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V. Kokilashvili and S. Samko

Sobolev Theorem for Potentials on Carleson Curves in Variable Lebesgue Spaces

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Let $\Gamma = \{t \in \mathbb{C} : t = t(s),\ 0 \le s \le \ell\}$ be a simple rectifiable curve with arc-length measure ν. Let p be a measurable function on Γ such that $p : \Gamma \to (1, \infty)$.

Assume that p satisfies the conditions

$$1 < p_- := \operatorname*{ess\,inf}_{t\in\Gamma} p(t) \le \operatorname*{ess\,sup}_{t\in\Gamma} p(t) =: p_+ < \infty, \tag{1}$$

$$|p(t) - p(\tau)| \le \frac{A}{\ln \frac{1}{|t-\tau|}}, \qquad t \in \Gamma, \quad \tau \in \Gamma, \qquad |t-\tau| \le \frac{1}{2}. \tag{2}$$

The generalized Lebesgue space with variable exponent is defined via the modular

$$\rho_p(f) := \int_\Gamma |f(t)|^{p(t)}\, d\nu$$

by the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

By $L^{p(\cdot)}_w$ we denote the weighted Banach space of all measurable functions $f : \Gamma \to \mathbb{C}$ such that

$$\|f\|_{p(\cdot),w} := \|wf\|_{p(\cdot)} < \infty.$$

By definition, Γ is a Carleson curve (or a regular curve) if there exists a constant $c > 0$ not depending on t and r such that

$$\nu(\Gamma \cap B(t,r)) \le cr$$

for all the balls $B(t,r)$, $t \in \Gamma$.

We consider – along Carleson curves – the potential type operator

$$I^{\alpha(\cdot)} f(t) = \int_\Gamma \frac{f(\tau)\, d\nu(\tau)}{|t-\tau|^{1-\alpha(\tau)}}. \tag{3}$$

When the order α is a constant, the following result is known [1].

Theorem A. *Let $0 < \alpha < 1$, $1 < p < \frac{1}{\alpha}$, and let $\frac{1}{q} = \frac{1}{p} - \alpha$. Then the operator I^α is bounded from L^p to L^q if and only if Γ is a Carleson curve.*

On the other hand, in the Euclidean space R^n an analogue of the well-known Hardy–Littlewood–Stein–Weiss theorem in $L^{p(\cdot)}$ spaces looks as

Theorem B ([2]). *Let Ω be a bounded domain in R^n and $x_0 \in \overline{\Omega}$, let p satisfy the conditions* (1) *and* (2), *where instead of t we mean $x \in \Omega$.*

Assume that

$$\inf \alpha(x) > 0 \quad \textit{and} \quad \sup_{x\in\Omega} \alpha(x)p(x) < n,$$

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and

$$|\alpha(x)-\alpha(y)| \le \frac{A}{\ln\frac{1}{|x-y|}} \quad \textit{for all} \quad x,y\in\overline{\Omega} \quad \textit{with} \quad |x-y|<\frac{1}{2},$$

A does not depend on x and y.

Then the operator

$$I^{\alpha(\cdot)}f(x)=\int\limits_{R^n}\frac{f(y)}{|x-y|^{n-\alpha}}\,dy, \quad 0<\alpha<n,$$

acts boundedly from $L^p_{|x-x_0|^\gamma}$ onto $L^q_{|x-x_0|^\mu}$ if

$$\frac{1}{q(x)}=\frac{1}{p(x)}-\frac{\alpha(x)}{n},$$

$$\alpha(x_0)p(x_0)-n<\gamma<n[p(x_0)-1]$$

and

$$\mu=\frac{q(x_0)}{p(x_0)}\,\gamma.$$

The following theorems are valid.

Theorem 1. *Let*

i) *Γ be a simple Carleson curve of finite length;*

ii) *p satisfy the conditions* (1)–(2)*;*

iii) *w be a power weight $w(t)=|t-t_0|^{\beta(t)}$, where $t_0\in\Gamma$ and $\beta(t)$ is a real valued function on Γ satisfying the condition* (2)*;*

iv) *the order $\alpha(t)$ satisfy the condition* (2) *and the assumptions*

$$0<\inf_{t\in\Gamma}\alpha(t)\le\sup_{t\in\Gamma}\alpha(t)<1 \qquad \textit{and} \qquad \sup_{t\in\Gamma}\alpha(t)p(t)<1. \tag{4}$$

Then the operator $I^{\alpha(\cdot)}$ is bounded from the space $L^{p(\cdot)}_w(\Gamma)$ into the space $L^{q(\cdot)}_w(\Gamma)$ with $\frac{1}{q(t)}=\frac{1}{p(t)}-\alpha(t)$ if

$$-\frac{1}{q(t_0)}<\beta(t_0)<\frac{1}{p'(t_0)}.$$

Theorem 2. *Let Γ be a simple Carleson curve. Let p satisfy the conditions* (1)–(2) *and let there exist a ball $B(0,R)$ such that $p(t)=\text{const}$ for $t\in\Gamma\backslash(\Gamma\cap B(0,R))$. Then for a constant α the operator I^α is bounded from the space $L^{p(\cdot)}(\Gamma)$ into the space $L^{q(\cdot)}(\Gamma)$, where $\frac{1}{q(t)}=\frac{1}{p(t)}-\alpha$.*

References

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Authors' addresses:

V. Kokilashvili
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, Aleksidze St., Tbilisi 0193
Georgia
E-mail: kokil@rmi.acnet.ge

S. Samko
University of Algarve
Portugal
E-mail: ssamko@ualg.pt