sum_{i=0}^{p}(\widehat{\gamma}_{i+1}^{-}-\widehat{\gamma}_i^{-})f_i^{-}]+$$

$$+\sum_{p+1}^{s}\psi(\gamma_i-)f_i^{-}\dot{\gamma}_i^{-}+\chi(\widetilde{t}_0)\dot{\widetilde{\varphi}}(\widetilde{t}_0),$$

$$\pi\widetilde{Q}_{t_1} \geq -\psi(\widetilde{t}_1)\Big[\sum_{j=1}^{k}A_j(\widetilde{t}_1)\dot{\widetilde{x}}(\eta_j(\widetilde{t}_1-))+f_{s+1}^{-}\Big];$$

c) *the condition for the solution* $\chi(t)$*,* $\psi(t)$

$$\pi\widetilde{Q}_{x_0}=-\chi(\widetilde{t}_0), \quad \pi\widetilde{Q}_{x_1}=\psi(\widetilde{t}_1)=\chi(\widetilde{t}_1).$$

*Here*

$$\widehat{\gamma}_0^{-}=1, \ \widehat{\gamma}_i^{-}=\dot{\gamma}_i^{-}, \quad i=1,\ldots,p, \quad \widehat{\gamma}_{p+1}^{-}=0;$$

**Theorem 2.** *Let the element* $\widetilde{\lambda}=(\widetilde{t}_0,\widetilde{t}_1,\widetilde{x}_0,\widetilde{\varphi},\widetilde{u})\in E_0$*,* $\widetilde{t}_1<b$ *be locally optimal and the condition* 1) *of Theorem* 1 *and the following conditions be fulfilled:*

4) *there exists a number* $\delta>0$ *such that*

$$\gamma_1(t)\leq\cdot\leq\gamma_p(t), \quad t\in[\widetilde{t_0},\widetilde{t_0}+\delta);$$

5) *there exist the finite limits :* $\dot{\gamma}_i^{+}=\dot{\gamma}_i(\widetilde{t}_0+)$*,* $i=1,\ldots,s$*,* $\dot{\widetilde{x}}(\eta_j(\widetilde{t}_1+))$*,* $j=1,\ldots,k$*;*

$$\lim_{\omega\to\sigma_i}\widetilde{f}(\omega)=f_i^{+}, \ \ \omega\in[\widetilde{t}_0,\widetilde{t}_0+\delta)\times O^s, \ \ i=0,\ldots,p,$$

$$\lim_{(\omega_1,\omega_2)\to(\sigma_i,\sigma_i^0)}\Big[\widetilde{f}(\omega_1)-\widetilde{f}(\omega_2)\Big]=f_i^{+}, \ \ \omega_1,\omega_2\in[\gamma_i,\gamma_i+\delta)\times O^s, \quad i=p+1,\ldots,s$$

$$\lim_{\omega\to\sigma_{s+1}}\widetilde{f}\Big(\omega)=f_{s+1}^{+},\omega\in[\widetilde{t}_1,\widetilde{t}_1+\delta\Big)\times O^s.$$

*Then there exists a non-zero vector* $\pi=(\pi_0,\ldots,\pi_l),\pi_0\leq 0$ *and a solution* $\chi(t)$*,* $\psi(t)$ *of the system* (5) *such that the conditions* a) *and* c) *are fulfilled. Moreover,*

$$\pi\widetilde{Q}_{t_0} \leq -\psi(\widetilde{t}_0+)[\dot{\widetilde{\varphi}}(\widetilde{t}_0)-\sum_{j=1}^{k}A_j(\widetilde{t}_0)\dot{\widetilde{\varphi}}(\eta_j(\widetilde{t}_0))+\sum_{i=0}^{p}(\widehat{\gamma}_{i+1}^{+}-\widehat{\gamma}_i^{+})f_i^{+}]+$$

$$+\sum_{i=p+1}^{s}\psi(\gamma_i+)f_i^{+}\dot{\gamma}_i^{+}+\chi(\widetilde{t}_0)\dot{\widetilde{\varphi}}(\widetilde{t}_0),$$

$$\pi\widetilde{Q}_{t_0} \leq -\psi(\widetilde{t}_1)\left[\sum_{j=1}^{k}A_j(\widetilde{t}_1)\dot{\widetilde{x}}(\eta_j(\widetilde{t}_1+))+f_{s+1}^{+}\right].$$

*Here*

$$\widehat{\gamma}_0^+ = 1,\ \ \widehat{\gamma}_i^+ = \dot{\gamma}_i^+,\ \ i = 1,\dots,p,\ \ \widehat{\gamma}_{p+1}^+ = 0.$$

**Theorem 3.** *Let the element* $\widetilde{\lambda} = (\widetilde{t}_0, \widetilde{t}_1, \widetilde{x}_0, \widetilde{\varphi}, \widetilde{u}) \in E_0$*,* $\widetilde{t}_0, \widetilde{t}_1 \in (a,b)$ *be locally optimal and the conditions of Theorems* 1, 2 *and the following conditions be fulfilled: the functions* $\dot{\widetilde{x}}(\eta_j(\widetilde{t}_1))$*,* $j = 1,\dots,k$*, are continuous;*

$$\gamma_i, \widetilde{t}_0 \notin \big\{\eta_{k_1}(\eta_{k_2}(\dots(\eta_{k_e}(\widetilde{t}_1)),\dots,)) \in (a,\widetilde{t}_1) :\ e = 1,2,\dots,\ \ m = 1,\dots,e,$$
$$k_m = 1,\dots,k\big\},\quad i = p+1,\dots,s;$$
$$\sum_{i=0}^{p} (\widehat{\gamma}_{i+1}^- - \widehat{\gamma}_i^-) f_i^- = \sum_{i=0}^{p} (\widehat{\gamma}_{i+1}^+ - \widehat{\gamma}_i^+) f_i^+ = f_0,$$
$$f_i^- \dot{\gamma}_i^- = f_i^+ \dot{\gamma}_i^+ = f_i,\quad i = p+1,\dots,s,\quad f_{s+1}^- = f_{s+1}^+ = f_{s+1}.$$

*Then there exists a non-zero vector* $\pi = (\pi_0,\dots,\pi_l)$*,* $\pi_0 \le 0$ *and a solution* $\chi(t)$*,* $\psi(t)$ *of the system* (5) *such that the condition* a) *and* c) *are fulfilled. Moreover,*

$$\pi \widetilde{Q}_{t_0} = -\psi(\widetilde{t}_0)[\dot{\widetilde{\varphi}}(\widetilde{t}_0) - \sum_{j=1}^{k} A_j(\widetilde{t}_0)\dot{\widetilde{\varphi}}(\eta_j(\widetilde{t}_0)) + f_0] + \sum_{i=p+1}^{s} \psi(\gamma_i) f_i + \chi(\widetilde{t}_0)\dot{\widetilde{\varphi}}(\widetilde{t}_0),$$

$$\pi \widetilde{Q}_{t_0} = -\psi(\widetilde{t}_1)\bigg[\sum_{j=1}^{k} A_j(\widetilde{t}_1)\dot{\widetilde{x}}(\eta_j(\widetilde{t}_1)) + f_{s+1}\bigg].$$

Finally we note that the optimal control problems for various classes of delay and neutral differential equations with discontinuous initial condition are considered in [1]–[4].

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