Mem. Differential Equations Math. Phys. 38 (2006), 146–149

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ON BLOW–UP SOLUTIONS OF INITIAL CHARACTERISTIC PROBLEM FOR NONLINEAR HYPERBOLIC SYSTEMS WITH TWO INDEPENDENT VARIABLES

(Reported on November 28, 2006)

Let $n \geq 2$ be a positive integer, $\mathbb{R}^n$ be an n-dimensional Euclidean space, $0 < a < +\infty$, $0 < b < +\infty$,

$$\Omega = [0,a] \times [0,b],$$

$f : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ and $c_2 : [0,b] \to \mathbb{R}^n$ be continuous, and $c_1 : [0,a] \to \mathbb{R}^n$ be a continuously differentiable vector function. In the rectangle Ω consider the nonlinear hyperbolic equation

$$u_{xy} = f\big(x,y,u\big) \tag{1}$$

with the initial conditions

$$u(x,0) = c_1(x) \quad \text{for} \quad 0 \leq x \leq a, \quad u_y(0,y) = c_2(y) \quad \text{for} \quad 0 \leq y \leq b. \tag{2}$$

Global solvability of the problem (1),(2) was studied rather thoroughly (see, e.g., [1–9] and the literature quoted therein). In the present paper new sufficient conditions of existence and nonexistence of so called blow-up solutions to the problem (1),(2) are given.

To formulate the main results, we need to introduce the following notation and definitions.

$z = (z_i)_{i=1}^n \in \mathbb{R}^n$ is a vector with components $z_1, \dots, z_n$, and $\|z\|$ is its Euclidean norm.

$v \cdot w$ is the scalar product of the vectors v and $w \in \mathbb{R}^n$.

$\Omega_0(a_1,b_1) = \{(x,y) : 0 \leq x < a_1,\ 0 \leq y \leq b\} \cup \{(x,y) : 0 \leq x \leq a,\ 0 \leq y < b_1\}$.

$\overline{\Omega}_0(a_1,b_1)$ is the closure of the set $\Omega_0(a_1,b_1)$, i.e.,

$$\overline{\Omega}_0(a_1,b_1) = \big([0,a_1] \times [0,b]\big) \cup \big([0,a] \times [0,b_1]\big).^*$$

Definition 1. A vector function $u : \Omega_0(a_1,b_1) \to \mathbb{R}^n$ $\big(u : \overline{\omega}_0(a_1,b_1) \to \mathbb{R}^n\big)$ is called *a solution of the system* (1) defined on $\Omega_0(a_1,b_1) \to \mathbb{R}^n$ (defined on $\overline{\Omega}_0(a_1,b_1) \to \mathbb{R}^n$), if it has continuous partial derivatives u_x, u_y, u_{xy} and satisfies the system (1) at every point of the mentioned set. A solution of the system (1) satisfying the initial conditions (2) will be called *a solution of the problem* (1),(2).

Definition 2. A solution u of the problem (1),(2) is called *continuable*, if it is defined on $\Omega_0(a_1,b_1)$ and either of the following three conditions hold:
(*i*) $a_1 = a$, $b_1 \leq b$ and the problem (1),(2) has a solution $\overline{u}$ defined on Ω such that

$$\overline{u}(x,y) = u(x,y) \quad \text{for} \quad (x,y) \in \Omega_0(a_1,b_1); \tag{3}$$

(*ii*) $a_1 < a$, $b_1 = b$ and the problem (1),(2) has a solution $\overline{u}$ defined on Ω and satisfying the equality (3);
(*iii*) $a_1 < a$, $b_1 < b$ and there exist numbers $a_0 \in [a_1,a]$, $b_0 \in [b_1,b]$ such that $a_0 + b_0 >$

2000 *Mathematics Subject Classification.* 35L15, 35B30, 35B60.

Key words and phrases. Characteristic problem, nonlinear hyperbolic systems, blow-up solution.

*It is clear that $\overline{\Omega}_0(a,b_1) = \Omega$ for $0 < b_1 \leq b$, and $\overline{\Omega}_0(a_1,b) = \Omega$ for $0 < a_1 \leq a$.

$a_1 + b_1$ and the problem (1),(2) has a solution $\overline{u}$ defined on $\Omega_0(a_0, b_0)$ and satisfying the equality (3).

Definition 3. A solution u of the problem (1),(2) is called *non-continuable*, if either it is defined on Ω, or it is defined on $\Omega_0(a_1, b_1)$, where $0 < a_1 \le a$, $0 < b_1 \le b$ and all of the three conditions (i), (ii) and (iii) of Definition 2 are violated.

Definition 4. A solution u of the problem (1),(2) defined on $\Omega_0(a_1, b_1)$ is called a *blow-up solution*, if

$$\max\{\|u(x,y)\| : 0 \le y \le b\} \to +\infty \quad \text{for} \quad x \to a_1-$$

and

$$\max\{\|u(x,y)\| : 0 \le x \le a\} \to +\infty \quad \text{for} \quad y \to b_1- .$$

Let $a_0 > 0$, $b_0 > 0$, $g : [0, a_0] \times [0, b_0] \to \mathbb{R}_+$ be a Lebesgue integrable function, and $h : [0, +\infty) \to (0, +\infty)$ be a continuous nondecreasing function.

Lemma 1. *Let there exist a nonnegative number r_0 such that*

$$\lim_{t \to +\infty} h_0(t) > \int_0^{a_0}\int_0^{b_0} g(x,y)\, dx\, dy,$$

where

$$h_0(t) = \int_{r_0}^{t} \frac{ds}{h(s)}.$$

Then an arbitrary continuous function $v : [0, a_0) \times [0, b_0) \to \mathbb{R}_+$ satisfying the integral inequality

$$v(x,y) \le r_0 + \int_0^x \int_0^y g(s,t) h(v(s,t))\, ds\, dt \quad \textit{for} \quad (x,y) \in [0,a_0) \times [0,b_0)$$

admits the estimate

$$v(x,y) \le h_0^{-1}\Big(\int_0^x \int_0^y g(s,t)\, ds\, dt\Big) \quad \textit{for} \quad (x,y) \in [0,a_0) \times [0,b_0),$$

where h_0^{-1} is the function inverse to h_0.

Along with the system (1) consider the hyperbolic system depending on a parameter $\lambda \in [0,1]$

$$u_{xy} = \lambda f\big(x, y, u\big). \tag{4}$$

Theorem 1. *Let there exist numbers $a_1 \in (0, a]$, $b_1 \in (0, b]$ and $r > 0$ such that for any $\lambda \in [0,1]$ an arbitrary solution u of the problem* (4), (2) *defined on $\Omega_0(a_1, b_1)$ admits the estimate*

$$\|u(x,y)\| \le r \quad \textit{for} \quad (x,y) \in \Omega_0(a_1, b_1).$$

Then the problem (1), (2) *has at least one solution defined on $\overline{\Omega}_0(a_1, b_1)$.*

Set

$$c(x) = c_1(x) + \int_0^y c_2(t)\, dt.$$

According to Lemma 1, Theorem 1 yields

Corollary 1. *Let there exist numbers* $a_1 \in (0, a]$, $b_1 \in (0, b]$, $r_1 \le 0$, $r_2 \ge 0$, *an integrable function* $g : \Omega_0(a_1, b_1) \to \mathbb{R}_+$ *and a continuous nondecreasing function* $h : \mathbb{R}_+ \to (0, +\infty)$ *such that*

$$\|f(x,y,z)\| \le g(x,y)h(\|z\|) \quad \text{for} \quad (x,y) \in \Omega_0(a_1,b_1),\ z \in \mathbb{R}^n;$$

$$\|c(x,y)\| \le r_1 \quad \text{for} \quad (x,y) \in [0,a_1] \times [0,b]; \quad \|c(x,y)\| \le r_2 \quad \text{for} \quad (x,y) \in [0,a] \times [0,b_1]$$

and

$$\int\limits_{r_1}^{+\infty} \frac{ds}{h(s)} > \int\limits_0^{a_1} \int\limits_0^{b} g(x,y)\,dx\,dy, \qquad \int\limits_{r_2}^{+\infty} \frac{ds}{h(s)} > \int\limits_0^{a} \int\limits_0^{b_1} g(x,y)\,dx\,dy.$$

Then the problem (1), (2) *has at least one solution defined on* $\overline{\Omega}_0(a_1, b_1)$, *and has no blow-up solutions defined on* $\Omega_0(a_1, b_1)$.

On the basis of Corollary 1 one can prove

Theorem 2. *The problem* (1), (2) *has at least one non-continuable solution. Besides, an arbitrary non-continuable solution of this problem is either defined on* Ω, *or is a blow-up solution.*

Theorem 2′. *If* $f(x,y,z)$ *is locally Lipschitz continuous in* z, *then the problem* (1), (2) *has a unique non-continuable solution which is either defined on* Ω *or is a blow-up solution.*

Theorem 3. *Let there exist a positive number* r_0, *a nonzero vector* l *and a nondecreasing continuous function* $\varphi : [0, +\infty) \to (0, +\infty)$ *such that*

$$l \cdot f(x,y,z) \ge \varphi(|l \cdot z|) \quad \text{for} \quad (x,y) \in \Omega,\ z \in \mathbb{R}^n, |l \cdot z| \ge r_0$$

and

$$\int\limits_t^{+\infty} \frac{ds}{\Phi(s)} < +\infty \quad \text{for} \quad t > r_0,$$

where

$$\Phi(t) = \Big(\int\limits_{r_0}^{t} \varphi(s)\,ds \Big)^{\frac{1}{2}} \quad \text{for} \quad t \ge r_0.$$

Then there exists a number $r \ge r_0$ *such that every non-continuable solution of the problem* (1), (2) *is a blow-up solution provided that*

$$l \cdot c(x,y) > r \quad \text{for} \quad (x,y) \in \Omega.$$

As an example consider the problem

$$u_{ixy} = \sum_{k=1}^{n} p_{ik}(x,y)|u_k|^{\mu_{ik}(x,y)} + q_i(x,y) \quad (i = 1, \dots, n), \tag{5}$$

$$u_i(x,0) = c_{1i}(x) \ \text{ for } \ 0 \le x \le a, \ \ u_{iy}(0,y) = c_{2i}(y) \ \text{ for } \ 0 \le y \le b \ \ (i = 1, \dots, n), \tag{6}$$

where $\mu_{ik} : \Omega \to \mathbb{R}$, $p_{ik} : \Omega \to \mathbb{R}$, $q_i : \Omega \to \mathbb{R}$, $c_{2i} : [0,b] \to \mathbb{R}$ $(i, k = 1, \dots, n)$ are continuous, and $c_{1i} : [0,a] \to \mathbb{R}$ $(i = 1, \dots, n)$ are continuously differentiable functions.

Theorems 2′ and 3 imply

Corollary 2. *Let the inequalities*

$$\mu_{11}(x,y) > 1, \quad \mu_{ik}(x,y) \ge 1 \ \ (i, k = 1, \dots, n),$$
$$p_{11}(x,y) > 0, \quad p_{1k}(x,y) \ge 0 \ \ (k = 2, \dots, n)$$

hold on the rectangle Ω*. Then there exists a positive number* r *such that the problem* (5), (6) *has a unique non-continuable solution which is a blow-up solution provided that*

$$c_{11}(x) + \int_0^y c_{21}(t)\, dt \geq r \quad for \quad (x,y) \in \Omega.$$

Acknowledgement

This work was supported by INTAS (grant No. 03-51-5–7).

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