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# ON A PROPERTY OF HARMONIC FUNCTIONS FROM THE SMIRNOV CLASS

**Abstract.** It is proved that for harmonic functions from the Smirnov class $e(L_{1p}(\rho_1), L'_{2q}(\rho_2))$ (i.e., for functions satisfying the inequality (2)) in a simply connected domain with the Lyapunov boundary $L$ almost everywhere on $L$ there exist the angular boundary values which on the part $L_2$ of the boundary form an absolutely continuous function.

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**Key words and phrases**: Harmonic functions, Smirnov classes of harmonic functions, Zaremba's problem, absolute continuity.

**რეზიუმე.** ნაშრომში მტკიცდება, რომ ჰარმონიული ფუნქციებისათვის სმირნოვის $e(L_p(\rho_1), L'_{2q}(\rho_2))$ კლასიდან (ანუ ფუნქციებისათვის, რომლებიც (2) უტოლობას აკმაყოფილებენ) ლიაპუნოვის $L$ საზღვრის მქონე ცალადბმულ არეში თითქმის ყველგან $L$-ზე არსებობს კუთხური სასაზღვრო მნიშვნელობა, რომელიც საზღვრის $L_2$ ნაწილზე წარმოადგენს აბსოლუტურად უწყვეტ ფუნქციას.

The boundary value problems for harmonic functions are, as usual, considered in different functional classes and the character of their solvability depends considerably on the choice of a class of unknown functions.

When considering Zaremba's mixed boundary value problem, the boundary of the domain is divided into two parts $L_1$ and $L_2$ and it is required to find a harmonic function from a class $A$ such that on the portion $L_1$ the boundary function of that function and on the portion $L_2$ the boundary function of its normal derivative take preassigned values. In the capacity of the class $A$, one of the possible sets is the set of harmonic functions such that the integral $p$-means are bounded "near" $L_1$ and the integral $q$-means of their partial derivatives are bounded "near" $L_2$. Since in the role of $L_1$ and $L_2$ there appear finite unions of arcs, it is natural to consider weighted integral means with singularities at the ends of those arcs.

Proceeding from the above reasoning, in the works [1, 2] the authors, in connection with the study of Zaremba's problem, have introduced the classes $e(L_{1p}(\rho_1), L'_{2q}(\rho_2))$.

As far as the boundedness of integral means is taken as the basis in determining Smirnov classes of analytic functions, the above-introduced class is naturally called the Smirnov class of harmonic functions.

In [1, 2], the solution of the mixed boundary value problem, besides its belonging to the class $e(L_{1p}(\rho_1), L'_{2q}(\rho_2))$, is required to be absolutely continuous on $L_2$. However, it turns out that any function from the above-indicated class possesses the latter property. In the present paper we prove this fact. In Section $1^0$ we present the definition of the class $e(L_{1p}(\rho_1), L'_{2q}(\rho_2))$ and cite some properties of functions from that class established in [2] which will be needed in the sequel. In Section $2^0$ we prove absolute continuity on $L_2$ of the boundary function of the function from $e(L_{1p}(\rho_1), L'_{2q}(\rho_2))$.

$1^0$. Let $D$ be a simply connected domain bounded by a simple rectifiable curve $L$ and let $\mathcal{L}_k = (A_k, B_k)$, $k = \overline{1,m}$ be arcs lying separately on $L$. By $C_1, C_2, \dots, C_m$ we denote the points $A_k$, $B_k$ taken arbitrarily. Assume $L_1 = \bigcup_{k=1}^{m} \mathcal{L}_k$, $L_2 = L \backslash L_1$. $D_1, D_2, \dots, D_n$ denote the points on $L$ different from $C_k$; note that the points $D_1, \dots, D_{n_1}$ are located on $L_1$ while $D_{n_1+1} \dots D_n$ on $L_2$. Assume

$$\rho_1(z) = \prod_{k=1}^{n_1} |z - D_k|^{\alpha_k}, \quad \rho_2(z) = \prod_{k=1}^{2m} |z - C_k|^{\alpha_k} \prod_{k=n_1+1}^{n} |z - D_k|^{\beta_k}. \tag{1}$$

Let $z = z(w)$ be the conformal mapping of the unit circle $U = \{w : |w| < 1\}$ onto the domain $D$, and let $w = w(z)$ be the inverse mapping. Suppose $\Gamma_1 = w(L_1)$, $\Gamma_2 = w(L_2)$, $\Gamma_j(r) = \{w : w = re^{i\theta}, e^{i\theta} \in \Gamma_j\}$, $L_j(r) = z(\Gamma_j(r))$.

We say that a harmonic in the domain $D$ function $u(z)$, $z = x + iy = re^{i\theta}$ belongs to the class $e(L_{1p}(\rho_1), L'_{2q}(\rho_2))$ if

$$\sup_r \left[ \int_{L_1(r)} |u(z)\rho_1(z)|^p |dz| + \int_{L_2(r)} \left( \left|\frac{\partial u}{\partial x}(z)\right|^q + \left|\frac{\partial u}{\partial y}(z)\right|^q \right) \rho_2^q(z) |dz| \right] < \infty. \tag{2}$$

In the case where $D$ coincides with the unit circle, this class will be denoted by $h(\Gamma_{1p}(\rho_1),\Gamma'_{2q}(\rho_2))$. For $\Gamma_1=\gamma=\{t:|t|=1\}$ and $\rho_1\equiv 1$, we obtain the well-known class $h_p$ ([3], p. 373).

**Statement 1** (see [2]). *If $p>1$, $q>1$ and for the weights $\rho_1$ and $\rho_2$ we have $-\frac{1}{p}<\alpha_k<\frac{1}{p'}$, $-\frac{1}{q}<\gamma_k<\frac{1}{q'}$, $-\frac{1}{q}<\beta_k<\frac{1}{q'}$ ($p'=\frac{p}{p-1}$, $q'=\frac{q}{q-1}$) and $u\in h(\Gamma_{1p}(\rho_1),\Gamma'_{2q}(\rho_2))$, then:*

(i) *there exists $\sigma>1$ such that $u\in h_\sigma$;*

(ii) *if $v$ is the function harmonically conjugate to $u$, then $v\in h(\Gamma_{1p_1}(\rho_1),\Gamma'_{2q}(\rho_2))$, where $p_1=\frac{p\sigma}{p+\sigma}$;*

(iii) *if, however, $u\in e(L_{1p}(\rho_1),L'_{2q}(\rho_2))$, then the function $U(w)=u(z(w))$ belongs to the class $h(\Gamma_{1p}(\omega_1),\Gamma'_{2q}(\omega_2))$, where $\omega_1(w)=\rho_1(z(w))\times\times\sqrt[p]{|z'(w)|}$, $\omega_2(w)=\rho_2(z(w))\sqrt[q]{|z'(w)|}$.*

Due to this fact, if $u\in h(\Gamma_{1p}(\rho_1),\Gamma'_{2q}(\rho_2))$, then:

(a) almost everywhere on $\gamma$ there exist angular boundary values $u^+(t)$, and $u(re^{i\theta})$ can be represented by the Poisson integral of the function $u^+$;

(b) if $\phi(z)=u(z)+iv(z)$, then $\phi\in H^\sigma$ and

$$\sup_{|z|=r}\int\limits_{\Theta(\Gamma_2)}|\phi'(z)|^q\omega_2^q(z)|dz|<\infty,\Theta(\Gamma_2)=\{\theta:0\le\theta\le 2\pi,\ e^{i\theta}\in\Gamma_2\}\quad(3)$$

(for the definition of Hardy classes $H^6$ see [3], p. 388).

$2^0$. **Theorem.** *Let $p>1$, $q>1$, the weight functions $\rho_1$, $\rho_2$ be given by the equalities (1), where $\alpha_k\in(-\frac{1}{p},\frac{1}{p'})$, $\gamma_k,\beta_k\in(-\frac{1}{q},\frac{1}{q'})$, and let $u\in h(\Gamma_{1p}(\rho_1),\Gamma'_{2q}(\rho_2))$. Then the function $u$ can be continuously extended to every closed arc lying on $\Gamma_2$. Moreover, the boundary function $u^+(t)$ is such that there exist the limits*

$$u(A_k-)=\lim_{t\to A_k-}u^+(t),\quad u(B_{k-1}+)=\lim_{t\to B_{k-1}+}u^+(t),\quad k=\overline{2,m},$$

*and the obtained in such a way function is absolutely continuous on $\Gamma_2$. Moreover, $\frac{\partial u^+}{\partial\theta}\in L^q(\Gamma_2;\rho_2)$.*

*Proof.* It suffices to consider the case where $m=1$, i.e., we assume that $\gamma_{ab}$ is the arc of the circumference $\gamma$ with the ends $a$ and $b$, and

$$\sup_r\Big[\int\limits_{\Theta(\gamma\setminus\gamma_{ab})}(u(re^{i\theta})\rho_1(re^{i\theta}))^p d\theta+ \\ +\int\limits_{\Theta(\gamma_{ab})}\Big(\Big|\frac{\partial u}{\partial x}(re^{i\theta})\Big|^q+\Big|\frac{\partial u}{\partial y}(re^{i\theta})\Big|^q\Big)\rho_2(re^{i\theta})d\theta\Big]<\infty,\quad(4)$$

where $\Theta(E)=\{\theta:e^{i\theta}\in E,\ 0\le\theta\le 2\pi\}$.

Let the function $v$ be harmonically conjugate to the function $u$, and $\phi=u+iv$. According to Statement 1, $\phi\in H^\sigma\subset H^1$, and therefore $\phi(z)$ possesses angular boundary values almost everywhere on $\gamma$. Thus in arbitrarily small neighbourhoods of the points $a$ and $b$ there are the points

$\widetilde{a} = e^{i\widetilde{\alpha}}$, $\widetilde{b} = e^{i\widetilde{b}}$, $\widetilde{a}$, $\widetilde{b} \in \gamma_{ab}$ at which there exist angular boundary values $\phi^+(\widetilde{a})$, $\phi^+(\widetilde{b})$. Moreover,

$$\phi(z) = \frac{1}{2\pi} \int_0^{2\pi} u^+(\theta) \, \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta. \tag{5}$$

Consider now the domain $G \subset U$ which is bounded by the radii passing through the points $\widetilde{a}$, $\widetilde{b}$ and by the arc of the circumference $\gamma_{\widetilde{a}\widetilde{b}} \subset \gamma_{ab}$. Let us show that $\phi' \subset E^1(G)$ (for the definition of the classes $E^p(G)$, $p > 0$, see [3], p. 422). Towards this end, it is sufficient to construct a sequence of rectifiable curves $\gamma_n \subset G$ converging to the boundary for which

$$\sup_n \int_{\gamma_n} |\phi'(z)| \, |dz| < \infty \tag{6}$$

(see, e.g., [3], p. 422–423).

Let $\{\widetilde{a}_n\}$ and $\{\widetilde{b}_n\}$ be sequences of points on $\gamma_{ab}$, converging respectively to the points $\widetilde{a}$ and $\widetilde{b}$. Consider the curves $\gamma_{1n} = \{z : z = \rho e^{i\widetilde{\alpha}_n}, \frac{1}{n} < \rho < r_n = 1 - \frac{1}{n}\}$, $\gamma_{2n} = \{z : z = \rho e^{i\widetilde{\beta}_n}, \frac{1}{n} < \rho < r_n\}$, $\gamma_{3n} = \{z : z = \frac{1}{n} e^{i\alpha}, \widetilde{\alpha}_n < \alpha < \widetilde{\beta}_n\}$, $\gamma_{4n} = \{z : z = r_n e^{i\alpha}, \widetilde{\alpha_n} < \alpha < \widetilde{\beta}_n\}$, where we put $\widetilde{\alpha}_n = \arg \widetilde{a}_n$, $\widetilde{\beta}_n = \arg b_n$ and let $\gamma_n = \bigcup_{j=1}^{4} \gamma_{jn}$, $n > 2$. It is obvious that $\gamma_n$ converges to the boundary $G$. Let us prove that the inequality (6) is valid for $\gamma_n$.

Let $r < r_n < \rho$; choose a point $e^{i\alpha}$ between $a$ and $\widetilde{a}$ and a point $e^{i\beta}$ between $\widetilde{b}$ and $b$ with the condition that there exist $\phi^+(e^{i\alpha})$ and $\phi^+(e^{i\beta})$. We write $(-2\pi i \phi')$ in the form

$$\begin{aligned} -2\pi i \phi'(re^{i\varphi}) &= \int_\alpha^\beta \frac{\rho \phi(\rho e^{i\theta}) de^{i\theta}}{(\rho e^{i\theta} - re^{i\varphi})^2} + \int_{2\pi \setminus [\alpha,\beta]} \frac{\rho \phi(\rho e^{i\theta}) de^{i\theta}}{(\rho e^{i\theta} - re^{i\varphi})^2} = \\ &= \phi_1(re^{i\varphi}) + \phi_2(re^{i\varphi}). \end{aligned} \tag{7}$$

Since the distance from $\gamma_n$ to the arc $\gamma \setminus (e^{i\alpha}, e^{i\beta})$ is positive, we get

$$\sup_n \int_{\gamma_n} |\phi_2(z)| \, |dz| \le M_1 \sup_\rho \int_{\Theta(\gamma \setminus \gamma_{ab})} |\phi(\rho e^{i\theta})| d\theta < \infty. \tag{$7_1$}$$

Estimate now the integrals of $\phi_1$.

$$\int_{\gamma_n} |\phi_1(z)| \, |dx| \le \sum_{j=1}^{4} \int_{\gamma_{jn}} |\phi_1(z)| \, |dz| = \sum_{j=1}^{4} I_{jn}. \tag{$7_2$}$$

We have

$$I_{1n}=\int\limits_{1/n}^{r_n}\Big|\int\limits_{\alpha}^{\beta}\phi(\rho e^{i\theta})\frac{d}{d\theta}\frac{1}{\rho e^{i\theta}-re^{i\alpha_n}}\Big|dr=$$

$$=\int\limits_{1/n}^{r_n}\Big|\frac{\phi(\rho e^{i\beta})}{\rho e^{i\beta}-re^{i\alpha_n}}-\frac{\phi(\rho e^{i\alpha})}{\rho e^{i\alpha}-re^{i\alpha_n}}-\int\limits_{\alpha}^{\beta}\frac{\phi'(\rho e^{i\theta})}{\rho e^{i\theta}-re^{i\alpha_n}}de^{i\theta}\Big|\,dr. \qquad (8)$$

Since $\phi^+(e^{i\alpha})$ and $\phi^+(e^{i\beta})$ exist and the distance from the points $\rho e^{i\beta}$, $\rho e^{i\alpha}$ to $\gamma_{1n}$ is positive, it follows from (8) that

$$I_{1n}\leq M+\int\limits_{1/n}^{r_n}\Big|\frac{\phi'(\rho e^{i\theta})de^{i\theta}}{\rho e^{i\theta}-re^{i\alpha_n}}\Big|\,dr\leq$$

$$\leq M+\int\limits_{1/n}^{r_n}\Big|\int\limits_{\alpha}^{\beta}\frac{|\phi'(\rho e^{i\theta})|d\theta}{\sqrt{(\rho-r)^2+4\rho r\sin^2\frac{\theta-\alpha_n}{2}}}\Big|\,dr=M+J_n. \qquad (9)$$

Next, taking into account that $n\geq 3$ and $\sin x>\frac{2}{\pi}x$ for $|x|<\frac{\pi}{2}$, we have

$$J_n\leq\int\limits_{\alpha}^{\beta}|\phi'(\rho e^{i\theta})|\int\limits_{1/n}^{r_n}\frac{dr}{\sqrt{(\rho-r)^2+4\rho r\sin^2\frac{\theta-\alpha_n}{2}}}\leq$$

$$\leq\frac{\pi}{2\sqrt{\rho}}\int\limits_{\alpha}^{\beta}|\phi'(\rho e^{i\theta})|\int\limits_{1/n}^{r_n}\frac{dr}{\sqrt{r}\sqrt{(\frac{\rho-r}{\theta-\alpha_n})^2+1}}\,d\theta\leq$$

$$\leq M_1\int\limits_{\alpha}^{\beta}|\phi'(\rho e^{i\theta})|\int\limits_{1/n}^{r}\frac{dr}{\sqrt{(\frac{\rho-r}{\theta-\alpha_n})^2+1}}\,d\theta. \qquad (10)$$

Assuming $(\rho-r)|\theta-\alpha_n|^{-1}=x$, we obtain

$$\int\limits_{0}^{r}\frac{dr}{\sqrt{(\frac{\rho-r}{\theta-\alpha_n})^2+1}}=\frac{1}{|\theta-\alpha_n|}\int\limits_{\frac{\rho-r_n}{|\theta-\alpha_n|}}^{\frac{\rho}{|\theta-\alpha_n|}}\frac{|\theta-\alpha_n|dx}{\sqrt{x^2+1}}\leq$$

$$\leq\int\limits_{0}^{\frac{\rho}{|\theta-\alpha_n|}}\frac{dx}{\sqrt{x^2+1}}+\int\limits_{1}^{\frac{\rho}{|\theta-\alpha_n|}}\frac{dx}{\sqrt{x^2+1}}\leq 1+\ln\Big|\frac{1}{\theta-\alpha_n}\Big|.$$

The inequality (9) implies that

$$J_n\leq M_2\Big(\int\limits_{\alpha}^{\beta}|\phi'(\rho e^{i\theta})\rho_2(\rho e^{i\theta})|^q d\theta\Big)^{1/q}\Big(\int\limits_{\alpha}^{\beta}\frac{d\theta}{|\rho_2(\rho e^{i\theta})\ln|\theta-\alpha_n|\,|^{q'}}\Big)^{1/q'}.$$

Taking into account that $\gamma_k < \frac{1}{q'}$, the last inequality, (3) and (9) allow us to conclude that $\sup\limits_n I_{1n} < \infty$. Just in the same way we can establish that $\sup\limits_n I_{2n} < \infty$. The estimate for $I_{3n}$ is obvious.

Further,

$$I_{4n} = \int\limits_{\widetilde{\alpha}_n}^{\widetilde{\beta}_n} |\phi'(r_n e^{i\theta})| d\theta \le \int\limits_{\Theta(\gamma_{ab})} |\phi'(r_n e^{i\theta})| d\theta,$$

and from (3) it follows that $\sup\limits_n I_{4n} < \infty$.

Thus $\sup\limits_n I_{jn} < \infty$, $j = \overline{1,4}$, and therefore (7), $(7_1)$ and $(7_2)$ show that the inequality (6) is valid. In particular, we conclude that angular boundary values $\phi'(t)$ exist almost everywhere on $\gamma_{\widetilde{a}\widetilde{b}}$ for any $\widetilde{a}$, $\widetilde{b} \in \gamma_{ab}$ at which $\phi^+(\widetilde{a})$, $\phi^+(\widetilde{b})$ exist. Since such $\widetilde{a}$ and $\widetilde{b}$ lie arbitrarily close to $a$ and $b$, $\lim\limits_{r\to 1} \phi'(re^{i\theta})$ exists almost everywhere on $\Theta(\gamma_{ab})$. By Fatou's lemma, the expressions (3) yield

$$\int\limits_{\Theta(\gamma_{ab})} |\phi'(e^{i\theta})\rho_2(e^{i\theta})|^q d\theta < \infty. \tag{11}$$

In view of the inequalities $-\frac{1}{q} < \gamma_k < \frac{1}{q'}$, $-\frac{1}{q} < \beta_k < \frac{1}{q'}$, it is not difficult to establish the existence of $\varepsilon$, $\varepsilon > 0$, such that

$$\int\limits_{\Theta(\gamma_{ab})} |\phi'(e^{i\theta})|^{1+\varepsilon} d\theta \le M < \infty. \tag{12}$$

Since $\phi' \in E^1(G)$, the function $\phi(z)$ is continuous on $G$ and $\phi(t) = \phi^+(t)$ is absolutely continuous on the boundary of $G$ (see, e.g., [4], p. 208). Thus $\phi(t)$ is absolutely continuous on the arcs $\gamma_{\widetilde{a}\widetilde{b}}$ and, consequently, is such on every closed arc lying on $\gamma_{ab}$. Moreover,

$$\phi(e^{i\theta}) = \int\limits_{\widetilde{\alpha}}^{\theta} \phi'_\theta(e^{i\theta}) d\theta - \phi(e^{i\widetilde{\alpha}}), \quad \widetilde{\alpha} \le \theta \le \widetilde{\beta}. \tag{13}$$

From (12) and (13) it follows that the limits

$$\lim_{\theta\to(\arg a)+} \phi(e^{i\theta}) = \phi(a+), \qquad \lim_{\theta\to(\arg b)-} \phi(e^{i\theta}) = \phi(b-)$$

exist. Therefore the representation (13) is valid for any $\theta$, $e^{i\theta} \in \gamma_{ab}$ if we replace $\phi(e^{i\widetilde{\alpha}})$ by $\phi(a+)$. Hence $\phi(t)$ is absolutely continuous on $\overline{\gamma}_{ab}$. Moreover, the inequality (11) holds. Since $u(z) = \operatorname{Re}\phi(z)$, this implies that all the assertions of the theorem about the function $u(z)$ are true.

Incidentally, we have proved the following

**Statement 2.** *If $\phi \in H^1$ and for some $\varepsilon > 0$*

$$\sup_r \int_\alpha^\beta |\operatorname{Re} \phi'(re^{i\theta})|^{1+\varepsilon} d\theta < \infty, \quad 0 \le \alpha < \beta \le 2\pi,$$

*then $\phi(z)$ is continuously extendable to every closed arc lying on the arc $\gamma_{ab}$ with $a = e^{i\alpha}$, $b = e^{i\beta}$, there exist the limits*

$$\lim_{t \to a+} \phi^+(t) = \phi^+(a+), \quad \lim_{t \to b-} \phi^+(t) = \phi^+(b-)$$

*and the function $\phi^+(t)$ is absolutely continuous on $\overline{\gamma}_{ab}$.*

Let $z = t(s)$ be the equation of the curve $L$ with respect to the arc coordinate. Taking into account the property of the absolute continuity of the function $w(t(s))$ with respect to $s$ on $[0, l]$ and of the function $z(e^{i\theta})$ with respect to $\theta$ on $[0, 2\pi]$, due to the fact that in the case of Lyapunov curves we have $0 < m \le |z'(w)| \le M$ (see, e.g., [3], pp. 405, 407, 411), one can, using the above-proven theorem, establish that *the statement of the above theorem is valid for any functions of the class $e(L_{1p}(\rho_1), L'_{2q}(\rho_2))$ if $L$ is Lyapunov curve.*

## References


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