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CAPILLARITY PROBLEMS FOR COMPRESSIBLE FLUIDS

Abstract. The rise height problem for compressible fluid in a capillary tube is considered. A partial differential equation for the rise height is suggested which takes into account the bulk energy changes within the fluid. Some applications and comparison with previous results are also presented.

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რეზიუმე. ნაშრომში განხილულია კაპილარულ მილში კუმშვადი სითხის აწევის სიმაღლის ამოცანა. აწევის სიმაღლის ფუნქციისათვის შემოთავაზებულია კერძოწარმოებულიანი დიფერენციალური განტოლება, რომელიც ითვალისწინებს სითხის შიგნით ენერგიის ცვლილებებს. მოყვანილია მიღებული შედეგების ზოგიერთი გამოყენება და ადრინდელ შედეგებთან შედარება.

In memory of Professor G. Manjavidze,
on the occasion of his 80th Anniversary

One of the major outstanding problems studied by natural philosophers of the eighteenth century was the determination of the rise height at the center of a circular capillary tube. Specifically, one dips a circular cylindrical tube of homogeneous solid material into a bath of liquid, in the presence of a downward gravity field, as indicated in Figure 1. It had been observed that for specific solid and liquid combinations (such as glass and water), the liquid would rise in the tube relative to the rest level of the liquid bath, to heights that increase with decreasing tube radius. It was desired to obtain a specific formula for the rise height at the center of the tube, in terms of the physical properties of the materials and the tube radius.

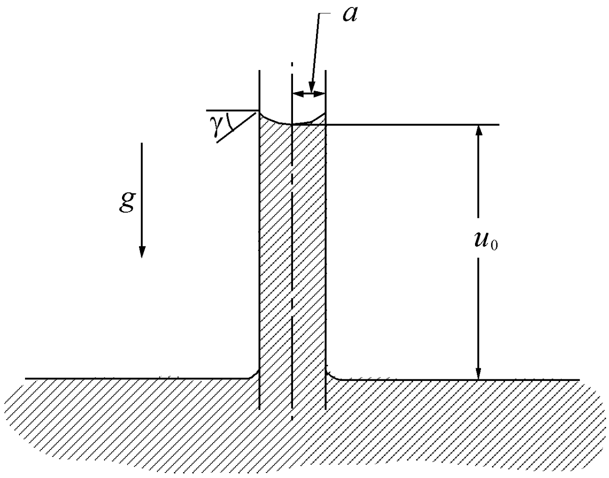

Figure 1. Capillary tube dipped into infinite bath; no volume constraint

Although some of the most celebrated scientists of the time wrote on the problem, it was not solved during that century. The first approximation in reasonable agreement with experiment was given by Young in 1805. In 1806, Laplace produced the equation

$$\operatorname{div} Tu = \frac{\rho g}{\sigma}\, u + \lambda, \quad Tu \equiv \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \tag{1}$$

for the rise height $u(x, y)$ in a general configuration, and he improved Young's estimate for a circular tube. Here r is density change across the free surface, σ the surface tension, g the gravitational acceleration, and λ a Lagrange parameter depending on an eventual volume constraint. On the boundary Σ of the domain Ω of definition, the condition

$$\nu \cdot Tu = \cos\gamma \tag{2}$$

is imposed, expressing the requirement that the fluid is to meet vertical cylindrical walls over Σ in a prescribed angle γ, depending only on the particular materials and in no other way on the conditions of the problem or on the shape of the solution surface, see Figures 1 and 3. In (2), ν is the unit exterior normal on Σ.

The reasonings both of Young and of Laplace were based on force balance considerations, which were imprecisely defined and not fully convincing. The bulk properties of the fluid were not addressed explicitly, although some hypotheses, such as incompressibility, were implicit in the discussions.

In 1830 Gauss re-derived the relations (1) and (2) in a more appealing way, based on the Principle of Virtual Work of Johann Bernoulli. This entailed evaluating energy variations throughout the bulk medium, which he did in great detail, in an impressive work essentially anticipating modern measure theory, potential theory, and thermodynamics. He was led to elaborate expressions that could not be readily applied, but he had an easy way out. He simply wrote that the bulk internal energy of the fluid is proportional to its volume, and in a practical problem would be unvaried due to the volume constraint and could thus be neglected in the variational procedure.

In fact, in the central problem of his time (see Figure 1) there is no volume constraint. Nevertheless, Gauss did achieve the identical equation (1) and boundary condition (2) by a basically new method, which must certainly have been convincing as to the correctness of the result.

Again in the Gauss reasoning, incompressibility is assumed, this time as an explicit hypothesis. But still further hypotheses may be implicit in the procedure. The following example ("exotic container") is taken from the papers [1,2,3]:

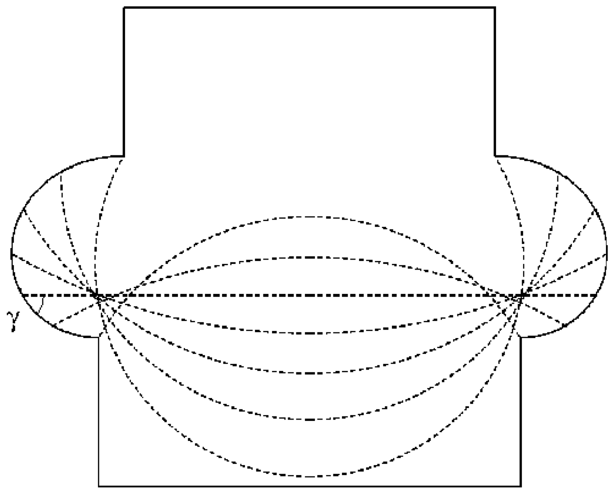

Figure 2: exotic container, continuum of solution surfaces

Here the solid lines indicate the meridional curve of a rotationally symmetric solid container. The dashed curves show seven profiles of a continuum of symmetric solutions of (1), corresponding to an incompressible fluid, that are equivalent from the Gauss point of view. All surfaces make the same contact angle with the bounding walls, bound the same volume of fluid below the surface, and have the same total mechanical energy as calculated in the Gauss procedure. Which if any of these surfaces would be observed physically, and on the basis of what criteria?

We obtain some insight by recalling a discovery of Young, that the pressure change δp across each such surface satisfies $\delta p = 2\sigma H$, where H is the mean curvature of the surface. If we assume fluid below the surfaces and vacuum above them, we observe a great variation in fluid pressures, from large positive to large negative. One would expect such pressure changes to be associated with bulk energy changes within the fluid (perhaps arising from small changes in density). None of the procedures, of Young, of Laplace, nor of Gauss, take any account of such energy changes.

2. An initial step toward constructing a theory accounting for bulk energy variations was taken in [4]. A pressure-density relation of the form $\rho = \rho_0 + \chi(p - p_0)$ was assumed, with p_0, ρ_0 the pressure and density at a fixed fluid level (both assumed unvaried), and χ a physical constant. A downward gravity field is supposed, with the fluid below the surface. One finds that the density satisfies a relation

$$\rho = \rho_0 e^{-\chi g h} \tag{3}$$

in terms of the height h above the reference level; as a consequence, the density cannot vanish, a result lending indirect support to the physical validity of the procedure. The expression (3) leads to an explicit energy term that can be joined with those arising in the classical theory. From the vanishing of the first variation of energy we obtain the equation

$$\operatorname{div} Tu = \frac{\rho_0 g}{\sigma}\, u - \chi g(1 + \cos\omega) + \lambda \tag{4}$$

generalizing (1), under the same boundary condition (2). Here ω is the angle between the upward surface normal and the upward vertical direction. The Lagrange parameter λ vanishes in the case of a tube dipped into an infinite bath as indicated in Figure 1; however the formulation (4) accounts also for configurations with a mass constraint (replacing the volume constraint of the classical theory), as occurs for example in a tube closed at the bottom and partially filled with liquid, see Figure 3.

In contrast to what occurs for an incompressible fluid, the mass cannot be prescribed arbitrarily; one finds the restriction:

A necessary condition for existence of a solution surface is that the mass M satisfy

$$M < \rho_0 |\Omega| / \chi g. \tag{5}$$

Presumably, if (5) fails then the free surface rises indefinitely in the tube and disappears to infinity.

In [4] it is shown that *in the particular case of a tube of circular section, the condition* (5) *is sufficient for the existence of a solution to* (4, 2). *The* *solution is uniquely determined, and the effect of changing the mass is simply to move the interface rigidly up or down.* For values of M close to the upper bound in (5) small changes in M will yield large changes in height.

In a work [5] joint with M. Athanassenas currently in preparation, the unconstrained problem as indicated in Figure 1 is studied. It is shown that *in the absence of mass constraint, the restriction to circular section can be replaced by the weaker requirement that Ω be smooth, of class $C^{2+\alpha}$.* This latter result leans heavily on methods introduced by Ural'tseva [7,8] in the case of the classical equation (1); the basic lemma is an a-priori bound on $|\nabla u|$ at the boundary Σ, depending only on the boundary curvature, the angle γ, and an a-priori bound for $|u|$.

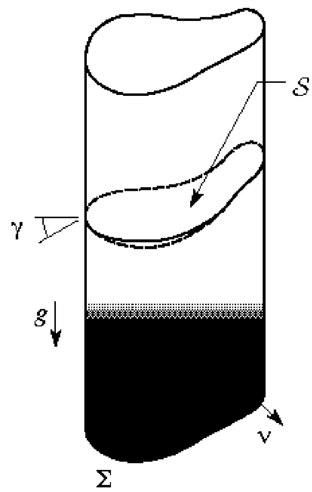

Figure 3. Capillary tube closed at bottom, partly filled with compressible fluid.

3. Solutions of (4) admit a remarkable a-priori bound, depending only on distance to the domain boundary: We may normalize to the case $\lambda = 0$, by adding a constant to u.

If $\lambda = 0$, then any solution of (4) *defined over a disk B_δ of radius δ satisfies*

$$-\frac{2\sigma}{\rho_0 g\delta} - \delta < u < \frac{\chi\sigma}{\rho_0} + \frac{2\sigma}{\rho_0 g\delta} + \delta \tag{6}$$

throughout B_δ. It follows that solutions of (4) *cannot be unbounded at an isolated singular point.*

We consider the behavior of capillary surfaces in domains with corner points. This question has a special interest, in view of the idiosyncratic behavior of the solutions at such points (see, e.g., [6] for a general outline and for references). The existence theorem in its present form does not extend to such domains. Nor does the uniqueness proof extend to that case. We are, however, able to state a "near uniqueness" property in that generality. *If* u^1, u^2 *are two solutions of* (4) *in a piecewise smooth domain, with* $\cos\gamma^1 \leq \cos\gamma^1$ *on* $\Sigma = \partial\Omega$, *then* $u^1 < u^2 + \sigma\chi/\rho_0$ *throughout* Ω. *This result holds also in configurations for which* u^1 *may be unbounded near a boundary point.*

The behavior at a protruding corner point $P \in \partial\Omega$ *depends in a discontinuous way on the opening angle. We consider a corner of opening* 2α, $0 < 2\alpha < \pi$. *If* $\alpha \geq |(\pi/2) - \gamma|$. *then* (6) *holds in the domain* Ω_δ *of Figure* 4. *If* $\alpha < |(\pi/2) - \gamma|$, *then* u *grows near* P *as* $O(1/r)$, *where* r *is distance to* P.

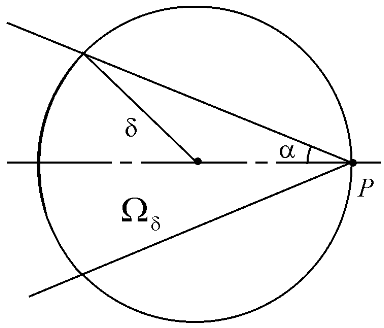

Figure 4. Protruding corner point P, subdomain Wd

4. The discussion in Sections 2 and 3 took account of the change of density with height, but omitted the energy associated with expansion of fluid due to pressure change when raising fluid elements from the reference height to the actual height. It is possible to account for such energy terms, and we are led to the equation

$$\operatorname{div} Tu = \frac{\rho_0 - \chi p_0}{\sigma\chi}\,(e^{\chi g u} - 1) + \chi g(1 - \cos\omega) + \Lambda \tag{7}$$

which exhibits significantly new features, notably in the bound from below of the term involving the solution explicitly. We see immediately that *if physically reasonable behavior for the solutions is to be expected, one must assume* $\rho_0 - \chi p_0 > 0$. The necessary condition (5) for the constrained problem remains unchanged; however sufficiency conditions differ somewhat from those that apply to (4). Notably, *for the unconstrained problem*$\Lambda = 0$ *existence may fail when* $\gamma > \pi/2$, even in the simplest case of a circular disk domain. To see that, we integrate (7) over a disk of radius r and apply the divergence theorem to obtain

$$-2\cos\gamma < \left(\frac{\rho_0 - \chi p_0}{\sigma\chi} + \chi g\right) r \tag{8}$$

which yields a contradiction if $\gamma > \pi/2$ and r is small enough. Presumably, if a sufficiently narrow tube is dipped into an infinite bath of liquid having the assumed properties, the liquid will descend in the tube, in an ideal sense to negative infinity, in practical terms to the bottom of the tube.

We do find, in partial analogy with the behavior of solutions of (4), that *if* $\Omega \in C^{2+\alpha}$ *and if* $\gamma \le \pi/2$, *then if* $\rho - \chi p_0 > 0$, *there is a unique solution of* (7) *corresponding to any prescribed* Λ, *and satisfying the boundary condition* (2).

Upper bounds on solutions of (7) hold that are analogous to those of Section 3, however lower bounds require further hypotheses. This circumstance relates closely to the nonexistence example indicated above.

5. In contrast to the situation with (4), the existence proof for (7) as it presently stands does not guaranty a solution with prescribed fluid mass. From a physical point of view, the result provides only the existence of a solution for a tube of general section dipped into an infinite bath; existence with prescribed mass in a tube closed at the bottom has not yet been proved. The reason for this difficulty is that the Lagrange parameter Λ can no longer be determined explicitly from the (prescribed) fluid mass. For the special case of rotational symmetry, the difficulty was overcome in a joint work of the author with Kevin Luli, now in preparation [9]. There it is shown by a continuity reasoning that *for prescribed mass* M *subject to* (5) *and contact angle* γ, *with* $0 \le \gamma < \pi$, *if* $\rho_0 - \chi p_0 > 0$, *then there exists at least one solution* $u(r;\gamma)$ *of the symmetric form*

$$(r \sin\psi)_r = \left(\frac{\rho_0 - \chi p_0}{\sigma\chi} \left(e^{\chi g u} - 1\right) + \chi g (1 - \cos\psi) + \Lambda \right)^r \qquad (9)$$

of (7). *The solution is unique among symmetric solutions if* $0 \le \gamma \le \pi/2$. Here, ψ is the inclination angle of the profile curve $u(r;\gamma)$ of the solution surface, relative to increasing r and the outward directed horizontal direction.

Conditions are established in [9] ensuring that the solution surface will lie above the reference level $u = 0$, so that it will have physical meaning for which the fluid covers the base domain.

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