Mem. Differential Equations Math. Phys. 31(2004), 117–122

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ON SOME BOUNDARY VALUE PROBLEMS FOR NONLINEAR DEGENERATE HYPERBOLIC EQUATIONS OF HIGHER ORDER

(Reported on June 23, 2003)

The Dirichlet problem and other nonclassical boundary value problems for linear hyperbolic equations with two independent variables have been the subject of various studies (see, e.g., [1]–[8], [10]–[16] and the references therein). However such problems for higher order nonlinear hyperbolic equations still remain unstudied. The present paper is an attempt to fulfill the existing gap.

Let m and n be natural numbers, $0 < a, b < +\infty$, $\Omega = (0,a)\times(0,b)$ and $f : \Omega\times\mathbb{R}^{mn} \to \mathbb{R}$ be a continuous function. Consider the nonlinear hyperbolic equation

$$u^{(2m,2n)} = f\big(x, y, u, \dots, u^{(0,n-1)}, \dots, u^{(m-1,0)}, \dots, u^{(m-1,n-1)}\big) \tag{1}$$

either with the boundary conditions

$$\begin{gathered} u^{(i,0)}(0,y) = u^{(i,0)}(a,y) = 0 \quad (i = 0,\dots,m-1) \quad \text{for} \quad 0 < y < b, \\ u^{(0,k)}(x,0) = u^{(0,k)}(x,b) = 0 \quad (k = 0,\dots,n-1) \quad \text{for} \quad 0 < x < a, \end{gathered} \tag{2_1}$$

or with the boundary conditions

$$\begin{gathered} u^{(i,0)}(0,y) = u^{(m+i,0)}(a,y) = 0 \quad (i = 0,\dots,m-1) \quad \text{for} \quad 0 < y < b, \\ u^{(0,k)}(x,0) = u^{(0,n+k)}(x,b) = 0 \quad (k = 0,\dots,n-1) \quad \text{for} \quad 0 < x < a, \end{gathered} \tag{2_2}$$

where $u^{(j,k)}(x,y) = \frac{\partial^{j+k}u(x,y)}{\partial x^j \partial y^k}$. Mainly we are interested in the case, where for arbitrarily fixed $z_{ik} \in \mathbb{R}$ $(i = 0,\dots,m-1;\ k = 0,\dots,n-1)$ the function

$$f(\cdot,\cdot,z_{00},\dots,z_{0\,n-1},\dots,z_{m-1\,0},\dots,z_{m-1\,n-1}) : \Omega \to \mathbb{R}$$

is nonintegrable in the rectangle Ω having singularities on its boundary, i.e., the case where equation (1) is degenerated on the boundary of Ω.

Below we state theorems on existence and uniqueness of solutions to problems (1), (2_1) and (1), (2_2) and theorems on stability of those solutions with respect to small perturbation of the right-hand member of equation (1). Analogous results for ordinary differential equations are obtained in [9].

We make use of the following notation.

$$\mu_{in} = 2^{2n-i}\Big(\prod_{j=1}^{n}(4j-3)\Big)^{-\frac{1}{2}}\Big(\prod_{j=1}^{n-i}(4j-3)\Big)^{-\frac{1}{2}} \quad (i = 0,\dots,n-1),$$

$$\nu_{in} = 2^{2n-i}\Big(\prod_{j=1}^{n}(2j-1)\Big)^{-1}\Big(\prod_{j=1}^{n-i}(2j-1)\Big)^{-1} \quad (i = 0,\dots,n-1),$$

$$\varphi_{1jk}(x,y) = [x(a-x)]^{m-j-1/2}[y(b-y)]^{n-k-1/2} \quad (j = 0,\dots,m-1;\ k = 0,\dots,n-1),$$

$$\varphi_{2jk}(x,y) = x^{m-j-1/2}y^{n-k-1/2} \quad (j = 0,\dots,m-1;\ k = 0,\dots,n-1),$$

2000 *Mathematics Subject Classification.* 35L35, 35L80.

Key words and phrases. Nonlinear degenerate hyperbolic equation, the Dirichlet problem, nonclassical boundary value problems.

$$f^*_{i\rho}(x,y) = \max\big\{|f(x,y,z_{00},\dots,z_{0\,n-1},\dots,z_{m-1\,0},\dots,z_{m-1\,n-1})| : |z_{jk}| \le \rho\varphi_{ijk}(x,y)$$
$$(j=0,\dots,m-1;\ k=0,\dots,n-1)\big\} \quad (i=1,2).$$

$C^{k,l}(\Omega)$ is the space of functions $z:\Omega\to\mathbb{R}$, having the continuous partial derivatives $z^{(i,j)}$ $(i=0,\dots,k;\ j=0,\dots,l)$.

$H^{k,l}(\Omega)$ is the space of functions $z\in L^2(\Omega)$, having the generalized partial derivatives $z^{(i,j)}\in L^2(\Omega)$ $(i=0,\dots,k;\ j=0,\dots,l)$.

A solution of problem $(1),(2_1)$ will be sought in the class

$$M_1(\Omega) \overset{def}{=} C^{2m,2n}(\Omega)\cap H^{m,n}(\Omega),$$

while a solution of problem $(1),(2_2)$ will be sought in the class

$$M_2(\Omega) \overset{def}{=} C^{2m,2n}(\Omega)\cap H^{m,n}(\Omega)\cap C^{2m-1,2n-1}((0,a]\times(0,b]).$$

Along with equation (1) consider the perturbed equation

$$u^{(2m,2n)} = f\big(x,y,u,\dots,u^{(0,n-1)},\dots,u^{(m-1,0)},\dots,u^{(m-1,n-1)}\big) + h(x,y), \tag{3}$$

wher $h:\Omega\to\mathbb{R}$ is a continuous function.

Introduce the following definitions.

Definition 1_1. A solution u of problem $(1),(2_1)$ is called stable with respect to small perturbation of the right-hand member of equation (1) if there exists a positive constant ρ_0 such that for an arbitrary continuous function $h:\Omega\to\mathbb{R}$ satisfying the condition

$$\int_0^a\int_0^b [x(a-x)]^{2m}[y(b-y)]^{2n}h^2(x,y)\,dx\,dy < +\infty,$$

problem $(3),(2_1)$ is uniquely solvable in the class $M_1(\Omega)$ and its solution $\overline{u}$ satisfies the inequality

$$\int_0^a\int_0^b \big|\overline{u}^{(m,n)}(x,y) - u^{(m,n)}(x,y)\big|^2 dxdy \le$$
$$\le \rho_0 \int_0^a\int_0^b [x(a-x)]^{2m}[y(b-y)]^{2n}h^2(x,y)dxdy.$$

Definition 1_2. A solution u of problem $(1),(2_2)$ is called stable with respect to small perturbation of the right-hand member of equation (1), if there exists a positive constant ρ_0 such that for an arbitrary continuous function $h:\Omega\to\mathbb{R}$ satisfying the condition

$$\int_0^a\int_0^b x^{2m}y^{2n}h^2(x,y)\,dx\,dy < +\infty,$$

problem $(3),(2_2)$ is uniquely solvable in the class $M_2(\Omega)$ and its solution $\overline{u}$ satisfies the inequality

$$\int_0^a\int_0^b \big|\overline{u}^{(m,n)}(x,y) - u^{(m,n)}(x,y)\big|^2\,dx\,dy \le \rho_0\int_0^a\int_0^b x^{2m}y^{2n}h^2(x,y)\,dx\,dy.$$

Theorem 1_1. *Let*

$$\int_0^b [x(a-x)]^m[y(b-y)]^n f^*_{1\rho}(x,y)\,dx\,dy < +\infty \quad \textit{for} \quad 0<\rho<+\infty \tag{4_1}$$

and the inequality

$$(-1)^{m+n} f(x,y,z_{00},\dots,z_{0\,n-1},\dots,z_{m-1\,0},\dots,z_{m-1\,n-1})\operatorname{sgn} z_{00} \le$$
$$\le \sum_{k=0}^{n-1}\sum_{i=0}^{m-1} \frac{l_{ik}|z_{ik}|}{[x(a-x)]^{2m-i}[y(b-y)]^{2n-k}} + l(x,y)$$

hold in $\Omega\times\mathbb{R}^{mn}$, *where* l_{ij} $(i=0,\dots,m-1;\ k=0,\dots,n-1)$ *are nonnegative constants and* $l:\Omega\to[0,+\infty)$ *is a continuous function such that*

$$\sum_{i=0}^{m-1}\sum_{k=0}^{n-1} \frac{\mu_{im}\mu_{kn}l_{ik}}{a^{2m-i}b^{2n-k}} < 1, \tag{5_1}$$

$$\int_0^a\int_0^b [x(a-x)]^{2m}[y(b-y)]^{2n} l^2(x,y)\,dx\,dy < +\infty. \tag{6_1}$$

Then problem $(1),(2_1)$ *has at least one solution in the class* $M_1(\Omega)$.

Theorem 1_2. *Let*

$$\int_0^b x^m y^n f^*_{2\rho}(x,y)\,dx\,dy < +\infty \quad \text{for} \quad 0<\rho<+\infty \tag{4_2}$$

and the inequality

$$(-1)^{m+n} f(x,y,z_{00},\dots,z_{0\,n-1},\dots,z_{m-1\,0},\dots,z_{m-1\,n-1})\operatorname{sgn} z_{00} \le$$
$$\le \sum_{k=0}^{n-1}\sum_{i=0}^{m-1} \frac{l_{ik}|z_{ik}|}{x^{2m-i}y^{2n-k}} + l(x,y)$$

hold in $\Omega\times\mathbb{R}^{mn}$, *where* l_{ij} $(i=0,\dots,m-1;\ k=0,\dots,n-1)$ *are nonnegative constants and* $l:\Omega\to[0,+\infty)$ *is a continuous function such that*

$$\sum_{i=0}^{m-1}\sum_{k=0}^{n-1} \nu_{im}\nu_{kn}l_{ik} < 1, \tag{5_2}$$

$$\int_0^a\int_0^b x^{2m}y^{2n} l^2(x,y)\,dx\,dy < +\infty. \tag{6_2}$$

Then problem $(1),(2_2)$ *has at least one solution in the class* $M_2(\Omega)$.

Theorem 2_1. *Let the condition*

$$(-1)^{m+n}\big(f(x,y,z_{00},\dots,z_{m-1\,n-1}) - f(x,y,\overline{z}_{00},\dots,\overline{z}_{m-1\,n-1})\big)\operatorname{sgn}(z_{00}-\overline{z}_{00}) \le$$
$$\le \sum_{k=0}^{n-1}\sum_{i=0}^{m-1} \frac{l_{ik}|z_{ik}-\overline{z}_{ik}|}{[x(a-x)]^{2m-i}[y(b-y)]^{2n-k}}$$

hold in $\Omega\times\mathbb{R}^{mn}$, *where* l_{ij} $(i=0,\dots,m-1;\ k=0,\dots,n-1)$ *are nonnegative constants satisfying inequality* (5_1). *Then problem* $(1),(2_1)$ *has at most one solution in the class* $M_1(\Omega)$.

Theorem 2_2. *Let the condition*

$$(-1)^{m+n}\big(f(x,y,z_{00},\dots,z_{m-1\,n-1}) - f(x,y,\overline{z}_{00},\dots,\overline{z}_{m-1\,n-1})\big)\operatorname{sgn}(z_{00}-\overline{z}_{00}) \le$$
$$\le \sum_{k=0}^{n-1}\sum_{i=0}^{m-1} \frac{l_{ik}|z_{ik}-\overline{z}_{ik}|}{x^{2m-i}y^{2n-k}}$$

hold in $\Omega\times\mathbb{R}^{mn}$, *where* l_{ij} $(i=0,\dots,m-1;\ k=0,\dots,n-1)$ *are nonnegative constants satisfying inequality* (5_2). *Then problem* $(1),(2_2)$ *has at most one solution in the class* $M_2(\Omega)$.

Theorem 3. *Let along with the conditions of Theorem* 2_1 (*Theorem* 2_2) *conditions* (4_1) *and* (6_1) (*conditions* (4_2) *and* (6_2)) *hold, where* $l(x,y)=|f(x,y,0,\dots,0)|$. *Then problem* $(1),(2_1)$ (*problem* $(1),(2_2)$) *is uniquely solvable in the class* $M_1(\Omega)$ (*in the class* $M_2(\Omega)$) *and its solution is stable with respect to small perturbation of the right-hand member of equation* (1).

As an example consider the nonlinear hyperbolic equation

$$u^{(2m,2n)}=\sum_{k=1}^{k_0}p_k(x,y)|u|^{\lambda_k}\operatorname{sgn}u+q(x,y), \tag{7}$$

where λ_k $(k=1,\dots,k_0)$ are positive constants, and $p_k:\Omega\to\mathbb{R}$ $(k=1,\dots,k_0)$ and $q:\Omega\to\mathbb{R}$ are continuous functions with singularities on the boundary of the rectangle Ω. Moreover, let

$$(-1)^{m+n}p_k(x,y)\le 0\quad (k=1,\dots,k_0)\quad\text{for}\quad (x,y)\in\Omega \tag{8}$$

and either the inequalities

$$\int_0^a\int_0^b[x(a-x)]^{m+\left(m-\frac12\right)\lambda_k}[y(b-y)]^{n+\left(n-\frac12\right)\lambda_k}|p_k(x,y)|\,dx\,dy<+\infty \tag{9_1}$$

$$(k=1,\dots,k_0),$$

$$\int_0^a\int_0^b[x(a-x)]^{2m}[y(b-y)]^{2n}q^2(x,y)\,dx\,dy<+\infty, \tag{10_1}$$

or the inequalities

$$\int_0^a\int_0^b x^{m+\left(m-\frac12\right)\lambda_k}y^{n+\left(n-\frac12\right)\lambda_k}|p_k(x,y)|\,dx\,dy<+\infty\quad (k=1,\dots,k_0), \tag{9_2}$$

$$\int_0^a\int_0^b x^{2m}y^{2n}q^2(x,y)\,dx\,dy<+\infty \tag{10_2}$$

hold.

Corollary 1. *Let along with* (8) *conditions* (9_1) *and* (10_1) (*conditions* (9_2) *and* (10_2)) *hold. Then problem* $(7),(2_1)$ (*problem* $(7),(2_2)$) *is uniquely solvable in the class* $M_1(\Omega)$ (*in the class* $M_2(\Omega)$) *and its solution is stable with respect to small perturbation of right-hand member of equation* (7).

Finally let us state a corollary of Theorem 3 concerning the linear hyperbolic equation

$$u^{(2m,2n)}=\sum_{i=0}^{m-1}\sum_{k=0}^{n-1}p_{ik}(x,y)u^{(i,k)}+q(x,y) \tag{11}$$

with the continuous coefficients $p_{ik}:\Omega\to\mathbb{R}$ $(i=0,\dots,m-1;\ k=0,\dots,n-1)$ and $q:\Omega\to\mathbb{R}$ having singularities on the boundary of the rectangle Ω.

We are interested in the case, where the functions p_{ik} $(i=0,\dots,m-1;\ k=0,\dots,n-1)$ satisfy either the conditions

$$\int\limits_0^a\int\limits_0^b [x(a-x)]^{2m-\frac{1}{2}}[y(b-y)]^{2n-\frac{1}{2}}|p_{00}(x,y)|\,dx\,dy<+\infty, \tag{12_1}$$

$$(-1)^{m+n}[x(a-x)]^{2m}[y(b-y)]^{2n}p_{00}(x,y)\le l_{00}\quad\text{for}\quad (x,y)\in\Omega, \tag{13_1}$$

$$[x(a-x)]^{2m-i}[y(b-y)]^{2n-k}|p_{ik}(x,y)|\le l_{ik} \tag{14_1}$$

$$(i=0,\dots,m-1;\ k=0,\dots,n-1;\ i+k\neq 0)\quad\text{for}\quad (x,y)\in\Omega,$$

or the conditions

$$\int\limits_0^a\int\limits_0^b x^{2m-\frac{1}{2}}y^{2n-\frac{1}{2}}|p_{00}(x,y)|\,dx\,dy<+\infty, \tag{12_2}$$

$$(-1)^{m+n}x^{2m}y^{2n}p_{00}(x,y)\le l_{00}\quad\text{for}\quad (x,y)\in\Omega, \tag{13_2}$$

$$x^{2m-i}y^{2n-k}|p_{ik}(x,y)|\le l_{ik}$$

$$(i=0,\dots,m-1;\ k=0,\dots,n-1;\ i+k\neq 0)\quad\text{for}\quad (x,y)\in\Omega, \tag{14_2}$$

where l_{ik} $(i=0,\dots,m-1;\ k=0,\dots,n-1)$ are nonnegative constants.

Corollary 2. *If along with* $(12_1),(13_1),(14_1)$ (*along with* $(12_2),(13_2),(14_2)$) *conditions* (5_1) *and* (10_1) (*conditions* (5_2) *and* (10_2)) *hold, then problem* $(11),(2_1)$ (*problem* $(11),(2_2)$) *is uniquely solvable in the class* $M_1(\Omega)$ (*in the class* $M_2(\Omega)$) *and its solution is stable with respect to small perturbation of the right-hand member of equation* (11).

Acknowledgment

This work was supported by INTAS (Grant No. 00136).

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