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A BOUNDARY VARIATIONAL INEQUALITY APPROACH TO UNILATERAL CONTACT WITH HEMITROPIC MATERIALS

Abstract. We study unilateral frictionless contact problems with hemitropic materials in the theory of linear elasticity. We model these problems as unilateral (Signorini type) boundary value problems, give their variational formulation as spatial variational inequalities, and transform them to boundary variational inequalities with the help of the potential method for hemitropic materials. Using the self-adjointness of the Steklov–Poincaré operator, we obtain the equivalence of the boundary variational inequality formulation and the corresponding minimization problem. Based on our variational inequality approach we derive existence and uniqueness theorems. Our investigation includes the special particular case of only traction-contact boundary conditions without prescribing the displacement and microrotation vectors along some part of the boundary of the hemitropic elastic body.

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რეზიუმე. ნაშრომში შესწავლილია წრფივი დრეკადობის თეორიის ცალმხრივი კონტაქტის ამოცანა ჰემიტროპული სხეულისათვის. ეს ამოცანა, როგორც ცალმხრივ შეზღუდვებიანი სასაზღვრო ამოცანა, ფორმულირებულია სპეციალური სივრცითი ვარიაციული უტოლობის სახით, რომელიც პოტენციალთა მეთოდის საშუალებით დაიყვანება სასაზღვრო ვარიაციულ უტოლობაზე. სტეკლოვ–პუანკარეს ოპერატორის გამოყენებით ჩვენ ვღებულობთ ექვივალენტურ სასაზღვრო ვარიაციულ უტოლობას და შესაბამის მინიმიზაციის ამოცანას. მტკიცდება ამოცანის ამონახსნის არსებობისა და ერთადერთობის თეორემები. გამოკვლეულია ის სპეციალური შემთხვევაც, როდესაც ჰემიტროპული დრეკადი სხეული საზღვრის გასწვრივ არ არის ჩამაგრებული.

1. Introduction

In recent years continuum mechanical theories in which the deformation is described not only by the usual displacement vector field, but by other scalar, vector or tensor fields as well, have been the object of intensive research. Classical elasticity associates only the three translational degrees of freedom to material points of the body and all the mechanical characteristics are expressed by the corresponding displacement vector. On the contrary, micropolar theory, by including intrinsic rotations of the particles, provides a rather complex model of an elastic body that can support body forces and body couple vectors as well as force stress vectors and couple stress vectors at the surface. Consequently, in this case all the mechanical quantities are written in terms of the displacement and microrotation vectors.

The origin of the rational theories of polar continua goes back to brothers E. and F. Cosserat [6] who gave a development of the mechanics of continuous media in which each material point has the six degrees of freedom of a rigid body (for the history of the problem see [28], [33], [22], [9], and the references therein).

A micropolar solid which is not isotropic with respect to inversion is called *hemitropic, noncentrosymmetric*, or *chiral*. Materials may exhibit chirality on the atomic scale, as in quartz and in biological molecules, as well as on a large scale, as in composites with helical or screw-shaped inclusions.

Mathematical models describing the hemitropic properties of elastic materials have been proposed by Aero and Kuvshinski [1], [2]. We note that the governing equations in this model become very involved and generate 6×6 matrix partial differential operator of second order. Evidently, the corresponding 6×6 matrix boundary differential operators describing the force stress and couple stress vectors have also an involved structure in comparison with the classical case.

In [29], [30], [31], the fundamental matrices of the associated systems of partial differential equations of statics and steady state oscillations have been constructed explicitly in terms of elementary functions and the basic boundary value problems of hemitropic elasticity have been studied by the potential method for smooth and non-smooth Lipschitz domains.

Particular problems of the elasticity theory of hemitropic continuum have been considered in [10], [23], [24], [25], [26], [33], [34], [35], [39] (see also [3], [4] and the references therein for electromagnetic scattering by a homogeneous chiral obstacle).

The main goal of the present paper is the study of unilateral frictionless contact problems for hemitropic elastic material, their mathematical modelling as *unilateral* boundary value problems of Signorini type and their analysis with the help of the spatial and boundary variational inequality technique.

In classical elasticity similar problems have been considered in many monographs and papers (see, e.g., [8], [11], [12], [13], [15], [16], [17], [18], [19], [20], [21], [36], and the references therein).

Due to the complexity of the physical model of a hemitropic continuum we need a special mathematical interpretation of the unilateral mechanical constraints related to the microrotation vector and the couple stress vector. We note that the displacement vector and the force stress vector are subject to the usual classical unilateral conditions. However, we have to pay special attention to the fact that the force stress and couple stress vectors depend on the displacement and microrotation vectors.

Here in this paper we can present a reasonable mathematical model for the unilateral constraints that apply to hemitropic material in contact. We transform the unilateral boundary value problem to the corresponding spatial variational inequality (SVI) equivalently. Furthermore by boundary integral techniques, we can reduce the SVI to an equivalent boundary variational inequality (BVI). Applying the potential method we establish some coercivity properties of the boundary bilinear forms involved in the BVI and thus prove uniqueness and existence for the original unilateral problems.

2. Basic Equations and Green Formulae

2.1. **Field equations.** Let $\Omega^+ \subset \mathbb{R}^3$ be a bounded domain with a smooth connected boundary $S := \partial\Omega^+$, $\overline{\Omega^+} = \Omega^+ \cup S$; $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$. We assume that $\overline{\Omega} \in \{\overline{\Omega^+}, \overline{\Omega^-}\}$ is occupied by an elastic material possessing the hemitropic properties.

Denote by $u = (u_1, u_2, u_3)^\top$ and $\omega = (\omega_1, \omega_2, \omega_3)^\top$ the displacement vector and the microrotation vector, respectively; here and in what follows the symbol $(\cdot)^\top$ denotes transposition. Note that the microrotation vector in the hemitropic elasticity theory is kinematically distinct from the macrorotation vector $\frac{1}{2}$ curl u.

In the linear theory of hemitropic elasticity we have the following constitutive equations for the force stress tensor $\{\tau_{pq}\}$ and the couple stress tensor $\{\mu_{pq}\}$

$$\begin{aligned}\tau_{pq} = \tau_{pq}(U) :&= (\mu+\alpha)\,\frac{\partial u_q}{\partial x_p} + (\mu-\alpha)\,\frac{\partial u_p}{\partial x_q} + \lambda\delta_{pq}\operatorname{div} u + \\ &+\delta\,\delta_{pq}\operatorname{div}\omega + (\kappa+\nu)\,\frac{\partial \omega_q}{\partial x_p} + (\kappa-\nu)\,\frac{\partial \omega_p}{\partial x_q} - \\ &-2\alpha\sum_{k=1}^{3}\varepsilon_{pqk}\,\omega_k, \end{aligned} \tag{2.1}$$

$$\mu_{pq} = \mu_{pq}(U) := \delta\,\delta_{pq}\operatorname{div} u + (\kappa+\nu)\Big[\frac{\partial u_q}{\partial x_p} - \sum_{k=1}^{3}\varepsilon_{pqk}\omega_k\Big] +$$

$$+\beta\delta_{pq}\,\mathrm{div}\,\omega+(\kappa-\nu)\Big[\frac{\partial u_p}{\partial x_q}-\sum_{k=1}^{3}\varepsilon_{qpk}\omega_k\Big]+$$
$$+(\gamma+\varepsilon)\,\frac{\partial\omega_q}{\partial x_p}+(\gamma-\varepsilon)\,\frac{\partial\omega_p}{\partial x_q}, \tag{2.2}$$

where $U=(u,\omega)^\top$, δ_{pq} is the Kronecker delta, ε_{pqk} is the permutation (Levi–Civitá) symbol, and α, β, γ, δ, λ, μ, ν, κ, and ε are the material constants (see [1]).

The components of the force stress vector $\tau^{(n)}=(\tau_1^{(n)},\tau_2^{(n)},\tau_3^{(n)})^\top$ and the coupled stress vector $\mu^{(n)}=(\mu_1^{(n)},\mu_2^{(n)},\mu_3^{(n)})^\top$, acting on a surface element with the normal vector $n=(n_1,n_2,n_3)$, read as

$$\tau_q^{(n)}=\sum_{p=1}^{3}\tau_{pq}n_p,\quad \mu_q^{(n)}=\sum_{p=1}^{3}\mu_{pq}n_p,\quad q=1,2,3. \tag{2.3}$$

Let us introduce the generalized stress operator (6×6 matrix differential operator)

$$T(\partial,n)=\begin{bmatrix}T^{(1)}(\partial,n) & T^{(2)}(\partial,n)\\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n)\end{bmatrix}_{6\times 6},\quad T^{(j)}=\big[T_{pq}^{(j)}\big]_{3\times 3},\quad j=\overline{1,4}, \tag{2.4}$$

where $\partial=(\partial_1,\partial_2,\partial_3)$, $\partial_j=\partial/\partial x_j$,

$$\begin{aligned}
T_{pq}^{(1)}(\partial,n)&=(\mu+\alpha)\delta_{pq}\,\frac{\partial}{\partial n}+(\mu-\alpha)n_q\,\frac{\partial}{\partial x_p}+\lambda n_p\,\frac{\partial}{\partial x_q}\,,\\
T_{pq}^{(2)}(\partial,n)&=(\kappa+\nu)\delta_{pq}\,\frac{\partial}{\partial n}+(\kappa-\nu)n_q\frac{\partial}{\partial x_p}+\delta n_p\,\frac{\partial}{\partial x_q}-\\
&\quad-2\alpha\sum_{k=1}^{3}\varepsilon_{pqk}n_k,\\
T_{pq}^{(3)}(\partial,n)&=(\kappa+\nu)\delta_{pq}\,\frac{\partial}{\partial n}+(\kappa-\nu)n_q\,\frac{\partial}{\partial x_p}+\delta n_p\,\frac{\partial}{\partial x_q},\\
T_{pq}^{(4)}(\partial,n)&=(\gamma+\varepsilon)\delta_{pq}\,\frac{\partial}{\partial n}+(\gamma-\varepsilon)n_q\,\frac{\partial}{\partial x_p}+\beta n_p\,\frac{\partial}{\partial x_q}-\\
&\quad-2\,\nu\sum_{k=1}^{3}\varepsilon_{pqk}\,n_k\,.
\end{aligned} \tag{2.5}$$

In view of (2.1), (2.2), and (2.3) it can easily be checked that

$$(\tau^{(n)},\mu^{(n)})^\top=T(\partial,n)U.$$

The static equilibrium equations for hemitropic elastic bodies are written as

$$\sum_{p=1}^{3}\partial_p\tau_{pq}(x)+\varrho F_q(x)=0,$$

$$\sum_{p=1}^{3} \partial_p \, \mu_{pq}(x) + \sum_{l,r=1}^{3} \varepsilon_{qlr} \tau_{lr}(x) + \varrho \, G_q(x) = 0, \quad q = 1,2,3,$$

where $F = (F_1, F_2, F_3)^\top$ and $G = (G_1, G_2, G_3)^\top$ are the given body force and body couple vectors per unit mass. Using the relations (2.1)–(2.2) we can rewrite the above equations in terms of the displacement and microrotation vectors:

$$\begin{aligned} (\mu+\alpha)\Delta u(x) + (\lambda+\mu-\alpha)\,\mathrm{grad\,div}\,u(x) + (\kappa+\nu)\Delta\omega(x) + \\ +(\delta+\kappa-\nu)\,\mathrm{grad\,div}\,\omega(x) + 2\alpha\,\mathrm{curl}\,\omega(x) + \varrho F(x) = 0, \\ (\kappa+\nu)\Delta u(x) + (\delta+\kappa-\nu)\,\mathrm{grad\,div}\,u(x) + 2\alpha\,\mathrm{curl}\,u(x) + \\ +(\gamma+\varepsilon)\Delta\omega(x) + (\beta+\gamma-\varepsilon)\,\mathrm{grad\,div}\,\omega(x) + 4\nu\,\mathrm{curl}\,\omega(x) - \\ -4\alpha\omega(x) + \varrho G(x) = 0, \end{aligned} \tag{2.6}$$

where Δ is the Laplace operator.

Let us introduce the matrix differential operator corresponding to the system (2.6):

$$L(\partial) := \begin{bmatrix} L^{(1)}(\partial), & L^{(2)}(\partial) \\ L^{(3)}(\partial), & L^{(4)}(\partial) \end{bmatrix}_{6\times 6}, \tag{2.7}$$

where

$$\begin{aligned} L^{(1)}(\partial) &:= (\mu+\alpha)\Delta I_3 + (\lambda+\mu-\alpha)Q(\partial), \\ L^{(2)}(\partial) = L^{(3)}(\partial) &:= (\kappa+\nu)\Delta I_3 + (\delta+\kappa-\nu)Q(\partial) + 2\alpha R(\partial), \\ L^{(4)}(\partial) &:= \big[(\gamma+\varepsilon)\Delta - 4\alpha\big] I_3 + (\beta+\gamma-\varepsilon)Q(\partial) + 4\nu R(\partial). \end{aligned} \tag{2.8}$$

Here I_k stands for the $k \times k$ unit matrix and

$$R(\partial) := \begin{bmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix}_{3\times 3}, \quad Q(\partial) := \big[\partial_k \, \partial_j\big]_{3\times 3}. \tag{2.9}$$

Note that obviously

$$R(\partial)u = \begin{bmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{bmatrix} = \mathrm{curl}\,u, \quad Q(\partial)u = \mathrm{grad\,div}\,u\,. \tag{2.10}$$

Due to the above notation, the equations (2.6) can be rewritten in matrix form as

$$\begin{gathered} L(\partial)U(x) = \Phi(x), \\ U = (u,\omega)^\top, \quad \Phi = (\Phi^{(1)}, \Phi^{(2)})^\top := (-\varrho F, -\varrho G)^\top. \end{gathered} \tag{2.11}$$

Let us remark that the operator $L(\partial)$ is formally self-adjoint, i.e., $L(\partial) = [L(-\partial)]^\top$.

2.2. **Green's formulae.** For real–valued vectors $U := (u,\omega)^\top$, $U' := (u',\omega')^\top \in [C^2(\overline{\Omega^+})]^6$ there holds the following Green formula [29]

$$\int\limits_{\Omega^+} \big[L(\partial)U\cdot U' + E(U,U')\big]\,dx = \int\limits_{\partial\Omega^+} \big[T(\partial,n)U\big]^+\cdot[U']^+\,dS, \tag{2.12}$$

where n is the outward unit normal vector to $S=\partial\Omega^+$, the symbols $[\,\cdot\,]^\pm$ denote the limits on S from $\Omega^\pm$, $E(\cdot\,,\cdot)$ is the bilinear form associated with the potential energy density defined by

$$E(U,U') = E(U',U) =$$
$$= \sum_{p,q=1}^{3} \Big\{(\mu+\alpha)u'_{pq}u_{pq} + (\mu-\alpha)u'_{pq}u_{qp} + (\kappa+\nu)(u'_{pq}\omega_{pq}+\omega'_{pq}u_{pq}) +$$
$$+(\kappa-\nu)(u'_{pq}\omega_{qp}+\omega'_{pq}u_{qp}) + (\gamma+\varepsilon)\omega'_{pq}\omega_{pq} +$$
$$+(\gamma-\varepsilon)\omega'_{pq}\omega_{qp} + \delta(u'_{pp}\omega_{qq}+\omega'_{qq}u_{pp}) + \lambda u'_{pp}u_{qq} + \beta\omega'_{pp}\omega_{qq}\Big\} \tag{2.13}$$

with

$$u_{pq} = u_{pq}(U) = \partial_p u_q - \sum_{k=1}^{3}\varepsilon_{pqk}\omega_k,$$
$$\omega_{pq} = \omega_{pq}(U) = \partial_p\omega_q, \quad p,q=1,2,3, \tag{2.14}$$

and $u'_{pq}=u'_{pq}(U')$ and $\omega'_{pq}=\omega'_{pq}(U')$ represented analogously by means of U'. Here and in what follows $a\cdot b$ denotes the usual scalar product of two vectors a, $b\in\mathbb{R}^m$: $a\cdot b = \sum\limits_{j=1}^{m} a_j b_j$.

The expressions $u_{pq}(U)$ and $\omega_{pq}(U)$ are called *generalized strains* corresponding to the vector $U=(u,\omega)^\top$.

From (2.13) and (2.14) we get

$$E(U,U') =$$
$$= \frac{3\lambda+2\mu}{3}\Big(\operatorname{div}u + \frac{3\delta+2\kappa}{3\lambda+2\mu}\operatorname{div}\omega\Big)\Big(\operatorname{div}u' + \frac{3\delta+2\kappa}{3\lambda+2\mu}\operatorname{div}\omega'\Big) +$$
$$+\frac{1}{3}\Big(3\beta+2\gamma-\frac{(3\delta+2\kappa)^2}{3\lambda+2\mu}\Big)(\operatorname{div}\omega)(\operatorname{div}\omega') +$$
$$+\frac{\mu}{2}\sum_{k,j=1,\,k\neq j}^{3}\Big[\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}+\frac{\kappa}{\mu}\Big(\frac{\partial\omega_k}{\partial x_j}+\frac{\partial\omega_j}{\partial x_k}\Big)\Big]\times$$
$$\times\Big[\frac{\partial u'_k}{\partial x_j}+\frac{\partial u'_j}{\partial x_k}+\frac{\kappa}{\mu}\Big(\frac{\partial\omega'_k}{\partial x_j}+\frac{\partial\omega'_j}{\partial x_k}\Big)\Big] +$$
$$+\frac{\mu}{3}\sum_{k,j=1}^{3}\Big[\frac{\partial u_k}{\partial x_k}-\frac{\partial u_j}{\partial x_j}+\frac{\kappa}{\mu}\Big(\frac{\partial\omega_k}{\partial x_k}-\frac{\partial\omega_j}{\partial x_j}\Big)\Big]\times$$
$$\times\Big[\frac{\partial u'_k}{\partial x_k}-\frac{\partial u'_j}{\partial x_j}+\frac{\kappa}{\mu}\Big(\frac{\partial\omega'_k}{\partial x_k}-\frac{\partial\omega'_j}{\partial x_j}\Big)\Big] +$$

$$+\Big(\gamma-\frac{\kappa^2}{\mu}\Big)\sum_{k,j=1,\,k\neq j}^{3}\Big[\frac{1}{2}\Big(\frac{\partial\omega_k}{\partial x_j}+\frac{\partial\omega_j}{\partial x_k}\Big)\Big(\frac{\partial\omega'_k}{\partial x_j}+\frac{\partial\omega'_j}{\partial x_k}\Big)+$$
$$+\frac{1}{3}\Big(\frac{\partial\omega_k}{\partial x_k}-\frac{\partial\omega_j}{\partial x_j}\Big)\Big(\frac{\partial\omega'_k}{\partial x_k}-\frac{\partial\omega'_j}{\partial x_j}\Big)\Big]+$$
$$+\alpha\Big(\operatorname{curl}u+\frac{\nu}{\alpha}\operatorname{curl}\omega-2\omega\Big)\cdot\Big(\operatorname{curl}u'+\frac{\nu}{\alpha}\operatorname{curl}\,\omega'-2\omega'\Big)+$$
$$+\Big(\varepsilon-\frac{\nu^2}{\alpha}\Big)\operatorname{curl}\omega\cdot\operatorname{curl}\omega'. \tag{2.15}$$

The potential energy density $E(U,U)$ is positive definite with respect to the variables $u_{pq}(U)$ and $\omega_{pq}(U)$ (see (2.14)), i.e. there exists $c_0>0$ depending only on the material constants such that

$$E(U,U)\geq c_0\sum_{p,q=1}^{3}\big[u_{pq}^2+\omega_{pq}^2\big]. \tag{2.16}$$

From (2.16) it follows that the material constants satisfy the inequalities (cf. [2], [29])

$$\begin{gathered}\mu>0,\quad \alpha>0,\quad 3\lambda+2\mu>0,\quad \mu\gamma-\kappa^2>0,\quad \alpha\,\varepsilon-\nu^2>0,\\ (3\lambda+2\mu)(3\beta+2\gamma)-(3\delta+2\kappa)^2>0,\end{gathered} \tag{2.17}$$

whence we easily derive

$$\begin{gathered}\gamma>0,\quad \varepsilon>0,\quad \lambda+\mu>0,\quad \beta+\gamma>0,\\ d_1:=(\mu+\alpha)(\gamma+\varepsilon)-(\kappa+\nu)^2>0,\\ d_2:=(\lambda+2\mu)(\beta+2\gamma)-(\delta+2\kappa)^2>0.\end{gathered} \tag{2.18}$$

Due to [29] we can characterize the kernel of the energy bilinear form as follows.

Lemma 2.1. *Let $U=(u,\omega)^\top\in[C^1(\Omega)]^6$. Then $E(U,U)=0$ in Ω is equivalent to*

$$u(x)=[a\times x]+b,\quad \omega(x)=a,\quad x\in\Omega, \tag{2.19}$$

where a and b are arbitrary three-dimensional constant vectors and where the symbol $\times$ denotes the cross product of two vectors.

We call vectors of the type $U(x)=([a\times x]+b,\,a)^\top$ *generalized rigid displacement vectors.* It is evident that if a generalized rigid displacement vector vanishes at one point then it is zero vector, i.e., $a=b=0$.

Throughout the paper L_2, $W^s=W_2^s$, and $H^s=H_2^s$ with $s\in\mathbb{R}$ stand for the well-known Lebesgue, Sobolev–Slobodetskiĭ, and Bessel potential spaces, respectively (see, [37], [38], [27]). Note that $H^s=W^s$ for $s\geq 0$. We denote the associated norm by $\|\cdot\|_{H^s}$.

From the positive definiteness of the energy density it easily follows that there exist positive constants c_1 and c_2, depending only on the material constants, such that

$$\mathcal{B}(U,U):=\int\limits_{\Omega^+} E(U,U)dx \geq$$

$$\geq c_1 \int\limits_{\Omega^+} \Big\{ \sum_{p,q=1}^{3} \big[(\partial_p u_q)^2+(\partial_p \omega_q)^2\big] + \sum_{p=1}^{3}[u_p^2+\omega_p^2] \Big\}\, dx -$$

$$-c_2 \int\limits_{\Omega^+} \sum_{p=1}^{3}[u_p^2+\omega_p^2]\, dx \tag{2.20}$$

for an arbitrary real–valued vector $U\in[C^1(\overline{\Omega^+})]^6$, i.e., for any $U\in[H^1(\Omega^+)]^6$ the following Korn's type inequality holds (cf. [11], Part I, § 12, [5], Section 6.3)

$$\mathcal{B}(U,U) \geq c_1 \|U\|^2_{[H^1(\Omega^+)]^6} - c_2 \|U\|^2_{[H^0(\Omega^+)]^6}, \tag{2.21}$$

where $\|\cdot\|_{[H^s(\Omega^+)]^6}$ denotes the norm in the space $[H^s(\Omega^+)]^6$.

Remark 2.2. From (2.12) it follows that

$$\int\limits_{\Omega^+}\big[L(\partial)U\cdot U' - U\cdot L(\partial)U'\big]\, dx = \int\limits_{\partial\Omega^+} \Big\{[TU]^+\cdot[U']^+ - [U]^+\cdot[TU']^+\Big\}\, dS \tag{2.22}$$

for arbitrary $U:=(u,\omega)^\top$, $U':=(u',\omega')^\top \in [C^2(\overline{\Omega^+})]^6$.

Remark 2.3. By standard arguments, Green's formula (2.12) can be extended to Lipschitz domains and to the case of vector functions $U \in [H^1(\Omega^+)]^6$ and $U' \in [H^1(\Omega^+)]^6$ and $L(\partial)U \in [L_2(\Omega^+)]^6$ (cf. [32], [27])

$$\int\limits_{\Omega^+} \big[L(\partial)U\cdot U' + E(U,U')\big]\, dx = \big\langle [T(\partial,n)U]^+, [U']^+ \big\rangle_{\partial\Omega^+}, \tag{2.23}$$

where $\langle\cdot,\cdot\rangle_{\partial\Omega^+}$ denotes the duality between the spaces $[H^{-1/2}(\partial\Omega^+)]^6$ and $[H^{1/2}(\partial\Omega^+)]^6$, which extends the usual "real" L_2-scalar product , i.e., for $f,g\in[L_2(S)]^6$ we have

$$\langle f,g\rangle_S = \sum_{k=1}^{6}\int\limits_S f_k g_k\, dS =: (\,f,\,g\,)_{L_2(S)}.$$

Clearly, in the general case the functional $T(\partial,n)U \in [H^{-1/2}(\partial\Omega^+)]^6$ is well defined by the relation (2.23).

3. Formulation of Unilateral Problems and Main Existence Results

3.1. **Mechanical description of the problem.** Mathematical aspects of the mechanical unilateral problems (Signorini type problems) in the framework of the classical elasticity theory are well described in many works (see, e.g., [12], [19], [20], and the references therein). Here we will apply the analogous arguments and construct a mathematical model for the unilateral problems in the theory of elasticity of hemitropic materials. The main difficulties appear in the reasonable interpretation of the unilateral restrictions for the microrotation vector and/or the couple stress vector.

We assume that a hemitropic elastic body, in its reference configuration, occupies the closure of a domain $\Omega^+ \in \mathbb{R}^3$. The boundary $S = \partial\Omega^+$ is divided into three parts $S = \overline{S_D} \cup \overline{S_T} \cup \overline{S_C}$ with disjoint S_D, S_T, and S_N. For definiteness and simplicity in what follows we assume (if not otherwise stated) that $\overline{S_D}$ has a positive surface measure, $\overline{S_D} \cap \overline{S_C} = \varnothing$ and, moreover, S, ∂S_C, ∂S_T, ∂S_D are C^∞-smooth.

The hemitropic elastic body is fixed along the subsurface S_D, i.e., the displacement vector u and the microrotation vector ω vanish on S_D. The surface force stress vector and the surface couple stress vector are applied to the portion S_T, i.e., $[\tau^{(n)}(U)]^+ = \psi^{(1)}$ and $[\mu^{(n)}(U)]^+ = \psi^{(2)}$ on S_T, where $\psi^{(j)} = (\psi_1^{(j)}, \psi_2^{(j)}, \psi_3^{(j)})^\top$, $j = 1, 2$, are given vector functions on S_T.

The motion of the elastic body is restricted by the so called *foundation* $\mathcal{F}$ which is a rigid and absolutely fixed body. We are interested in the deformation of the hemitropic body brought about due to its motion from its reference configuration to another configuration when some portion of the material surface of the body comes in contact with the foundation $\mathcal{F}$. Note that in the case of statics we ignore the dependence of all mechanical and geometrical characteristics involved on time t. The actual surface on which the body comes in contact with the foundation is not known in advance but is contained in the portion S_C of S.

We confine our attention to infinitesimal generalized deformations of the body (see (2.14)). Moreover, we assume that the foundation surface $\partial\mathcal{F}$ is *frictionless* and no force and couple stresses are applied on S_C. Therefore, by the standard arguments (for details see, e.g., [20], Chapter 2) we arrive at the following *linearized* conditions on S_C for the displacement vector u and the force stress vector $\tau^{(n)}(U)$:

(i) the so called non-penetration condition,

$$[u \cdot n]^+ \le \varphi, \tag{3.1}$$

where n is the outward unit normal vector to $\partial\Omega^+$ and φ is a given scalar function characterizing the *initial gap* between the foundation and the elastic body;

(ii) the conditions describing that a compressive normal force stress must be developed at the points of contact, while the normal component of the force stress vector is zero if no contact occurs,

$$\big[\tau^{(n)}(U) \cdot n\big]^+ \le 0, \quad \big[\tau^{(n)}(U) \cdot n\big]^+ \big([u \cdot n]^+ - \varphi\big) = 0; \tag{3.2}$$

(iii) the condition showing that the tangential components of the force stress vector vanish on S_C,

$$\big[\tau^{(n)}(U)\big]^+ - n\big[\tau^{(n)}(U) \cdot n\big]^+ = 0. \tag{3.3}$$

Further, we make a fundamental observation: since the surface $\partial\mathcal{F}$ is frictionless, the microrotation vector ω is not restricted by any condition on S_C even at the points where the contact with the rigid foundation $\mathcal{F}$ occurs and, therefore, it is very natural to require that the couple stress vector $\mu^{(n)}(U)$ is zero on S_C,

$$\big[\mu^{(n)}(U)\big]^+ = 0 \ \text{ on } \ S_C. \tag{3.4}$$

Now we are in a position to formulate mathematically the unilateral problem that corresponds to the equilibrium state of a hemitropic body for given data: body force and body couple vectors $\Phi^{(1)} = (\Phi_1^{(1)}, \Phi_2^{(1)}, \Phi_3^{(1)})^\top$ and $\Phi^{(2)} = (\Phi_1^{(2)}, \Phi_2^{(2)}, \Phi_3^{(2)})^\top$ in Ω^+, surface force stress and surface couple stress vectors $\psi^{(1)}$ and $\psi^{(2)}$ on S_T, and initial gap φ on S_C.

Problem (UP)**.** We have to find a vector $U = (u,\omega)^\top) \in [H^1(\Omega^+)]^6$ satisfying the following system of equations and inequalities:

$$L(\partial)U = -\Phi \ \text{ in } \ \Omega^+, \tag{3.5}$$

$$[u]^+ = 0, \quad [\omega]^+ = 0 \ \text{ on } \ S_D, \tag{3.6}$$

$$\left.\begin{aligned} [\tau^{(n)}(U)]^+ &= \big[T^{(1)}u + T^{(2)}\omega\big]^+ = \psi^{(1)} \\ [\mu^{(n)}(U)]^+ &= \big[T^{(3)}u + T^{(4)}\omega\big]^+ = \psi^{(2)} \end{aligned}\right\} \ \text{ on } \ S_T, \tag{3.7}$$

$$\left.\begin{aligned} &[u\cdot n]^+ \le \varphi \\ &\big[\tau^{(n)}(U)\cdot n\big]^+ \le 0 \\ &\big\langle r_{S_C}\big[\tau^{(n)}(U)\cdot n\big]^+, r_{S_C}[u\cdot n]^+ - \varphi\big\rangle_{S_C} = 0 \\ &[\tau^{(n)}(U)]^+ - n\big[\tau^{(n)}(U)\cdot n\big]^+ = 0 \\ &[\mu^{(n)}(U)]^+ = 0 \end{aligned}\right\} \ \text{ on } \ S_C, \tag{3.8}$$

where r_Σ denotes the restriction operator to Σ,

$$\Phi = (\Phi^{(1)}, \Phi^{(2)})^\top \in [L_2(\Omega^+)]^6, \quad \Phi^{(j)} = \big(\Phi_1^{(j)}, \Phi_2^{(j)}, \Phi_3^{(j)}\big)^\top, \tag{3.9}$$

equation (3.5) is understood in the weak sense, i.e.,

$$\int\limits_{\Omega^+} E(U,V)\,dx = \int\limits_{\Omega^+} \Phi\cdot V\,dx \tag{3.10}$$

for arbitrary infinitely differentiable function $V \in [C_0^\infty(\Omega^+)]^6$ with compact support in Ω. Due to the well-known interior regularity results for solutions of strongly elliptic systems (see, e.g., [12]) we conclude that the equation (3.5), with Φ as in (3.9), holds pointwise almost everywhere in Ω^+.

The condition (3.6) and the first inequality in (3.8) are understood in the usual trace sense, while (3.7) and the fourth and fifth equations in (3.8) are understood in the generalized functional sense described in Remark 2.3,

$$\begin{aligned}\Psi : &= (\psi^{(1)}, \psi^{(2)})^\top \in [L_2(S_T)]^6,\\ \psi^{(j)} &= \big(\psi_1^{(j)}, \psi_2^{(j)}, \psi_3^{(j)}\big)^\top, \quad j=1,2,\end{aligned} \tag{3.11}$$

$$\varphi \in H^{\frac12}(S_C). \tag{3.12}$$

Throughout the paper $H^s(\Sigma) := \big\{ r_{\Sigma} f : \ f \in H^s(S) \big\}$ is the space of restrictions to $\Sigma \subset S$ of functions from the space $H^s(S)$, while $\widetilde{H}^s(\Sigma) := \big\{ f \in H^s(S) : \ \operatorname{supp} f \subset \overline{\Sigma} \big\}$ for $s \in \mathbb{R}$. We recall that $\widetilde{H}^s(\Sigma)$ and $H^{-s}(\Sigma)$ are mutually adjoint function spaces and $L_2(\Sigma)$ is continuously embedded in $\widetilde{H}^{-1/2}(\Sigma)$ (for details see, e.g., [27], [37]). In the third equation of (3.8) the symbol $\langle \cdot , \cdot \rangle_{S_C}$ denotes the duality brackets between the spaces $\widetilde{H}^{-1/2}(S_C)$ and $H^{1/2}(S_C)$, which is well-defined due to the embedding (3.11) and the boundary condition (3.7) implying that

$$r_{\Sigma} [\tau^{(n)}(U) \cdot n]^+ \in r_{\Sigma} [\widetilde{H}^{-1/2}(\Sigma)]^3 \ \text{ for } \ \Sigma \in \{ S_D, S_T, S_C \}.$$

The second inequality in (3.8) means that

$$\big\langle r_{S_C} [\tau^{(n)}(U) \cdot n]^+, \psi \big\rangle_{S_C} \leq 0 \ \text{ for all } \ \psi \in H^{1/2}(S_C), \ \ \psi \geq 0.$$

Note that due the unilateral conditions (3.8) the problem (UP) is nonlinear since the coincidence set of the foundation $\mathcal{F}$ and the elastic body (i.e., the subset of S_C where $[\, u \cdot n \,]^+ = \varphi$) is not known in advance. We have the following uniqueness result.

Theorem 3.1. *The problem* (UP) *possesses at most one solution.*

Proof. Let $U^{(1)} = (u^{(1)}, \omega^{(1)})^\top$ and $U^{(2)} = (u^{(2)}, \omega^{(2)})^\top$ be two solutions of the problem (UP) and denote $U = (u, \omega)^\top := U^{(1)} - U^{(2)}$. It is evident that U satisfies the homogeneous differential equation (3.5) with $\Phi = 0$, the homogeneous Dirichlet boundary conditions (3.6), the homogeneous Neumann boundary conditions (3.7) with $\Psi := (\psi^{(1)}, \psi^{(2)})^\top = 0$. Moreover, the tangential components of the force stress vector $[\tau^{(n)}(U)]^+ - n[\tau^{(n)}(U) \cdot n]^+$ and the couple stress vector $[\mu^{(n)}(U)]^+$ vanish on S_C due to the fourth and fifth conditions in (3.8). Therefore, by Green's formula (2.23) and using the third equality in (3.8) we get

$$\begin{aligned}\int\limits_{\Omega^+} E(U,U)\, dx &= \big\langle [T(\partial,n)U]_S^+, [U]_S^+ \big\rangle_S = \big\langle [\tau^{(n)}(U)]_S^+, [u]_S^+ \big\rangle_S = \\ &= \Big\langle \big[\tau^{(n)}(U^{(1)}) \cdot n\big]_{S_C}^+ - \big[\tau^{(n)}(U^{(2)}) \cdot n\big]_{S_C}^+, [u^{(1)} \cdot n]_{S_C}^+ - \varphi - [u^{(2)} \cdot n]_{S_C}^+ + \varphi \Big\rangle_{S_C} = \\ &= -\Big\langle \big[\tau^{(n)}(U^{(1)}) \cdot n\big]_{S_C}^+, [u^{(2)} \cdot n]_{S_C}^+ - \varphi \Big\rangle_{S_C} - \\ &\quad - \Big\langle \big[\tau^{(n)}(U^{(2)}) \cdot n\big]_{S_C}^+, [u^{(1)} \cdot n]_{S_C}^+ - \varphi \Big\rangle_{S_C} \leq 0,\end{aligned}$$

due to the first and second inequalities in (3.8). Thus, $E(U,U) = 0$ in accordance with (2.16) and by Lemma 2.1 it then follows that $U = 0$ in Ω^+ since $S_D \neq \varnothing$. This completes the proof. □

Below, in the subsequent subsections we will give some different formulations of the above unilateral problem (UP) with the help of variational inequalities.

3.2. **Spatial variational inequality formulation.** Let us put

$$\mathbf{H}(\Omega^+; S_D) := \Big\{ V = (v,w)^\top \in [H^1(\Omega^+)]^6 : \ \ r_{_{S_D}}[V]^+ = 0 \Big\} \tag{3.13}$$

and for $\varphi \in H^{1/2}(S_C)$ introduce the convex closed set of vector functions

$$\mathbf{K}_\varphi := \Big\{ V = (v,w)^\top \in \mathbf{H}(\Omega^+; S_D) : \ \ r_{_{S_C}}([v]^+ \cdot n) \le \varphi \Big\}. \tag{3.14}$$

Let us consider the following *spatial variational inequality* (SVI):

Find $U = (u,\omega)^\top \in \mathbf{K}_\varphi$ such that

$$\int\limits_{\Omega^+} E(U, V-U)\, dx \ge$$

$$\ge \int\limits_{\Omega^+} \Phi \cdot (V-U)\, dx + \langle \Psi, r_{_{S_T}}[V-U]^+ \rangle_{S_T} \ \text{ for all } \ V \in \mathbf{K}_\varphi, \tag{3.15}$$

where $E(\cdot\,,\cdot)$ is given by (2.15) and Φ, Ψ and φ are as in (3.9), (3.11) and (3.12). Note that the duality relation in the right-hand side of (3.15) can be written as a usual Lebesgue integral over the subsurface S_T due to (3.11) and Remark 2.3.

Further we show that the SVI (3.15) and the unilateral problem (UP) are equivalent.

Theorem 3.2. *If a vector function U solves the* SVI (3.15)*, then U is a solution to the unilateral problem* (3.5)–(3.8)*, and vice versa.*

Proof. First, we assume that $U = (u,\omega)^\top \in \mathbf{K}_\varphi$ solves the SVI (3.15) and prove that it solves then the unilateral problem (3.5)–(3.8). It is evident that

$$V = U \pm V' \in \mathbf{K}_\varphi \ \text{ for arbitrary } \ V' \in [C_0^\infty(\Omega^+)]^6. \tag{3.16}$$

From (3.15) it then follows that

$$\int\limits_{\Omega^+} E(U, V')\, dx = \int\limits_{\Omega^+} \Phi \cdot V'\, dx \ \text{ for all } \ V' \in [C_0^\infty(\Omega^+)]^6,$$

which shows that U is a weak solution of the equation (3.5). Due to the interior regularity results the equation (3.5) holds point wise almost everywhere in Ω^+ since $\Phi \in [L_2(\Omega^+)]^6$.

The Dirichlet type condition (3.6) is automatically satisfied due to the inclusion $U \in \mathbf{K}_\varphi$ since $\mathbf{K}_\varphi \subset \mathbf{H}(\Omega^+; S_D)$.

For the solution vector U and for any $V\in\mathbf{K}_\varphi$ in accordance with Green's formula (2.23) we have

$$\int\limits_{\Omega^+} E(U,V-U)\,dx=\int\limits_{\Omega^+}\Phi\cdot(V-U)\,dx+\big\langle[T(\partial,n)U]^+,[V-U]^+\big\rangle_{\partial\Omega^+}. \tag{3.17}$$

From (3.15) and (3.16) we conclude

$$\big\langle[T(\partial,n)U]^+,[V-U]^+\big\rangle_{\partial\Omega^+}\geq\langle\Psi,r_{S_T}[V-U]^+\rangle_{S_T}\ \text{ for all }\ V\in\mathbf{K}_\varphi. \tag{3.18}$$

In virtue of (3.11) and

$$\left.\begin{aligned}&r_\Sigma[T(\partial,n)U]^+\in r_\Sigma[\widetilde{H}^{-1/2}(\Sigma)]^6\\&r_\Sigma[V-U]^+\in[H^{1/2}(\Sigma)]^6\end{aligned}\right\}\ \text{ for }\ \Sigma\in\{S_D,S_T,S_C\}, \tag{3.19}$$

and since elements from $\mathbf{K}_\varphi$ vanish on S_D we can decompose the duality relations in (3.18) as follows

$$\begin{aligned}\Big\langle r_{S_T}[T(\partial,n)U]^+,r_{S_T}[V-U]^+\Big\rangle_{S_T}+\Big\langle r_{S_C}[T(\partial,n)U]^+,r_{S_C}[V-U]^+\Big\rangle_{S_C}\geq\\ \geq\langle\Psi,r_{S_T}[V-U]^+\rangle_{S_T}\ \text{ for all }\ V\in\mathbf{K}_\varphi.\end{aligned} \tag{3.20}$$

Further, let us choose the vector V as in (3.16) but now with arbitrary $V'\in[H^1(\Omega^+)]^6$ such that $[V']^+_S\in[\widetilde{H}^{1/2}(S_T)]^6$. It is clear that $V\in\mathbf{K}_\varphi$ and from (3.20) we easily derive

$$\begin{aligned}\Big\langle r_{S_T}[T(\partial,n)U]^+,r_{S_T}[V']^+\Big\rangle_{S_T}=\\ =\langle\Psi,r_{S_T}[V']^+\rangle_{S_T}\ \text{ for all }\ [V']^+_S\in[\widetilde{H}^{1/2}(S_T)]^6,\end{aligned} \tag{3.21}$$

whence the conditions (3.7) on S_T follow immediately.

Note that the first condition in (3.8) is satisfied automatically since $U=(u,\omega)^\top\in\mathbf{K}_\varphi$.

Now, from (3.20) we conclude

$$\Big\langle r_{S_C}[T(\partial,n)U]^+,r_{S_C}[V-U]^+\Big\rangle_{S_C}\geq0\ \text{ for all }\ V\in\mathbf{K}_\varphi, \tag{3.22}$$

i.e.,

$$\Big\langle r_{S_C}[\tau^{(n)}(U)]^+,r_{S_C}[v-u]^+\Big\rangle_{S_C}+\Big\langle r_{S_C}[\mu^{(n)}(U)]^+,r_{S_C}[w-\omega]^+\Big\rangle_{S_C}\geq0 \tag{3.23}$$

for all $V=(v,w)^\top\in\mathbf{K}_\varphi$.

If we take $v=u$ and $w=\omega\pm\chi$ with arbitrary $\chi\in[H^1(\Omega^+)]^3$ such that $[\chi]^+_S\in[\widetilde{H}^{1/2}(S_C)]^3$, then $V=(v,w)^\top\in\mathbf{K}_\varphi$ and from (3.23) we see that $[\mu^{(n)}(U)]^+=0$ on S_C, i.e., the fifth condition in (3.8) is satisfied.

From (3.23) then we have

$$\Big\langle r_{S_C}[\tau^{(n)}(U)]^+,r_{S_C}[v-u]^+\Big\rangle_{S_C}\geq0 \tag{3.24}$$

for all $V = (v, 0)^\top \in \mathbf{K}_\varphi$. To show the remaining three conditions in (3.8) we proceed as follows. First we rewrite (3.24) in the form

$$\Big\langle r_{S_C}[\tau^{(n)}(U)]^+_n, r_{S_C}[v-u]^+_n\Big\rangle_{S_C} + \Big\langle r_{S_C}[\tau^{(n)}(U)]^+_t, r_{S_C}[v-u]^+_t\Big\rangle_{S_C} \geq 0, \quad (3.25)$$

where the subscripts n and t denote the normal and tangential components of the corresponding vectors defined on the boundary S: $a_n = a \cdot n$ and $a_t = a - n\, a_n$ for arbitrary $a \in \mathbb{R}^3$. We set $V = (v, 0)^\top$ with $v = u \pm g$ where $g \in [H^1(\Omega^+)]^3$, $[g]^+_S \in \widetilde{H}^{1/2}(S_C)$, and $[g]^+_n = 0$ on S_C. These restrictions imply that $V = (v, 0)^\top \in \mathbf{K}_\varphi$ and substitution into the inequality (3.25) leads to the equation

$$\Big\langle r_{S_C}[\tau^{(n)}(U)]^+_t, r_{S_C}[g]^+_t\Big\rangle_{S_C} = 0. \quad (3.26)$$

Since $[g]^+_t = [g]^+$ is arbitrary we conclude that the fourth condition in (3.8) is satisfied. Therefore, (3.25) yields

$$\Big\langle r_{S_C}[\tau^{(n)}(U)]^+_n, r_{S_C}[v-u]^+_n\Big\rangle_{S_C} \geq 0, \quad (3.27)$$

for all $V = (v, 0)^\top \in \mathbf{K}_\varphi$.

Let us put $V = (v, 0)^\top$ with $v = u - g$ where $g \in [H^1(\Omega^+)]^3$, $[g]^+_S = n\vartheta$, $\vartheta \geq 0$, and $\vartheta \in \widetilde{H}^{1/2}(S_C)$. Then $V = (v, 0)^\top \in \mathbf{K}_\varphi$ and therefore from (3.27) we get

$$\Big\langle r_{S_C}[\tau^{(n)}(U)]^+_n, r_{S_C}\vartheta\Big\rangle_{S_C} \leq 0, \quad (3.28)$$

which shows that the second condition in (3.8) holds.

Finally, if in (3.27) we substitute $V = (v, 0)^\top \in \mathbf{K}_\varphi$ with $r_{S_C}[v]^+_S = n\,\varphi$ on S_C, we arrive at the inequality

$$\Big\langle r_{S_C}[\tau^{(n)}(U)]^+_n, \varphi - r_{S_C}[u\cdot n]^+\Big\rangle_{S_C} \geq 0. \quad (3.29)$$

On the other hand, since $\varphi - r_{S_C}[u \cdot n]^+ \geq 0$ and $[\tau^{(n)}(U)]^+_n \leq 0$ on S_C we have

$$\Big\langle r_{S_C}[\tau^{(n)}(U)]^+_n, \varphi - r_{S_C}[u\cdot n]^+\Big\rangle_{S_C} \leq 0, \quad (3.30)$$

which along with (3.29) implies that the third condition in (3.8) is fulfilled as well.

Now, we prove the inverse assertion. Let $U = (u, \omega)^\top$ be a solution to the unilateral problem (3.5)–(3.8) with data satisfying the assumptions (3.9), (3.11), and (3.12). We have to show that then U solves the SVI (3.15). It is clear that for the solution vector U and for any $V \in \mathbf{K}_\varphi$ the formula (3.17) holds in accordance with Green's formula (2.23). Note that the assumption (3.11) implies by embedding $L^2(\Sigma) \subset \widetilde{H}^{-1/2}(\Sigma)$ that $r_{S_T}[T(\partial, n)U]^+ = \Psi \in r_{S_T}\widetilde{H}^{-1/2}(S_T)$ and by the trace theorem that $r_\Sigma[V - U]^+ \in [H^{1/2}(\Sigma)]^6$ for

$\Sigma \in \{S_T, S_C\}$. Therefore in view of the Dirichlet homogeneous condition (3.6) we can rewrite (3.17) as follows

$$\int\limits_{\Omega^+} E(U, V-U)\,dx = \int\limits_{\Omega^+} \Phi\cdot(V-U)\,dx + \Big\langle \Psi, r_{S_T}[V-U]^+\Big\rangle_{S_T} + \\ + \Big\langle r_{S_C}[T(\partial,n)U]^+, r_{S_C}[V-U]^+\Big\rangle_{S_C}. \tag{3.31}$$

With the help of unilateral conditions (3.8) we derive

$$\Big\langle r_{S_C}[T(\partial,n)U]^+, r_{S_C}[V-U]^+\Big\rangle_{S_C} = \Big\langle r_{S_C}[\tau^{(n)}(U)]^+_n, r_{S_C}[v-u]^+_n\Big\rangle_{S_C} + \\ + \Big\langle r_{S_C}[\tau^{(n)}(U)]^+_t, r_{S_C}[v-u]^+_t\Big\rangle_{S_C} + \Big\langle r_{S_C}[\mu^{(n)}(U)]^+, r_{S_C}[w-\omega]^+\Big\rangle_{S_C} = \\ = \Big\langle r_{S_C}[\tau^{(n)}(U)]^+_n, r_{S_C}[v-u]^+_n\Big\rangle_{S_C} = \\ = \Big\langle r_{S_C}[\tau^{(n)}(U)]^+_n, r_{S_C}[v]^+_n - \varphi\Big\rangle_{S_C} - \Big\langle r_{S_C}[\tau^{(n)}(U)]^+_n, r_{S_C}[u]^+_n - \varphi\Big\rangle_{S_C} = \\ = \Big\langle r_{S_C}[\tau^{(n)}(U)]^+_n, r_{S_C}[v]^+_n - \varphi\Big\rangle_{S_C} \geq 0$$

due to the second inequality in (3.8) and since $r_{S_C}\,[v]^+_n - \varphi \leq 0$ in accordance with (3.14). Now, (3.31) completes the proof. □

Thus we have shown that the unilateral problem (UP) (3.5)–(3.8) and the SVI (3.15) are absolutely equivalent.

Let us remark that the bilinear form $\mathcal{B} : [H^1(\Omega^+)]^6 \times [H^1(\Omega^+)]^6 \to \mathbb{R}$ with

$$\mathcal{B}(U, V) := \int\limits_{\Omega^+} E(U, V)\,dx \tag{3.32}$$

is bounded on $[H^1(\Omega^+)]^6 \times [H^1(\Omega^+)]^6$ and strictly coercive on $\mathbf{H}(\Omega^+; S_D)$ (for details see [32]), i.e., there are positive constants c_3 and c_4 such that

$$\mathcal{B}(U, V) \leq c_3 \|U\|^2_{[H^1(\Omega^+)]^6} \|V\|^2_{[H^1(\Omega^+)]^6} \ \text{ for all } \ U, V \in [H^1(\Omega^+)]^6, \tag{3.33}$$

$$\mathcal{B}(U, U) \geq c_4 \|U\|^2_{[H^1(\Omega^+)]^6} \ \text{ for all } \ U \in \mathbf{H}(\Omega^+; S_D). \tag{3.34}$$

It is easy to see that the linear functional $\mathcal{P} : [H^1(\Omega^+)]^6 \to \mathbb{R}$ with

$$\mathcal{P}(V) := \int\limits_{\Omega^+} \Phi\cdot V\,dx + \langle \Psi, r_{S_T}[V]^+\rangle_{S_T} \ \text{ for all } \ V \in \mathbf{K}_\varphi, \tag{3.35}$$

where Φ and Ψ are as in (3.9) and (3.11), is bounded due to the Schwarz inequality and the trace theorem.

Therefore, due to the general theory of variational inequalities in Hilbert spaces (see, e.g., [12], [8], [14]) we have the following uniqueness and existence results for the variational inequality (3.15) which can be written now as

$$\mathcal{B}(U, V-U) \geq \mathcal{P}(V-U) \ \text{ for all } \ V \in \mathbf{K}_\varphi. \tag{3.36}$$

Theorem 3.3. *The* SVI (3.15) (*i.e., the variational inequality* (3.36)) *is uniquely solvable.*

As an easy consequence of Theorems 3.2 and 3.3 we have

Corollary 3.4. *The unilateral problem* (UP) (3.5)–(3.8) *is uniquely solvable.*

It is also well-known that, in turn, the variational inequality (3.36) (that is, (3.15)) is equivalent to the following *minimization problem*: Find a minimum on the convex closed set $\mathbf{K}_\varphi$ of the energy functional (see, e.g., [12])

$$J(V) := 2^{-1}\mathcal{B}(V,V) - \mathcal{P}(V), \tag{3.37}$$

i.e., find $U \in \mathbf{K}\,\varphi$ such that

$$J(U) = \min_{V\in\mathbf{K}_\varphi} J(V). \tag{3.38}$$

It is evident that the minimization problem is also uniquely solvable.

Let us remark that we can reduce equivalently the above unilateral problem (UP) to the case when the right-hand side vector function in (3.5) vanishes. To this end, consider the auxiliary boundary value problem (BVP)

$$\begin{gathered} L(\partial)U_0 = -\Phi \ \text{ in } \ \Omega^+, \\ [U_0]^+ = 0 \ \text{ on } \ S_D, \quad [T(\partial,n)U_0]^+ = 0 \ \text{ on } \ S\setminus\overline{S_D}, \end{gathered} \tag{3.39}$$

where $U_0 = (u_0,\omega_0)^\top \in [H^1(\Omega^+)]^6$ and Φ is as in (3.5).

This mixed BVP is uniquely solvable (for details see [29]). Therefore, if we denote $U^* := U - U_0$, where U solves the unilateral problem (3.5)–(3.8) and U_0 is the solution vector of the auxiliary BVP, then we see that the vector U^* is a solution to the unilateral problem (UP) with homogeneous differential equation (3.5) (i.e., with $\Phi = 0$) and with $\varphi^* := \varphi - r_{S_C}[u_0]_n^+$ in the place of φ. Therefore, in what follows, we assume that $\Phi = 0$ without loss of generality.

3.3. **Boundary variational inequality formulation.** Here we reduce the unilateral problem (UP) (3.5)–(3.8) (with $\Phi = 0$) to an equivalent *boundary variational inequality* (BVI). To this end, for a given vector $f = (f^{(1)}, f^{(2)})^\top \in [H^{1/2}(S)]^6$, $f^{(j)} = (f_1^{(j)}, f_2^{(j)}, f_3^{(j)})^\top$, $j = 1,2$, let us consider the vector function

$$U(x) = V(\mathcal{H}^{-1}f)(x), \ \ x \in \Omega^+, \tag{3.40}$$

where $V(\,\cdot\,)$ is the single layer potential operator and $\mathcal{H}$ is the boundary integral operator on $S = \partial\Omega^+$ generated by the single layer potential (see the Appendix, formulas (A.2), (A.11), and Theorems 4.2 and 4.3). It can easily be verified that the vector (3.40) solves the Dirichlet BVP

$$\begin{gathered} L(\partial)U = 0 \ \text{ in } \ \Omega^+, \quad U = (u,\omega)^\top \in [H^1(\Omega^+)]^6, \\ [U]^+ = f \ \text{ on } \ S. \end{gathered} \tag{3.41}$$

Let us introduce the so called Steklov–Poincaré operator $\mathcal{A}$ relating the Dirichlet and Neumann data for a vector (3.40) (see the Appendix, Theorem 4.2)

$$\mathcal{A}f := \big[T(\partial,n)V(\mathcal{H}^{-1}f)\big]^{+} = [-2^{-1}I_6+\mathcal{K}]\mathcal{H}^{-1}f. \tag{3.42}$$

With the help of the equalities stated in the Appendix, Theorem 4.3.iii, it can easily be shown that

$$\mathcal{A} = \mathcal{H}^{-1}[-2^{-1}I_6+\mathcal{K}^*] = \mathcal{L} - [-2^{-1}I_6+\mathcal{K}]\mathcal{H}^{-1}[-2^{-1}I_6+\mathcal{K}^*], \tag{3.43}$$

whence it follows that the operator

$$\mathcal{A} : [H^{1/2}(S)]^6 \to [H^{-1/2}(S)]^6 \tag{3.44}$$

is an elliptic, self-adjoint pseudodifferential operator of order 1.

Denote by $X_S\{\Lambda^{(1)},\Lambda^{(2)},\dots,\Lambda^{(6)}\}$ the linear span of generalized rigid displacement vectors on S (for more details we refer to the Appendix A.1).

Theorem 3.5. *The Steklov–Poincaré operator* (3.42) *possesses the following properties:*

$$\langle \mathcal{A}f,f\rangle_S \geq 0 \;\; \textit{for all} \;\; f\in[H^{1/2}(S)]^6, \tag{3.45}$$

$$\langle \mathcal{A}f,f\rangle_S = 0 \;\; \textit{if and only if} \;\; f = r_{S}([a\times x]+b,a)^\top,\;\; a,\,b\in\mathbb{R}^3, \tag{3.46}$$

$$\textit{i.e.,} \;\; f\in X_S\{\Lambda^{(1)},\Lambda^{(2)},\dots,\Lambda^{(6)}\},$$

$$\langle \mathcal{A}f,g\rangle_S = \langle \mathcal{A}g,f\rangle_S \;\; \textit{for all} \;\; f,\,g\in[H^{1/2}(S)]^6, \tag{3.47}$$

$$\ker\mathcal{A} = X_S\big\{\Lambda^{(1)},\Lambda^{(2)},\dots,\Lambda^{(6)}\big\}, \tag{3.48}$$

i.e., for an arbitrary generalized rigid displacement vector $\chi(x) = ([a\times x]+b,a)^\top$, $x\in S$, *we have* $\mathcal{A}\chi = 0$ *on* S.

Proof. The relations (3.45) and (3.46) follow from inequality (2.16), Lemma 2.1, and Green's identity

$$\int\limits_{\Omega^+} E(U,U)\,dx = \langle \mathcal{A}f,f\rangle_S \;\text{ with }\; U = V(\mathcal{H}^{-1}f). \tag{3.49}$$

The equality (3.47) is an easy consequence of (3.43), while (3.48) can be established with the help of Theorem 5.3.ii in the Appendix. □

Further, let

$$\mathbb{H}(S,S_D) : = \Big\{g = (g^{(1)},g^{(2)})^\top, g^{(j)}\in[H^{1/2}(S)]^3,\; j=1,2:\; r_{S_D}g=0\Big\} \equiv$$
$$\equiv \big[\widetilde{H}^{1/2}(S\setminus\overline{S_D})\big]^6, \tag{3.50}$$

and for $\varphi\in H^{1/2}(S_C)$ introduce the convex closed set of vector functions

$$\mathbb{K}_\varphi := \Big\{g = (g^{(1)},g^{(2)})^\top \in \mathbb{H}(S,S_D):\; r_{S_C}(g^{(1)}\cdot n)\leq\varphi\Big\}. \tag{3.51}$$

Let us consider the following *boundary variational inequality* (BVI):

Find $f = (f^{(1)}, f^{(2)})^\top \in \mathbb{K}_\varphi$ such that

$$\big\langle \mathcal{A}f, g - f \big\rangle_S \ge \langle \Psi, r_{_{S_T}}(g - f)\rangle_{S_T} \;\text{ for all }\; g = (g^{(1)}, g^{(2)})^\top \in \mathbb{K}_\varphi, \quad (3.52)$$

where $\mathcal{A}$ is the Steklov–Poincaré operator, Ψ and φ are as in (3.11) and (3.12). Note that the duality relation in the right-hand side of (3.52) can be written as a usual Lebesgue integral over the subsurface S_T due to (3.11) and Remark 2.3.

First we establish the following strict coercivity property of the operator $\mathcal{A}$.

Theorem 3.6. *The Steklov–Poincaré operator* (3.42) *is strictly coercive on* $\mathbb{H}(S, S_D)$*, i.e., there is a positive constant* C_1 *such that*

$$\langle \mathcal{A}f, f\rangle_S \ge C_1 \|f\|^2_{[H^{1/2}(S)]^6} \;\text{ for all }\; f \in \mathbb{H}(S, S_D). \quad (3.53)$$

Proof. The proof coincides word for word with the proof of Lemma 4.2 in [13]. □

Thus, the bilinear form $\langle \mathcal{A}f\,,\, g\rangle_S$ is strictly coercive on the subspace $\mathbb{H}(S, S_D)$ and bounded on $[H^{1/2}(S)]^6 \times [H^{1/2}(S)]^6$, i.e. there is a positive constant C_2 such that

$$\langle \mathcal{A}f, g\rangle_S \le C_2 \|f\|_{[H^{1/2}(S)]^6} \|h\|_{[H^{1/2}(S)]^6} \;\text{ for all }\; f,\, g \in [H^{1/2}(S)]^6. \quad (3.54)$$

Therefore, the BVI (3.52) is uniquely solvable due to the general theory of variational inequalities in Hilbert spaces (see, e.g., Theorems 2.1 and 2.2 in [14]).

Further we show that the boundary variational inequality (3.52) is equivalent to the unilateral problem (UP).

Theorem 3.7. (i) *If* $f \in \mathbb{K}_\varphi$ *solves the* BVI (3.52) *then the vector* U *given by* (3.40) *is a solution to the unilateral problem* (UP) (3.5)–(3.8) *with* $\Phi = 0$.

(ii) *If* $U \in \mathbf{K}_\varphi$ *is a solution of the problem* (UP) *with* $\Phi = 0$*, then* $f = [U]^+_S$ *is a solution of the* BVI (3.52).

Proof. Let $f \in \mathbb{K}_\varphi$ be a solution of the BVI (3.52) and construct the vector U by (3.40). It is evident that $U \in [H^1(\Omega^+)]^6$ and $L(\partial)U = 0$ in Ω^+ in accordance with Theorem 4.2 in the Appendix. Moreover, since $[U]^+ = ([u]^+, [\omega]^+)^\top = f \in \mathbb{K}_\varphi$ on S we see that the conditions (3.5) with $\Phi = 0$, (3.6), and the first inequality in (3.8) are satisfied.

Note that

$$\mathcal{A}f = [T(\partial, n)U]^+ = \big([\tau^{(n)}(U)]^+, [\mu^{(n)}(U)]^+\big)^\top \in [H^{-1/2}(S)]^6. \quad (3.55)$$

Further, let $g = f \pm h$ where $h = (h^{(1)}, h^{(2)})^\top \in [\widetilde{H}^{1/2}(S_T)]^6$. Evidently, $g \in \mathbb{K}_\varphi$ and from (3.52) we get

$$\big\langle \mathcal{A}f, h \big\rangle_S = \langle \Psi, r_{_{S_T}} h\rangle_{S_T} \;\text{ for all }\; h = (h^{(1)}, h^{(2)})^\top \in [\widetilde{H}^{1/2}(S_T)]^6.$$

Consequently,

$$r_{_{S_T}}\mathcal{A}f = r_{_{S_T}}[T(\partial, n)U]^+ = \Psi \;\text{ on }\; S_T \quad (3.56)$$

and the conditions (3.7) hold.

Now in view of the embedding (3.11) and equality (3.56) we see that

$$r_{\Sigma}\mathcal{A}f \in r_{\Sigma}[\widetilde{H}^{-1/2}(\Sigma)]^6 \ \text{ with } \ \Sigma \in \{S_D, S_T, S_C\}. \tag{3.57}$$

Therefore, we can decompose the duality brackets in the left-hand side of the inequality (3.52) as a sum corresponding to the division of the surface $S = \overline{S_D} \cup \overline{S_T} \cup \overline{S_C}$. With the help of (3.56) we then obtain

$$\begin{aligned}\big\langle r_{S_C}\mathcal{A}f, r_{S_C}(g-f)\big\rangle_{S_C} &= \Big\langle r_{S_C}[\tau^{(n)}(U)]^+, r_{S_C}\,(g^{(1)}-f^{(1)})\Big\rangle_{S_C} + \\ &+\Big\langle r_{S_C}[\mu^{(n)}(U)]^+, r_{S_C}\,(g^{(2)}-f^{(2)})\Big\rangle_{S_C} \geq 0 \\ &\text{for all } \quad g = (g^{(1)}, g^{(2)})^\top \in \mathbb{K}_\varphi. \end{aligned} \tag{3.58}$$

Let us substitute $g = f \pm h$ with $h = (0, h^{(2)})^\top \in [\widetilde{H}^{1/2}(S_C)]^6$ in (3.58) to obtain

$$\begin{aligned}\Big\langle r_{S_C}\mathcal{A}f, r_{S_C}h\Big\rangle_{S_C} &= \\ = \Big\langle r_{S_C}[\mu^{(n)}(U)]^+, r_{S_C}h^{(2)}\Big\rangle_{S_C} &= 0 \ \text{ for all } \ h^{(2)} \in [\widetilde{H}^{1/2}(S_C)]^3.\end{aligned}$$

This implies that $r_{S_C}[\mu^{(n)}(U)]^+ = 0$, i.e., the fifth condition in (3.8) is satisfied. Then from (3.58) in view of (3.55) we derive

$$\Big\langle r_{S_C}[\tau^{(n)}(U)]^+, r_{S_C}(g^{(1)}-f^{(1)})\Big\rangle_{S_C} \geq 0 \ \text{ for all } \ (g^{(1)}, 0)^\top \in \mathbb{K}_\varphi. \tag{3.59}$$

In turn, this inequality can be rewritten in the following equivalent form

$$\begin{aligned}&\Big\langle r_{S_C}[\tau^{(n)}(U)]^+_n, r_{S_C}(g^{(1)}_n - f^{(1)}_n)\Big\rangle_{S_C} + \\ &+\Big\langle r_{S_C}[\tau^{(n)}(U)]^+_t, r_{S_C}(g^{(1)}_t - f^{(1)}_t)\Big\rangle_{S_C} \geq 0,\end{aligned} \tag{3.60}$$

where the subscripts n and t denote the normal and tangential components of the corresponding vectors.

Substituting here $g^{(1)} = f^{(1)} \pm h^{(1)}$, where $h^{(1)} \in [\widetilde{H}^{1/2}(S_C)]^3$ with $h^{(1)}_n = 0$, we arrive at the equation

$$\Big\langle r_{S_C}[\tau^{(n)}(U)]^+_t, r_{S_C}h^{(1)}_t\Big\rangle_{S_C} = 0 \tag{3.61}$$

for arbitrary tangential vector $h^{(1)}_t = h^{(1)} \in [\widetilde{H}^{1/2}(S_C)]^3$. This shows that the fourth equation in (3.8) is fulfilled.

Further, let $g^{(1)} = f^{(1)} - n\vartheta$, where $\vartheta \in \widetilde{H}^{1/2}(S_C)$ with $\vartheta \geq 0$. Clearly, $(g^{(1)}, 0) \in \mathbb{K}_\varphi$ and from (3.60) we then obtain

$$\Big\langle r_{S_C}[\tau^{(n)}(U)]^+_n, r_{S_C}\vartheta\Big\rangle_{S_C} \leq 0 \ \text{ for all } \ \vartheta \in \widetilde{H}^{1/2}(S_C), \ \ \vartheta \geq 0, \tag{3.62}$$

implying that $[\tau^{(n)}(U)]^+_n \leq 0$, i.e., the second condition in (3.8) holds.

The last inequality yields that

$$\Big\langle r_{S_C}[\tau^{(n)}(U)]_n^+, r_{S_C}[u\cdot n]^+ - \varphi\Big\rangle_{S_C} \geq 0, \tag{3.63}$$

since $r_{S_C}[u\cdot n]^+ - \varphi \leq 0$.

On the other hand, let $g^{(1)} = f_t^{(1)} + n\theta$, where $\theta \in H^{1/2}(S)$ with $r_{S_C}\theta = \varphi$ on S_C and $r_{S_D}\theta = 0$ on S_D. We have then $(g^{(1)}, 0) \in \mathbb{K}_\varphi$ and from (3.60) we conclude

$$\Big\langle r_{S_C}[\tau^{(n)}(U)]_n^+, \varphi - r_{S_C}[u\cdot n]^+\Big\rangle_{S_C} \geq 0, \tag{3.64}$$

since $[u^+] = f^{(1)}$. From the inequalities (3.63) and (3.64) the third condition in (3.8) follows. This completes the proof of item (i).

Now, let $U \in \mathbf{K}_\varphi$ be a solution to the unilateral problem (3.5)–(3.8) with $\Phi = 0$. It is evident that U can be represented in the form (3.40) with $f = (f^{(1)}, f^{(2)})^\top := [U]_S^+$. Due to the boundary conditions (3.6) and the first inequality in (3.8) we see that $f = [U]_S^+ \in \mathbb{K}_\varphi$. Further, by Theorem 3.2 the vector U is a solution to the spatial variational inequality (3.15) with $\Phi = 0$:

$$\int\limits_{\Omega^+} E(U, V-U)\,dx \geq \big\langle \Psi, r_{S_T}[V-U]^+\big\rangle_{S_T} \ \text{ for all } \ V \in \mathbf{K}_\varphi. \tag{3.65}$$

With the help of Green's formula the left-hand side integral in (3.65) can be rewritten as

$$\int\limits_{\Omega^+} E(U, V-U)\,dx = \Big\langle [T(\partial,n)U]_S^+, [V]_S^+ - [U]_S^+\Big\rangle_S = \big\langle \mathcal{A}f, g-f\big\rangle_S, \tag{3.66}$$

where $g := [V]_S^+$. Since $V \in \mathbf{K}_\varphi$ is arbitrary, from (3.65) and (3.66) it follows that

$$\big\langle \mathcal{A}f, g-f\big\rangle_S \geq \langle \Psi, r_{S_T}(g-f)\rangle_{S_T} \ \text{ for all } \ g = (g^{(1)}, g^{(2)})^\top \in \mathbb{K}_\varphi.$$

Thus, the vector $f = [U]_S^+$ is a solution of the boundary variational inequality (3.52). □

Note that if f solves the BVI (3.52) then it is also a solution to the following *minimization problem*: Find a minimum on the convex closed set $\mathbb{K}_\varphi$ of the energy functional

$$\mathcal{E}(g) := 2^{-1}\big\langle \mathcal{A}g, g\big\rangle_S - \big\langle \Psi, r_{S_T}g\big\rangle_{S_T}, \tag{3.67}$$

i.e., find $f \in \mathbb{K}_\varphi$ such that

$$\mathcal{E}(f) = \min_{g\in\mathbb{K}_\varphi} \mathcal{E}(g). \tag{3.68}$$

In accordance with Theorems 3.3 and 3.7, and Corollary 3.4 we have the evident existence and equivalence result.

Corollary 3.8. *The unilateral problem* (UP) (3.5)–(3.8)*,* SVI (3.15)*,* BVI (3.52)*, and the minimization problems* (3.38) *and* (3.68) *are equivalent in the sense described in Theorem* 3.7 *provided* $\Phi = 0$*. All this problems are uniquely solvable.*

3.4. **A special particular case.** In this subsection we consider the unilateral problem (3.5)–(3.8) when $S_D = \varnothing$, i.e., $\partial\Omega^+ = \overline{S_T} \cup \overline{S_C}$. In addition, we assume that the submanifold S_C is *neither rotational nor ruled* (i.e., S_C is not a part of some rotational or ruled surface). Thus, now we have to find a vector function $\widetilde{U} = (\widetilde{u}, \widetilde{\omega})^\top \in [H^1(\Omega^+)]^6$ by the following conditions

$$L(\partial)\widetilde{U} = \Phi \ \text{ in } \ \Omega^+, \tag{3.69}$$

$$\big[T(\partial, n)\widetilde{U}\big]^+ = \Psi \ \text{ on } \ S_T, \tag{3.70}$$

$$\left.\begin{aligned} &[\widetilde{u} \cdot n]^+ \le \varphi \\ &\big[\tau^{(n)}(\widetilde{U}) \cdot n\big]^+ \le 0 \\ &\Big\langle r_{S_C}\big[\tau^{(n)}(\widetilde{U}) \cdot n\big]^+, r_{S_C}[\widetilde{u} \cdot n]^+ - \varphi\Big\rangle_{S_C} = 0 \\ &\big[\tau^{(n)}(\widetilde{U})\big]^+ - n\big[\tau^{(n)}(\widetilde{U}) \cdot n\big]^+ = 0 \\ &\big[\mu^{(n)}(\widetilde{U})\big]^+ = 0 \end{aligned}\right\} \ \text{on } S_C, \tag{3.71}$$

with Φ, Ψ, and φ as in (3.9), (3.11), and (3.12).

As we will see, in contrast to the above considered case, the problem (3.69)–(3.71) is not always solvable and the uniqueness theorem does not hold as well. Below we derive the necessary and sufficient conditions of solvability for the problem (3.69)–(3.71) and study the structure of the set of solutions.

To this end we reduce equivalently the problem under consideration to the form which is more convenient for further analysis.

Let us consider the auxiliary boundary value problem

$$L(\partial)U_0 = \Phi \ \text{ in } \ \Omega^+, \quad U_0 := (u_0, \omega_0)^\top \in [H^1(\Omega^+)]^6, \tag{3.72}$$

$$[T(\partial, n)U_0]^+ = 0 \ \text{ on } \ S_T, \tag{3.73}$$

$$\left.\begin{aligned} &[u_0 \cdot n]^+ = \varphi \\ &[\tau^{(n)}(U_0)]^+ - n[\tau^{(n)}(U_0) \cdot n]^+ = 0 \\ &[\mu^{(n)}(U_0)]^+ = 0 \end{aligned}\right\} \ \text{on } S_C. \tag{3.74}$$

This problem has a unique solution for arbitrary Φ and φ if the submanifold S_C is neither rotational nor ruled. In fact, the uniqueness theorem is shown in the Appendix, Subsection A.2, while the existence is a consequence of the Korn's type inequality (2.21). Evidently,

$$[T(\partial,n)U_0]^+_S \in [\widetilde{H}^{-1/2}(S_C)]^6. \tag{3.75}$$

Denote $U := \widetilde{U} - U_0$. Then it is easy to check that U solves the following unilateral problem

$$L(\partial)U = 0 \ \text{ in } \ \Omega^+, \tag{3.76}$$

$$[T(\partial,n)U]^+ = \Psi \ \text{ on } \ S_T, \tag{3.77}$$

$$\left.\begin{aligned} &[u\cdot n]^+ \le 0 \\ &[\tau^{(n)}(U)\cdot n]^+ \le \sigma \\ &\big\langle\, r_{S_C}[\tau^{(n)}(U)\cdot n]^+ - \sigma, r_{S_C}[u\cdot n]^+\big\rangle_{S_C} = 0 \\ &[\tau^{(n)}(U)]^+ - n[\tau^{(n)}(U)\cdot n]^+ = 0 \\ &[\mu^{(n)}(U)]^+ = 0 \end{aligned}\right\} \ \text{on } \ S_C, \tag{3.78}$$

where

$$\sigma = -r_{S_C}[\tau^{(n)}(U_0)\cdot n]^+_S \in r_{S_C}[\widetilde{H}^{-1/2}(S_C)]^6. \tag{3.79}$$

Due to the inclusions (3.11) and (3.79) we have $r_{S_C}[\tau^{(n)}(U)]^+_S \in r_{S_C}[\widetilde{H}^{-1/2}(S_C)]^6$ which implies that the duality relation in (3.78) is correctly defined since $r_{S_C}[U]^+_S \in [H^{1/2}(S_C)]^6$.

Now, let us introduce the convex closed cone of vector functions

$$\mathbf{K}_0 := \Big\{V = (v,w)^\top \in [H^1(\Omega^+)]^6 : \ \ r_{S_C}([v]^+\cdot n) \le 0\Big\} \tag{3.80}$$

and consider the *spatial variational inequality*: Find $U = (u,\omega)^\top \in \mathbf{K}_0$ such that

$$\int\limits_{\Omega^+} E(U, V-U)\,dx \ge$$

$$\ge \langle \Psi, r_{S_T}[V-U]^+\rangle_{S_T} + \langle \sigma, r_{S_C}[v-u]^+\cdot n\rangle_{S_C} \ \text{ for all } \ V\in\mathbf{K}_0. \tag{3.81}$$

By the same arguments as in the proof of Theorem 3.2 we can show that if a vector function U solves the SVI (3.81), then U is a solution to the unilateral problem (3.76)–(3.78), and vice versa.

Further, let

$$\mathbb{K}_0 := \Big\{g = (g^{(1)}, g^{(2)})^\top \in [H^{1/2}(S)]^6 : \ \ r_{S_C}(g^{(1)}\cdot n) \le 0\Big\} \tag{3.82}$$

and consider the *boundary variational inequality*:

Find $f = (f^{(1)}, f^{(2)})^\top \in \mathbb{K}_0$ such that

$$\big\langle \mathcal{A}f, g-f\big\rangle_S \ge \langle \Psi, r_{S_T}(g-f)\rangle_{S_T} + \big\langle \sigma, r_{S_C}(g^{(1)}-f^{(1)})\cdot n\big\rangle_{S_C} \tag{3.83}$$

$$\text{for all } \ g = (g^{(1)}, g^{(2)})^\top \in \mathbb{K}_0,$$

where $\mathcal{A}$ is the Steklov–Poincaré operator. Clearly, $\mathbb{K}_0$ is a convex closed cone.

By the same arguments as in Theorem 3.7 we can prove that if $f \in \mathbb{K}_0$ solves the BVI (3.83) then the vector U given by (3.40) is a solution to the unilateral problem (3.76)–(3.78), and vice versa, if $U \in \mathbf{K}_0$ is a solution of the problem (3.76)–(3.78), then $[U]_S^+ = f$ is a solution of the BVI (3.83).

Similarly, we can show also that if f solves the BVI (3.83) then it is also a solution to the following *minimization problem*: Find a minimum on the convex closed cone $\mathbb{K}_0$ of the energy functional

$$\mathcal{E}_0(g) := 2^{-1}\big\langle \mathcal{A}g, g\big\rangle_S - \langle \Psi, r_{S_T} g\rangle_{S_T} - \langle \sigma, r_{S_C} g^{(1)} \cdot n\rangle_{S_C}. \tag{3.84}$$

Thus, the BVP (3.76)–(3.78), SVI (3.81), BVI (3.83), and the minimization problem (3.84) are equivalent in the sense described above.

In accordance with the general theory (see, e.g., [12], Part 2, Section 10) we have to construct the set $\mathcal{R}$ of vector functions $g \in [H^{1/2}(S)]^6$ satisfying the equation $\langle \mathcal{A}g, g\rangle_S = 0$. By Theorem 3.5

$$\mathcal{R} = \Big\{\chi = (\chi^{(1)}, \chi^{(2)})^\top \in [H^{1/2}(S)]^6 : \\ \chi^{(1)} = [a \times x] + b, \;\; \chi^{(2)} = a, \;\; x \in S, \;\; a,\, b \in \mathbb{R}^3\Big\},$$

i.e., $\mathcal{R}$ is the restriction on S of the space of generalized rigid displacements.

Further, we set

$$\begin{aligned} \mathcal{R}_1 :&= \mathbb{K}_0 \cap \mathcal{R} = \big\{\chi \in \mathcal{R} : \; r_{S_C}\chi^{(1)} \cdot n \le 0\big\}, \\ \mathcal{R}_1^* :&= \big\{\chi \in \mathcal{R}_1 : \; r_{S_C}\chi^{(1)} \cdot n = 0\big\} = \{0\}. \end{aligned} \tag{3.85}$$

The last equality is a consequence of the fact that if $([a\times x]+b)\cdot n(x) = 0$ for $x \in S_C$, then S_C is *either rotational or ruled.*

Let $f = (f^{(1)}, f^{(2)})^\top \in \mathbb{K}_0$ be a solution of the boundary variational inequality (3.83). Then for arbitrary $\chi = (\chi^{(1)}, \chi^{(2)})^\top \in \mathcal{R}_1$ we see that $\widetilde{f} = f + \chi = (f^{(1)} + \chi^{(1)}, f^{(2)} + \chi^{(2)})^\top \in \mathbb{K}_0$. Therefore, from (3.83) with $g = \widetilde{f}$ we get

$$\big\langle \mathcal{A}f, \chi\big\rangle_S \ge \langle \Psi, r_{S_T}\chi\rangle_{S_T} + \langle \sigma, r_{S_C}\chi^{(1)} \cdot n\rangle_{S_C} \;\text{ for all }\; \chi \in \mathcal{R}_1. \tag{3.86}$$

Since $\mathcal{A}$ is self-adjoint, by Theorem 3.5 it follows that $\langle \mathcal{A}f, \chi t\rangle_S = \langle \mathcal{A}\chi, f\rangle_S = 0$ for every $\chi \in \mathcal{R}$. Therefore we arrive at the following necessary condition for the BVI (3.83) to be solvable,

$$\langle \Psi, r_{S_T}\chi\rangle_{S_T} + \langle \sigma, r_{S_C}\chi^{(1)} \cdot n\rangle_{S_C} \le 0 \;\text{ for all }\; \chi \in \mathcal{R}_1. \tag{3.87}$$

On the other hand, if the condition (3.87) holds in the *strong sense*, which means that in (3.87) the equality appears if and only if $\chi \in \mathcal{R}_1^*$ (i.e., if and only if $\chi = 0$), then the minimization problem for the functional (3.84) and consequently the BVI (3.83) are solvable due to Theorem 10.1 in the reference [12], Part 2, Section 10. Thus we have the following assertion.

Theorem 3.9. *The inequality* (3.87) *is a necessary condition for the* BVI (3.83) *to be solvable. If* (3.87) *holds in the strong sense, then the* BVI (3.83) *is solvable.*

Let us study the uniqueness question. If $f = (f^{(1)}, f^{(2)})^\top \in \mathbb{K}_0$ and $h = (h^{(1)}, h^{(2)})^\top \in \mathbb{K}_0$ are solutions of the BVI (3.83), then we have

$$\langle \mathcal{A}f, h-f\rangle_S \geq \big\langle \Psi, r_{S_T}(h-f)\big\rangle_{S_T} + \big\langle \sigma, r_{S_C}(h^{(1)}-f^{(1)})\cdot n\big\rangle_{S_C}, \quad (3.88)$$

$$\langle \mathcal{A}h, f-h\rangle_S \geq \big\langle \Psi, r_{S_T}(f-h)\big\rangle_{S_T} + \big\langle \sigma, r_{S_C}(f^{(1)}-h^{(1)})\cdot n\big\rangle_{S_C}. \quad (3.89)$$

By adding of these inequalities and with the help of the property (3.45) we derive

$$\langle \mathcal{A}(h-f), h-f\rangle_S = 0,$$

which implies that $\widetilde{\chi} = (\widetilde{\chi}^{(1)}, \widetilde{\chi}^{(2)})^\top := h-f \in \mathcal{R}$ due to Theorem 3.5. Note that $h = f + \widetilde{\chi} \in \mathbb{K}_0$. Since $\langle \mathcal{A}f, \widetilde{\chi}\rangle_S = 0$ and $\langle \mathcal{A}h, \widetilde{\chi}\rangle_S = 0$ we get from (3.88) and (3.89)

$$\langle \Psi, r_{S_T}\widetilde{\chi}\rangle_{S_T} + \big\langle \sigma, r_{S_C}\widetilde{\chi}^{(1)}\cdot n\big\rangle_{S_C} \leq 0, \quad \langle \Psi, r_{S_T}\widetilde{\chi}\rangle_{S_T} + \big\langle \sigma, r_{S_C}\widetilde{\chi}^{(1)}\cdot n\big\rangle_{S_C} \geq 0,$$

that is,

$$\langle \Psi, r_{S_T}\widetilde{\chi}\rangle_{S_T} + \big\langle \sigma, r_{S_C}\widetilde{\chi}^{(1)}\cdot n\big\rangle_{S_C} = 0. \quad (3.90)$$

Thus we have shown the following assertion.

Theorem 3.10. *Let f be a solution of the* BVI (3.83). *Then f is defined modulo a generalized rigid displacement vector $\widetilde{\chi} = (\widetilde{\chi}^{(1)}, \widetilde{\chi}^{(2)})^\top \in \mathcal{R}$ such that $f + \widetilde{\chi} \in \mathbb{K}_0$ and the condition* (3.90) *holds.*

Remark 3.11. It is evident that if $\sigma := 0$ on S_C and $\langle \Psi, r_{S_T}\chi\rangle_{S_T} = 0$ for arbitrary rigid displacement vector χ, i.e., Ψ satisfies the generalized equilibrium conditions on S_T, then the necessary condition (3.87) holds but not in the strong sense. In this case, we can not say anything about the solvability of the variational inequality (3.83). Therefore, the condition $S_D \neq \varnothing$ is crucial for existence and uniqueness results formulated in Corollary 3.8.

4. Appendix

A.1. **Properties of potentials and boundary pseudodifferential operators**. The fundamental matrix $\Gamma(x)$ for the differential operator $L(\partial)$ reads as follows (this matrix can be obtained by standard limiting procedure, as the frequency parameter $\sigma \to 0$, from the fundamental matrix of the pseudo-oscillation equations constructed in [29]):

$$\begin{aligned} \Gamma(x) &= \begin{Vmatrix} \Gamma^{(1)}(x) & \Gamma^{(2)}(x) \\ \Gamma^{(3)}(x) & \Gamma^{(4)}(x) \end{Vmatrix}_{6\times 6}, \\ \Gamma^{(m)}(x) &= \big\|\Gamma^{(m)}_{kj}(x)\big\|_{3\times 3}, \quad m = 1,2,3,4, \end{aligned} \quad (A.1)$$

where

$$\Gamma^{(1)}_{kj}(x)=-\frac{1}{4\pi}\bigg\{\Big[\frac{\gamma+\varepsilon}{d_1}\,\frac{1}{|x|}+\sum_{l=1}^{2}k_l^2c_{1l}\,\frac{e^{-k_l|x|}-1}{|x|}\Big]\delta_{kj}-$$
$$-\frac{\partial^2}{\partial x_k\partial x_j}\Big[\frac{\lambda+\mu}{\mu(\lambda+2\,\mu)}\,\frac{|x|}{2}+\sum_{l=1}^{3}c_{1l}\,\frac{e^{-k_l\,|x|}-1}{|x|}\Big]+$$
$$+\sum_{l,p=1}^{3}c_{2l}\varepsilon_{kjp}\,\frac{\partial}{\partial x_p}\,\frac{e^{-k_l|x|}-1}{|x|}\bigg\},$$

$$\Gamma^{(2)}_{kj}(x)=\Gamma^{(3)}_{kj}(x)=-\frac{1}{4\pi}\bigg\{-\Big[\frac{\kappa+\nu}{d_1}\,\frac{1}{|x|}+\sum_{l=1}^{2}k_l^2c_{3l}\,\frac{e^{-k_l\,|x|}-1}{|x|}\Big]\delta_{kj}+$$
$$+\frac{\partial^2}{\partial x_k\partial x_j}\sum_{l=1}^{3}c_{3l}\,\frac{e^{-k_l|x|}-1}{|x|}+\sum_{l,p=1}^{3}c_{4l}\varepsilon_{kjp}\,\frac{\partial}{\partial x_p}\,\frac{e^{-k_l|x|}-1}{|x|}\bigg\},$$

$$\Gamma^{(4)}_{kj}(x)=-\frac{1}{4\pi}\bigg\{\Big[\frac{\mu+\alpha}{d_1}\,\frac{1}{|x|}+\sum_{l=1}^{2}k_l^2c_{5l}\,\frac{e^{-k_l|x|}-1}{|x|}\Big]\delta_{kj}-$$
$$-\frac{\partial^2}{\partial x_k\partial x_j}\sum_{l=1}^{3}c_{5l}\,\frac{e^{-k_l\,|x|}-1}{|x|}+\sum_{l,p=1}^{3}c_{6l}\varepsilon_{kjp}\,\frac{\partial}{\partial x_p}\,\frac{e^{-k_l|x|}-1}{|x|}\bigg\}.$$

Here d_1 and d_2 are defined by (2.18), and

$$c_{1l}=\frac{c_l}{d_1}\,(k_l^2-k_3^2)\Big\{\big[(\gamma+\varepsilon)k_l^2-4\alpha\big](k_l^2-\lambda_1^2)+4\nu d_3k_l^2\Big\},$$
$$c_{2l}=\frac{c_lk_l^2}{d_1}\,(k_l^2-k_3^2)\Big\{\big[d_3(\gamma+\varepsilon)-4\nu\big]k_l^2+\frac{16\alpha^2\kappa}{d_1}\Big\},$$
$$c_{3l}=\frac{c_lk_l^2}{d_1}\,(k_l^2-k_3^2)\big[(k_l^2-\lambda_1^2)(\kappa+\nu)-2\alpha d_3\big],$$
$$c_{4l}=\frac{c_lk_l^2}{d_1}\,(k_l^2-k_3^2)\big[2\alpha(k_l^2-\lambda_1^2)-d_3k_l^2(\kappa+\nu)\big],$$
$$c_{5l}=\frac{c_lk_l^2}{d_1}\,(\mu+\alpha)(k_l^2-k_3^2)(k_l^2-\lambda_1^2),$$
$$c_{6l}=\frac{c_ld_3k_l^4}{d_1}\,(\mu+\alpha)(k_l^2-k_3^2),\quad l=1,2,$$
$$c_{13}=-\frac{(\delta+2\kappa)^2}{4\alpha(\lambda+2\mu)}\,,\quad c_{33}=-\frac{\delta+2\kappa}{4\alpha(\lambda+2\mu)^2}\,,$$
$$c_{53}=-\frac{1}{4\alpha}\,,\quad c_{23}=c_{43}=c_{63}=0,$$
$$c_q=\frac{1}{k_q^4(k_{q+1}^2-k_q^2)(k_{q+2}^2-k_q^2)}\,,\quad q=1,2,3,\quad k_4:=k_1,\quad k_5:=k_2,$$
$$d_3=\frac{4(\mu\nu-\alpha\kappa)}{d_1}\,,\quad \lambda_1^2=\frac{4\alpha\mu}{d_1}\,,\quad k_3^2=\frac{4\alpha(\lambda+2\mu)}{d_2}\,,$$

$$k_1^2+k_2^2=2\lambda_1^2-d_3^2,\quad k_1^2k_2^2=\lambda_1^4,\quad k_1=\overline{k_2}=a+ib,\quad a>0,\ b\in\mathbb{R}.$$

With the help of the relations

$$\sum_{l=1}^{3}c_l k_l^2=\lambda^{-4}k_3^{-2},\quad \sum_{l=1}^{3}c_l k_l^{2m}=0\ \text{ for }\ m=2,3,$$

$$\sum_{l=1}^{3}c_l k_l^8=1,\quad \sum_{l=1}^{3}c_{1l}k_l^2=-\frac{\beta+2\gamma}{d_2}+\frac{\gamma+\varepsilon}{d_1}+\frac{\lambda+\mu}{\mu(\lambda+2\,\mu)},$$

$$\sum_{l=1}^{3}c_{3l}k_l^2=-\frac{\delta+2\kappa}{d_2}+\frac{\nu+\kappa}{d_1}\,,\quad \sum_{l=1}^{3}c_{5l}k_l^2=-\frac{\lambda+2\mu}{d_2}+\frac{\mu+\alpha}{d_1}\,,$$

it can easily be checked that for sufficiently large $|x|$ (as $|x|\to+\infty$) and for arbitrary multi-index $\sigma=(\sigma_1,\sigma_2,\sigma_3)$ we have the following asymptotic behaviour

$$\partial^\sigma\Gamma_{kj}(x)=\begin{cases}\mathcal{O}(|x|^{-1-|\sigma|}) & \text{for }\ k,j=1,2,3,\\ \mathcal{O}(|x|^{-2-|\sigma|}) & \text{for either }\ k\geq 4\ \text{ or }\ j\geq 4.\end{cases}$$

Here σ_j are nonnegative integers, $\partial^\sigma=\partial_1^{\sigma_1}\partial_2^{\sigma_2}\partial_3^{\sigma_3}$, and $|\sigma|=\sigma_1+\sigma_2+\sigma_3\geq 0$.

We remark also that

$$\Gamma(y-x)=[\,\Gamma(x-y)\,]^\top.$$

The corresponding single layer and double layer potentials, and the Newton type volume potential read as follows [29]

$$V(\varphi)(x)=\int\limits_S\Gamma(x-y)\varphi(y)\,dS_y,\quad x\in\mathbb{R}^3\setminus S, \tag{A.2}$$

$$W(\varphi)(x)=\int\limits_S\big[T(\partial_y,n(y))\Gamma(y-x)\big]^\top\varphi(y)\,dS_y,\quad x\in\mathbb{R}^3\setminus S, \tag{A.3}$$

$$N_\Omega(\psi)(x)=\int\limits_\Omega\Gamma(x-y)\psi(y)\,dy,\quad x\in\mathbb{R}^3, \tag{A.4}$$

where $T(\partial,n)$ is the stress operator of the theory of hemitropic elasticity (see (2.4)), $\varphi=(\varphi_1,\dots,\varphi_6)^\top$ is a density vector-function defined on $S=\partial\Omega$, while a density vector-function $\psi=(\psi_1,\dots,\psi_6)^\top$ is defined in $\Omega\in\{\Omega^+,\Omega^-\}$.

These potentials have the jump and mapping properties described by the following theorems (for details see [29]).

Theorem 4.1. *Let $U\in[H^1(\Omega^+)]^6$ with $L(\partial)U\in[L_2(\Omega^+)]^6$. Then there holds the following integral representation formula*

$$W([U]^+)(x)-V([TU]^+)(x)+N_{\Omega^+}(L(\partial)U)(x)=$$
$$=\begin{cases}U(x) & \textit{for }\ x\in\Omega^+,\\ 0 & \textit{for }\ x\in\Omega^-.\end{cases} \tag{A.5}$$

Theorem 4.2. *Let $S \in C^{k+1,\tau}$ where $k \geq 0$ is an integer, $0 < \tau \leq 1$, and let $0 < t < \tau$. Then the operators*

$$V : [C^{k,t}(S)]^6 \to \big[C^{k+1,t}(\overline{\Omega^\pm})\big]^6, \quad W : [C^{k,t}(S)]^6 \to \big[C^{k,t}(\overline{\Omega^\pm})\big]^6, \tag{A.6}$$

are bounded.

For any $g \in C^{k,t}(S)$ and any $x \in S$

$$[V(g)(x)]^\pm = V(g)(x) = \mathcal{H}g(x), \tag{A.7}$$

$$[T(\partial_x, n(x))V(g)(x)]^\pm = [\mp 2^{-1} I_6 + \mathcal{K}]g(x), \tag{A.8}$$

$$[W(g)(x)]^\pm = [\pm 2^{-1} I_6 + \mathcal{K}^*]g(x), \tag{A.9}$$

$$[T(\partial_x, n(x))W(g)(x)]^+ = [T(\partial_x, n(x))W(g)(x)]^- = \mathcal{L}g(x), \tag{A.10}$$

where

$$\mathcal{H}g(x) := \int\limits_S \Gamma(x-y)g(y)\, dS_y, \tag{A.11}$$

$$\mathcal{K}g(x) := \int\limits_S T(\partial_x, n(x))\Gamma(x-y)g(y)\, dS_y, \tag{A.12}$$

$$\mathcal{K}^*\, g(x) := \int\limits_S \big[T(\partial_y, n(y))\Gamma(y-x)\big]^\top g(y)\, dS_y, \tag{A.13}$$

$$\mathcal{L}g(x) := \lim_{\Omega^\pm \ni z \to x \in S} T(\partial_z, n(x)) \int\limits_S \big[T(\partial_y, n(y))\Gamma(y-z)\big]^\top g(y)\, dS_y. \tag{A.14}$$

The operators V and W can be extended by continuity to the bounded mappings

$$V : [H^{-1/2}(S)]^6 \to [H^1(\Omega^+)]^6 \quad \Big[[H^{-1/2}(S)]^6 \to [H^1_{loc}(\Omega^-)]^6\Big],$$

$$W : [H^{1/2}(S)]^6 \to [H^1(\Omega^+)]^6 \quad \Big[[H^{1/2}(S)]^6 \to [H^1_{loc}(\Omega^-)]^6\Big].$$

The jump relations (A.7)–(A.10) *on S remain valid for the extended operators in the corresponding functional spaces.*

Denote by $X_{\Omega^+}\{\Lambda^{(1)}, \Lambda^{(2)}, \dots, \Lambda^{(6)}\}$ the linear span of vectors of generalized rigid displacements in a region Ω^+, where, for definiteness, we assume that

$$\Lambda^{(1)} = (0, -x_3, x_2, 1, 0, 0)^\top, \quad \Lambda^{(2)} = (x_3, 0, -x_1, 0, 1, 0)^\top,$$
$$\Lambda^{(3)} = (-x_2, x_1, 0, 0, 0, 1)^\top, \quad \Lambda^{(4)} = (1, 0, 0, 0, 0, 0)^\top,$$
$$\Lambda^{(5)} = (0, 1, 0, 0, 0, 0)^\top, \quad \Lambda^{(6)} = (0, 0, 1, 0, 0, 0)^\top.$$

The restriction of the space $X_{\Omega^+}\{\Lambda^{(1)}, \Lambda^{(2)}, \dots, \Lambda^{(6)}\}$ onto the boundary $S = \partial\Omega$ we denote by $X_S\{\Lambda^{(1)}, \Lambda^{(2)}, \dots, \Lambda^{(6)}\}$. Clearly, the vectors $\{\Lambda^{(j)}\}_{j=1}^6$ are basis in both spaces X_{Ω^+} and X_S.

Theorem 4.3. *Let S, k, t, and τ be as in Theorem* 6.1. *Then the operators*

$$\mathcal{H} : [C^{k,t}(S)]^6 \to [C^{k+1,t}(S)]^6 \quad \Big[[H^{-1/2}(S)]^6 \to [H^{1/2}(S)]^6 \Big], \tag{A.15}$$

$$\mathcal{K} : [C^{k,t}(S)]^6 \to [C^{k,t}(S)]^6 \quad \Big[[H^{-1/2}(S)]^6 \to [H^{-1/2}(S)]^6 \Big], \tag{A.16}$$

$$\mathcal{K}^* : [C^{k,t}(S)]^6 \to [C^{k,t}(S)]^6 \quad \Big[[H^{1/2}(S)]^6 \to [H^{1/2}(S)]^6 \Big], \tag{A.17}$$

$$\mathcal{L} : [C^{k+1,t}(S)]^6 \to [C^{k,t}(S)]^6 \quad \Big[[H^{1/2}(S)]^6 \to [H^{-1/2}(S)]^6 \Big] \tag{A.18}$$

are bounded.

Moreover,

(i) *$\mathcal{H}$, $\pm 2^{-1} I_6 + \mathcal{K}$, $\pm 2^{-1} I_6 + \mathcal{K}^*$, and $\mathcal{L}$ are elliptic pseudodifferential operators of order* -1, 0, 0, *and* 1, *respectively;*

(ii) *$\pm 2^{-1} I_6 + \mathcal{K}$ and $\pm 2^{-1} I_6 + \mathcal{K}^*$ are mutually adjoint singular integral operators of normal type with index equal to zero. The operators $\mathcal{H}$, $2^{-1} I_6 + \mathcal{K}$ and $2^{-1} I_6 + \mathcal{K}^*$ are invertible. The inverse of $\mathcal{H}$*

$$\mathcal{H}^{-1} : [C^{k+1,t}(S)]^6 \to [C^{k,t}(S)]^6 \quad \Big[[H^{1/2}(S)]^6 \to [H^{-1/2}(S)]^6 \Big]$$

is a singular integro-differential operator.

The null space of the operator $-2^{-1} I_6 + \mathcal{K}^*$ is $X_S\{\Lambda^{(1)}, \Lambda^{(2)}, \dots, \Lambda^{(6)}\}$;

(iii) *$\mathcal{L}$ is a singular integro-differential operator and the following equalities hold in appropriate function spaces:*

$$\mathcal{K}^* \mathcal{H} = \mathcal{H} \mathcal{K}, \;\; \mathcal{L} \mathcal{K}^* = \mathcal{K} \mathcal{L}, \;\; \mathcal{H} \mathcal{L} = -4^{-1} I_6 + (\mathcal{K}^*)^2, \;\; \mathcal{L} \mathcal{H} = -4^{-1} I_6 + \mathcal{K}^2;$$

(iv) *The operators $-\mathcal{H}$ and $\mathcal{L}$ are self-adjoint and non-negative elliptic pseudodifferential operators with positive definite principal symbol matrices and with index equal to zero. Moreover, $\langle h, -\mathcal{H} h \rangle_S \geq c_0 \|h\|_{[H^{-1/2}(S)]^6}$ for all $h \in [H^{-1/2}(S)]^6$ and $\langle \mathcal{L} g, g \rangle_S \geq 0$ for all $g \in [H^{1/2}(S)]^6$ with equality only for*

$$g = ([a \times x] + b, a)^\top, \tag{A.19}$$

where $a, b \in \mathbb{R}^3$ are arbitrary constant vectors; here $\langle \cdot\,, \cdot \rangle_S$ denotes the duality between the spaces $[H^{-1/2}(S)]^6$ and $[H^{1/2}(S)]^6$ which extends the usual $[L_2(S)]^6$-scalar product;

(v) *a general solution of the homogeneous equations $[-2^{-1} I_6 + \mathcal{K}^*] g = 0$ and $\mathcal{L} g = 0$ is given by* (A.19) (*i.e., the operators $\mathcal{L}$, $-2^{-1} I_6 + \mathcal{K}^*$, and $-2^{-1} I_6 + \mathcal{K}$ have six dimensional null-spaces*).

A.2. **Uniqueness theorem for the** BVP (3.72)–(3.74).

Theorem 4.4. *The homogeneous* BVP (3.72)–(3.74) *has only the trivial solution if the submanifold S_C is neither rotational nor ruled.*

Proof. Let $U_0 := (u_0,\, \omega_0)^\top$ be a solution to the homogeneous BVP (3.72)–(3.74) with $\Phi = 0$ and $\varphi = 0$. From Green's formula (2.12) with $U = U' =$

U_0 and Lemma 2.1 then we have

$$u_0(x) = [a \times x] + b, \quad \omega_0(x) = a, \ \ x \in \Omega^+, \tag{A.20}$$

where a, $b \in \mathbb{R}^3$ are arbitrary three-dimensional constant vectors, i.e., U_0 is a generalized rigid displacement vector. It is evident that all conditions of the homogeneous BVP (3.72)–(3.74) are automatically satisfied except the first equation in (3.74) with $\varphi = 0$, implying the following restriction

$$u_0(x) \cdot n(x) = ([a \times x] + b) \cdot n(x) = 0, \ \ x \in S_C, \tag{A.21}$$

where, as above, $n(x)$ is the exterior unit normal vector at the point $x \in S$. In what follows we shall show that if (A.21) holds and $|a| + |b| \neq 0$, then S_C is either rotational or ruled submanifold. We prove it in several steps.

Step 1. First we assume that $a = 0$ and $b \neq 0$. Without loss of generality we set $b_3 \neq 0$. From (A.21) then we get

$$b \cdot n(x) = b_1 n_1(x) + b_2 n_2(x) + b_3 n_3(x) = 0, \ \ x \in S_C. \tag{A.22}$$

Let $\zeta(x_1, x_2, x_3) = 0$ be an equation of the submanifold S_C in some neighbourhood of a point $x_0 \in S_C$. Then $n(x)$ is parallel to the vector $\nabla_x \zeta(x) = (\partial_1 \zeta(x), \partial_2 \zeta(x), \partial_3 \zeta(x))$. Therefore, by (A.22) we arrive at the partial differential equation of the first order

$$b_1 \frac{\partial \zeta(x)}{\partial x_1} + b_2 \frac{\partial \zeta(x)}{\partial x_2} + b_3 \frac{\partial \zeta(x)}{\partial x_3} = 0. \tag{A.23}$$

The corresponding system of characteristic equations is (see, e.g., [7])

$$\frac{dx_1}{b_1} = \frac{dx_2}{b_2} = \frac{dx_3}{b_3}\,. \tag{A.24}$$

Evidently, the first integrals of the system of ordinary differential equations (A.24) are the functions

$$x_1 - \alpha x_3 = C_1, \quad x_2 - \beta\, x_3 = C_2,$$

where $\alpha = b_1/b_3$, $\beta = b_2/b_3$, and C_1 and C_2 are arbitrary constants.

Therefore, $\widetilde{\zeta}(x) := F(x_1 - \alpha x_3, x_2 - \beta\, x_3)$ with an arbitrary differentiable function $F(\cdot\,,\cdot)$ is a general solution of the differential equation (A.23). Moreover, it is evident that if $\nabla F \neq 0$, then $\widetilde{\zeta}(x) = F(x_1 - \alpha x_3, x_2 - \beta x_3) = 0$ defines a two-dimensional manifold. Performing the linear transformation,

$$x_1 = x_1' + \alpha x_3', \quad x_2 = x_2' + \beta x_3', \quad x_3 = x_3',$$

it is easy to see that $\widetilde{\zeta}(x) = F(x_1 - \alpha x_3, x_2 - \beta x_3) = F(x_1', x_2') = 0$ describes a cylindrical surface with directrix parallel to the x_3' axis, i.e., parallel to the vector $(\alpha, \beta, 1) = (1/b_3) b$. Thus, the above mentioned equation $\zeta(x) = 0$ of the submanifold S_C, as a particular case of the general equation $\widetilde{\zeta}(x) = 0$, defines a cylindrical manifold with a directrix parallel to b (note that $\nabla \zeta(x) \neq 0$). Since S_C is not ruled submanifold we conclude that $b = 0$.

Step 2. Now, let $a \neq 0$. Represent b as $b = c + d$, where $c,\ d \in \mathbb{R}^3$ and c is parallel to a, while d is perpendicular to a. Then there is a point x_0 such that $d = -[a \times x_0]$ and due to (A.20) we have

$$u_0(x) = [a \times (x - x_0)] + c, \quad x \in \Omega^+. \tag{A.25}$$

Then from (A.21)

$$[a \times (x - x_0)] \cdot n(x) + c \cdot n(x) = 0, \quad x \in S_C. \tag{A.26}$$

Sub-step 2.1. We first consider the case $c \neq 0$ and make an orthogonal transformation of the co-ordinate system, $x = \Lambda\,\xi + x_0$ such that the new axis ξ_1 is directed parallel to the vector a and the new origin coincides with the point x_0. Clearly, $\Lambda = [\lambda_{kj}]_{3\times 3}$ is an orthogonal matrix with the properties

$$\det \Lambda = 1, \ \ \Lambda^{-1} = \Lambda^\top, \ \ \lambda_{kj} = \lambda^*_{kj}, \ \ a = \Lambda(|a|, 0, 0)^\top, \ \ c = \Lambda(|c|, 0, 0)^\top, \tag{A.27}$$

where λ^*_{kj} is the co-factor of the element λ_{kj} , $|a|$ and $|c|$ stand for the lengths of the vectors a and c respectively. Set $\widetilde{a} := (|a|, 0, 0)^\top$ and $\widetilde{c} := (|c|, 0, 0)^\top$. Note that $[\Lambda z \times \Lambda y] = \Lambda[z \times y]$ for arbitrary $z,\ y \in \mathbb{R}^3$ and for an arbitrary orthogonal matrix Λ. Applying the above identities from (A.26) we have

$$\begin{aligned}
&\Big\{[a \times (x - x_0)] \cdot n(x) + c \cdot n(x)\Big\}_{x=\Lambda\xi + x_0} = \\
&= \Big\{\Lambda\big[\Lambda^\top a \times \Lambda^\top (x - x_0)\big] \cdot n(x) + \Lambda\Lambda^\top c \cdot n(x)\Big\}_{x=\Lambda\xi + x_0} = \\
&= \Big\{\big[\Lambda^\top a \times \Lambda^\top\,(x - x_0)\big] \cdot \Lambda^\top n(x) + \Lambda^\top c \cdot \Lambda^\top n(x)\Big\}_{x=\Lambda\xi + x_0} = \\
&= \Big\{\Big([\widetilde{a} \times \xi] + \widetilde{c}\Big) \cdot \Lambda^\top n(x)\Big\}_{x=\Lambda\xi + x_0} = \\
&= \ell(\xi) \cdot \widetilde{n}(\xi) = 0, \quad \xi \in S_C,
\end{aligned} \tag{A.28}$$

where $\ell(\xi) := [\widetilde{a} \times \xi] + \widetilde{c} = (|c|, -|a|\xi_3, |a|\xi_2)$ and $\widetilde{n}(\xi) := \Lambda^\top n(\Lambda\xi + x_0)$ is the outward unit normal vector at the point $\xi \in S_C$.

Further, let $\zeta(\xi_1, \xi_2, \xi_3) = 0$ be an equation of the submanifold S_C in some neighbourhood of a point $\xi_0 \in S_C$. Then $\widetilde{n}(\xi)$ is parallel to the vector $\nabla_\xi \zeta(\xi) = (\partial_1\zeta(\xi), \partial_2\zeta(\xi), \partial_3\zeta(\xi))$ and by (A.26) and (A.28) we arrive at the partial differential equation of the first order

$$|c|\,\frac{\partial\zeta(\xi)}{\partial\xi_1} - |a|\xi_3\,\frac{\partial\zeta(\xi)}{\partial\xi_2} + |a|\xi_2\,\frac{\partial\zeta(\xi)}{\partial\xi_3} = 0, \tag{A.29}$$

i.e.,

$$e\,\frac{\partial\zeta(\xi)}{\partial\xi_1} - \xi_3\,\frac{\partial\zeta(\xi)}{\partial\xi_2} + \xi_2\,\frac{\partial\zeta(\xi)}{\partial\xi_3} = 0 \ \text{ with } \ e = \frac{|c|}{|a|}\,. \tag{A.30}$$

The simultaneous characteristic equations are

$$\frac{d\xi_1}{e} = -\frac{d\xi_2}{\xi_3} = \frac{d\xi_3}{\xi_2}\,. \tag{A.31}$$

It can easily be shown that

$$\begin{aligned}\eta_1(\xi) : &= -\xi_2 \sin\frac{\xi_1}{e} + \xi_3 \cos\frac{\xi_1}{e} = C_1,\\ \eta_2(\xi) : &= \xi_2 \cos\frac{\xi_1}{e} + \xi_3 \sin\frac{\xi_1}{e} = C_2,\end{aligned} \tag{A.32}$$

where C_1 and C_2 are arbitrary constants, are the first integrals of the system of ordinary differential equations (A.31). Therefore, a function

$$\widetilde{\zeta}(\xi) := F(\eta_1(\xi), \eta_2(\xi)) = F\Big(- \xi_2 \sin\frac{\xi_1}{e} + \xi_3 \cos\frac{\xi_1}{e}\, , \xi_2 \cos\frac{\xi_1}{e} + \xi_3 \sin\frac{\xi_1}{e}\Big)$$

with an arbitrary differentiable function $F(\cdot\,,\cdot)$ is a general solution of the differential equation (A.30).

Note that the equation $\zeta(\xi_1, \xi_2, \xi_3) = 0$ of the submanifold S_C is a particular case of the general equation $\widetilde{\zeta}(\xi) := F(\eta_1(\xi), \eta_2(\xi)) = 0$ with $\nabla F(\eta_1, \eta_2) \neq 0$. This relation defines a two-dimensional manifold $\mathcal{M}$ in a neighbourhood of some point $\xi_0 \in \mathcal{M}$. Further, by the implicit function theorem we derive that either $\eta_1(\xi) = h_1(\eta_2(\xi))$ or $\eta_2(\xi) = h_2(\eta_1(\xi))$, i.e., either

$$-\xi_2 \sin\frac{\xi_1}{e} + \xi_3 \cos\frac{\xi_1}{e} = h_1\Big(\xi_2 \cos\frac{\xi_1}{e} + \xi_3 \sin\frac{\xi_1}{e}\Big)$$

or

$$\xi_2 \cos\frac{\xi_1}{e} + \xi_3 \sin\frac{\xi_1}{e} = h_2\Big(- \xi_2 \sin\frac{\xi_1}{e} + \xi_3 \cos\frac{\xi_1}{e}\Big).$$

We easily see that in both cases the functions $h_k(\,\cdot\,)$ are linear functions in ξ_2 and ξ_3, since the second order partial derivatives of the functions h_k with respect to the variables ξ_2 and ξ_3 vanish identically. Therefore, the intersection of the manifold $\mathcal{M}$ with the plane $\xi_1 = \text{const}$ is a straight line segment defined by the above linear relationship between the variables ξ_2 and ξ_3. This proves that $\mathcal{M}$ is a ruled manifold. Since, by assumption S_C is not ruled, we conclude that $c = 0$ in (A.25), and consequently in (A.26).

Sub-step 2.2. From (A.26) with $c = 0$, by the same arguments as above we arrive at the equation (A.29) with $|c| = 0$, that is,

$$-\xi_3 \,\frac{\partial\zeta(\xi)}{\partial\xi_2} + \xi_2 \,\frac{\partial\zeta(\xi)}{\partial\xi_3} = 0, \tag{A.33}$$

due to the assumption $|a| \neq 0$. Here $\zeta(\xi_1, \xi_2, \xi_3) = 0$ is again an equation of the submanifold S_C in some neighbourhood of a point $\xi_0 \in S_C$. The characteristic equations now read as follows

$$\frac{d\xi_1}{0} = -\frac{d\xi_2}{\xi_3} = \frac{d\xi_3}{\xi_2}\,. \tag{A.34}$$

The corresponding first integrals are

$$\xi_1 = C_1, \quad \xi_2^2 + \xi_3^2 = C_2,$$

where C_1 and C_2 are arbitrary constants. A function

$$\widetilde{\zeta}(\xi) := F(\xi_1, \xi_2^2 + \xi_3^2)$$

with an arbitrary differentiable function $F(\cdot\,,\cdot)$, is then a general solution of the differential equation (A.33). If we assume that $\nabla F(\eta_1,\eta_2)\neq 0$ then the equation $\widetilde{\zeta}(\xi)=F(\xi_1,\xi_2^2+\xi_3^2)=0$ defines a rotational submanifold $\mathcal{M}$ with the axis of rotation parallel to the co-ordinate axis ξ_1, i.e., to the vector a.

Therefore, the function $\zeta(\xi)$, as a particular solution of the equation (A.33), belongs to the family of functions of the form $F(\xi_1,\xi_2^2+\xi_3^2)$, and consequently, the equation $\zeta(\xi_1,\xi_2,\xi_3)=0$ describes a rotational surface. Since the submanifold S_C is not rotational, we conclude that $a=0$.

Thus, we have shown that if S_C is neither rotational nor ruled submanifold, then $a=b=0$. Due to (A.20) this proves that the homogeneous BVP (3.72)–(3.74) possesses only the trivial solution. □

References

1. É. L. Aéro and E. V. Kuvshinski, Continuum theory of asymmetric elasticity. The problem of "internal" rotation. (Russian) *Fiz. Tverd. Tela* **5**, 2591–2598; English transl.: *Soviet Physics Solid State* **5**(1963), 1892–1897.
2. É. L. Aéro and E. V. Kuvshinski, Continuum theory of asymmetric elasticity. Equilibrium of an isotropic body. (Russian) *Fiz. Tverd. Tela* **6**, 2689–2699; English transl.: *Soviet Physics Solid State* **6**(1965), 2141–2148.
3. C. Athanasiadis, P. A. Martin, and I. G. Stratis, Electromagnetic scattering by a homogeneous chiral obstacle: scattering relations and the far-field operator. *Math. Methods Appl. Sci.* **22**(1999), No. 14, 1175–1188.
4. C. Athanasiadis, P. A. Martin, and I. G. Stratis, Electromagnetic scattering by a homogeneous chiral obstacle: boundary integral equations and low-chirality approximations. *SIAM J. Appl. Math.* **59**(1999), No. 5, 1745–1762 (electronic).
5. P. G. Ciarlet, Mathematical elasticity. Vol. I. Three-dimensional elasticity. *Studies in Mathematics and its Applications,* 20*, North-Holland Publishing Co., Amsterdam*, 1988.
6. E. Cosserat and F. Cosserat, Sur les équations de la théorie de l'élasticité. *C.R. Acad. Sci., Paris* **126**(1898), 1129–1132.
7. R. Courant and D. Hilbert, Methods of mathematical physics. Vol. II: Partial differential equations. (Vol. II by R. Courant.) *Interscience Publishers* (*a division of John Wiley* & *Sons*)*, New York–London*, 1962.
8. G. Duvaut and J. L. Lions, Les inéquations en mécanique et en physique. *Travaux et Recherches Mathématiques*, No. 21, *Dunod, Paris*, 1972.
9. J. Dyszlewicz, Micropolar theory of elasticity. *Lecture Notes in Applied and Computational Mechanics,* 15*, Springer-Verlag, Berlin*, 2004.
10. M. J. Elphinstone and A. Lakhtakia, Planewave response of an elastic chiral solid slab sandwiched between achiral fluid halfspaces. *Proc. Indian Natl. Sci. Acad., Part A* **60**(1994), No. 4, 593–605.
11. G. Fichera, Problemi elastostatici con vincoli unilaterali: Il problema di Signorini con ambigue condizioni al contorno. *Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. I (8)* **7**(1963/1964), 91–140.
12. G. Fichera, Existence theorems in linear and semi-linear elasticity. *Z. Angew. Math. Mech.* 54*, Sonderheft,* 1974, T24–T36.
13. A. Gachechiladze and D. Natroshvili, Boundary variational inequality approach in the anisotropic elasticity for the Signorini problem. *Georgian Math. J.* **8**(2001), No. 3, 469–492.
14. R. Glowinski, J.-L. Lions, and R. Trémolières, Numerical analysis of variational inequalities. *Studies in Mathematics and its Applications,* 8*, North-Holland Publishing Co., Amsterdam-New York*, 1981.

15. D. Goeleven and J. Gwinner, On semicoerciveness, a class of variational inequalities, and an application to von Kármán plates. *Math. Nachr.* **244**(2002), 89–109.
16. J. Gwinner and E. P. Stephan, A boundary element procedure for contact problems in plane linear elastostatics. *RAIRO Model. Math. Anal. Numer.* **27**(1993), No. 4, 457–480.
17. H. D. Han, A boundary element procedure for the Signorini problem in three-dimensional elasticity. *Numer. Math. J. Chinese Univ.* (*English Ser.*) **3**(1994), No. 1, 104–117.
18. H. D. Han and G. C. Hsiao, The boundary element method for a contact problem. *Theory and applications of boundary element methods, Proc.* 2nd *China-Jap. Symp., Beijing/China,* 1988, 1990, 33–38.
19. I. Hlaváček, J. Haslinger, J. Nečas, and J. Lovíček, Solution of variational inequalities in mechanics. *Applied Mathematical Sciences,* 66*, Springer-Verlag, New York*, 1988.
20. N. Kikuchi and J. T. Oden, Contact problems in elasticity: a study of variational inequalities and finite element methods. *SIAM Studies in Applied Mathematics,* 8*, Society for Industrial and Applied Mathematics* (*SIAM*)*, Philadelphia, PA*, 1988.
21. D. Kinderlehrer, Remarks about Signorini's problem in linear elasticity. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **8**(1981), No. 4, 605–645.
22. V. D. Kupradze, T. G. Gegelia, M. O. Basheleishvili, and T. V. Burchuladze, Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity. (Russian) *Nauka, Moscow*, 1976; English transl.: *North-Holland Series in Applied Mathematics and Mechanics,* 25*, North-Holland Publishing Co., Amsterdam–New York*, 1979.
23. R. Lakes, Elastic and viscoelastic behavior of chiral materials. *Int. J. Mech. Sci.* **43**(2001), No. 7, 1579–1589.
24. R. S. Lakes and R. L.Benedict, Noncentrosymmetry in micropolar elasticity. *Int. J. Eng. Sci.* **20**(1982), 1161–1167.
25. A. Lakhtakia, Varadan V.K. and Varadan V.V., Elastic wave propagation in noncentrosymmetric isotropic media: dispersion and field equations. *J. Appl. Phys.* **64**(1988), 5246–5250.
26. A. Lakhtakia,V. V. Varadan, and V. K. Varadan, Elastic wave scattering by an isotropic noncentrosymmetric sphere. *J. Acoust. Soc. Am.* **91**(1992), 680–684.
27. J.-L. Lions and E. Magenes, Problèmes aux limites non homogènes et applications. Vol. 1. *Travaux et Recherches Mathématiques*, No. 17 *Dunod, Paris*, 1968.
28. R. D. Mindlin, Micro-structure in linear elasticity. *Arch. Rational Mech. Anal.* **16**(1964), 51–78.
29. D. Natroshvili, L. Giorgashvili, and I. G. Stratis, Mathematical problems of the theory of elasticity of chiral materials. *Appl. Math. Inform. Mech.* **8**(2003), No. 1, 47–103.
30. D. Natroshvili, L. Giorgashvili, and Sh. Zazashvili, Steady state oscillation problems in the theory of elasticity for chiral materials. *J. Integral Equations Appl.* **17**(2005), No. 1, 19–69.
31. D. Natroshvili and I. G. Stratis, Mathematical problems of the theory of elasticity of chiral materials for Lipschitz domains. *Math. Methods Appl. Sci.* **29**(2006), No. 4, 445–478.
32. J. Nečas, Les méthodes directes en théorie des équations elliptiques. *Masson et Cie, Éditeurs, Paris; Academia, Éditeurs, Prague*, 1967.
33. W. Nowacki, Theory of asymmetric elasticity. *Pergamon Press, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1986.
34. J. P. Nowacki, Green functions for a hemitropic micropolar continuum. *Bull. Acad. Polon. Sci. Ser. Sci. Tech.* **25**(1977), No. 8, 619–625.
35. J. P. Nowacki and W. Nowacki, Some problems of hemitropic micropolar continuum. *Bull. Acad. Polon. Sci. Ser. Sci. Tech.* **25**(1977), No. 4, 297–305.

36. J.-F. Rodrigues, Obstacle problems in mathematical physics. *North-Holland Mathematics Studies*, 134, *North-Holland Publishing Co., Amsterdam*, 1987.
37. H. Triebel, Interpolation theory, function spaces, differential operators. *North-Holland Mathematical Library,* 18, *North-Holland Publishing Co., Amsterdam-New York*, 1978.
38. H. Triebel, Theory of function spaces. *Monographs in Mathematics,* 78, *Birkhäuser Verlag, Basel*, 1983.
39. Y. Weitsman, Initial stress and skin effects in a hemitropic Cosserat continuum. *J. Appl. Math.* **349**(1967), 160–164.

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