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**LARGE TIME ASYMPTOTICS
OF SOLUTIONS TO A NONLINEAR
INTEGRO-DIFFERENTIAL EQUATION**

Abstract. The large time asymptotic behavior of solutions to a nonlinear integro-differential equation associated with the penetration of a magnetic field into a substance is studied. The rates of convergence are given as well.

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1. INTRODUCTION

The process of diffusion of the magnetic field into a substance is modelled by Maxwell's system of partial differential equations [1]. As it is shown in [2], if the coefficients of thermal heat capacity and electroconductivity of the substance depend on temperature, then Maxwell's system can be rewritten in the integro-differential form. The equation

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x} \left[a \left(\int_0^t \left(\frac{\partial U}{\partial x} \right)^2 d\tau \right) \frac{\partial U}{\partial x} \right], \quad (1.1)$$

where the function $a = a(S)$ is defined for $S \in [0, \infty)$, is the scalar analogue of that system.

The integro-differential equation of the type (1.1) is complex and still yields to investigation only for special cases [2]–[7].

The existence and uniqueness of solution of initial-boundary value problems for the equations of type (1.1) are studied in the works [2], [3] and subsequently in a number of other works as well (see, for example, [4]–[7]). It should be noted that theorems on existence of solutions have been proved only for power-like functions $a(S) = (1 + S)^p$ so far, and even that under the restriction $0 < p \leq 1$ on the exponent, but these cases are important as it was noted in the works [2], [4]. The existence theorems that are proved in [2], [3] are based on a priori estimates, Galerkin's method and compactness arguments as in [8], [9] for nonlinear parabolic equations.

Naturally, the importance of investigation of asymptotic behavior of solutions of boundary value problems for the equation (1.1) have arisen. In this direction research was made in the work [10], where asymptotic behavior of the solution under the homogeneous Dirichlet conditions in the space H^1 was given. The purpose of this note is to continue investigation of the large time asymptotic behavior as $t \rightarrow \infty$ of the solutions of the first boundary value problems for the equation (1.1). The attention is again paid to the case $a(S) = (1 + S)^p$, $0 < p \leq 1$. The rest of the paper is organized as follows. In the second section we discuss the initial-boundary value problem with zero lateral boundary data. Here stabilization results of the solution are proved in the space C^1 . Section 3 is devoted to the study of the problem with non-zero boundary data on one side of the lateral boundary. The asymptotic property for this case is also proved in the space C^1 . Mathematical results that are given below show difference between stabilization rates of solutions with homogeneous and nonhomogeneous boundary conditions.

2. THE PROBLEM WITH ZERO BOUNDARY CONDITIONS

In the domain $Q = (0, 1) \times (0, \infty)$, let us consider the following initial-boundary value problem:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x} \left[a(S) \frac{\partial U}{\partial x} \right], \quad (x, t) \in Q, \quad (2.1)$$

$$U(0, t) = U(1, t) = 0, \quad t \geq 0, \quad (2.2)$$

$$U(x, 0) = U_0(x), \quad x \in [0, 1], \quad (2.3)$$

where

$$S(x, t) = \int_0^t \left(\frac{\partial U}{\partial x} \right)^2 d\tau,$$

$a(S) = (1 + S)^p$, $0 < p \leq 1$, $U_0 = U_0(x)$ is a given function.

The existence and uniqueness of solution of the problem (2.1)–(2.3) in suitable classes have been proved in [2], [3].

Let us introduce the usual L_2 -inner product and the norm:

$$(u, v) = \int_0^1 u(x)v(x) dx, \quad \|u\| = (u, u)^{1/2}.$$

It is not difficult to prove the validity of the following statement.

Lemma 2.1. *For the solution of the problem (2.1)–(2.3) the following estimate takes place*

$$\|U\| \leq C \exp(-t).$$

Here and below in this section C , C_i and c denote positive constants dependent on U_0 and independent of t .

Note that Lemma 2.1 gives exponential stabilization of the solution of the problem (2.1)–(2.3) in the norm of the space $L_2(0, 1)$. The purpose of this section is to show that the stabilization is also achieved in the norm of the space $C^1(0, 1)$. First we formulate a result on the stabilization in the Sobolev space $H^1(0, 1)$ [10].

Theorem 2.1. *If $U_0 \in H^2(0, 1)$, $U_0(0) = U_0(1) = 0$, then for the solution of the problem (2.1)–(2.3) the following estimate is true*

$$\left\| \frac{\partial U}{\partial x} \right\| + \left\| \frac{\partial U}{\partial t} \right\| \leq C \exp\left(-\frac{t}{2}\right).$$

Now let us prove the following main statement of this section.

Theorem 2.2. *If $U_0 \in H^2(0, 1)$, $U_0(0) = U_0(1) = 0$, then for the solution of the problem (2.1)–(2.3) the following relation holds*

$$\left| \frac{\partial U(x, t)}{\partial x} \right| \leq C \exp\left(-\frac{t}{2}\right).$$

In order to prove Theorem 2.2, we will obtain some auxiliary estimates.

Lemma 2.2. *For the function S the following estimates are true:*

$$c\varphi^{\frac{1}{1+2p}}(t) \leq 1 + S(x, t) \leq C\varphi^{\frac{1}{1+2p}}(t),$$

where

$$\varphi(t) = 1 + \int_0^t \int_0^1 \sigma^2 dx d\tau \quad (2.4)$$

and $\sigma = (1 + S)^p \frac{\partial U}{\partial x}$.

Proof. From the definition of the function S it follows that

$$\frac{\partial S}{\partial t} = \left(\frac{\partial U}{\partial x} \right)^2, \quad S(x, 0) = 0. \quad (2.5)$$

Let us multiply (2.5) by $(1 + S)^{2p}$:

$$\frac{1}{1 + 2p} \frac{\partial(1 + S)^{1+2p}}{\partial t} = \left(\frac{\partial U}{\partial x} \right)^2 (1 + S)^{2p}.$$

Note that the equation (2.1) can be rewritten as

$$\frac{\partial U}{\partial t} = \frac{\partial \sigma}{\partial x}. \quad (2.6)$$

We have

$$\frac{1}{1 + 2p} \frac{\partial(1 + S)^{1+2p}}{\partial t} = \sigma^2, \quad (2.7)$$

$$\sigma^2(x, t) = \int_0^1 \sigma^2(y, t) dy + 2 \int_0^1 \int_y^x \sigma(\xi, t) \frac{\partial U(\xi, t)}{\partial t} d\xi dy. \quad (2.8)$$

From Theorem 2.1 and the relations (2.4), (2.7), (2.8) we get

$$\begin{aligned} \frac{1}{1 + 2p} (1 + S)^{1+2p} &= \int_0^t \sigma^2 d\tau + \frac{1}{1 + 2p} = \int_0^t \int_0^1 \sigma^2(y, \tau) dy d\tau + \\ &+ 2 \int_0^t \int_0^1 \int_y^x \sigma(\xi, \tau) \frac{\partial U(\xi, \tau)}{\partial \tau} d\xi dy d\tau + \frac{1}{1 + 2p} \leq \\ &\leq 2 \int_0^t \int_0^1 \sigma^2(y, \tau) dy d\tau + \int_0^t \int_0^1 \left(\frac{\partial U(x, \tau)}{\partial \tau} \right)^2 dx d\tau + \frac{1}{1 + 2p} \leq \\ &\leq 2 \int_0^t \int_0^1 \sigma^2(y, \tau) dy d\tau + C_1 \int_0^t \exp(-\tau) d\tau + \frac{1}{1 + 2p} \leq C_2 \varphi(t), \end{aligned}$$

i.e.,

$$1 + S(x, t) \leq C \varphi^{\frac{1}{1+2p}}(t). \quad (2.9)$$

Analogously,

$$\frac{1}{1 + 2p} (1 + S)^{1+2p} =$$

$$\begin{aligned}
&= \int_0^t \int_0^1 \sigma^2(y, \tau) dy d\tau + 2 \int_0^t \int_0^1 \int_y^x \sigma(\xi, \tau) \frac{\partial U(\xi, \tau)}{\partial \tau} d\xi dy d\tau + \frac{1}{1+2p} \geq \\
&\geq \frac{1}{2} \int_0^t \int_0^1 \sigma^2(y, \tau) dy d\tau - C_2 = \frac{1}{2} \varphi(t) - C_3. \tag{2.10}
\end{aligned}$$

We have

$$C_3(1+S)^{1+2p} \geq C_3. \tag{2.11}$$

From (2.10) and (2.11) we get

$$\left(\frac{1}{1+2p} + C_3 \right) (1+S)^{1+2p} \geq \frac{1}{2} \varphi(t),$$

that is,

$$1+S(x, t) \geq c\varphi^{\frac{1}{1+2p}}(t). \tag{2.12}$$

Finally, from (2.9) and (2.12) it follows Lemma 2.2.

Taking into account the relation (2.4), Lemma 2.2 and Theorem 2.1, we have

$$\frac{d\varphi(t)}{dt} = \int_0^1 (1+S)^{2p} \left(\frac{\partial U}{\partial x} \right)^2 dx \leq C\varphi^{\frac{2p}{1+2p}}(t) \exp(-t),$$

that is,

$$\frac{d}{dt} (\varphi^{\frac{1}{1+2p}}(t)) \leq C \exp(-t).$$

Integrating from 0 to t and keeping in mind the definition (2.4), we get

$$1 \leq \varphi(t) \leq C.$$

From this, using Lemma 2.2, we receive

$$1 \leq 1+S(x, t) \leq C. \tag{2.13}$$

In view of (2.13) and Theorem 2.1, the equality (2.8) gives

$$\sigma^2(x, t) \leq 2 \int_0^1 (1+S)^{2p} \left(\frac{\partial U}{\partial x} \right)^2 dx + \int_0^1 \left(\frac{\partial U}{\partial t} \right)^2 dx \leq C \exp(-t),$$

that is,

$$|\sigma(x, t)| \leq C \exp\left(-\frac{t}{2}\right).$$

This estimate along with (2.13) and the relation $\sigma = (1+S)^p \frac{\partial U}{\partial x}$ completes the proof of Theorem 2.2. \square

3. THE PROBLEM WITH NON-ZERO DATA ON ONE SIDE OF THE LATERAL BOUNDARY

In the domain Q let us consider the following initial-boundary value problem:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x} \left[a(S) \frac{\partial U}{\partial x} \right], \quad (x, t) \in Q, \quad (3.1)$$

$$U(0, t) = 0, \quad U(1, t) = \psi, \quad t \geq 0, \quad (3.2)$$

$$U(x, 0) = U_0(x), \quad x \in [0, 1], \quad (3.3)$$

where

$$S(x, t) = \int_0^t \left(\frac{\partial U}{\partial x} \right)^2 d\tau,$$

$a(S) = (1 + S)^p$, $0 < p \leq 1$, $\psi = \text{const} > 0$, $U_0 = U_0(x)$ is a given function.

The main purpose of this section is to prove the following statement.

Theorem 3.1. *If $U_0 \in H^2(0, 1)$, $U_0(0) = 0$, $U_0(1) = \psi$, then for the solution of the problem (3.1)–(3.3) the following estimate is true*

$$\left| \frac{\partial U(x, t)}{\partial x} - \psi \right| \leq Ct^{-1-p}, \quad t \geq 1.$$

In this section C , C_i and c denote positive constants dependent on ψ , U_0 and independent of t .

The proof of Theorem 3.1 is based on a priori estimates which are obtained below.

Lemma 3.1. *For the solution of the problem (3.1)–(3.3) the following estimate takes place*

$$\int_0^t \int_0^1 \left(\frac{\partial U}{\partial \tau} \right)^2 dx d\tau \leq C.$$

Proof. Let us differentiate the equation (3.1) with respect to t

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial}{\partial x} \left[\frac{\partial(1 + S)^p}{\partial t} \frac{\partial U}{\partial x} + (1 + S)^p \frac{\partial^2 U}{\partial t \partial x} \right] = 0$$

and multiply scalarly by $\partial U / \partial t$. Using the formula of integrating by parts and the boundary conditions (3.2), we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^1 \left(\frac{\partial U}{\partial t} \right)^2 dx + \int_0^1 (1 + S)^p \left(\frac{\partial^2 U}{\partial t \partial x} \right)^2 dx + \\ & + p \int_0^1 (1 + S)^{p-1} \left(\frac{\partial U}{\partial x} \right)^3 \frac{\partial^2 U}{\partial t \partial x} dx = 0. \end{aligned} \quad (3.4)$$

From (3.4), taking into account Poincaré's inequality, we have

$$\begin{aligned} & \frac{d}{dt} \int_0^1 \left(\frac{\partial U}{\partial t} \right)^2 dx + 2 \int_0^1 \left(\frac{\partial U}{\partial t} \right)^2 dx + \\ & + \frac{p}{2} \int_0^1 (1+S)^{p-1} \frac{\partial}{\partial t} \left(\frac{\partial U}{\partial x} \right)^4 dx \leq 0. \end{aligned} \quad (3.5)$$

Let us integrate the relation (3.5) from 0 to t

$$\begin{aligned} & \int_0^1 \left(\frac{\partial U}{\partial t} \right)^2 dx + 2 \int_0^t \int_0^1 \left(\frac{\partial U}{\partial \tau} \right)^2 dx d\tau + \\ & + \frac{p}{2} \int_0^t \int_0^1 (1+S)^{p-1} \frac{\partial}{\partial \tau} \left(\frac{\partial U}{\partial x} \right)^4 dx d\tau \leq C. \end{aligned}$$

Integration by parts gives

$$\int_0^1 \left(\frac{\partial U}{\partial t} \right)^2 dx + 2 \int_0^t \int_0^1 \left(\frac{\partial U}{\partial \tau} \right)^2 dx d\tau \leq C. \quad (3.6)$$

Therefore, Lemma 3.1 is proved. \square

Note that from Lemma 3.1, according to the scheme applied in the second section, we get the validity of Lemma 2.2 for the problem (3.1)–(3.3) as well.

Lemma 3.2. *The following estimates are true:*

$$c\varphi^{\frac{2p}{1+2p}}(t) \leq \int_0^1 \sigma^2(x, t) dx \leq C\varphi^{\frac{2p}{1+2p}}(t).$$

Proof. Taking into account Lemma 2.2, we get

$$\begin{aligned} \int_0^1 \sigma^2 dx &= \int_0^1 (1+S)^{2p} \left(\frac{\partial U}{\partial x} \right)^2 dx \geq c\varphi^{\frac{2p}{1+2p}}(t) \int_0^1 \left(\frac{\partial U}{\partial x} \right)^2 dx \geq \\ &\geq c\varphi^{\frac{2p}{1+2p}}(t) \left(\int_0^1 \frac{\partial U}{\partial x} dx \right)^2 = \psi^2 c\varphi^{\frac{2p}{1+2p}}(t), \end{aligned}$$

that is,

$$\int_0^1 \sigma^2(x, t) dx \geq c\varphi^{\frac{2p}{1+2p}}(t). \quad (3.7)$$

From (3.6) it follows that

$$\int_0^1 \left(\frac{\partial U}{\partial t} \right)^2 dx \leq C. \quad (3.8)$$

Let us multiply the equation (3.1) scalarly by U . Using the boundary conditions (3.2), we have

$$\int_0^1 U \frac{\partial U}{\partial t} dx + \int_0^1 (1+S)^p \left(\frac{\partial U}{\partial x} \right)^2 dx = \psi \sigma(1, t).$$

Using this equality, Lemma 2.2, the relations (2.6), (2.8), (3.7) and (3.8), and the maximum principle

$$|U(x, t)| \leq \max_{0 \leq y \leq 1} |U_0(y)|, \quad 0 \leq x \leq 1, \quad t \geq 0,$$

we get

$$\begin{aligned} \left\{ \int_0^1 \sigma^2(x, t) dx \right\}^2 &\leq C_1 \varphi^{\frac{2p}{1+2p}}(t) \left[\int_0^1 (1+S)^p \left(\frac{\partial U}{\partial x} \right)^2 dx \right]^2 \leq \\ &\leq 2C_1 \varphi^{\frac{2p}{1+2p}}(t) \left[\left(\psi \sigma(1, t) \right)^2 + \left(\int_0^1 U \frac{\partial U}{\partial t} dx \right)^2 \right] \leq \\ &\leq 2C_1 \varphi^{\frac{2p}{1+2p}}(t) \left[2\psi^2 \int_0^1 \sigma^2 dx + \psi^2 \int_0^1 \left(\frac{\partial \sigma}{\partial x} \right)^2 dx + \int_0^1 U^2 dx \int_0^1 \left(\frac{\partial U}{\partial t} \right)^2 dx \right] \leq \\ &\leq 2C_1 \varphi^{\frac{2p}{1+2p}}(t) \left[2\psi^2 \int_0^1 \sigma^2 dx + \psi^2 \int_0^1 \left(\frac{\partial U}{\partial t} \right)^2 dx + \left(\max_{0 \leq y \leq 1} |U_0(y)| \right)^2 \int_0^1 \left(\frac{\partial U}{\partial t} \right)^2 dx \right] \leq \\ &\leq 2C_1 \varphi^{\frac{2p}{1+2p}}(t) \left(C_2 \int_0^1 \sigma^2 dx + \frac{C_3}{\varphi^{\frac{2p}{1+2p}}(t)} \int_0^1 \sigma^2 dx \right). \end{aligned}$$

Hence, taking into account the relation $\varphi(t) \geq 1$, we get

$$\int_0^1 \sigma^2(x, t) dx \leq C \varphi^{\frac{2p}{1+2p}}(t).$$

So Lemma 3.2 is proved. \square

From Lemma 3.2 and (2.4) we have the following estimates:

$$c \varphi^{\frac{2p}{1+2p}}(t) \leq \frac{d\varphi(t)}{dt} \leq C \varphi^{\frac{2p}{1+2p}}(t). \quad (3.9)$$

Lemma 3.3. *The derivative $\partial U/\partial t$ satisfies the inequality*

$$\int_0^1 \left(\frac{\partial U}{\partial t} \right)^2 dx \leq C \varphi^{-\frac{2}{1+2p}}(t).$$

Proof. The equality (3.4) yields

$$\begin{aligned} \frac{d}{dt} \int_0^1 \left(\frac{\partial U}{\partial t} \right)^2 dx + \int_0^1 (1+S)^p \left(\frac{\partial^2 U}{\partial t \partial x} \right)^2 dx &\leq \\ &\leq p^2 \int_0^1 (1+S)^{p-2} \left(\frac{\partial U}{\partial x} \right)^6 dx. \end{aligned} \quad (3.10)$$

Now using Lemmas 2.2 and 3.2, the relation $\sigma = (1+S)^p \frac{\partial U}{\partial x}$ and the identity

$$\int_0^1 \left(\frac{\partial \sigma}{\partial x} \right)^2 dx = - \int_0^1 \sigma \frac{\partial^2 \sigma}{\partial x^2} dx,$$

from (3.10) we get

$$\begin{aligned} \frac{d}{dt} \int_0^1 \left(\frac{\partial U}{\partial t} \right)^2 dx + c \varphi^{\frac{p}{1+2p}}(t) \int_0^1 \left(\frac{\partial^2 U}{\partial t \partial x} \right)^2 dx &\leq C_1 \varphi^{-\frac{5p+2}{1+2p}}(t) \int_0^1 \sigma^6 dx \leq \\ &\leq C_1 \varphi^{-\frac{5p+2}{1+2p}}(t) \int_0^1 \sigma^2(x, t) dx \left[\max_{0 \leq x \leq 1} \sigma^2(x, t) \right]^2 \leq \\ &\leq C_2 \varphi^{-\frac{3p+2}{1+2p}}(t) \left\{ \int_0^1 \sigma^2 dx + 2 \left[\int_0^1 \sigma^2 dx \right]^{1/2} \left[\int_0^1 \left(\frac{\partial \sigma}{\partial x} \right)^2 dx \right]^{1/2} \right\}^2 \leq \\ &\leq C_2 \varphi^{-\frac{3p+2}{1+2p}}(t) \left\{ \int_0^1 \sigma^2 dx + 2 \left[\int_0^1 \sigma^2 dx \right]^{3/4} \left[\int_0^1 \left(\frac{\partial^2 \sigma}{\partial x^2} \right)^2 dx \right]^{1/4} \right\}^2 \leq \\ &\leq C_3 \varphi^{\frac{p-2}{1+2p}}(t) + C_4 \varphi^{-\frac{3p+2}{1+2p}}(t) \varphi^{\frac{3p}{1+2p}}(t) \left[\int_0^1 \left(\frac{\partial^2 U}{\partial t \partial x} \right)^2 dx \right]^{1/2} \leq \\ &\leq C_3 \varphi^{\frac{p-2}{1+2p}}(t) + C_5 \varphi^{-\frac{p+4}{1+2p}}(t) + \frac{C}{2} \varphi^{\frac{p}{1+2p}}(t) \int_0^1 \left(\frac{\partial^2 U}{\partial t \partial x} \right)^2 dx. \end{aligned}$$

Note that in our case $p - 2 > -p - 4$. So the last relation, in view of Poincaré's inequality, gives

$$\frac{d}{dt} \int_0^1 \left(\frac{\partial U}{\partial t} \right)^2 dx + \frac{c}{2} \varphi^{\frac{p}{1+2p}}(t) \int_0^1 \left(\frac{\partial U}{\partial t} \right)^2 dx \leq C \varphi^{\frac{p-2}{1+2p}}(t).$$

From Gronwall's inequality we get

$$\begin{aligned} & \frac{d}{dt} \int_0^1 \left(\frac{\partial U}{\partial t} \right)^2 dx \leq \exp \left(-\frac{c}{2} \int_0^t \varphi^{\frac{p}{1+2p}}(\tau) d\tau \right) \times \\ & \times \left[\int_0^1 \left(\frac{\partial U}{\partial t} \right)^2 dx \Big|_{t=0} + C \int_0^t \exp \left(\frac{c}{2} \int_0^\tau \varphi^{\frac{p}{1+2p}}(\xi) d\xi \right) \varphi^{\frac{p-2}{1+2p}}(\tau) d\tau \right]. \end{aligned} \quad (3.11)$$

Noting that $\varphi(t) \geq 1$, and applying L'Hopital's rule and the estimate (3.9), we have

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{\int_0^t \exp \left(\frac{c}{2} \int_0^\tau \varphi^{\frac{p}{1+2p}}(\xi) d\xi \right) \varphi^{\frac{p-2}{1+2p}}(\tau) d\tau}{\exp \left(\frac{c}{2} \int_0^t \varphi^{\frac{p}{1+2p}}(\tau) d\tau \right) \varphi^{-\frac{2}{1+2p}}(t)} = \\ & = \lim_{t \rightarrow \infty} \frac{\exp \left(\frac{c}{2} \int_0^t \varphi^{\frac{p}{1+2p}}(\tau) d\tau \right) \varphi^{\frac{p-2}{1+2p}}(t)}{\exp \left(\frac{c}{2} \int_0^t \varphi^{\frac{p}{1+2p}}(\tau) d\tau \right) \left(\frac{c}{2} \varphi^{\frac{p-2}{1+2p}}(t) - \frac{2}{1+2p} \varphi^{\frac{-3-2p}{1+2p}}(t) \frac{d\varphi}{dt} \right)} \leq \\ & \leq \lim_{t \rightarrow \infty} \frac{1}{\frac{c}{2} - \frac{2C}{1+2p} \varphi^{-\frac{p+1}{1+2p}}(t)} \leq C. \end{aligned} \quad (3.12)$$

Therefore, the validity of Lemma 3.3 follows from (3.11) and (3.12). Let us now estimate $\partial S / \partial x$ in $L_1(0, 1)$.

Lemma 3.4. *For $\partial S / \partial x$ the following estimate is true:*

$$\int_0^1 \left| \frac{\partial S}{\partial x} \right| dx \leq C \varphi^{-\frac{p}{1+2p}}(t).$$

Proof. Let us differentiate (2.7) with respect to x

$$\frac{\partial}{\partial t} \left[(1+S)^{2p} \frac{\partial S}{\partial x} \right] = 2\sigma \frac{\partial \sigma}{\partial x}. \quad (3.13)$$

From Lemmas 3.2 and 3.3 we obtain

$$\int_0^1 \left| \sigma \frac{\partial U}{\partial t} \right| dx \leq C \varphi^{\frac{p}{1+2p}}(t) \varphi^{-\frac{1}{1+2p}}(t) = C \varphi^{\frac{p-1}{1+2p}}(t). \quad (3.14)$$

Finally, from Lemma 2.2 and the relations (2.6), (3.9), (3.13) and (3.14), we have

$$\begin{aligned} (1+S)^{2p} \frac{\partial S}{\partial x} &= \int_0^t 2\sigma \frac{\partial U}{\partial \tau} d\tau, \\ \int_0^1 \left| \frac{\partial S}{\partial x} \right| dx &\leq C\varphi^{-\frac{2p}{1+2p}}(t) \int_0^t \varphi^{\frac{p-1}{1+2p}}(\tau) d\tau \leq C_1\varphi^{-\frac{2p}{1+2p}}(t) \int_0^t \varphi^{-\frac{p+1}{1+2p}} d\varphi = \\ &= C_2\varphi^{-\frac{2p}{1+2p}}(t) \int_0^t d\varphi^{\frac{p}{1+2p}} = C_2\varphi^{-\frac{2p}{1+2p}}(t) (\varphi^{\frac{p}{1+2p}}(t) - 1) \leq C\varphi^{-\frac{p}{1+2p}}(t). \end{aligned}$$

Thus, Lemma 3.4 is proved. \square

We are ready to prove Theorem 3.1. Using Lemma 3.2 and the relations (2.8) and (3.14), we arrive at

$$\begin{aligned} \sigma^2(x, t) &\leq \int_0^1 \sigma^2(y, t) dy + 2 \int_0^1 \left| \sigma(y, t), \frac{\partial U(y, t)}{\partial t} \right| dy \leq \\ &\leq C_1\varphi^{\frac{2p}{1+2p}}(t) + C_2\varphi^{\frac{p-1}{1+2p}}(t). \end{aligned}$$

Hence we get

$$|\sigma(x, t)| \leq C\varphi^{\frac{p}{1+2p}}(t).$$

Now, taking into account Lemmas 2.2, 3.3 and 3.4, the equality (2.6), the definition of σ and the last estimate, we derive

$$\begin{aligned} &\int_0^1 \left| \frac{\partial^2 U(x, t)}{\partial x^2} \right| dx \leq \\ &\leq \int_0^1 \left| \frac{\partial U}{\partial t} (1+S)^{-p} \right| dx + p \int_0^1 \text{Big} |\sigma(1+S)^{-p-1} \frac{\partial S}{\partial x}| dx \leq \\ &\leq \left[\int_0^1 (1+S)^{-2p} dx \right]^{1/2} \left[\int_0^1 \left| \frac{\partial U}{\partial t} \right|^2 dx \right]^{1/2} + p \int_0^1 \left| \sigma(1+S)^{-p-1} \frac{\partial S}{\partial x} \right| dx \leq \\ &\leq C_1\varphi^{-\frac{p}{1+2p}}(t)\varphi^{-\frac{1}{1+2p}}(t) + C_2\varphi^{-\frac{p+1}{1+2p}}(t)\varphi^{\frac{p}{1+2p}}(t) \int_0^1 \left| \frac{\partial S}{\partial x} \right| dx \leq \\ &\leq C_3\varphi^{-\frac{p+1}{1+2p}}(t). \end{aligned}$$

Hence, we have

$$\int_0^1 \left| \frac{\partial^2 U(x, t)}{\partial x^2} \right| dx \leq C\varphi^{-\frac{p+1}{1+2p}}(t).$$

From this estimate, taking into account the relation

$$\frac{\partial U(x, t)}{\partial x} = \int_0^1 \frac{\partial U(y, t)}{\partial y} dy + \int_0^1 \int_y^x \frac{\partial^2 U(\xi, t)}{\partial \xi^2} d\xi dy,$$

we derive

$$\begin{aligned} \left| \frac{\partial U(x, t)}{\partial x} - \psi \right| &= \left| \int_0^1 \int_y^x \frac{\partial^2 U(\xi, t)}{\partial \xi^2} d\xi dy \right| \leq \\ &\leq \int_0^1 \left| \frac{\partial^2 U(y, t)}{\partial y^2} \right| dy \leq C\varphi^{-\frac{p+1}{1+2p}}(t). \end{aligned} \quad (3.15)$$

From (3.9) it is easy to show that

$$ct^{1+2p} \leq \varphi(t) \leq Ct^{1+2p}, \quad t \geq 1.$$

From here, taking into account the estimate (3.15), we get the validity of Theorem 3.1.

Note that in this section we have used the scheme similar to that of the work [11] in which the adiabatic shearing of incompressible fluids with temperature-dependent viscosity is studied.

The existence of globally defined solutions of the problems (2.1)–(2.3) and (3.1)–(3.3) now can be reobtained by a routine procedure, proving first the existence of the local solutions on a maximal time interval and then using the derived a priori estimates to show that these solutions cannot escape in a finite time.

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