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UNIQUE SOLVABILITY OF SOME TWO-POINT BOUNDARY VALUE PROBLEMS FOR LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH SINGULARITIES

Dedicated to Professor Ivan Kiguradze on the occasion of his 70th birthday

Abstract. We obtain new conditions sufficient for the unique solvability of a singular mixed type two-point boundary value problem for systems of linear functional differential equations of the second order.

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1. Problem Setting and Introduction

In this paper, we prove several theorems on the unique solvability of the two-point mixed type boundary value problem

$$u_k(a) = c_{0k}, \quad k = 1, 2, \dots, n, \tag{1.1}$$

$$u'_k(\tau) = c_{1k}, \quad k = 1, 2, \dots, n, \tag{1.2}$$

for the linear functional differential system

$$u''_k(t) = (l_k u)(t) + f_k(t), \quad t \in [a, b], \quad k = 1, 2, \dots, n, \tag{1.3}$$

where $-\infty < a < b < +\infty$, $\tau \in [a, b]$, $\{c_{0k}, c_{1k} \mid k = 1, 2, \dots, n\} \subset \mathbb{R}$, the functions f_k, $k = 1, 2, \dots, n$, are locally integrable and have certain additional properties, and the linear mapping $l = (l_k)_{k=1}^n : C([a, b], \mathbb{R}^n) \to L_{1;\mathrm{loc}}((a, b), \mathbb{R}^n)$ is assumed to transform the space $C([a, b], \mathbb{R}^n)$ into a certain function class wider than $L_1([a, b], \mathbb{R}^n)$ and narrower than the linear manifold of all locally integrable functions. More precisely, we assume that $l = (l_k)_{k=1}^n$ is regular in the sense of Definition 2.3.

The setting (1.3) covers, in particular, the differential system with argument deviations

$$u''_k(t) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ikj}(t) u_j(\omega_{ikj}(t)) + f_k(t), \quad t \in [a, b], \quad k = 1, 2, \dots, n, \tag{1.4}$$

where $m \geq 1$, $n \geq 1$, $\omega_{ikj} : [a, b] \to [a, b]$, $i = 1, 2, \dots, m$, $k, j = 1, 2, \dots, n$, are arbitrary measurable transformations, and the locally integrable functions $p_{ikj} : [a, b] \to \mathbb{R}$, $i = 1, 2, \dots, m$, $k, j = 1, 2, \dots, n$, may have singularities at the points a and b. The possibility to choose τ inside the given interval corresponds to the mixed type problems (1.3), (1.1), (1.2) possessing solutions with the second derivative of type (2.1).

The notion of a solution of the boundary value problem (1.3), (1.1), (1.2) is understood in the sense of the following definition.

Definition 1.1. By a solution of the problem (1.3), (1.1), (1.2) with $a < \tau < b$ (resp., $\tau = b$, $\tau = a$), we mean a continuous vector function $u = (u_k)_{k=1}^n : [a, b] \to \mathbb{R}^n$ satisfying the conditions (1.1) and (1.2), possessing almost everywhere on $[a, b]$ the second derivative $u'' = (u''_k)_{k=1}^n$ belonging to $L_{1;\mathrm{loc}}((a, b), \mathbb{R}^n)$ (resp., $L_{1;\mathrm{loc}}((a, b], \mathbb{R}^n)$, $L_{1;\mathrm{loc}}([a, b), \mathbb{R}^n)$) and such that

$$\max_{k=1,2,\dots,n} \left(\int_a^\tau (s - a)|u''_k(s)|\, ds + \int_\tau^b (b - s)|u''_k(s)|\, ds \right) < +\infty,$$

and satisfying the equalities (1.3) at almost every point t of the interval $[a, b]$.

Our present study is motivated by recent investigations on second order equations of the form

$$u''(t) = p(t)u(\omega(t)) + f(t), \quad t \in [a, b].$$

We refer, in particular, to [1], [2], [3], [4], [5], [6], [7], [8], [9] for some bibliography.

The proofs of the theorems obtained here are based upon our previous results on differential inequalities related to the initial value problem for first order functional differential systems [10], [11], [12].

2. Notation and Definitions

We need some definitions and notation.

(1) $L_1([a,b],\mathbb{R}^n)$ is the Banach space of functions $u=(u_k)_{k=1}^n : [a,b] \to \mathbb{R}^n$ with the Lebesgue integrable components $(u_k)_{k=1}^n$.

(2) $L_{1;\mathrm{loc}}((a,b),\mathbb{R}^n)$ (resp., $L_{1;\mathrm{loc}}([a,b),\mathbb{R}^n)$, $L_{1;\mathrm{loc}}((a,b],\mathbb{R}^n)$) is the set of vector functions $u:(a,b)\to\mathbb{R}^n$ (resp., $u:[a,b)\to\mathbb{R}^n$, $u:(a,b]\to\mathbb{R}^n$) such that $u|_{[a+\varepsilon,b-\varepsilon]} \in L_1([a+\varepsilon,b-\varepsilon],\mathbb{R}^n)$ (resp., $u|_{[a,b-\varepsilon]} \in L_1([a,b-\varepsilon],\mathbb{R}^n)$, $u|_{[a+\varepsilon,b]} \in L_1([a+\varepsilon,b],\mathbb{R}^n)$) for arbitrary $\varepsilon \in (0,(b-a)/2)$ (resp., $\varepsilon \in (0,b-a)$).

(3) $C([a,b],\mathbb{R}^n)$ is the Banach space of continuous functions from $[a,b]$ to $\mathbb{R}^n$.

(4) Given an operator $l=(l_k)_{k=1}^n$ from $C([a,b],\mathbb{R}^n)$ to $L_{1;\mathrm{loc}}((a,b),\mathbb{R}^n)$ (resp., to $L_{1;\mathrm{loc}}([a,b),\mathbb{R}^n)$, $L_{1;\mathrm{loc}}((a,b],\mathbb{R}^n)$), the mappings $l_k : C([a,b],\mathbb{R}^n) \to L_{1;\mathrm{loc}}((a,b),\mathbb{R})$ (resp., to $L_{1;\mathrm{loc}}([a,b),\mathbb{R})$, $L_{1;\mathrm{loc}}((a,b],\mathbb{R})$), $k=1,2,\dots,n$, are called the components of l.

Definition 2.1. By the symbol $\widetilde{L}_1([a,b),\mathbb{R})$ we denote in the sequel the set of Lebesgue measurable functions $x:[a,b)\to\mathbb{R}$ such that $x\in L_{1;\mathrm{loc}}([a,b),\mathbb{R})$ and

$$\int_a^b (b-s)|x(s)|\,ds < +\infty.$$

By $\widetilde{L}_1((a,b],\mathbb{R})$ we denote the set of Lebesgue measurable functions $x:(a,b]\to\mathbb{R}$ such that $x\in L_{1;\mathrm{loc}}((a,b],\mathbb{R})$ and

$$\int_a^b (s-a)|x(s)|\,ds < +\infty.$$

Finally, the symbol $\widetilde{L}_1((a,b),\mathbb{R})$ stands for the set of functions $x\in L_{1;\mathrm{loc}}((a,b),\mathbb{R})$ such that

$$\int_a^{\frac{a+b}{2}} (s-a)|x(s)|\,ds + \int_{\frac{a+b}{2}}^b (b-s)|x(s)|\,ds < +\infty.$$

Using Definition 2.1, we introduce the set $\widetilde{L}_1([a,b),\mathbb{R}^n)$ as the linear manifold of all the vector functions $x=(x_k)_{k=1}^n : [a,b)\to\mathbb{R}^n$ such that $x_k \in \widetilde{L}_1([a,b),\mathbb{R})$ for all $k=1,2,\dots,n$. The sets $\widetilde{L}_1((a,b],\mathbb{R}^n)$ and $\widetilde{L}_1((a,b),\mathbb{R}^n)$ are defined by analogy.

Lemma 2.2. *The following assertions are true:*

(1) *The set* $\widetilde{L}_1([a,b),\mathbb{R})$ *(resp.,* $\widetilde{L}_1((a,b],\mathbb{R})$*,* $\widetilde{L}_1((a,b),\mathbb{R})$*) is a linear manifold in* $L_{1;\mathrm{loc}}([a,b),\mathbb{R})$ *(resp.,* $L_{1;\mathrm{loc}}((a,b],\mathbb{R})$*,* $L_{1;\mathrm{loc}}((a,b),\mathbb{R})$*);*

(2) *Each of the sets* $\widetilde{L}_1([a,b),\mathbb{R})$*,* $\widetilde{L}_1((a,b],\mathbb{R})$*, and* $\widetilde{L}_1((a,b),\mathbb{R})$ *contains non-integrable elements.*

Proof. The assertion (1) follows immediately from Definition 2.1. In order to verify the validity of the assertion (2), it is sufficient to consider, e. g., the function

$$x(t) := \begin{cases} \dfrac{1}{t-a} & \text{for } t \in \Big[a, \dfrac{1}{2}\,(a+b)\Big), \\ \dfrac{1}{b-t} & \text{for } t \in \Big(\dfrac{1}{2}\,(a+b), b\Big], \end{cases} \tag{2.1}$$

which is a non-integrable element of $\widetilde{L}_1((a,b),\mathbb{R})$. $\square$

For the sake of convenience, we introduce the following definition which is a slightly modified version of Definition 1.1 from [3].

Definition 2.3. We say that a linear mapping l from $C([a,b],\mathbb{R}^n)$ to $L_{1;\mathrm{loc}}([a,b),\mathbb{R})$ (resp., $L_{1;\mathrm{loc}}((a,b],\mathbb{R})$, $L_{1;\mathrm{loc}}((a,b),\mathbb{R})$) is *regular* if there exist some non-negative functions h_j, $j = 1,2,\dots,n$, belonging to the set $\widetilde{L}_1([a,b),\mathbb{R})$ (resp., $\widetilde{L}_1((a,b],\mathbb{R})$, $\widetilde{L}_1((a,b),\mathbb{R})$) and such that the inequality

$$|(lu)(t)| \le \sum_{j=1}^{n} h_j(t) \max_{s\in[a,b]} |u_j(s)| \tag{2.2}$$

holds for all $u = (u_k)_{k=1}^n$ from $C([a,b],\mathbb{R}^n)$ and almost every t from $[a,b]$.

A linear mapping $l = (l_k)_{k=1}^n$ from the space $C([a,b],\mathbb{R}^n)$ to $L_{1;\mathrm{loc}}([a,b),\mathbb{R}^n)$ (resp., $L_{1;\mathrm{loc}}((a,b],\mathbb{R}^n)$, $L_{1;\mathrm{loc}}((a,b),\mathbb{R}^n)$) is said to be regular if each of its components l_k, $k = 1,2,\dots,n$, possesses the property indicated.

It is easy to see that the range every regular linear mapping from the space $C([a,b],\mathbb{R}^n)$ to $L_{1;\mathrm{loc}}([a,b),\mathbb{R}^n)$ (resp., $L_{1;\mathrm{loc}}((a,b],\mathbb{R}^n)$, $L_{1;\mathrm{loc}}((a,b),\mathbb{R}^n)$) is in $\widetilde{L}_1([a,b),\mathbb{R}^n)$ (resp., $\widetilde{L}_1((a,b],\mathbb{R}^n)$, $\widetilde{L}_1((a,b),\mathbb{R}^n)$).

Definition 2.4. Let τ be an arbitrary point from the interval $[a,b]$. We say that a mapping l from $C([a,b],\mathbb{R}^n)$ to one of the sets $L_{1;\mathrm{loc}}([a,b),\mathbb{R})$, $L_{1;\mathrm{loc}}((a,b],\mathbb{R})$, and $L_{1;\mathrm{loc}}((a,b),\mathbb{R})$ is τ*-positive* if the relation

$$\operatorname*{vrai\,min}_{t\in[a,b]} (lu)\,(t) \operatorname{sign}(t-\tau) \ge 0$$

holds for an arbitrary vector function $u = (u_k)_{k=1}^n \in C([a,b],\mathbb{R}^n)$ possessing the property

$$\min_{k=1,2,\dots,n} \min_{t\in[a,b]} u_k(t) \ge 0. \tag{2.3}$$

Similarly, a mapping $l = (l_k)_{k=1}^n$ from the space $C([a,b],\mathbb{R}^n)$ to $L_{1;\mathrm{loc}}([a,b),\mathbb{R}^n)$, $L_{1;\mathrm{loc}}((a,b],\mathbb{R}^n)$, or $L_{1;\mathrm{loc}}((a,b),\mathbb{R}^n)$) is said to be τ-positive if each of its components l_k, $k = 1,2,\dots,n$, has this property.

An a-positive operator is simply called positive, whereas a b-positive operator is referred to as negative.

3. General Statements

The following theorem on solvability of the boundary value problem (1.3), (1.1), (1.2) is true.

Theorem 3.1. *Assume that* $a < \tau < b$ *and* l_k, $k = 1, 2, \dots, n$, *in the system* (1.3) *admit the decomposition in the form*

$$l_k = l_k^+ - l_k^-, \tag{3.1}$$

where l_k^+ *and* l_k^-, $k = 1, 2, \dots, n$, *are certain* τ*-positive regular linear mappings from* $C([a, b], \mathbb{R}^n)$ *to* $L_{1;\mathrm{loc}}((a, b), \mathbb{R})$. *Furthermore, let there exist a constant* $\varepsilon \in (0, 1)$ *and a vector function* $y = (y_k)_{k=1}^n : [a, b] \to \mathbb{R}^n$ *with absolutely continuous components* y_k, $k = 1, 2, \dots, n$, *satisfying the conditions*

$$y_k(a) = 0, \quad k = 1, 2, \dots, n, \tag{3.2}$$

$$y_k(t) > 0, \quad k = 1, 2, \dots, n, \quad t \in (a, b], \tag{3.3}$$

and such that the functional differential inequalities

$$\varepsilon y_k'(t) \geq \int\limits_\tau^t \left[(l_k^+ y)(\xi) + (l_k^- y)(\xi) \right] d\xi \tag{3.4}$$

are true for all $k = 1, 2, \dots, n$ *and almost every* $t \in [a, b]$.

Then the problem (1.3), (1.1), (1.2) *has a unique solution for arbitrary* $\{c_{0k}, c_{1k} \mid k = 1, 2, \dots, n\} \subset \mathbb{R}$ *and* $\{f_k \mid k = 1, 2, \dots, n\} \subset \widetilde{L}_1((a, b), \mathbb{R})$.

It is essential that the positive constant ε appearing in the relation (3.4) should be strictly less than 1 because, as is seen from Example 3.3 below, the weakened version

$$\min_{k=1,2,\dots,n} \operatorname*{vrai\,min}_{t \in [a,b]} \left(y_k'(t) - \int\limits_\tau^t \left[(l_k^+ y)(\xi) + (l_k^- y)(\xi) \right] d\xi \right) \geq 0$$

of the condition (3.4) does not guarantee the validity of the assertion of Theorem 3.1.

Theorem 3.2. *Assume that* $a < \tau < b$ *and the linear mapping* $l = (l_k)_{k=1}^n : C([a, b], \mathbb{R}^n) \to L_{1;\mathrm{loc}}((a, b), \mathbb{R}^n)$ *is regular and* τ*-positive. Let there exist a constant* $\varepsilon \in (0, 1)$ *and a function* $y = (y_k)_{k=1}^n : [a, b] \to \mathbb{R}^n$ *whose components* y_k, $k = 1, 2, \dots, n$, *are absolutely continuous functions satisfying the conditions* (3.2) *and* (3.3) *and such that the functional differential inequalities*

$$\varepsilon y_k'(t) \geq \int\limits_\tau^t (l_k y)(\xi) \, d\xi \tag{3.5}$$

are true for all $k = 1, 2, \dots, n$ *and almost every* $t \in [a, b]$.

Then the problem (1.3), (1.1), (1.2) *has a unique solution for arbitrary* $\{c_{0k}, c_{1k} \mid k = 1,2,\dots,n\} \subset \mathbb{R}$ *and* $\{f_k \mid k = 1,2,\dots,n\} \subset \widetilde{L}_1((a,b),\mathbb{R})$. *Moreover, the unique solution* $u = (u_k)_{k=1}^n$ *of this problem possesses the property* (2.3) *provided that* c_{0k}, c_{1k}, *and* f_k, $k = 1,2,\dots,n$, *satisfy the relations*

$$\int_a^t \left(\int_\tau^\xi f_k(s)\,ds \right) d\xi \geq -c_{0k} - c_{1k}(t-a), \quad t \in [a,b], \quad k = 1,2,\dots,n. \tag{3.6}$$

Similarly to Theorem 3.1, it should be noted that the condition (3.5) of Theorem 3.2 cannot be replaced by the weaker condition

$$\min_{k=1,2,\dots,n} \operatorname*{vrai\,min}_{t\in[a,b]} \left(y_k'(t) - \int_\tau^t (l_k y)(\xi)\,d\xi \right) \geq 0 \tag{3.7}$$

because the assertion of the latter theorem is therewith lost.

Example 3.3. Consider the homogeneous problem

$$u(a) = 0, \tag{3.8}$$

$$u'(\tau) = 0 \tag{3.9}$$

for the equation

$$u''(t) = -\frac{2\chi_\tau(t)}{(\tau-a)^2}\,u(\tau), \quad t \in [a,b], \tag{3.10}$$

where $\tau \in (a,b]$ and the function $\chi_\tau : [a,b] \to \{0,1\}$ is defined by the formula

$$\chi_\tau(t) := \begin{cases} 1 & \text{if } t \in [a,\tau), \\ 0 & \text{if } t \in [\tau,b]. \end{cases}$$

Let us define the linear mapping $l_1 : C([a,b],\mathbb{R} \to L_{1;\mathrm{loc}}((a,b),\mathbb{R})$ by putting

$$(l_1 u)(t) := -\frac{2\chi_\tau(t)}{(\tau-a)^2}\,u(\tau), \quad t \in [a,b], \tag{3.11}$$

for any u from $C([a,b],\mathbb{R})$. Then the equation (3.10) takes the form (1.3) for $n = 1$. It is easy to see that the operator (3.11) is regular and τ-positive in the sense of Definitions 2.3 and 2.4.

One can verify that the function defined by the formula

$$u(t) = (t^2 - 2\tau t + 2\tau a - a^2)\,\chi_\tau(t) - (\tau-a)^2(1-\chi_\tau(t)), \quad t \in [a,b],$$

is a non-trivial solution of the problem (3.8), (3.9), (3.10). However, for $y_1 = u$ and the operator l_1 given by the formula (3.11), the condition (3.7) is satisfied in the form of an equality.

In the case where $\tau = b$ and, hence, the condition (1.2) has the form

$$u_k'(b) = c_{1k}, \quad k = 1,2,\dots,n, \tag{3.12}$$

the following statements are true.

Theorem 3.4. *Assume that l_k, $k = 1, 2, \dots, n$, admit the decomposition* (3.1)*, where l_k^+ and l_k^-, $k = 1, 2, \dots, n$, are negative regular linear mappings from $C([a,b],\mathbb{R}^n)$ to $L_{1;\mathrm{loc}}((a,b],\mathbb{R})$. Furthermore, let there exist a constant $\varepsilon \in (0,1)$ and a vector function $y = (y_k)_{k=1}^n : [a,b] \to \mathbb{R}^n$ with absolutely continuous components y_k, $k = 1, 2, \dots, n$, satisfying the conditions* (3.2), (3.3) *and such that the functional differential inequalities*

$$\varepsilon y_k'(t) \geq \int\limits_t^b \left|(l_k^+ y)(\xi) + (l_k^- y)(\xi)\right| d\xi \tag{3.13}$$

are true for all $k = 1, 2, \dots, n$ and almost every $t \in [a,b]$.

Then the problem (1.3), (1.1), (3.12) *has a unique solution for arbitrary $\{c_{0k}, c_{1k} \mid k = 1, 2, \dots, n\} \subset \mathbb{R}$ and $\{f_k \mid k = 1, 2, \dots, n\} \subset \widetilde{L}_1((a,b],\mathbb{R})$.*

The non-negativity of the solution of the problem (1.3), (1.1), (3.12) is guaranteed by

Theorem 3.5. *Assume that in the system* (1.3) *the linear mapping $l = (l_k)_{k=1}^n : C([a,b],\mathbb{R}^n) \to L_{1;\mathrm{loc}}((a,b],\mathbb{R}^n)$ is regular and negative. Let there exist a constant $\varepsilon \in (0,1)$ and a function $y = (y_k)_{k=1}^n : [a,b] \to \mathbb{R}^n$ whose components y_k, $k = 1, 2, \dots, n$, are absolutely continuous functions satisfying the conditions* (3.2) *and* (3.3) *and such that the functional differential inequalities*

$$\varepsilon y_k'(t) \geq \int\limits_t^b \left|(l_k y)(\xi)\right| d\xi \tag{3.14}$$

are true for all $k = 1, 2, \dots, n$ and almost every $t \in [a,b]$.

Then the problem (1.3), (1.1), (3.12) *has a unique solution for arbitrary $\{c_{0k}, c_{1k} \mid k = 1, 2, \dots, n\} \subset \mathbb{R}$ and $\{f_k \mid k = 1, 2, \dots, n\} \subset \widetilde{L}_1((a,b],\mathbb{R})$. Moreover, the unique solution $u = (u_k)_{k=1}^n$ of this problem possesses the property* (2.3) *provided that c_{0k}, c_{1k}, and f_k, $k = 1, 2, \dots, n$, satisfy the relations*

$$c_{0k} + c_{1k}(t-a) \geq \int\limits_a^t \Bigg(\int\limits_\xi^b f_k(s)\, ds \Bigg) d\xi, \quad t \in [a,b], \quad k = 1, 2, \dots, n.$$

Finally, in the case where $\tau = a$, the condition (1.2) has the form

$$u_k'(a) = c_{1k}, \quad k = 1, 2, \dots, n, \tag{3.15}$$

and the following statements are true.

Theorem 3.6. *Assume that l_k, $k = 1, 2, \dots, n$, admit the decomposition* (3.1) *where l_k^+ and l_k^-, $k = 1, 2, \dots, n$, are positive regular linear mappings from $C([a,b],\mathbb{R}^n)$ to $L_{1;\mathrm{loc}}([a,b),\mathbb{R})$. Furthermore, let there exist a constant $\varepsilon \in (0,1)$ and a vector function $y = (y_k)_{k=1}^n : [a,b] \to \mathbb{R}^n$ with absolutely* *continuous components y_k, $k = 1, 2, \dots, n$, satisfying the conditions* (3.2), (3.3) *and such that the functional differential inequalities*

$$\varepsilon y_k'(t) \geq \int\limits_a^t \big[(l_k^+ y)(\xi) + (l_k^- y)(\xi)\big]\, d\xi \tag{3.16}$$

are true for all $k = 1, 2, \dots, n$ and almost every $t \in [a, b]$.

Then the Cauchy problem (1.3), (1.1), (3.15) *has a unique solution for arbitrary $\big\{c_{0k}, c_{1k} \mid k = 1, 2, \dots, n\big\} \subset \mathbb{R}$ and $\big\{f_k \mid k = 1, 2, \dots, n\big\} \subset \widetilde{L}_1([a, b), \mathbb{R})$.*

The non-negativity of the solution of the Cauchy problem (1.3), (1.1), (3.15) is guaranteed by

Theorem 3.7. *Assume that in the system* (1.3) *the linear mapping $l = (l_k)_{k=1}^n : C([a, b], \mathbb{R}^n) \to L_{1;\mathrm{loc}}([a, b), \mathbb{R}^n)$ is regular and positive. Let there exist a constant $\varepsilon \in (0, 1)$ and a function $y = (y_k)_{k=1}^n : [a, b] \to \mathbb{R}^n$ whose components y_k, $k = 1, 2, \dots, n$, are absolutely continuous functions satisfying the conditions* (3.2) *and* (3.3) *and such that the functional differential inequalities*

$$\varepsilon y_k'(t) \geq \int\limits_a^t (l_k y)(\xi)\, d\xi \tag{3.17}$$

are true for all $k = 1, 2, \dots, n$ and almost every $t \in [a, b]$.

Then the Cauchy problem (1.3), (1.1), (3.15) *has a unique solution for arbitrary $\{c_{0k}, c_{1k} \mid k = 1, 2, \dots, n\} \subset \mathbb{R}$ and $\{f_k \mid k = 1, 2, \dots, n\} \subset \widetilde{L}_1([a, b), \mathbb{R})$. Moreover, the unique solution $u = (u_k)_{k=1}^n$ of this problem possesses the property* (2.3) *provided that c_{0k}, c_{1k}, and f_k, $k = 1, 2, \dots, n$, satisfy the relations*

$$\int\limits_a^t (t - s) f_k(s)\, ds \geq -c_{0k} - c_{1k}(t - a), \quad t \in [a, b], \quad k = 1, 2, \dots, n.$$

The proofs of the results presented in Sections 3 and 4 are postponed till Section 5.2.

4. Theorems for Differential Equations with Deviations

The theorems given above allow one to obtain conditions sufficient for the unique solvability of the mixed two-point boundary value problem (1.1), (1.2) for the system of differential equations with argument deviations (1.4).

Theorem 4.1. *Let*

$$\big\{p_{ikj} \mid i = 1, 2, \dots, m;\ k, j = 1, 2, \dots, n\big\} \subset L_{1;\mathrm{loc}}((a, b], \mathbb{R})$$

satisfy the condition

$$\sum_{i=1}^{m}\sum_{k=1}^{n}\sum_{j=1}^{n}\int_a^b (\xi-a)|p_{ikj}(\xi)|\,d\xi < +\infty \tag{4.1}$$

and, moreover, there exist some $\{\alpha_1,\alpha_2,\dots,\alpha_n\} \subset (0,+\infty)$ *and* $\{\gamma_1,\gamma_2,\dots,\gamma_n\} \subset (0,+\infty)$ *such that*

$$\sup_{t\in(a,b]} \frac{1}{(t-a)^{\alpha_k-1}} \sum_{i=1}^{m}\sum_{j=1}^{n}\gamma_j \int_t^b (\omega_{ikj}(\xi)-a)^{\alpha_j}|p_{ikj}(\xi)|\,d\xi < \gamma_k\alpha_k \tag{4.2}$$

for all $k=1,2,\dots,n$.

Then the boundary value problem (1.4)*,* (1.1)*,* (3.12) *is uniquely solvable for arbitrary* $\{c_{0k},c_{1k} \mid k=1,2,\dots,n\}\subset\mathbb{R}$ *and* $\{f_k \mid k=1,2,\dots,n\}\subset L_{1;\mathrm{loc}}((a,b],\mathbb{R})$ *possessing the property*

$$\sum_{j=1}^{n}\int_a^b (\xi-a)|f_j(\xi)|\,d\xi < +\infty. \tag{4.3}$$

Theorem 4.1 implies

Corollary 4.2. *Assume that* $\{p_{ikj} \mid i=1,2,\dots,m,\ k,j=1,2,\dots,n\}\subset L_{1;\mathrm{loc}}((a,b],\mathbb{R})$ *and for certain* $\{\gamma_1,\gamma_2,\dots,\gamma_n\}\subset(0,+\infty)$ *the measurable functions* $\omega_{ikj}:[a,b]\to[a,b]$*,* $i=1,2,\dots,m,\ k,j=1,2,\dots,n$*, satisfy the condition*

$$\max_{k=1,2,\dots,n}\frac{1}{\gamma_k}\sum_{i=1}^{m}\sum_{j=1}^{n}\gamma_j\int_a^b(\omega_{ikj}(\xi)-a)|p_{ikj}(\xi)|\,d\xi < 1. \tag{4.4}$$

Assume moreover that the relation

$$\operatorname*{vrai\,max}_{t\in[a,b]\setminus\Gamma}\frac{t-a}{\omega_{ikj}(t)-a} < +\infty, \tag{4.5}$$

where

$$\Gamma := \bigcup_{i=1}^{m}\bigcup_{k=1}^{n}\bigcup_{j=1}^{n}\omega_{ikj}^{-1}(a), \tag{4.6}$$

is true for all $i=1,2,\dots,m$ *and* $k,j=1,2,\dots,n$.

Then the boundary value problem (1.4)*,* (1.1)*,* (3.12) *is uniquely solvable for arbitrary* $\{c_{0k},c_{1k} \mid k=1,2,\dots,n\}\subset\mathbb{R}$ *and* $\{f_k \mid k=1,2,\dots,n\}\subset L_{1;\mathrm{loc}}((a,b],\mathbb{R})$ *possessing property* (4.3).

Let us also formulate similar statements for the Cauchy problem (1.4), (1.1), (3.15).

Theorem 4.3. *Let*

$$\{p_{ikj} \mid i=1,2,\dots,m;\ k,j=1,2,\dots,n\}\subset L_{1;\mathrm{loc}}([a,b),\mathbb{R})$$

satisfy the condition

$$\sum_{i=1}^{m}\sum_{k=1}^{n}\sum_{j=1}^{n}\int_a^b (b-\xi)|p_{ikj}(\xi)|\,d\xi < +\infty \tag{4.7}$$

and, moreover, there exist some $\{\alpha_1, \alpha_2, \dots, \alpha_n\} \subset (0, +\infty)$ *and* $\{\gamma_1, \gamma_2, \dots, \gamma_n\} \subset (0, +\infty)$ *such that*

$$\sup_{t\in(a,b]} \frac{1}{(t-a)^{\alpha_k-1}} \sum_{i=1}^{m}\sum_{j=1}^{n} \gamma_j \int_a^t (\omega_{ikj}(\xi)-a)^{\alpha_j}|p_{ikj}(\xi)|\,d\xi < \gamma_k\alpha_k \tag{4.8}$$

for all $k = 1, 2, \dots, n$.

Then the Cauchy problem (1.4), (1.1), (3.15) *is uniquely solvable for arbitrary* $\{c_{0k}, c_{1k} \mid k = 1, 2, \dots, n\} \subset \mathbb{R}$ *and* $\{f_k \mid k = 1, 2, \dots, n\} \subset L_{1;\mathrm{loc}}([a, b), \mathbb{R})$ *possessing the property*

$$\sum_{j=1}^{n}\int_a^b (b-\xi)|f_j(\xi)|\,d\xi < +\infty. \tag{4.9}$$

Theorem 4.3 implies the following corollary.

Corollary 4.4. *Assume that* $\{p_{ikj} \mid i = 1, 2, \dots, m,\ k, j = 1, 2, \dots, n\} \subset L_{1;\mathrm{loc}}([a, b), \mathbb{R})$ *and for certain* $\{\gamma_1, \gamma_2, \dots, \gamma_n\} \subset (0, +\infty)$ *the measurable functions* $\omega_{ikj} : [a, b] \to [a, b]$, $i = 1, 2, \dots, m$, $k, j = 1, 2, \dots, n$, *satisfy the condition* (4.4). *Assume moreover that the relation*

$$\operatorname*{vrai\,max}_{t\in[a,b]\setminus\Gamma} \frac{b-t}{\omega_{ikj}(t)-a} < +\infty, \tag{4.10}$$

where Γ *is the subset of* $[a, b]$ *defined by the formula* (4.6), *is true for all* $i = 1, 2, \dots, m$ *and* $k, j = 1, 2, \dots, n$.

Then the Cauchy problem (1.4), (1.1), (3.15) *is uniquely solvable for arbitrary* $\{c_{0k}, c_{1k} \mid k = 1, 2, \dots, n\} \subset \mathbb{R}$ *and* $\{f_k \mid k = 1, 2, \dots, n\} \subset L_{1;\mathrm{loc}}([a, b), \mathbb{R})$ *possessing the property* (4.9).

Note that similar results for regular Cauchy problems have been obtained in [13]. The statements given above are, in fact, particular cases of the Theorem 4.5.

Theorem 4.5. *Let us assume that*

$$\{p_{ikj} \mid i = 1, 2, \dots, m,\ k, j = 1, 2, \dots, n\} \subset L_{1;\mathrm{loc}}((a, b), \mathbb{R})$$

satisfies

$$\sum_{i=1}^{m}\sum_{k=1}^{n}\sum_{j=1}^{n} \max\left\{ \int_a^\tau (\xi-a)|p_{ikj}(\xi)|\,d\xi,\ \int_\tau^b (b-\xi)|p_{ikj}(\xi)|\,d\xi \right\} < +\infty \tag{4.11}$$

and there exist $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset (0, +\infty)$ *and* $\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \subset (0, +\infty)$ *such that the inequality*

$$\sup_{t\in(a,b]} \frac{\operatorname{sign}(t-\tau)}{(t-a)^{\alpha_k-1}} \sum_{i=1}^{m}\sum_{j=1}^{n} \gamma_j \int_{\tau}^{t} |p_{ikj}(\xi)|(\omega_{ikj}(\xi)-a)^{\alpha_j}\, d\xi < \gamma_k \alpha_k \tag{4.12}$$

is satisfied for every $k = 1, 2, \ldots, n$.

Then the boundary value problem (1.4), (1.1), (1.2) *is uniquely solvable for arbitrary* $\{c_{0k}, c_{1k} \mid k = 1, 2, \ldots, n\} \subset \mathbb{R}$ *and* $\{f_k \mid k = 1, 2, \ldots, n\} \subset L_{1;\mathrm{loc}}((a,b), \mathbb{R})$ *possessing the property*

$$\sum_{j=1}^{n} \max\left\{ \int_a^{\tau} (\xi-a)|f_j(\xi)|\, d\xi,\ \int_{\tau}^{b} (b-\xi)|f_j(\xi)|\, d\xi \right\} < +\infty.$$

If, moreover, the inequality

$$p_{ikj}(t)\operatorname{sign}(t-\tau) \geq 0 \tag{4.13}$$

is satisfied for all $i = 1, 2, \ldots, m$, $k, j = 1, 2, \ldots, n$, *and a. e.* $t \in [a, b]$, *then the condition* (3.6) *ensures the non-negativity of the solution of the problem indicated.*

5. Lemmas and Proofs

This section is devoted to the substantiation of the results stated above.

5.1. **Auxiliary statements.**

The proofs of the theorems stated in the preceding sections are based upon several auxiliary propositions.

5.1.1. *General properties.* Here we establish properties of certain mappings associated with the boundary value problems under consideration.

Lemma 5.1. *Let* $a < \tau < b$ *(resp.,* $\tau = a$, $\tau = b$*) and* $l = (l_k)_{k=1}^n$ *be a regular linear mapping from* $C([a,b], \mathbb{R}^n)$ *to* $L_{1;\mathrm{loc}}((a,b), \mathbb{R}^n)$ *(resp.,* $L_{1;\mathrm{loc}}([a,b), \mathbb{R}^n)$, $L_{1;\mathrm{loc}}((a,b], \mathbb{R}^n)$*). For an arbitrary continuous vector function* $u : [a,b] \to \mathbb{R}^n$, *we put*

$$(\widehat{l}_{\tau,k}u)(t) := \int_{\tau}^{t} (l_k u)(\xi)\, d\xi, \quad t \in [a,b], \quad k = 1, 2, \ldots, n, \tag{5.1}$$

and

$$(\widehat{l}_{\tau}u)(t) := \begin{pmatrix} (\widehat{l}_{\tau,1}u)(t) \\ (\widehat{l}_{\tau,2}(u)(t) \\ \vdots \\ (\widehat{l}_{\tau,n}u)(t) \end{pmatrix}, \quad t \in [a,b]. \tag{5.2}$$

Then the formula (5.2) *determines a bounded linear operator* $\widehat{l}_{\tau}$ *from* $C([a,b], \mathbb{R}^n)$ *to* $L_1([a,b], \mathbb{R}^n)$.

Proof. We assume that $a < \tau < b$; the other two cases are considered in a similar way.

Let $(u_k)_{k=1}^n$ be an arbitrary element of $C([a,b],\mathbb{R}^n)$ not equal identically to zero. We can assume, without loss of generality, that

$$\max_{k=1,2,\dots,n}\max_{t\in[a,b]}|u_k(t)| = 1. \tag{5.3}$$

Let us put

$$J_{k1}(u) := \int_a^\tau \left|\int_\tau^t (l_k u)(\xi)\,d\xi\right| dt \tag{5.4}$$

and

$$J_{k2}(u) := \int_\tau^b \left|\int_\tau^t (l_k u)(\xi)\,d\xi\right| dt \tag{5.5}$$

for any $k = 1,2,\dots,n$. It follows immediately from (5.4) that

$$J_{k1}(u) \le \int_a^\tau \left(\int_t^\tau |(l_k u)(\xi)|\,d\xi\right) dt. \tag{5.6}$$

By assumption, each of the operators l_k, $k = 1,2,\dots,n$, is regular. According to Definition 2.3, this means the existence of a set $\{h_{kj} \mid k,j = 1,2,\dots,n\} \subset \widetilde{L}_1((a,b),\mathbb{R})$ such that the inequality

$$|(l_k u)(t)| \le \sum_{j=1}^n h_{kj}(t)\max_{s\in[a,b]}|u_j(s)| \tag{5.7}$$

is true for any $k = 1,2,\dots,n$ and almost all t from $[a,b]$. Due to (5.3) and (5.7), the estimate (5.6) implies that

$$J_{k1}(u) \le \sum_{j=1}^n \int_a^\tau \left(\int_t^\tau h_{kj}(\xi)\,d\xi\right) dt = \sum_{j=1}^n \int_a^\tau (\xi - a)h_{kj}(\xi)\,d\xi \tag{5.8}$$

for all $k = 1,2,\dots,n$. Similarly, the relations (5.3), (5.5), and (5.7) yield

$$\begin{aligned} J_{k2}(u) &\le \int_\tau^b \left(\int_\tau^t |(l_k u)(\xi)|\,d\xi\right) dt \le \sum_{j=1}^n \int_\tau^b \left(\int_\tau^t h_{kj}(\xi)\,d\xi\right) dt = \\ &= \sum_{j=1}^n \int_\tau^b (b-\xi)h_{kj}(\xi)\,d\xi. \end{aligned} \tag{5.9}$$

Combining (5.8) with (5.9) and taking (5.4) and (5.5) into account, we conclude that for any $k = 1, 2, \ldots, n$ the estimate

$$\int_a^b \left| \int_\tau^t (l_k u)(\xi)\, d\xi \right| dt = J_{k1}(u) + J_{k2}(u) \leq C_k \tag{5.10}$$

is true, where

$$C_k := \sum_{j=1}^n \left(\int_a^\tau (\xi - a) h_{kj}(\xi)\, d\xi + \int_\tau^b (b - \xi) h_{kj}(\xi)\, d\xi \right) \tag{5.11}$$

for $k = 1, 2, \ldots, n$. Since all the functions h_{kj}, $k, j = 1, 2, \ldots, n$, belong to $\widetilde{L}_1((a, b), \mathbb{R})$, it follows that each of the values (5.11) is finite. Recalling that the estimate (5.10) holds for an arbitrary continuous function $u = (u_k)_{k=1}^n : [a, b] \to \mathbb{R}^n$ with the property (5.3), we conclude that each of the linear mappings

$$C([a, b], \mathbb{R}^n) \ni u = (u_k)_{k=1}^n \longmapsto \left| \int_\tau^{\cdot} (l_i u)(\xi)\, d\xi \right|, \quad i = 1, 2, \ldots, n,$$

is a bounded operator from $C([a, b], \mathbb{R}^n)$ to $L_1([a, b], \mathbb{R})$ and, hence, according to the formulae (5.1) and (5.2), $\widehat{l_\tau} : C([a, b], \mathbb{R}^n) \to L_1([a, b], \mathbb{R}^n)$ is a bounded linear operator. □

Lemma 5.2. *Let l be a τ-positive regular linear mapping from the space $C([a, b], \mathbb{R}^n)$ to $L_{1;\mathrm{loc}}((a, b), \mathbb{R}^n)$, $L_{1;\mathrm{loc}}([a, b), \mathbb{R}^n)$, or $L_{1;\mathrm{loc}}((a, b], \mathbb{R}^n)$.*

Then the corresponding operator $\widehat{l_\tau} : C([a, b], \mathbb{R}^n) \to L_1([a, b], \mathbb{R}^n)$ is positive.

Proof. Let $u = (u_k)_{k=1}^n \in C([a, b], \mathbb{R}^n)$ be an arbitrary function having the property (2.3) and let $x_k = l_k u$. According to Definition 2.4, the relation

$$(l_k u)(\xi) \operatorname{sign}(\xi - \tau) \geq 0$$

holds for any $k = 1, 2, \ldots, n$ and almost all ξ from $[a, b]$. Therefore,

$$\int_{\min\{\tau, t\}}^{\max\{\tau, t\}} (l_k u)(\xi) \operatorname{sign}(\xi - \tau)\, d\xi \geq 0$$

for almost every t from $[a, b]$. Since $\operatorname{sign}(\xi - \tau) = \operatorname{sign}(t - \tau)$ for any ξ lying between τ and t, we have thus shown that, for any function $u = (u_k)_{k=1}^n \in C([a, b], \mathbb{R}^n)$ satisfying the condition (2.3), the relation

$$\int_\tau^t (l_k u)(\xi)\, d\xi \geq 0$$

is true. According to the notation (5.2) and Definition 2.4, this means the a-positivity of the operator $\widehat{l_\tau} : C([a, b], \mathbb{R}^n) \to L_1([a, b], \mathbb{R}^n)$. □

5.1.2. *The mapping $S_{\tau,\sigma}$.* Given arbitrary $\tau \in [a,b]$ and $\sigma \in \{-1,1\}$, we put

$$(S_{\tau,\sigma}x)(t) := \max\left\{\sigma x(t)\,\mathrm{sign}\,(t-\tau), 0\right\}\mathrm{sign}\,(t-\tau) \tag{5.12}$$

for any $x : [a,b] \to \mathbb{R}$ and t from $[a,b]$.

Lemma 5.3. (1) *The equalities*

$$(S_{\tau,1}x)(t) - (S_{\tau,-1}x)(t) = x(t) \tag{5.13}$$

and

$$(S_{\tau,1}x)(t) + (S_{\tau,-1}x)(t) = |x(t)|\,\mathrm{sign}\,(t-\tau) \tag{5.14}$$

are true for any $x : [a,b] \to \mathbb{R}$ and $t \in [a,b] \setminus \{\tau\}$.

(2) *The inequality*

$$(S_{\tau,\sigma}x)(t)\,\mathrm{sign}\,(t-\tau) \geq 0$$

is satisfied for any $x : [a,b] \to \mathbb{R}$, $\sigma \in \{-1,1\}$ and $t \in [a,b]$.

Proof. It follows from the equality (5.12) that

$$(S_{\tau,1}x)(t) = \begin{cases} x_+(t) & \text{for } t \in (\tau,b], \\ -x_-(t) & \text{for } t \in [a,\tau), \end{cases} \tag{5.15}$$

and

$$(S_{\tau,-1}x)(t) = \begin{cases} x_-(t) & \text{for } t \in (\tau,b], \\ -x_+(t) & \text{for } t \in [a,\tau) \end{cases} \tag{5.16}$$

where, by definition, $x_+(t) := \frac{1}{2}(|x(t)| + x(t))$ and $x_-(t) := \frac{1}{2}(|x(t)| - x(t))$ for all $x : [a,b] \to \mathbb{R}$ and $t \in [a,b]$. Taking (5.15) and (5.16) into account, we arrive immediately at the assertions desired. □

5.1.3. *Linear inner superposition operators.* Let us fix some point $\tau \in [a,b]$ and, for all $u = (u_k)_{k=1}^n \in C([a,b],\mathbb{R}^n)$, put

$$(l_k u)(t) := \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ikj}(t)\, u_j(\omega_{ikj}(t)), \quad t \in [a,b], \quad k = 1,2,\ldots,n, \tag{5.17}$$

where $m \geq 1$, $n \geq 1$, $\omega_{ikj} : [a,b] \to [a,b]$, $i = 1,2,\ldots,m$, $k,j = 1,2,\ldots,n$, are arbitrary measurable transformations, and $\left\{p_{ikj} \mid i = 1,2,\ldots,m,\ k,j = 1,2,\ldots,n\right\} \subset L_{1;\mathrm{loc}}((a,b),\mathbb{R})$.

Lemma 5.4. *If the inequality* (4.13) *is satisfied for all $i = 1,2,\ldots,m$, $k,j = 1,2,\ldots,n$, and a. e. $t \in [a,b]$, then the mappings $l_k : C([a,b],\mathbb{R}^n) \to L_{1;\mathrm{loc}}((a,b),\mathbb{R})$, $k = 1,2,\ldots,n$, given by the formula* (5.17) *are τ-positive in the sense of Definition 2.4.*

The assertion of Lemma 5.4 is obvious.

Lemma 5.5. *Each of the mappings* $l_k : C([a,b],\mathbb{R}^n) \to L_{1;\mathrm{loc}}((a,b),\mathbb{R})$ *defined by the formula* (5.17) *admits the decomposition in the form* (3.1) *where the mappings* l_k^+ *and* l_k^-, $k = 1,2,\dots,n$, *given by the formulae*

$$(l_k^+ u)(t) := \sum_{i=1}^{m}\sum_{j=1}^{n}(S_{\tau,1}p_{ikj})(t)\,u_j(\omega_{ikj}(t)),\ \ t\in[a,b], \tag{5.18}$$

$$(l_k^- u)(t) := \sum_{i=1}^{m}\sum_{j=1}^{n}(S_{\tau,-1}p_{ikj})(t)\,u_j(\omega_{ikj}(t)),\ \ t\in[a,b], \tag{5.19}$$

are τ*-positive linear operators from* $C([a,b],\mathbb{R}^n)$ *to* $L_{1;\mathrm{loc}}((a,b),\mathbb{R})$.

Recall that the linear mappings $S_{\tau,\sigma} : L_{1;\mathrm{loc}}((a,b),\mathbb{R}) \to L_{1;\mathrm{loc}}((a,b),\mathbb{R})$, where $\sigma \in \{-1,1\}$, are introduced by the formula (5.12).

Proof of Lemma 5.5. The equality (3.1) for the mappings (5.18), (5.19), and (5.17) follows from the relation (5.13) of Lemma 5.3, assertion (1). The τ-positivity of the mappings (5.18) and (5.19), in view of Definition 2.4, is a consequence of the assertion (2) of the lemma indicated. □

Lemma 5.6. *If the inclusion*

$$\big\{p_{ikj} \mid i=1,2,\dots,m;\ k,j=1,2,\dots,n\big\} \subset \widetilde{L}_1((a,b),\mathbb{R}) \tag{5.20}$$

holds, then each of the operators l_k, l_k^+, l_k^- : $C([a,b],\mathbb{R}^n) \to L_{1;\mathrm{loc}}((a,b),\mathbb{R})$, $k=1,2,\dots,n$, *given by the formulae* (5.17), (5.18), *and* (5.19) *is regular.*

Proof. Let $u = (u_k)_{k=1}^n \in C([a,b],\mathbb{R}^n)$ be arbitrary. In view of (5.18), the estimate

$$\begin{aligned}|(l_k^+ u)(t)| &= \Big|\sum_{i=1}^{m}\sum_{j=1}^{n}(S_{\tau,1}p_{ikj})(t)\,u_j(\omega_{ikj}(t))\Big| \le \\ &\le \sum_{j=1}^{n}\Big(\sum_{i=1}^{m}|p_{ikj}(t)|\Big)\max_{\xi\in[a,b]}|u_j(\xi)|\end{aligned}$$

is true for a. e. $t \in [a,b]$ and all $k = 1,2,\dots,n$. Therefore, for every single $k = 1,2,\dots,n$, the relation (2.2) holds with l replaced by l_k^+ and

$$h_j(t) := \sum_{i=1}^{m}|p_{ikj}(t)|,\ \ t\in[a,b],\ \ j=1,2,\dots,n. \tag{5.21}$$

By virtue of the assumption (5.20) and the assertion (1) of Lemma 2.2, each of the functions (5.21) belongs to the set $\widetilde{L}_1((a,b),\mathbb{R})$ and thus, due to the arbitrariness of u, it remains to refer to Definition 2.3.

The regularity of $l_k^- : C([a,b],\mathbb{R}^n)\to L_{1;\mathrm{loc}}((a,b),\mathbb{R})$, $k=1,2,\dots,n$, is proved analogously and that of the mappings $l_k : C([a,b],\mathbb{R}^n)\to L_{1;\mathrm{loc}}((a,b),\mathbb{R})$, $k = 1,2,\dots,n$, is a consequence of their decomposition (3.1) ensured by Lemma 5.5. □

Similar statements are true with $L_{1;\mathrm{loc}}((a,b),\mathbb{R})$ replaced by $L_{1;\mathrm{loc}}((a,b],\mathbb{R})$ and $L_{1;\mathrm{loc}}([a,b),\mathbb{R})$; we do not formulate them explicitly.

5.2. **Proofs.** In this section, the proofs of the results of Sections 3 and 4 are given.

5.2.1. *Proof of Theorems* 3.2, 3.5 *and* 3.7. To establish the assertion of Theorem 3.2, we need a result of [10]. Theorem 5.7 below is a slightly modified version of Theorem 2 from [10, p. 1857].

Theorem 5.7. *Let $h_k : C([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R})$, $k = 1,2,\dots,n$, be positive operators, and let there exist a function $y = (y_k)_{k=1}^n : [a,b] \to \mathbb{R}^n$ all the components of which are absolutely continuous, satisfy the conditions* (3.2) *and* (3.3), *and, moreover, are such that the inequality*

$$\min_{k=1,2,\dots,n} \operatorname*{vrai\,min}_{t\in[a,b]} \big(y_k'(t) - \varrho\,(h_k y)(t)\big) \geq 0 \tag{5.22}$$

is true with a certain $\varrho \in (1,+\infty)$.

Then the initial value problem

$$u_k(a) = c_k, \quad k = 1,2,\dots,n, \tag{5.23}$$

for the system

$$u_k'(t) = (h_k u)(t) + q_k(t), \quad t \in [a,b], \quad k = 1,2,\dots,n, \tag{5.24}$$

is uniquely solvable for arbitrary $\{q_k \mid k = 1,2,\dots,n\} \subset L_1([a,b],\mathbb{R})$ and real c_k, $k = 1,2,\dots,n$. If, moreover, the constants c_k and the functions q_k, $k = 1,2,\dots,n$, have the property

$$\min_{t\in[a,b]} \int_a^t q_k(\xi)\,d\xi \geq -c_k, \quad k = 1,2,\dots,n, \tag{5.25}$$

then each of the components $u_1, u_2, \dots, u_n$ of the unique solution of the problem (5.24), (5.23) *is non-negative on $[a,b]$.*

Let us now turn to the proof of Theorem 3.2. Given the operators l_k, $k = 1,2,\dots,n$, we put

$$h_k := \widehat{l}_{\tau,k}, \quad k = 1,2,\dots,n, \tag{5.26}$$

where $\widehat{l}_{\tau,k}$, $k = 1,2,\dots,n$, are the mappings defined according to the formula (5.1). In view of Lemma 5.1, the regularity of the linear mappings $l_k : C([a,b],\mathbb{R}^n) \to L_{1;\mathrm{loc}}((a,b),\mathbb{R})$, $k = 1,2,\dots,n$, guarantees that the corresponding mapping $\widehat{l}_\tau$ is a bounded linear operator from $C([a,b],\mathbb{R}^n)$ to $L_1([a,b],\mathbb{R}^n)$. By assumption, the mapping l is τ-positive in the sense of Definition 2.4 and, therefore, Lemma 5.2 ensures the positivity of the corresponding operator $\widehat{l}_\tau$. Furthermore, taking (5.1) and (5.26) into account, we conclude that the condition (3.5) implies (5.22) with $\varrho = \frac{1}{\varepsilon}$. Finally, if the collection of functions $\{f_k \mid k = 1,2,\dots,n\} \subset \widetilde{L}_1((a,b),\mathbb{R})$ and the constants c_{0k}, c_{1k}, $k = 1, 2, \ldots, n$, possess the property (3.6), then the corresponding functions

$$q_k(t) := \int_{\tau}^{t} f_k(s)\, ds + c_{1k}, \quad t \in [a, b], \quad k = 1, 2, \ldots, n,$$

and the constants $c_k := c_{0k}$, $k = 1, 2, \ldots, n$, satisfy the condition (5.25) for all t from $[a, b]$ and $k = 1, 2, \ldots, n$. Applying Theorem 5.7 to the problem (5.24), (5.23) with the above definitions of h_k, q_k, and c_k, $k = 1, 2, \ldots, n$, we arrive at the conclusion desired.

Theorems 3.5 and 3.7 are proved analogously.

5.2.2. *Proof of Theorems* 3.1, 3.4 *and* 3.6. In order to prove Theorems 3.1, 3.4 and 3.6 we need the following assertion (see Theorem 3.1 in [14]).

Theorem 5.8. *Let the operators $h_k : C([a, b], \mathbb{R}^n) \to L_1([a, b], \mathbb{R})$, $k = 1, 2, \ldots, n$, and a function $y = (y_k)_{k=1}^n : [a, b] \to \mathbb{R}^n$ fulfil the assumptions of Theorem 5.7. Let, moreover, $h_k^* : C([a, b], \mathbb{R}^n) \to L_1([a, b], \mathbb{R})$, $k = 1, 2, \ldots, n$, be linear bounded operators such that*

$$(h_k^* v)(t) \operatorname{sign} v_k(t) \le (h_k |v|)(t) \tag{5.27}$$

for a. e. $t \in [a, b]$ and all $v = (v_k)_{k=1}^n \in C([a, b], \mathbb{R}^n)$, where $|v| = (|v_k|)_{k=1}^n$.

Then the initial value problem

$$u_k(a) = c_k^*, \quad k = 1, 2, \ldots, n, \tag{5.28}$$

for the system

$$u_k'(t) = (h_k^* u)(t) + q_k^*(t), \quad t \in [a, b], \quad k = 1, 2, \ldots, n, \tag{5.29}$$

is uniquely solvable for arbitrary $\{q_k^* \mid k = 1, 2, \ldots, n\} \subset L_1([a, b], \mathbb{R})$ and real c_k^*, $k = 1, 2, \ldots, n$.

Proof. According to the Fredholm property of the problem (5.29), (5.23) (see, e. g., [15]), it is sufficient to show that the homogeneous problem

$$u_k'(t) = (h_k^* u)(t), \quad u_k(a) = 0, \quad t \in [a, b], \quad k = 1, 2, \ldots, n,$$

has only the trivial solution. Indeed, let $u = (u_k)_{k=1}^n$ be a solution of the problem indicated. Then, using (5.27), we get

$$|u_k(t)|' = (h_k^* u)(t) \operatorname{sign} u_k(t) \le (h_k |u|)(t) \tag{5.30}$$

for a. e. $t \in [a, b]$, where $|u| = (|u_k|)_{k=1}^n$. Therefore, $|u|$ is a solution of the problem (5.24), (5.23) with $c_k = 0$ and

$$q_k(t) = |u_k(t)|' - (h_k |u|)(t)$$

for a. e. $t \in [a, b]$ and all $k = 1, 2, \ldots, n$. By virtue of (5.30), Theorem 5.7 ensures that $|u_k(t)| \le 0$ for all $t \in [a, b]$, $k = 1, 2, \ldots, n$, and thus $u_k \equiv 0$ for $k = 1, 2, \ldots, n$. □

Now we are in a position to prove Theorem 3.1. Given the operators l_k^+ and l_k^-, $k = 1, 2, \ldots, n$, we put

$$h_k^* := \widehat{l}_{\tau,k}^+ - \widehat{l}_{\tau,k}^+, \quad k = 1, 2, \ldots, n, \tag{5.31}$$

where $\widehat{l}_{\tau,k}^+$ (resp., $\widehat{l}_{\tau,k}^-$), $k = 1, 2, \ldots, n$, are the mappings defined according to the formula (5.1) with l replaced by l_k^+ (resp., l_k^-). In view of Lemma 5.1, the regularity of the linear mappings $l_k^\pm : C([a, b], \mathbb{R}^n) \to L_{1;\mathrm{loc}}((a, b), \mathbb{R})$, $k = 1, 2, \ldots, n$, guarantee that the corresponding mappings h_k^* are bounded linear operators from $C([a, b], \mathbb{R}^n)$ to $L_1([a, b], \mathbb{R})$. By assumption, the operators l^+ and l^- are τ-positive and, therefore, Lemma 5.2 ensures the positivity of the operators $\widehat{l}_\tau^+$ and $\widehat{l}_\tau^-$. Consequently, the condition (5.27) is satisfied, where

$$h_k := \widehat{l}_{\tau,k}^+ + \widehat{l}_{\tau,k}^+, \quad k = 1, 2, \ldots, n. \tag{5.32}$$

It is clear that $h_k : C([a, b], \mathbb{R}^n) \to L_1([a, b], \mathbb{R})$, $k = 1, 2, \ldots, n$, are positive operators. Moreover, taking (5.1) and (5.32) into account, we conclude that the condition (3.4) implies (5.22) with $\varrho = \frac{1}{\varepsilon}$. Applying Lemma 5.8 to the problem (5.29), (5.28) with the above definitions of h_k^*, the functions q_k^*, $k = 1, 2, \ldots, n$, given by the formula

$$q_k^*(t) := \int_\tau^t f_k(s)\, ds + c_{1k}, \quad t \in [a, b],$$

and $c_k^* := c_{0k}$, $k = 1, 2, \ldots, n$, we arrive at the conclusion desired.

Theorems 3.4 and 3.6 can be proved analogously.

5.2.3. *Proof of Theorem* 4.1. Let us define the function $y = (y_k)_{k=1}^n : [a, b] \to \mathbb{R}^n$ by setting

$$y_k(t) := \gamma_k (t - a)^{\alpha_k}, \quad t \in [a, b], \quad k = 1, 2, \ldots, n. \tag{5.33}$$

Clearly, the functions (5.33) satisfy the conditions (3.2), (3.3) and are absolutely continuous.

According to the assumption (4.2), there exists a constant $\varepsilon \in (0, 1)$ such that

$$\sum_{i=1}^m \sum_{j=1}^n \gamma_j \int_t^b |p_{ikj}(s)| \, (\omega_{ikj}(s) - a)^{\alpha_j}\, ds \le \varepsilon \gamma_k \alpha_k (t - a)^{\alpha_j - 1} \tag{5.34}$$

for all $t \in (a, b]$ and $k = 1, 2, \ldots, n$.

It is clear that the system (1.4) can be represented in the form (1.3) with l_k, $k = 1, 2, \ldots, n$, given by the equalities (5.17). By virtue of Lemma 5.5, each of the operators mentioned admit the decomposition (3.1), where l_k^+ and l_k^-, $k = 1, 2, \ldots, n$, are the negative linear mappings defined according to formulae (5.18) and (5.19) with $\tau = b$. The formulae (5.18), (5.19), (5.33), and the equality (5.14) of Lemma 5.3 imply that, in this case,

$$\begin{gathered}\left|(l_k^+ y)(t) + (l_k^- y)(t)\right| = -(l_k^+ y)(t) - (l_k^- y)(t) = \\ = -\sum_{i=1}^{m}\sum_{j=1}^{n}(S_{b,1}p_{ikj})(t)\, y_j(\omega_{ikj}(t)) - \sum_{i=1}^{m}\sum_{j=1}^{n}(S_{b,-1}p_{ikj})(t)\, y_j(\omega_{ikj}(t)) = \\ = \sum_{i=1}^{m}\sum_{j=1}^{n}|p_{ikj}(t)|\, y_j(\omega_{ikj}(t)) = \sum_{i=1}^{m}\sum_{j=1}^{n}|p_{ikj}(t)|\, \gamma_j(\omega_{ikj}(t) - a)^{\alpha_j}\end{gathered}$$

for all $k = 1, 2, \ldots, n$ and a. e. $t \in [a, b]$. Therefore, we find

$$\begin{gathered}\varepsilon y_k'(t) - \int_t^b \left|(l_k^+ y)(s) + (l_k^- y)(s)\right| ds = \\ = \varepsilon\gamma_k\alpha_k(t-a)^{\alpha_k - 1} - \sum_{i=1}^{m}\sum_{j=1}^{n}\gamma_j \int_t^b |p_{ikj}(s)|\, (\omega_{ikj}(s) - a)^{\alpha_j}\, ds\end{gathered}$$

for all $k = 1, 2, \ldots, n$ and $t \in [a, b]$. Taking the last equality into account, we conclude that the condition (5.34) ensures the fulfilment of the condition (3.4) assumed in Theorem 3.4.

The condition (4.11), in view of Definition 2.1, guarantees that the inclusion (5.20) holds for the functions p_{ikj}, $i = 1, 2, \ldots, m$, $k, j = 1, 2, \ldots, n$. Consequently, by Lemma 5.6, the operators (5.18) and (5.19) are regular. Rewriting the system (1.4) in the form (1.3) and applying Theorem 3.4, we arrive at the desired assertion on the unique solvability of the problem (1.4), (1.1), (3.12).

5.2.4. *Proof of Corollary* 4.2. In view of (4.4), the condition (4.2) holds with $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$. Furthermore, due to (4.5), the assumption (4.4) ensures that the relation (4.1) is also satisfied. Applying Theorem 4.1, we obtain the required statement.

5.2.5. *Proof of Theorem* 4.3. The validity of the theorem follows from Theorem 3.6. The proof is analogous to the proof of Theorem 4.1, with the difference that $\tau = a$ has to be taken therein.

5.2.6. *Proof of Corollary* 4.4. By virtue of (4.4), the condition (4.8) holds with $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$. Furthermore, in view of (4.10), the assumption (4.4) guarantees the validity of the relation (4.7). Applying Theorem 4.3, we obtain the required statement.

5.2.7. *Proof of Theorem* 4.5. If $\tau = a$ (resp., $\tau = b$), then the validity of the first assertion of the theorem follows immediately from Theorem 4.1 (resp., Theorem 4.3).

Assume that $a < \tau < b$. Then the first assertion of the theorem can be derived from Theorem 3.1 analogously to the proof of Theorem 4.1.

Let now $\tau \in [a, b]$. If the assumption (4.13) holds, then, according to Lemma 5.4, each of the operators l_k, $k = 1, 2, \dots, n$, given by the equalities (5.17) is τ-positive. Therefore, the validity of the last assertion of the theorem follows from Theorems 3.2, 3.5, and 3.7.

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