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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS CLOSE TO LINEAR EQUATIONS

Abstract. The differential equation

$$y'' = \alpha_0 p(t) y |\ln |y||^{\sigma},$$

is considered in a finite or infinite interval $[a, \omega[$, where $\alpha_0 \in \{-1, 1\}$, $\sigma \in \mathbb{R}$, and $p : [a, \omega[\to]0, +\infty[$ is a continuous function. Asymptotic representations of solutions of this equation is obtained as $t \to \omega$.

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[illegible] $[a, \omega[$ [illegible]

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[illegible] $\alpha_0 \in \{-1, 1\}$, $\sigma \in \mathbb{R}$, [illegible] $p : [a, \omega[\to]0, +\infty[$ [illegible]. [illegible], [illegible] $t \to \omega$.

We will consider the differential equation

$$y'' = \alpha_0 p(t) y |\ln|y||^\sigma, \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $\sigma \in \mathbb{R}$, $p : [a, \omega[\to]0, +\infty[$ is a continuous function, $a < \omega \leq +\infty$, $a > 1$ if $\omega = +\infty$ and $a > \omega - 1$ if $\omega < +\infty$. It belongs to the category of differential equations of the type

$$y'' = \alpha_0 p(t) \varphi(y), \tag{2}$$

where $\varphi : \Delta_Y \to]0, +\infty[$ (Δ_Y is a one-sided neighborhood of Y, Y being either zero or $\pm\infty$) is a twice continuously differentiable function satisfying the conditions

$$\lim_{\substack{y \to Y \\ y \in \Delta_Y}} \varphi(y) = \begin{cases} \text{either } \ 0, \\ \text{or } \ +\infty, \end{cases} \qquad \lim_{\substack{y \to \Delta_Y \\ Y \in \Delta_Y}} \frac{y \varphi''(y)}{\varphi'(y)} = \mu.$$

The question about asymptotically vanishing and unbounded as $t \uparrow \omega$ solutions of the equation (2) has been considered in the papers [1]–[4]. But it is not studied enough for the case $\mu = 0$. Its peculiarity is that the equation is somehow close to the linear differential equation and requires advancement of the analyzis scheme proposed for $\mu \neq 0$. The differential equation (1) refers just to this case and this paper is devoted exactly to it.

A solution y of the equation (1) defined on some interval $[t_y, \omega[\subset [a, \omega[$ will be called $P_\omega(\lambda_0)$-solution if it satisfies the conditions:

$$\lim_{t \uparrow \omega} y^{(k)}(t) = \begin{cases} \text{either } \ 0, \\ \text{or } \ \pm\infty \end{cases} \qquad (k = 0, 1), \quad \lim_{t \uparrow \omega} \frac{(y'(t))^2}{y''(t) y(t)} = \lambda_0. \tag{3}$$

The purpose of this paper is to obtain necessary and sufficient conditions for the equation (1) to have $P_\omega(\pm\infty)$-solutions as well as asymptotic representations as $t \uparrow \omega$ for all such solutions and their first-order derivatives.

Let us introduce the auxiliary notation

$$\pi_\omega(t) = \begin{cases} t, & \text{if } \ \omega = +\infty, \\ t - \omega, & \text{if } \ \omega < +\infty, \end{cases}$$

$$q(t) = p(t) \pi_\omega^2(t) \big| \ln|\pi_\omega(t)| \big|^\sigma, \quad Q(t) = \int_a^t p(\tau) \pi_\omega(\tau) \big| \ln|\pi_\omega(\tau)| \big|^\sigma \, d\tau.$$

The following statements are true for the equation (1).

Theorem 1. *For existence of a $P_\omega(\pm\infty)$-solution of the equation* (1) *it is necessary and sufficient that the following conditions be satisfied*

$$\lim_{t \uparrow \omega} q(t) = 0, \quad \lim_{t \uparrow \omega} Q(t) = \infty. \tag{4}$$

Moreover, for these conditions there is a one-parameter family of $P_\omega(\pm\infty)$-solutions and each of them assumes the following asymptotic representations

$$\begin{aligned} \ln|y(t)| &= \ln|\pi_\omega(t)| + \alpha_0 Q(t)[1 + o(1)], \\ \ln|y'(t)| &= \alpha_0 Q(t)[1 + o(1)] \ \text{ as } \ t \uparrow \omega. \end{aligned} \tag{5}$$

Theorem 2. *If the function $p : [a,\omega[\to]0,+\infty[$ is continuously differentiable, the conditions* (4) *are fulfilled and there exists* (*finite or equal to* $\pm\infty$) $\lim\limits_{t\uparrow\omega} \frac{\pi_\omega(t)q'(t)}{q(t)}$, *then for each $P_\omega(\pm\infty)$-solution of the equation* (1) *the asymptotic representations*

$$\begin{aligned} \ln|y(t)| &= \ln|\pi_\omega(t)| + \alpha_0 Q(t)[1+o(1)], \\ \frac{y'(t)}{y(t)} &= \frac{1}{\pi_\omega(t)}\big[1+\alpha_0 q(t)[1+o(1)]\big] \quad \textit{as} \ \ t\uparrow\omega \end{aligned} \tag{6}$$

are valid.

Theorem 3. *Let the function $p : [a,\omega[\to]0,+\infty[$ be continuously differentiable and along with* (4) *the following conditions be satisfied*

$$\int\limits_a^\omega |q'(t)|\,dt < +\infty, \quad \int\limits_a^\omega \frac{q^2(t)}{|\pi_\omega(t)|}\,dt < +\infty, \quad \int\limits_a^\omega \frac{q(t)|Q(t)|}{\pi_\omega(t)\ln|\pi_\omega(t)|}\,dt < +\infty. \tag{7}$$

Then for any $c\neq 0$ there exists a $P_\omega(\pm\infty)$-solution of the equation (1) *which assumes the asymptotic representations*

$$\begin{aligned} y(t) &= \pi_\omega(t)\exp[\alpha_0 Q(t)]\,[c+o(1)], \\ y'(t) &= \exp[\alpha_0 Q(t)]\,[c+o(1)] \quad \textit{as} \ \ t\uparrow\omega. \end{aligned} \tag{8}$$

Proof of Theorem 1. *Necessity.* Let $y : [t_y,\omega[\to\mathbb{R}$ be a $P_\omega(\pm\infty)$-solution of the equation (1). Then the first of the conditions (3) is satisfied and

$$\lim_{t\uparrow\omega}\frac{y''(t)y(t)}{(y'(t))^2} = 0.$$

Without restriction of generality we can assume that $y'(t)$ and $\ln|y(t)|$ are different from zero for $t\in[t_y,\omega[$. Hence in view of the identity

$$\frac{y''(t)y(t)}{(y'(t))^2} = \Big(\frac{y'(t)}{y(t)}\Big)'\Big(\frac{y'(t)}{y(t)}\Big)^{-2} + 1$$

it follows that

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)y'(t)}{y(t)} = 1, \quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)y''(t)}{y'(t)} = 0. \tag{9}$$

Due to the first limit relation (9) it follows that $\ln|y(t)|\sim\ln|\pi_\omega(t)|$ as $t\uparrow\omega$, and in view of (1)

$$y''(t) = \alpha_0 p(t)\pi_\omega(t)\big|\ln|\pi_\omega(t)|\big|^\sigma y'(t)[1+o(1)] \ \text{ as } \ t\uparrow\omega. \tag{10}$$

Hence in view of the second limit relation (9) it follows that

$$p(t)\pi_\omega^2(t)|\ln|\pi_\omega(t)||^\sigma \to 0 \ \text{ as } \ t\uparrow\omega.$$

Thus the first condition (4) is satisfied. Now dividing (10) by $y'(t)$ and taking integral from t_y to t, we conclude due to the first condition (4) that

$\int\limits_{t_y}^{\omega} p(t)\pi_\omega(t)|\ln|\pi_\omega(t)||^\sigma\,dt = \infty$ and the asymptotic representation

$$\ln|y'(t)| = \alpha_0 \int\limits_a^t p(\tau)\pi_\omega(\tau)\big|\ln|\pi_\omega(\tau)|\big|^\sigma\,d\tau[1+o(1)] \ \text{ as } \ t\uparrow\omega$$

is valid. So the second condition (4) and the second asymptotic representation (5) are satisfied.

The validity of the first asymptotic representation (5) follows from the second one if we note that according to (9) $y'(t) \sim \frac{y(t)}{\pi_\omega(t)}$ as $t\uparrow\omega$.

Sufficiency. Suppose that the conditions (4) are true. The equation (1) by the transformation

$$\ln|y(t)| = [1+v_1(\tau)]\ln|\pi_\omega(t)|, \quad \frac{y'(t)}{y(t)} = \frac{1+v_2(\tau)}{\pi_\omega(t)}, \quad \tau = \beta\ln|\pi_\omega(t)|, \tag{11}$$

where

$$\beta = \begin{cases} 1, & \text{if } \ \omega = +\infty, \\ -1, & \text{if } \ \omega < +\infty, \end{cases}$$

is converted to the system of differential equations

$$\begin{cases} v_1' = \dfrac{1}{\tau}\,[v_2 - v_1], \\ v_2' = \beta\big[f(\tau) + \sigma f(\tau)v_1 - v_2 + V(\tau,v_1,v_2)\big], \end{cases} \tag{12}$$

in which

$$f(\tau)=f(\tau(t))=\alpha_0 q(t), \quad V(\tau,v_1,v_2)=-v_2^2+f(\tau)\big[|1+v_1|^\sigma-1-\sigma v_1\big].$$

This system of equations can be considered on the set $\Omega = [\tau_0,+\infty[\,\times \big\{(v_1,v_2) : \ |v_i| \le 1/2 \ (i=1,2)\big\}$, where $\tau_0 = \beta\ln|\pi_\omega(a)|$. On this set the right-hand sides of the system are continuous and, because of the first condition (4), $\lim\limits_{\tau\to+\infty} f(\tau) = \lim\limits_{t\uparrow\omega}\alpha_0 q(t) = 0$. Besides,

$$\frac{\partial V(\tau,v_1,v_2)}{\partial v_i} \longrightarrow 0 \ \text{ as } \ |v_1|+|v_2| \longrightarrow 0 \ \ (i=1,2)$$
$$\text{uniformly in } \ \tau\in[\tau_0,+\infty[\,.$$

Therefore, due to Theorem 1.3 (taking into account the points 1.1, 1.4, 1.5) from the work [5], the system of differential equations (12) has a one-parameter family of solutions $(v_1(\tau),v_2(\tau)) : [\tau_1,+\infty[\,\to\mathbb{R}^2$ $(\tau_1\ge\tau_0)$ tending to zero as $\tau\to+\infty$. By the transformation (11) each of them corresponds to a solution $y : [t_1,\omega[\,\to\mathbb{R}$ $(\tau_1 = \beta\ln|\pi_\omega(t_1)|)$ that admits the asymptotic representations

$$\ln|y(t)| = [1+o(1)]\ln|\pi_\omega(t)|, \quad \frac{y'(t)}{y(t)} = \frac{1}{\pi_\omega(t)}\,[1+o(1)] \ \text{ as } \ t\uparrow\omega.$$

The solution y, in view of these asymptotic relations and the second condition (4), as it was shown in the proof of necessity, admits the asymptotic representations (5). $\square$

Proof of Theorem 2. First of all we will show that

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)q'(t)}{q(t)} = 0. \tag{13}$$

Indeed, if this is not the case, then supposing $c(t) = \frac{\pi_\omega(t)q'(t)}{q(t)}$ and noting that there exists a limit of this function as $t \uparrow \omega$, we will obtain the relation

$$q'(t) = \frac{q(t)c(t)}{\pi_\omega(t)}, \quad \text{where} \quad \lim_{t\uparrow\omega} c(t) = \begin{cases} \text{either const} \neq 0, \\ \text{or } \ \pm\infty. \end{cases}$$

Hence, taking into account the second condition (4), we get

$$q(t) - q(a) = \int_a^t \frac{q(\tau)c(\tau)}{\pi_\omega(\tau)}\, d\tau \longrightarrow \infty \ \text{ as } \ t \uparrow \omega.$$

But this can not be true because due to the first condition (4) the left-hand side of this relation has a finite limit as $t \uparrow \omega$.

Since the conditions (4) are satisfied, according to Theorem 1 the equation (1) has a one-parameter family of $P_\omega(\pm\infty)$-solutions, each of them admitting the asymptotic representations (5).

Let $y : [t_y, \omega[\rightarrow \mathbb{R}$ be any of these solutions. Without restriction of generality we can assume that $\ln|y(t)|$ and $y'(t)$ are different from zero as $t \in [t_y, \omega[$. For this solution in view of (1) and (5) we have

$$y''(t) = \alpha_0 p(t)y(t)\big|\ln|\pi_\omega(t)|\big|^\sigma \left|1 + \frac{\alpha_0 Q(t)}{\ln|\pi_\omega(t)|}[1 + o(1)]\right|^\sigma \ \text{ as } \ t \uparrow \omega.$$

Hence, since by l'Hospital's rule

$$\lim_{t\uparrow\omega} \frac{Q(t)}{\ln|\pi_\omega(t)} = \lim_{t\uparrow} \frac{Q'(t)}{(\ln|\pi_\omega(t))'} = \lim_{t\uparrow\omega} q(t) = 0,$$

we get

$$\Big(\frac{y'(t)}{y(t)}\Big)' + \Big(\frac{y'(t)}{y(t)}\Big)^2 = \alpha_0 p(t)\big|\ln|\pi_\omega(t)|\big|^\sigma [1 + \varepsilon(t)] \ \text{ as } \ t \in [t_0, \omega[, \tag{14}$$

where t_0 is some number from the interval $[t_y, \omega[$ and $\varepsilon : [t_0, \omega[\rightarrow \mathbb{R}$ is a continuous function satisfying the condition

$$\lim_{t\uparrow\omega} \varepsilon(t) = 0. \tag{15}$$

Now introduce the function $z : [t_y, \omega[\rightarrow \mathbb{R}$ by

$$\frac{y'(t)}{y(t)} = \frac{1}{\pi_\omega(t)}\big[1 + \alpha_0 q(t)z(t)\big]. \tag{16}$$

Because of (14) this function on the interval $[t_0, \omega[$ is a solution of the differential equation

$$z' = \frac{1}{\pi_\omega(t)} \Big[-\frac{\pi_\omega(t) q'(t)}{q(t)} z - z - \alpha_0 q(t) z^2 + 1 + \varepsilon(t) \Big]. \tag{17}$$

Taking into account (4), (12) and (15), we note that the corresponding to this equation function is

$$B_c(t) = \frac{1}{\pi_\omega(t)} \Big[-\frac{\pi_\omega(t) q'(t)}{q(t)} c - c - \alpha_0 q(t) c^2 + 1 + \varepsilon(t) \Big].$$

For any $c \neq 1$ it preserves sign in some left neighbourhood of ω. Therefore, repeating word for word the proof from Lemma 2.2 in the work [6], we conclude that every solution of the equation (17) is given in a left neighbourhood of ω, so the function $z(t)$ too has a finite or equal to $\pm\infty$ limit as $t \uparrow \omega$. Further, by reason of this fact we see that the following from (16) relation

$$\ln|y(t)| = \ln|\pi_\omega(t)| + \alpha_0 \int\limits_{t_y}^{t} \frac{q(\tau) z(\tau)}{\pi_\omega(\tau)}\, d\tau + C$$

with C be some constant does not contradict the first asymptotic representation (5) only in the case where $\lim\limits_{t\uparrow\omega} z(t) = 1$. Therefore according to (16) the second asymptotic representation (6) has to be fulfilled. The theorem is proved. □

Proof of Theorem 3. Choosing arbitrarily a constant $c \neq 0$, we transform the equation (1) by the transformation

$$\begin{gathered} y(t) = \pi_\omega(t) \exp[\alpha_0 Q(t)]\, [c + v_1(\tau)], \\ y'(t) = \exp[\alpha_0 Q(t)]\, \big[c + v_2(\tau) - \alpha_0 q(t) v_1(\tau)\big], \\ \tau(t) = \beta \ln|\pi_\omega(t)|, \quad \text{where } \beta = \begin{cases} 1, & \text{if } \omega = +\infty, \\ -1, & \text{if } \omega < +\infty, \end{cases} \end{gathered} \tag{18}$$

to the system of differential equations

$$\begin{cases} v_1' = \beta\big[-\alpha_0 c h(\tau) - (1 + 2\alpha_0 h(\tau)) v_1 + v_2 \big], \\ v_2' = \alpha\beta h(\tau) \Big[f(\tau) + b(\tau) v_1 + \dfrac{\beta}{\tau} V(\tau, v_1) \Big], \end{cases} \tag{19}$$

in which

$$h(\tau(t)) = q(t), \quad f(\tau(t)) = -\alpha c q(t) + c\Big|1 + \frac{\alpha_0 Q(t) + \ln|c|}{\ln|\pi_\omega(t)|}\Big|^\sigma - c,$$

$$V(\tau(t), v_1) =$$

$$= (c+v_1) \ln|\pi_\omega(t)| \Big[\Big|1 + \frac{\alpha_0 Q(t) + \ln|c+v_1|}{\ln|\pi_\omega(t)|}\Big|^\sigma - \Big|1 + \frac{\alpha_0 Q(t) + \ln|c|}{\ln|\pi_\omega(t)|}\Big|^\sigma \Big] -$$

$$-\sigma v_1 \Big|1 + \frac{\alpha_0 Q(t) + \ln|c|}{\ln|\pi_\omega(t)|}\Big|^{\sigma-1},$$

$$
\begin{aligned}
b(\tau(t)) = &-1 - \alpha_0 q(t) + \frac{\pi_\omega(t) q'(t)}{q(t)} + \Big|1 + \frac{\alpha_0 Q(t) + \ln|c|}{\ln|\pi_\omega(t)|}\Big|^{\sigma} + \\
&+ \frac{\sigma}{\ln|\pi_\omega(t)|}\Big|1 + \frac{\alpha_0 Q(t) + \ln|c|}{\ln|\pi_\omega(t)|}\Big|^{\sigma-1}.
\end{aligned}
$$

Having chosen, on account of the conditions (4), a number $t_0 \in [a, \omega[$ such that for $t \in [t_0, \omega[$ the inequalities

$$\Big|\frac{\alpha_0 Q(t) + \ln|c|}{\ln|\pi_\omega(t)|}\Big| \le \frac{1}{2}, \quad \Big|\frac{\ln\frac{1}{2}}{\ln|\pi_\omega(t)|}\Big| \le \frac{1}{4}$$

are fulfilled, we will consider the system of differential equations (19) on the set $\Omega = [\tau_0, +\infty[\, \times D$, where

$$\tau_0 = \beta \ln|\pi_\omega(t_0)|, \quad D = \Big\{(v_1, v_2):\ |v_i| \le \frac{|c|}{2}\ \ (i = 1, 2)\Big\}.$$

On this set the right-hand sides of the system (19) are continuous. Besides, due to the conditions (4) and (7) we have

$$\lim_{\tau \to +\infty} h(\tau) = \lim_{t \uparrow \omega} q(t) = 0, \quad \int_{\tau_0}^{+\infty} |h(\tau) f(\tau)|\, d\tau = \int_{t_0}^{\omega} \frac{q(t)|f(\tau(t))|}{|\pi_\omega(t)|}\, dt < +\infty,$$

$$\int_{\tau_0}^{+\infty} |h(\tau) b(\tau)|\, d\tau = \int_{t_0}^{\omega} \frac{q(t)|b(\tau(t))|}{|\pi_\omega(t)|}\, dt < +\infty,$$

$$\int_{\tau_0}^{+\infty} \frac{h(\tau)}{\tau}\, d\tau = \int_{t_0}^{\omega} \frac{q(t)}{\pi_\omega(t)\ln|\pi_\omega(t)|}\, dt < +\infty,$$

and

$$\frac{\partial V(\tau, v_1)}{\partial v_1} \longrightarrow 0 \ \text{ as } \ v_1 \longrightarrow 0 \ \text{ evenly in } \ \tau \in [\tau_0, +\infty[\,.$$

Therefore, according to Theorem 1.3 (including Remarks 1.4 and 1.5) from the paper [5] the system of differential equations (19) has at least one solution $(v_1, v_2) : [\tau_1, +\infty[\, \to \mathbb{R}$ $(\tau_1 \ge \tau_0)$ tending to zero as $\tau \to +\infty$. In view of the transformation (18) this solution corresponds to a solution of differential equation (1) assuming the asymptotic representations (8). The theorem is proved. □

When $\sigma = 0$, the equation (1) is a linear differential equation of the type

$$y'' = \alpha_0 p(t) y. \tag{20}$$

In the case where the function $p : [a, \omega[\, \to]0, +\infty[$ is continuously differentiable and $\lim_{t \uparrow \omega} p'(t) p^{-\frac{3}{2}}(t)$ is finite or equal to $\pm\infty$, it is not simple to show that every nonoscillatory solution y of the equation (20) different from the solutions admitting one of the asymptotic representations $y(t) \sim c$ or $y(t) \sim c\pi_\omega(t)$ $(c \ne 0)$ as $t \uparrow \omega$ is certainly a $P_\omega(\lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$.

From Theorems 1–3 there follow the next conclusions for the equation (20).

Conclusion 1. For existence of $P_\omega(\pm\infty)$-solutions of the equation (20) it is necessary and sufficient that the conditions

$$\lim_{t\uparrow\omega} p(t)\pi_\omega^2(t) = 0, \quad \int\limits_a^\omega p(\tau)|\pi_\omega(\tau)|\,d\tau < +\infty \tag{21}$$

be fulfilled. Moreover, under these conditions there exists a one-parameter family of $P_\omega(\pm\infty)$- solutions and each of them assumes the following asymptotic representations

$$\ln|y(t)| = \ln|\pi_\omega(t)| + \alpha_0 \int\limits_a^t p(\tau)\pi_\omega(\tau)\,d\tau[1+o(1)],$$

$$\ln|y'(t)| = \alpha_0 \int\limits_a^t p(\tau)\pi_\omega(\tau)\,d\tau[1+o(1)] \ \text{ as } \ t\uparrow\omega.$$

Conclusion 2. If the function $p : [a,\omega[\,\to\,]0,+\infty[$ is continuously differentiable, the conditions (21) hold and there exists (finite or equal to $\pm\infty$) $\lim\limits_{t\uparrow\omega} \frac{(p(t)\pi_\omega^2(t))'}{p(t)\pi_\omega(t)}$, then for every $P_\omega(\pm\infty)$-solution of the equation (20) the asymptotic representations

$$\ln|y(t)| = \ln|\pi_\omega(t)| + \alpha_0 \int\limits_a^t p(\tau)\pi_\omega(\tau)\,d\tau[1+o(1)],$$

$$\frac{y'(t)}{y(t)} = \frac{1}{\pi_\omega(t)}\left[1 + \alpha_0 p(t)\pi_\omega^2(t)[1+o(1)]\right] \ \text{ as } \ t\uparrow\omega$$

are valid.

Conclusion 3. Let the function $p : [a,\omega[\,\to\,]0,+\infty[$ be continuously differentiable and along with (21) the conditions

$$\int\limits_a^\omega \left|(p(t)\pi_\omega^2(t))'\right| dt < +\infty, \quad \int\limits_a^\omega p^2(t)|\pi_\omega(t)|^3\,dt < +\infty,$$

$$\int\limits_a^\omega \frac{p(t)\pi_\omega(t)}{\ln|\pi_\omega(t)|} \left|\int\limits_a^t p(\tau)\pi_\omega(\tau)\,d\tau\right| dt < +\infty$$

be satisfied. Then for any $c \neq 0$ there exists a $P_\omega(\pm\infty)$-solution of the equation (20) assuming the asymptotic representations

$$y(t) = \pi_\omega(t)\exp\left[\alpha_0 \int\limits_a^t p(\tau)\pi_\omega(\tau)\,d\tau\right][c+o(1)],$$

$$y'(t) = \exp\left[\alpha_0 \int_a^t p(\tau)\pi_\omega(\tau)\,d\tau\right][c + o(1)] \ \text{ as } \ t \uparrow \omega.$$

These conclusions complete the results from the monograph [7] (Ch. 1, § 6).

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